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Polar wavelets and associated Littlewood–Paley theory

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CONTENTS

Abstract

We develop an almost orthogonal wavelet-type expansion in $\mathbb{R}^2$ which is adapted to polar coordinates. We start by defining a product Fourier–Hankel transform $\hat{f}$ and proving a sampling formula for f such that $\hat{f}$ is compactly supported. For general f, the sampling formula and a partition of unity lead to an identity of the form $f = \sum_{\mu,k,m} \langle f, \varphi_{\mu km} \rangle \psi_{\mu km}$, in which each function $\varphi_{\mu km}$ and $\psi_{\mu km}$ is concentrated near a certain annular sector, has compactly supported product Fourier–Hankel transform, and is smooth away from the origin.

We introduce polar function spaces $A_p^{\alpha q}$, analogous to the usual Littlewood–Paley spaces. We show that $A_p^{02} \approx L^p$, $1 < p < \infty$. We prove that $f \in A_p^{\alpha q}$ if and only if a certain size condition on the coefficients $\{\langle f, \varphi_{\mu km} \rangle\}_{\mu,k,m}$ holds. A certain class of almost diagonal operators is shown to be bounded on $A_p^{\alpha q}$, which yields a product Fourier–Hankel transform multiplier theorem. Using this, we identify a polar potential operator P^α which maps $A_p^{\beta q}$ isomorphically onto $A_p^{\alpha+\beta,q}$.

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1. Introduction and main results

In the last decade or so, there has been a rapid development of wavelet theory, as both a theoretical and practical method in areas as diverse as harmonic analysis, signal and image processing, numerical partial differential equations, turbulence theory, constructive quantum field theory, fluid mechanics, statistics, and fractal geometry. A few of the many references are [D2], [M2], [R], [Ch], and [BF]. In standard wavelet theory, a function f on $\mathbb{R}^n$ is written as a linear combination of representing functions of the form $\psi_{\mu k}(x) = 2^{\mu n/2}\psi(2^\mu x - k)$, where the scale parameter μ runs through $\mathbb{Z}$, and k, the location parameter, runs through $\mathbb{Z}^n$. The coefficient of $\psi_{\mu k}$ is $\langle f, \varphi_{\mu k}\rangle$, where the analyzing functions $\{\varphi_{\mu k}\}$ are obtained from a single φ by translation and dilation as for $\psi_{\mu k}$. We single out three critical features of the wavelet expansion. First, $\psi_{\mu k}$ and $\varphi_{\mu k}$ are spatially localized, decaying rapidly, on a scale of $2^{-\mu}$, away from the point $2^{-\mu}k$. Second, the Fourier transforms of $\psi_{\mu k}$ and $\varphi_{\mu k}$ are localized near the band where $|\xi| \approx 2^\mu$. Third, by methods based on Littlewood–Paley–Stein theory, f belongs to L^p, $1 < p < \infty$, if and only if the magnitudes of the wavelet coefficients satisfy a certain size condition (and similarly for other spaces, e.g. Hardy and Sobolev spaces). See [M2] or [FJW] for precise statements. Note that the standard wavelet decomposition is based on the dyadic grid, although there are variations based on more general lattices (see e.g. [Str]).

Recently there has been interest in wavelets tailored to specific situations. One example is the construction of wavelets on closed sets by Meyer ([M3]). Another is Battle and Federbush's construction ([BaF]) of divergence-free vector wavelets in $\mathbb{R}^3$. Federbush uses this ([F]) in finding solutions to Navier–Stokes equations.

In [EF], we consider the decomposition of radial functions or tempered distributions on $\mathbb{R}^3$ in terms of radial wavelets. This decomposition retains the three properties above of spatial localization (near annuli this time), frequency localization, and the existence of explicit function space norm characterizations. To obtain this decomposition, we were not able to mimic Meyer's original construction of orthonormal wavelets [M1] or its subsequent generalizations ([Ma], [D1]), since these all seem to rely on an underlying lattice structure. Instead we followed the slightly earlier approach related to the φ-transform [FJ1–3], which used sampling theory to obtain an almost orthogonal expansion of the same form as the wavelet expansion.

In this paper we present a polar coordinates wavelet decomposition in two dimensions which is an extension of the radial wavelet transform, and which is suitable for problems that are best treated in polar coordinates. In particular our polar wavelet decomposition satisfies appropriate analogues of the three basic properties noted above. Further, we develop a polar Littlewood–Paley theory which supplies the right context for certain precise relationships we obtain relating the properties of f to the size of its polar wavelet coefficients.

