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DERIVATIONS OF OPERATOR ALGEBRAS, AUTOMATIC CLOSABILITY

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Abstract. In this note we review briefly the subject of bounded and unbounded derivations of operator algebras (C*-and W*-algebras). In particular, we mention some results on closability and continuity of such derivations. This is an invitation to work on three problems in this subject.

1. Introduction. A C*-algebra is a Banach *-algebra A such that $\|xx^\star\| = \|x\|^2$, $x \in A$. If A is a C*-algebra which has a predual Banach space $A_\star$, then A is called a W*-*algebra*. Let A be a C*-algebra and let δ be a linear operator from a dense *-subalgebra $D(\delta)$ of A into A such that $\delta(xy) = \delta(x)y+x\delta(y)$ $(x, y \in D(\delta))$. Then δ is called a *derivation* of A. If moreover $\delta(x^\star) = \delta(x)^\star$ $(x \in D(\delta))$ then δ is called a *-*derivation*. A derivation δ is called *inner* if there exists an element h of A such that $\delta(x) = hx - xh$ $(x \in D(\delta))$. A derivation δ is called *bounded* if there is a positive real number M such that $\|\delta(x)\| \le M\|x\|$ $(x \in D(\delta))$. It is easy to see that every inner derivation is bounded. Let A be a Banach space and $D(\delta)$ be a subspace of A. A linear operator $\delta : D(\delta) \to A$ is *closable* if $\sigma(\delta) = \{0\}$, where $\sigma(\delta)$ is the *separating* space of δ, i.e. $\sigma(\delta) = \{b \in A :$ there is a sequence $\{a_n\}$ in $D(\delta)$ such that $a_n \to 0,\ \delta(a_n) \to b\}$. For a closable operator δ, the following extension $\overline{\delta}$, called the *closure* of δ, is well defined: $D(\overline{\delta})$ is the set of all $x \in A$ such that there exists a sequence $\{x_n\}$, $x_n \in D(\delta)$, and $y \in A$ satisfying $\lim x_n = x$, $\lim \delta(x_n) = y$, and then we take $\overline{\delta}(x) = y$. The operator δ is said to be *closed* if $\delta = \overline{\delta}$, i.e. if the graph of δ is closed. Let $D(\delta) = C^1([a, b])$, $A = C([a, b])$ and $\delta(f) = f'$. Then δ is a densely defined unbounded closed derivation of $C([a, b])$.

2. Continuity of derivations. The subject of continuous derivations was started in 1953 with Kaplansky [12]. Kaplansky showed that each *-derivation

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of a type I von Neumann algebra is inner. Sakai [21] proved in 1960 that an everywhere defined derivation of a C*-algebra is continuous. Sakai and Kadison ([22], [11]) showed in 1966 that each continuous derivation of a von Neumann algebra is inner. This subject was essentially finished in 1977 with work of Elliott, Akemann and Pedersen [9], [1]. S. Ota proved that if δ is a closed *-derivation on a C*-algebra A with identity acting on a Hilbert space H and if its domain $D(\delta)$ is closed under the square-root operation of positive elements, then δ is continuous. We showed [19] that the closedness condition in Ota's theorem is essential and we also proved the following result using the Gelfand theorem for commutative C*-algebras.

Theorem 2.1. *Let A be a commutative $C^\star$-algebra with identity* 1 *and δ a derivation in A. If its domain $D(\delta)$ is closed under the square-root operation of positive elements and if* $1 \in D(\delta)$, *then* $\delta \equiv 0$.

Corollary 2.2. *An everywhere defined derivation on a commutative $C^\star$-algebra A is identically zero.*

Problem I. For which Banach algebras A is every derivation $\delta : A \to A$ automatically continuous?

Some partial results can be found in [25] and [16]. We would like to mention that each derivation on a semisimple Banach algebra is continuous [10].

3. Closability of derivations. The subject of unbounded derivations was started by Sakai, Powers, Helemskii, Sinai, and Robinson around 1974, and had its maximal growth the following four years with contributions by Batty, Bratteli, Chi, Goodman, Jorgensen, Kishimoto, Lance, Longo, McGovern, McIntosh, Niknam, Ota, Takenouchi in addition to the five founders (see [7], [24], [23]). This theory is concerned with the theory of closability, generator properties and classifications of closed derivations. Some new results on the closed derivations of $C([0,1])$ can be found in Kurose [14].

Problem II. Suppose that δ_0 is a densely defined closed *-derivation of a C*-algebra A and δ is a *-derivation of A with $D(\delta) \supseteq D(\delta_0)$. Is δ closable?

Sakai conjectured that the answer to problem II is *yes* [24].