We follow the sampling theory approach noted above, except that the role usually played by the Fourier transform is now taken by what we call the *product Fourier–Hankel transform* (abbreviated FHT). For $f : \mathbb{R}^2 \to \mathbb{C}$, $\varrho \in [0,\infty)$, and $n \in \mathbb{Z}$, we define the FHT $\widehat{f}$ formally by

$$\widehat{f}(\varrho, n) = \int_0^\infty \int_0^{2\pi} f(se^{i\theta}) J_0(s\varrho) e^{-in\theta} \frac{d\theta}{2\pi} s\, ds, \tag{1.1}$$

where J_0 is the Bessel function of order 0 (see §2 for the definition). Note we write $se^{i\theta}$ for points in $\mathbb{R}^2 \approx \mathbb{C}$. If $f \in L^2(\mathbb{R}^2)$, then $\widehat{f}(\varrho, n) \in L^2([0,\infty) \times \mathbb{Z})$, defined with the natural inner product

$$\langle F, G \rangle = \sum_{n\in\mathbb{Z}} \int_0^\infty F(\varrho, n)\overline{G(\varrho, n)}\varrho\, d\varrho. \tag{1.2}$$

We reserve the notation $^\wedge$ throughout for the FHT; we use $\mathcal{F}$ to denote the Fourier transform $\mathcal{F}f(\xi) = \int_{\mathbb{R}^2} f(x)e^{-ix\cdot\xi}\, dx$. The inverse product Fourier–Hankel transform, denoted $^\vee$, is defined on $\mathbb{R}^2 \setminus \{0\}$ formally for $g : [0,\infty) \times \mathbb{Z} \to \mathbb{C}$ by

$$\check{g}(re^{i\theta}) = \sum_{n\in\mathbb{Z}} \int_0^\infty g(\varrho, n) J_0(\varrho r)\varrho\, d\varrho\, e^{in\theta}. \tag{1.3}$$

If $f \in L^2(\mathbb{R}^2)$ we have $f = (\widehat{f})^\vee$ in the L^2-sense. For this (and the precise definitions), see Lemma 2.3.

Since the FHT is not standard, but plays a central role in this paper, a few comments are in order. The FHT does have a close relative in classical Fourier analysis. Suppose we expand,

$$f(re^{i\theta}) = \sum_{n\in\mathbb{Z}} g_n(r)e^{in\theta} = \sum_{n\in\mathbb{Z}} \int_0^\infty F(\varrho, n) J_n(\varrho r)\varrho\, d\varrho\, e^{in\theta},$$

where J_n is the Bessel function of order n. Let $h_n(re^{i\theta}) = g_n(r)e^{in\theta}$. Then $\mathcal{F}h_n(\varrho\, e^{i\varphi}) = 2\pi(-i)^n e^{in\varphi} F(\varrho, n)$ (cf. [H], p. 81). Because of the simple interpretation of $F(\varrho, n)$ in terms of the Fourier transform on $\mathbb{R}^2$, it is tempting to work with the map $f \mapsto F$ instead of $f \mapsto \widehat{f}$. The problem is that we do not obtain a sampling formula (like our Theorem 3.1) connected with $f \mapsto F$, due to the mixing of the different order Bessel functions. The FHT, however, also has a simple interpretation. For $r > 0$,

$$g_n(r) = \int_0^{2\pi} f(re^{i\theta})e^{-in\theta}\, d\theta/2\pi$$

is the nth Fourier coefficient of the restriction of f to the circle $\{x \in \mathbb{R}^2 : |x| = r\}$. If g_n^{ext} is the radial extension of g_n to $\mathbb{R}^2$, then $\widehat{f}(\varrho, n) = \int_0^\infty g_n(r)J_0(r\varrho)r\, dr$ is the $\mathbb{R}^2$ Fourier transform of g_n^{ext} (see (2.3)). In particular, if f is radial to start with, then $g_0(r) = f(re^{i\theta})$ for any θ and $g_n(r) \equiv 0$ for $n \neq 0$, so $\widehat{f}(\varrho, 0) = \mathcal{F}f(\varrho e^{i\theta})$ for any θ. In general $\widehat{f}(\varrho, n)$ gives a joint analysis of the angular and radial variations of f. Note that the FHT simultaneously diagonalizes the differential operators $\partial/\partial\theta$ and

$$\Delta_r \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$$

(the radial part of the Laplacian). Indeed, for nice enough f, integration by parts shows that

$$\left(\frac{\partial f}{\partial\theta}\right)^{\wedge}(\varrho, n) = in\widehat{f}(\varrho, n)$$

and, since J_0 satisfies $J_0''(t) + \frac{1}{t}J_0'(t) = -J_0(t)$,

$$(\Delta_r f)^{\wedge}(\varrho, n) = -\varrho^2 \widehat{f}(\varrho, n).$$

Most sampling theorems require $\operatorname{supp}\mathcal{F}f$ to be compact. Thus our basic sampling result is perhaps of interest in its own right because it holds under the assumption that the FHT $\widehat{f}$ is compactly supported. For $B > 0$ and $N \in \mathbb{N}$, we say that $f : \mathbb{R}^2 \to \mathbb{C}$ is (B, N)-*bandlimited* if $\operatorname{supp}\widehat{f} \subseteq [0, B] \times \{n \in \mathbb{Z} : -N \le n \le N-1\}$. If $f \in L^2(\mathbb{R}^2)$, so that $\widehat{f} \in L^2([0,\infty) \times \mathbb{Z})$, then bandlimitedness of f implies $\widehat{f} \in L^1([0,\infty)\times\mathbb{Z})$. Then $(\widehat{f})^{\vee}(x)$ is defined and continuous at each $x \in \mathbb{R}^2 \setminus \{0\}$. In this case we identify f with the representative $(\widehat{f})^{\vee}$ of its L^2-equivalence class, so that the sample values of f away from 0 are defined. Let $j_1 < j_2 < \ldots < j_k < \ldots$ be the positive zeroes of J_0, and set $b_k = \sqrt{2}/J_1(j_k)$, where J_1 is the Bessel function of order 1 (see §2).

Theorem 3.1 (Polar sampling formula). *Let $B > 0$ and $N \in \mathbb{N}$. Suppose $f, g \in L^2(\mathbb{R}^2)$, and f and g are (B, N)-bandlimited. Then*

$$(1.4)\qquad (\widehat{f}\widehat{g})^{\vee}(re^{i\theta}) = \frac{1}{2NB^2}\sum_{k=1}^{\infty}\sum_{m=-N}^{N-1} b_k^2 f(j_k e^{i\pi m/N}/B) \sum_{n=-N}^{N-1}\int_0^B \widehat{g}(\varrho, n) J_0(j_k\varrho/B) J_0(r\varrho)\varrho\, d\varrho\, e^{in(\theta-\pi m/N)}.$$

The sum on the right side of (1.5) *converges uniformly on* $\mathbb{R}^2 \setminus \{0\}$ *to the left side. If also* $g \in L^1(\mathbb{R}^2)$, *the convergence is in* L^2 *as well.*

To see this as a sampling theorem, take $\widehat{g}$ identically 1 on $\operatorname{supp}\widehat{f}$, so the left side of (1.4) is just f. We obtain a formula analogous to the classical Shannon sampling theorem, with the sinc functions replaced by functions of the form

$$\sum_{n=-N}^{N-1}\int_0^B J_0(j_k\varrho/B) J_0(r\varrho)\varrho\, d\varrho\, e^{in(\theta-\pi m/N)}.$$

Then (1.4) shows that f is determined by its sample values

$$\{f(j_k\, e^{i\pi m/N}/B)\}_{k\in\mathbb{N}, -N\le m\le N-1}.$$

Note that these points form an approximate polar grid (see (2.7)). This formula should be compared with the result in [H], p. 81.

The φ-transform identity is obtained by applying a partition of unity to the Fourier transform, and using a standard rectangular sampling theorem (see [FJ1] or [FJW]). We do the same thing with the FHT and Theorem 3.1. It turns out to be convenient to construct the FHT partition of unity from the usual one for the Fourier transform. We select radial functions $\varPhi, \varPsi, \varphi, \psi \in \mathcal{S}(\mathbb{R}^2)$ such that

$$(1.5)\quad \begin{aligned} &\text{(i)} && \varphi, \psi, \varPhi, \varPsi \in \mathcal{S}(\mathbb{R}^2),\\ &\text{(ii)} && \varphi, \psi, \varPhi, \varPsi \text{ are radial},\\ &\text{(iii)} && \operatorname{supp} \mathcal{F}\varPhi, \mathcal{F}\varPsi \subseteq \{\xi : |\xi| \le 1\},\\ &\text{(iv)} && \operatorname{supp} \mathcal{F}\varphi, \mathcal{F}\psi \subseteq \{\xi : 1/4 \le |\xi| \le 1\},\\ &\text{(v)} && |(\mathcal{F}\varPhi)(\xi)|, |(\mathcal{F}\varPsi)(\xi)| \ge c > 0 \text{ if } |\xi| \le 5/6,\\ &\text{(vi)} && |(\mathcal{F}\varphi)(\xi)|, |(\mathcal{F}\psi)(\xi)| \ge c > 0 \text{ if } 3/10 \le |\xi| \le 5/6,\\ &\text{(vii)} && \overline{(\mathcal{F}\varPhi)(\xi)}(\mathcal{F}\varPsi)(\xi) + \sum_{\mu=1}^{\infty} \overline{(\mathcal{F}\varphi)(2^{-\mu}\xi)}(\mathcal{F}\psi)(2^{-\mu}\xi) = 1,\ \forall \xi \in \mathbb{R}^2. \end{aligned}$$

(This is possible; see e.g. [FJ3], §12.) Now regard $(\varrho, n) \in [0, \infty) \times \mathbb{Z}$ as a point in the right half-plane of $\mathbb{R}^2$, and define $\{\varphi_\mu\}_{\mu=0}^{\infty}$ and $\{\psi_\mu\}_{\mu=0}^{\infty}$ by

$$(1.6)\qquad \widehat{\varphi}_0(\varrho, n) = (\mathcal{F}\varPhi)(\varrho, n), \qquad \widehat{\psi}_0(\varrho, n) = (\mathcal{F}\varPsi)(\varrho, n),$$

and for $\mu \ge 1$,

$$(1.7)\qquad \widehat{\varphi}_\mu(\varrho, n) = (\mathcal{F}\varphi)(2^{-\mu}\varrho, 2^{-\mu}n),$$

$$(1.8)\qquad \widehat{\psi}_\mu(\varrho, n) = (\mathcal{F}\psi)(2^{-\mu}\varrho, 2^{-\mu}n).$$