Let δ and δ_0 be derivations of a C*-algebra A defined on the same domain, say D. Then δ is called δ_0-*bounded* if there is a number $M > 0$ such that $\|\delta(a)\| \leq M(\|a\| + \|\delta_0(a)\|)$ $(a \in D)$. It follows from [15] that if δ_0 is a closed *-derivation and δ is a *-derivation on A with $D(\delta) \supseteq D(\delta_0)$, then δ is δ_0-bounded. An easy argument shows the following

Theorem 3.1 [13]. *Let δ_0 and δ be *-derivations of a $C^\star$-algebra A such that δ_0 is closed and $D(\delta) = D(\delta_0)$. Then δ is closed if and only if δ_0 is δ-bounded.*

It is easy to see that δ_0–δ is δ_0-bounded. Hence there are two positive numbers N, M such that $\|(\delta_0 - \delta)(a)\| \leq N\|a\| + M\|\delta_0(a)\|$. If $M < 1$, then δ is closed.

S. Sakai [23] has asked: when is the range of a closed $\star$-derivation of a simple C*-algebra not dense? A result of ours implies the answer to a converse question:

THEOREM 3.2. *Let A be a simple $C^\star$-algebra and δ be a derivation of A. Then δ is closable if the range of δ is not dense in A.*

COROLLARY 3.3. *Let δ be a $\star$-derivation of a simple $C^\star$-algebra A. Then δ is closable if one of the two sets $\{a \pm \delta(a) : a \in D(\delta)\}$ is not dense in A.*

We can extend these results to more general simple Banach algebras (see [17]).

THEOREM 3.4. *Suppose that δ is a densely defined derivation of a simple Banach algebra A. Then either δ is closable or both of the sets $\{a \pm \delta(a) : a \in D(\delta)\}$ are dense A.*

COROLLARY 3.5. *Suppose that δ is a densely defined derivation of a simple Banach algebra A such that the set $\{a + \delta(a) : a \in D(\delta)\}$ is closed. Then either δ is closable or the map $a \to a + \delta(a)$ is onto.*

It follows that if δ is not closable then the differential equation $\delta(x) + x = b$ has a solution in A.

The following result may be helpful in working on both problems I and II (see [17]).

THEOREM 3.6. *Let $(A, \|\cdot\|)$ be a simple unital Banach algebra and δ be a densely defined derivation of A. Suppose that $(D(\delta), |\cdot|)$ is Banach algebra for some norm $|\cdot|$ defined on the domain of δ such that $\delta : (D(\delta), |\cdot|) \to (A, \|\cdot\|)$ is continuous. If the deficiency indices of δ are finite and not equal then $\delta : D(\delta) \to A$ is closable.*

An immediate consequence of Theorem 3.6 is the following:

COROLLARY 3.7. *Let δ, δ_0 be densely defined $\star$-derivations of a simple $C^\star$-algebra A such that δ_0 is closed and $D(\delta) = D(\delta_0)$. If one of the deficiency indices of δ is finite and non-zero, then δ is closable.*

Using the subharmonic methods of Aupetit ([2], [3]) we established in [17] a generalization of the Aupetit result and obtained some results on closability of derivations of semisimple Banach algebras.

THEOREM 3.8. *Let A, B be unital Banach algebras and let T be a linear mapping from a linear subspace $D(T)$ of A into B such that $SP(T(x)) \subseteq SP(x)$ for every element x in $D(T)$. If the range of T is of finite or countable codimension then either $\sigma(T)$ is not a subalgebra of B or $\sigma(T)$ consists of quasi-nilpotent elements.*

COROLLARY 3.9. *Let δ be a densely defined derivation from a unital semisimple Banach algebra A into A. Then each of the following conditions implies the closability of the derivation*:

(i) *The range of δ is closed and $\varrho(\delta(a)) \le \varrho(a)$ $(a \in D(\delta))$.*

(ii) *The range of* δ *is of finite or countable codimension and* $SP(\delta(a)) \subseteq SP(a)$ $(a \in D(\delta))$.

(iii) *The set* $\{a - \delta(A) : a \in D(\delta)\}$ *is of finite or countable codimension and* $SP(a - \delta(a)) \subseteq SP(a)$ $(a \in D(\delta))$

The following result is due to Chi and can be found in [8].

THEOREM 3.10. *Let* A *be a* $C^\star$*-algebra and let* $\delta : D(\delta) \to A$ *be a* **-derivation such that there exists a family* S *of states of* A *with the properties that* $w \circ \delta \in A^\star$ *for all* $w \in S$ *and* $\bigoplus_{w \in S} \pi_w$ *is faithful where* π_w *denotes the cyclic representation associated with* w. *Then* δ *is closable.*

4. Classification of all closable derivations on a given C*-algebra

PROBLEM III. Classify all closable derivations on a given C*-algebra.