Since $\widehat{\varphi}_\mu$ and $\widehat{\psi}_\mu$ are bounded with compact support, $\varphi_\mu = (\widehat{\varphi}_\mu)^\vee$ and $\psi_\mu = (\widehat{\psi}_\mu)^\vee$ are defined pointwise on $\mathbb{R}^2 \setminus \{0\}$. Our definitions easily imply that

$$(1.9)\qquad \operatorname{supp} \widehat{\varphi}_0, \operatorname{supp} \widehat{\psi}_0 \subseteq [0, 1] \times \{0\},$$

$$(1.10)\qquad \operatorname{supp} \widehat{\varphi}_\mu, \operatorname{supp} \widehat{\psi}_\mu \subseteq \{(\varrho, n) \in [0, \infty) \times \mathbb{Z} : 2^{\mu-2} \le (\varrho^2 + n^2)^{1/2} \le 2^\mu\},$$

for $\mu \ge 1$, and

$$(1.11)\qquad \sum_{\mu=0}^{\infty} \overline{\widehat{\varphi}_\mu(\varrho, n)} \widehat{\psi}_\mu(\varrho, n) = 1 \quad \text{for all } (\varrho, n) \in [0, \infty) \times \mathbb{Z}.$$

By (1.11) we have, formally,

$$(1.12)\qquad f = \sum_{\mu=0}^{\infty} (\overline{\widehat{\varphi}}_\mu \widehat{\psi}_\mu \widehat{f})^\vee.$$

This is our analogue of the Calderón formula. We will see (Lemma 2.10) that for $f \in L^p$, $1 < p < \infty$, (1.12) holds a.e. and in the sense of L^p convergence.

Let $A_\mu = \{n \in \mathbb{Z} : -2^\mu \le n \le 2^\mu - 1\}$. For $\mu \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$, $k \in \mathbb{N}$, $m \in A_\mu$, and $re^{i\theta} \in \mathbb{R}^2 \setminus \{0\}$, define

$$(1.13)\qquad \varphi_{\mu k m}(re^{i\theta}) = \frac{b_k 2^{-3\mu/2}}{2\sqrt{\pi}} \sum_{n \in \mathbb{Z}} \int_0^\infty \widehat{\varphi}_\mu(\varrho, n) J_0(r\varrho) J_0(j_k 2^{-\mu}\varrho)\varrho \, d\varrho \, e^{in(\theta - \pi 2^{-\mu} m)}$$

and similarly

$$(1.14)\qquad \psi_{\mu k m}(re^{i\theta}) = \frac{b_k 2^{-3\mu/2}}{2\sqrt{\pi}} \sum_{n \in \mathbb{Z}} \int_0^\infty \widehat{\psi}_\mu(\varrho, n) J_0(r\varrho) J_0(j_k 2^{-\mu}\varrho)\varrho \, d\varrho \, e^{in(\theta - \pi 2^{-\mu} m)}.$$

Note that the sums and integrals in (1.13)–(14) are actually finite, by (1.9)–(1.10). Define $\varphi_{\mu km}$ and $\psi_{\mu km}$ to be 0 at the origin, since they cannot be made continuous there. We will see (after Lemma 2.5) that for $r > 0$ and each $M > 0$, there exists $c_M > 0$ such that

$$(1.15) \quad |\varphi_{\mu km}(re^{i\theta})|, |\psi_{\mu km}(re^{i\theta})| \le c_M b_k 2^{3\mu/2}(1 + 2^\mu|e^{i\theta} - e^{i\pi 2^{-\mu}m}|)^{-M}(1 + 2^\mu|r - 2^{-\mu}j_k|)^{-M}/\max(2^\mu r, j_k).$$

It follows that

$$(1.16) \qquad \|\varphi_{\mu km}\|_{L^2}, \|\psi_{\mu km}\|_{L^2} \le c,$$

with c independent of μ, k, and m.

We obtain our basic identity by applying Theorem 3.1 to each term in (1.12). For convenience, we state an L^2 version, although later this will be generalized.

THEOREM 3.2 (Polar wavelet identity). *Suppose $f \in L^2(\mathbb{R}^2)$. Then*

$$(1.17) \qquad f = \sum_{\mu=0}^{\infty}\sum_{k=1}^{\infty}\sum_{m\in A_\mu}\langle f, \varphi_{\mu km}\rangle\psi_{\mu km},$$

where the sum over k converges uniformly on $\mathbb{R}^2 \setminus \{0\}$ and in L^2 for each μ, and the sum over μ converges a.e. and in L^2. If also $f \in L^p$, $1 < p < \infty$, the sum converges in L^p.

For $\mu \in \mathbb{N}_0$, $k \in \mathbb{N}$, and $m \in A_\mu$, define the *annular sector*

$$(1.18) \quad R_{\mu km} = \{re^{i\theta} \in \mathbb{R}^2 : 2^{-\mu}j_{k-1} < r < 2^{-\mu}j_k \text{ and } 2^{-\mu}\pi m < \theta < 2^{-\mu}\pi(m+1)\},$$

where we use the nonstandard convention that $j_0 = 0$ (so $R_{\mu 1m}$ is a fact a circular wedge rather than an annular sector). Then (1.15) shows that $\varphi_{\mu km}$ and $\psi_{\mu km}$ are concentrated near $R_{\mu km}$, decaying at a scale of $2^{-\mu}$ away from $R_{\mu km}$ in both the angular and radial directions. In fact they are C^∞ away from the origin and satisfy

$$(1.19) \quad \left|\frac{\partial^l}{\partial r^l}\frac{\partial^n}{\partial\theta^n}\varphi_{\mu km}(re^{i\theta})\right|, \left|\frac{\partial^l}{\partial r^l}\frac{\partial^n}{\partial\theta^n}\psi_{\mu km}(re^{i\theta})\right| \le c_M 2^{\mu(l+n)} b_k j_k^{-1} 2^{3\mu/2}(1 + 2^\mu|e^{i\theta} - e^{i\pi 2^{-\mu}m}|)^{-M}(1 + 2^\mu|r - 2^{-\mu}j_k|)^{-M}$$

for $r > 0$ and each $l \ge 0$, $n \ge 0$, and $M > 0$. This is the basic spatial localization property of the identity (1.17) that we regard as natural for polar analysis.

Observe that if we set $g_{\mu km}(\varrho, n) = J_0(j_k 2^{-\mu}\varrho)e^{-i\pi 2^{-\mu}mn}$, then

$$\varphi_{\mu km} = b_k 2^{-3\mu/2}(2\sqrt{\pi})^{-1}(\widehat{\varphi}_\mu g_{\mu km})^\vee \quad \text{and} \quad \psi_{\mu km} = b_k 2^{-3\mu/2}(2\sqrt{\pi})^{-1}(\widehat{\psi}_\mu g_{\mu km})^\vee.$$

Hence $\operatorname{supp}\widehat{\varphi}_{\mu km} \subseteq \operatorname{supp}\widehat{\varphi}_\mu$ and $\operatorname{supp}\widehat{\psi}_{\mu km} \subseteq \operatorname{supp}\widehat{\psi}_\mu$; i.e. by (1.9)–(1.10),

$$\operatorname{supp}\widehat{\varphi}_{0km}, \operatorname{supp}\widehat{\psi}_{0km} \subseteq [0,1] \times \{0\},$$

and

$$\operatorname{supp}\widehat{\varphi}_{\mu km}, \operatorname{supp}\widehat{\psi}_{\mu km} \subseteq \{(\varrho, n) \in [0,\infty) \times \mathbb{Z} : 2^{\mu-2} \le |(\varrho, n)| \le 2^\mu\}.$$

This is our polar analogue of the frequency localization property for standard wavelets. Another property of interest is that

$$(1.20) \qquad \int_{\mathbb{R}^2}\varphi_{\mu km} = 0 = \int_{\mathbb{R}^2}\psi_{\mu km} \quad \text{for all } \mu \ge 1,\ k \in \mathbb{N},\ \text{and } m \in A_\mu.$$

Now we turn to the key issue of how the function space properties of f are reflected in the size of its coefficients $\{\langle f, \varphi_{\mu km}\rangle\}_{\mu,k,m}$ in (1.17). In the standard φ-transform or wavelet case, results for many classical spaces (e.g. L^p, $1<p<\infty$, Hardy, and Sobolov spaces) can be generalized and treated systematically by considering the Triebel–Lizorkin spaces $F_p^{\alpha q}$ (see [T1] and [FJ3]). These are defined via the norm

$$\|f\|_{F_p^{\alpha q}(\mathbb{R}^n)} = \|\varPhi * f\|_{L^p} + \Big\|\Big(\sum_{\mu=1}^{\infty}(2^{\mu\alpha}|\varphi_\mu^* * f|)^q\Big)^{1/q}\Big\|_{L^p},$$

where $\varphi_\mu^*(x) = 2^{\mu n}\varphi(2^\mu x)$, with φ and $\varPhi$ as above. For example, $F_p^{02}\approx L^p$, $1<p<\infty$, by Littlewood–Paley methods (see [T1]). For the polar setting, the analogue of the convolution $\varphi_\mu^* * f$ is the FHT multiplier $(\widehat{\varphi}_\mu \widehat{f})^\vee$. For $\alpha\in\mathbb{R}$ and $0<p,\ q<\infty$, we define (initially for $f\in L^2(\mathbb{R}^2)$)

$$\|f\|_{A_p^{\alpha q}} = \Big\|\Big(\sum_{\mu=0}^{\infty}|2^{\mu\alpha}(\widehat{\varphi}_\mu\widehat{f})^\vee|^q\Big)^{1/q}\Big\|_{L^p}.$$

We define $A_p^{\alpha q}$ to be the completion of $\mathcal{A}_p^{\alpha q} = \{f\in L^2 : \|f\|_{A_p^{\alpha q}}<\infty\}$ with respect to $\|\cdot\|_{A_p^{\alpha q}}$. Then $A_p^{\alpha q}$ is a quasi-Banach space in general and a Banach space if $1\le p,\ q<\infty$.

We will show (Theorem 5.5) that $A_p^{\alpha q}$ is independent of the choice of $\{\varphi_\mu\}_{\mu=0}^\infty$, given that $\{\varphi_\mu\}_{\mu=0}^\infty$ is obtained from some φ and $\varPhi$ satisfying the conditions in (1.5). Also, the following shows that the $A_p^{\alpha q}$ scale does intersect the classical spaces.