We now mention some results on the classification of all closable derivations. First we give an example of a non-closable derivation.

EXAMPLE 4.1. Let X be the Cantor set in $[0,1]$ and let $C(X)$ be the set of all complex valued continuous functions on X. Then $C(X)$ is an abelian C*-algebra. Let δ be the first order differential operator defined in the obvious way on

$$D(\delta) = \left\{ f \in C(X) : \delta(f)(x) = \lim_{\substack{h \to 0 \\ x+h \in X}} \frac{f(x+h) - f(x)}{h} \in C(X) \text{ exists} \right\}$$

Then δ is a non-zero derivation, since the function $f(x) = x$ is contained in $D(\delta)$ and $\delta(f) = 1$. One can see that δ is not closable (see p. 10 of [7]).

This example may be extended to the non-commutative case [5].

We denote the set of all compact operators on the Hilbert space H by $K(H)$.

THEOREM 4.2. *Let* δ *be a derivation of* $K(H)$. *Then the following conditions are equivalent*:

(i) δ *is a closable derivation.*

(ii) *There exists a skew symmetric operator* h, *i.e.* $h \subseteq -h^\star$, *such that* $\delta \subseteq \mathrm{ad}(h)$ (see [7]).

THEOREM 4.3. *If* X *is a compact Hausdorff space with a dense*, *open*, *totally disconnected subset*, *then* $C(X)$ *does not admit non-zero closed derivations* (see [4], [5]).

We fix $I = [0,1]$ and state the following result which implies a complete classification of all closed derivations on $C(I)$ due to Kurose (see [13], [26], [14]).

THEOREM 4.4. *The following conditions are equivalent*:

(i) δ *is a closed derivation of* $C(I)$ *with the range* $R(\delta) = C(I)$ *and the kernel* $\delta = \mathbb{C} \cdot 1$.

(ii) *There exists a non-atomic signed measure* μ *on* I *with* $D(\delta) = \{g \in C(I)$: $\exists f \in C(I)$, $\exists \lambda \in \mathbb{C}$, *such that* $g(x) = \lambda + \int_0^x f\, d\mu\}$, *and* $\delta(g) = f$.

Whenever we have the derivation δ, the measure μ satisfying the condition in Theorem 4.4 is uniquely determined. There is a dense subset $U \subseteq I$ such that

$$\delta(g)(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{u(x+h)-u(x)} \quad (x \in U, g \in D(\delta)),$$

where $\mu(x) = \int_0^x du$, so we put $\delta = d/du$. More generally, suppose (A, B, μ) is a triple such that A is an open subset of I, B is a dense open subset of A and μ is a continuous function on B which has a finite total variation on each closed subinterval $E \subseteq B$. Define the derivation $\delta_\mu|_E$ on $C(E)$ as in Theorem 4.4. If f is an element of $C(I)$ and $f|_E$ belongs to $D(\delta_\mu|_E)$ for all such E, then define

$$\frac{df}{du}(x) = \begin{cases} \delta_\mu|_E(f)(x) & (x \in E \subseteq B), \\ 0 & (x \notin E). \end{cases}$$

This definition is independent of the choice of E. Define $D(\delta(A, B, u)) = \{f \in C(I) : df/du$ exists and has a continuous extension to $I\}$.

Then $\delta(A, B, u)(f)$ is this continuous extension of df/du and we have

THEOREM 4.5. *$\delta(A, B, u)$ is a closed derivation. Conversely, if δ is a closed derivation, then there exists a triple (A, B, μ) such that $\delta \subseteq \delta(A, B, \mu)$.*

Although Theorem 4.5 gives a complete classification of closed derivations of $C(I)$, some aspects of the structure of such derivations are not obvious from the theorem, as the following surprising result of Tomiyama shows:

THEOREM 4.6. *Every non-zero closed derivation of $C(I)$ has a proper closed extension.*

We now come to the end of this trip by the following result due to Nishio [20], and would like to mention that the classification of closed derivations of $C(I \times I)$ is an open problem yet.

THEOREM 4.7. *Let δ be a closed derivation on $C(I \times I)$ with range $R(\delta) = C(I \times I)$. Then for any open subset $V \subseteq I \times I$, there exists a non-empty open connected subset $U \subseteq V$ such that* $\operatorname{Ker} \delta_{\bar{U}}$ *contains a non-constant function in $D(\delta_{\bar{U}})$, where*

$$D(\delta_{\bar{U}} = \{f|_{\bar{U}} : f \in D(\delta)\}, \quad \delta_{\bar{U}}(f|_{\bar{U}}) = \delta(f)|_{\bar{U}}.$$

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