Theorem 8.8. *For* $1<p<\infty$, $L^p\approx A_p^{02}$.

Here $\approx$ means the spaces are isomorphic and the norms are equivalent.

We define sequence spaces $a_p^{\alpha q}$ associated with $A_p^{\alpha q}$ in analogy to the spaces $f_p^{\alpha q}$ associated to the Triebel spaces $F_p^{\alpha q}$ (see [FJ3]). For a sequence

$$s = \{s_{\mu km}\}_{\mu\in\mathbb{N}_0, k\in\mathbb{N}, m\in A_\mu},$$

define

$$\|s\|_{a_p^{\alpha q}} = \Big\|\Big(\sum_{\mu,k,m}(2^{\mu\alpha}|s_{\mu km}|\widetilde{\chi}_{\mu km})^q\Big)^{1/q}\Big\|_{L^p},$$

where $\widetilde{\chi}_{\mu km} = |R_{\mu km}|^{-1/2}\chi_{R\mu km}$. Clearly $a_p^{\alpha q}$ is a quasi-Banach space (Banach if $1\le p,q<\infty$).

Theorem 6.3. *Suppose* $\alpha\in\mathbb{R}$, $0<p,q<\infty$ *and* $f\in L^2(\mathbb{R}^2)$. *For* $\mu\in\mathbb{N}_0$, $k\in\mathbb{N}$, *and* $m\in A_\mu$, *let* $s_{\mu km} = \langle f,\varphi_{\mu km}\rangle$ *and let* $s=\{s_{\mu km}\}_{\mu,k,m}$. *Then*

$$\|f\|_{A_p^{\alpha q}} \approx \|s\|_{a_p^{\alpha q}}. \tag{1.21}$$

We can then extend this and Theorem 3.2 to all of $A_p^{\alpha q}$.

Theorem 6.4. *Suppose* $\alpha\in\mathbb{R}$, $0<p,q<\infty$ *and* $f\in A_p^{\alpha q}$. *Let* s *be as in Theorem* 6.3. *Then* (1.17) *and* (1.21) *hold. The equality* (1.17) *holds in the sense that if*

$$f_N = \sum_{\mu=0}^{N}\sum_{k=1}^{N}\sum_{m\in A_\mu}\langle f,\varphi_{\mu km}\rangle\psi_{\mu km},$$

then f_N *converges to* f *in the* $A_p^{\alpha q}$ *quasi-norm.*

Having this relationship between $A_p^{\alpha q}$ and the sequence space $a_p^{\alpha q}$ allows us to study linear operators on $A_p^{\alpha q}$ in terms of corresponding matrices, as in the rectangular case of $F_p^{\alpha q}$ and $f_p^{\alpha q}$ (see [FJ3] and [FJ4]). A matrix B acting on $a_p^{\alpha q}$ has the form $B=\{b_{\nu ln;\mu km}\}$, with $\nu,\mu\in\mathbb{N}_0$, $l,k\in\mathbb{N}$, $n\in A_\nu$ and $m\in A_\mu$. For $s=\{s_{\mu km}\}_{\mu,k,m}$, define

$$(Bs)_{\nu ln}=\sum_{\mu,k,m} b_{\nu ln;\mu km}s_{\mu km}$$

and $Bs=\{(Bs)_{\nu ln}\}$, if the series converges absolutely for each ν,l, and n. In [FJ3], there is a condition, called almost diagonality, on the decay of a matrix, that implies its boundedness on $f_p^{\alpha q}$. We have a similar result. For $\alpha\in\mathbb{R}$ and $0<p,\ q<\infty$ fixed, set $J=2/\min(1,p,q)$. Also, for $j\in\mathbb{Z}$, define χ_j on $\mathbb{N}$ by setting $\chi_j\equiv 0$ if $j<0$, while if $j\geq 0$, let $\chi_j(k)=1$ if $1\leq k\leq 2^j$ and $\chi_j(k)=0$ otherwise. For $\varepsilon>0$, define

$$\begin{aligned}(1.22)\qquad \omega_{\nu ln;\mu km}(\varepsilon)&=|R_{\nu ln}|^{1/2}|R_{\mu km}|^{-1/2}2^{(\mu-\nu)\alpha}\min(2^{(\nu-\mu)(J+\varepsilon)},2^{(\mu-\nu)\varepsilon})\\ &\times(1+\min(2^\mu,2^\nu)|e^{i\pi 2^{-\nu}n}-e^{i\pi 2^{-\mu}m}|)^{-1-\varepsilon-J/2}(1+\min(2^\mu,2^\nu)|2^{-\nu}j_l-2^{-\mu}j_k|)^{-J-\varepsilon}\\ &\times(1+(2^{(\nu-\mu)J/2}-1)\chi_{\mu-\nu}(k)).\end{aligned}$$

We say that a matrix $B=\{b_{\nu ln;\mu km}\}$ is *almost diagonal on* $a_p^{\alpha q}$ if there exists some $\varepsilon>0$ and $c>0$ such that

$$|b_{\nu ln;\mu km}|\leq c\ \omega_{\nu ln;\mu km}(\varepsilon)$$

for all ν,l,n,μ,k, and m.

THEOREM 4.2. *Suppose* $\alpha\in\mathbb{R}$, $0<p,\ q<\infty$, *and* B *is an almost diagonal matrix on* $a_p^{\alpha q}$. *Then* B *is bounded on* $a_p^{\alpha q}$.

Associated with a linear operator T on $A_p^{\alpha q}$ (which we assume is L^2-bounded, for simplicity) is the matrix $B=\{b_{\nu ln;\mu km}\}$, where

$$b_{\nu ln;\mu km}=\langle T\psi_{\mu km},\varphi_{\nu ln}\rangle.$$

Theorems 6.3 and 6.4 easily imply that if B is bounded on $a_p^{\alpha q}$, then T is bounded on $A_p^{\alpha q}$. We say that T is *almost diagonal on* $A_p^{\alpha q}$ if the associated matrix is almost diagonal on $a_p^{\alpha q}$. Such an operator T is bounded, by Theorem 4.2. We use this fact to obtain a criterion for boundedness of an FHT multiplier operator. A bounded function $m(\varrho,n)$, $\varrho\in[0,\infty)$, $n\in\mathbb{Z}$, is called an FHT multiplier for $A_p^{\alpha q}$ if the operator T_m defined by $T_mf=(m\widehat{f})^\vee$ satisfies $\|T_mf\|_{A_p^{\alpha q}}\leq c\|f\|_{A_p^{\alpha q}}$ for $f\in L^2$. If so, T_m extends to a bounded operator on all of $A_p^{\alpha q}$.

For $\mu\in\mathbb{N}_0$ and $g:[0,\infty)\times 2^{-\mu}\mathbb{Z}\to\mathbb{C}$, define the difference operator $D_{\mu,\xi}$ by

$$(D_{\mu,\xi}g)(\varrho,\xi)=2^\mu[g(\varrho,\xi)-g(\varrho,\xi-2^{-\mu})].$$

THEOREM 7.2. *Suppose* $\alpha\in\mathbb{R}$, $0<p,q<\infty$, $L\in 2\mathbb{Z}$, $L>J+2$, *and* $K>J/2+1$, *where* $J=2/\min(1,p,q)$. *Suppose* $m:[0,\infty)\times\mathbb{Z}\to\mathbb{C}$ *is given. Define* $m^*:\mathbb{R}^2\times\mathbb{Z}\to\mathbb{C}$ *by* $m^*(x,n)=m(|x|,n)$. *For* $\mu\in\mathbb{N}$, *let* $m^*_\mu(x,\xi)=m^*(2^\mu x,2^\mu\xi)$. *Assume also*

$$(1.23)\qquad \partial_x^\beta m^*(x,n)=\frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}}\frac{\partial^{\beta_2}}{\partial x_2^{\beta_2}}m^*(x,n)\ \textit{exists and is continuous for}\ x\in\mathbb{R}^2,$$

for each $n \in \mathbb{Z}$ *and each multi-index* $\beta = (\beta_1, \beta_2)$ *such that* $|\beta| = \beta_1 + \beta_2 \le L$,

$$\max_{|\beta|\le L} \int_{\{x:|x|\le 1\}} |\partial_x^\beta m^*(x,0)|\,dx < \infty, \tag{1.24}$$

and

$$\sup_{\mu\in\mathbb{N}} \max_{\substack{k=0,1,\dots,K\\ |\beta|\le L}} 2^{-\mu} \int_{\mathbb{R}^2} \sum_{\xi\in 2^{-\mu}\mathbb{Z}} |(D_{\mu,\xi})^k \partial_x^\beta m_\mu^*(x,\xi)| \chi(x,\xi)\,dx < \infty, \tag{1.25}$$

where $\chi(x,\xi) = 1$ *if* $1/4 \le (|x|^2+\xi^2)^{1/2} \le 1$ *and* 0 *otherwise. Then* T_m *is almost diagonal, hence* m *is an FHT multiplier, on* $A_p^{\alpha q}$.

The conditions (1.23)–(1.24) are analogous to the L^1-Mikhlin condition for bounded Fourier multipliers on $F_p^{\alpha q}$ in [FJ3], Example 9.19. We have adapted the procedure there because it is relatively simple and yields a result satisfactory for most purposes. However, in §10 of [FJ3] there are some sharper results in the $F_p^{\alpha q}$ context, including the familiar Hörmander result. Probably these methods can be adapted to the FHT setting to sharpen Theorem 7.2.

We can now describe the meaning of the α index for the spaces $A_p^{\alpha q}$. In the rectangular case, the Bessel potential J_α defined by $(\mathcal{F}J_\alpha f)(\xi) = (1+|\xi|^2)^{-\alpha/2}\mathcal{F}f(\xi)$ is an isomorphism from $F_p^{\beta q}$ to $F_p^{\alpha+\beta,q}$. For the $A_p^{\alpha q}$ spaces, J_α is replaced by a potential operator P^α defined via the FHT by

$$(P^\alpha f)^\wedge(\varrho,n) = (1+\varrho^2+n^2)^{-\alpha/2}\widehat{f}(\varrho,n).$$

We use a boundedness result that follows from Theorem 7.2 to establish the following.

THEOREM 7.3. *Suppose* $\alpha,\beta\in\mathbb{R}$ *and* $0<p,q<\infty$. *Then*

$$P^\alpha : A_p^{\beta q} \to A_p^{\alpha+\beta,q}$$

is a topological isomorphism.

As noted above, if f is radial, then $\widehat{f}(\varrho,n) = 0$ for $n\ne 0$ and $\widehat{f}(\varrho,0) = \mathcal{F}f(\varrho e^{i\theta})$. Hence in this case $P^\alpha f = J_\alpha f$. Then Theorems 7.3 and 8.8 imply that $\|f\|_{A_p^{\alpha 2}} \approx \|f\|_{L_\alpha^p}$ for $\alpha\in\mathbb{R}$ and $1<p<\infty$ if f is radial.

The remainder of the paper is organized as follows. Section 2 contains a number of basic facts and technical estimates that are needed throughout the paper. The basic sampling theorem and polar wavelet identity are proved in §3. In §4 we prove the boundedness of almost diagonal matrices on the sequence spaces $a_p^{\alpha q}$. We obtain an analogue for $A_p^{\alpha q}$ of Peetre's maximal function characterization of $F_p^{\alpha q}$ ([P]) in §5. This result is used to prove the main norm-equivalence result in §6. The FHT multiplier theorem is proved in §7, as is Theorem 7.3 concerning P^α. In §8 we prove that $A_p^{02} \approx L^p$ for $1<p<\infty$. Finally, in §9 we describe some open problems and directions for further research.

2. Preliminaries

We begin with some basic facts about Bessel functions which can be found in many places, e.g., [W], [SW], or summarized in [EF].

Let $d\sigma_t$ denote (unnormalized) Lebesgue measure on the circle $\{z \in \mathbb{R}^2 : |z| = t\}$, i.e., so that $\int d\sigma_t = 2\pi t$. Let J_0 be the Bessel function of order 0, defined for $x \in \mathbb{R}$ by

$$J_0(x) = \frac{1}{\pi} \int_{-1}^{1} (1-t^2)^{-1/2} e^{ixt}\, dt. \tag{2.1}$$

Then for $\xi \in \mathbb{R}^2$,

$$(\mathcal{F}\sigma_t)(\xi) = 2\pi t\, J_0(t|\xi|). \tag{2.2}$$

If $f : \mathbb{R}^2 \to \mathbb{C}$ is radial, i.e. $f(x) = f_0(|x|)$, then

$$(\mathcal{F}f)(\xi) = 2\pi \int_0^\infty f_0(\varrho) J_0(\varrho|\xi|)\varrho\, d\varrho. \tag{2.3}$$

This suggests the definition of the 0th order *Hankel transform* H. For $f_0 : [0,\infty) \to \mathbb{C}$, define $Hf_0 : [0,\infty) \to \mathbb{C}$ (initially on $L^1([0,\infty), rdr)$ and extended to L^2 as below) by

$$Hf_0(r) = \int_0^\infty f_0(\varrho) J_0(\varrho r)\varrho\, d\varrho. \tag{2.4}$$

If we define $f : \mathbb{R}^2 \to \mathbb{C}$ by $f(x) = f_0(|x|)$, then by (2.3) we have, for any $\theta \in \mathbb{R}$,

$$Hf_0(r) = (2\pi)^{-1}(\mathcal{F}f)(re^{i\theta}) = 2\pi(\mathcal{F}^{-1}f)(re^{i\theta}), \tag{2.5}$$

and, hence,

$$H \circ H \text{ is the identity operator.} \tag{2.6}$$

To be precise about this, note that $f_0 \in L^2([0,\infty), \varrho\, d\varrho)$ if and only if $f \in L^2(\mathbb{R}^2)$, and similarly for L^1. So for f_0, Hf_0 is the limit in L^2 as $\varepsilon \to 0$ of $\int_\varepsilon^{1/\varepsilon} f_0(\varrho)J_0(\varrho r)\varrho\, d\varrho$ (which is defined pointwise for each $\varepsilon > 0$) and (2.6) holds on $L^2([0,\infty), \varrho\, d\varrho)$.

Let $j_1 < j_2 < \ldots$ be the positive zeroes of J_0 (usually denoted $j_{0,k}$ instead of j_k). Then

$$j_k = (k - 1/4)\pi + O(1/k). \tag{2.7}$$

For $k \in \mathbb{N}$, let $b_k = \sqrt{2}/J_1(j_k)$, where J_1 is the Bessel function of order 1, and let

$$h_k(x) = b_k J_0(j_k x). \tag{2.8}$$

The collection $\{h_k\}_{k=1}^\infty$ forms a complete orthonormal basis for $L^2([0,1], \varrho\, d\varrho)$. The constants $\{b_k\}_{k=1}^\infty$ satisfy (see [EF])

$$|b_k| \approx j_k^{1/2}, \tag{2.9}$$

where $\approx$ means that there are constants $c_1, c_2 > 0$, independent of k, such that $c_1 \le |b_k|/j_k^{1/2} \le c_2$ for all k.

We also need the following estimate.

LEMMA 2.1. *Suppose $f : \mathbb{R}^2 \to \mathbb{C}$ satisfies $|f(x)| \le (1+|x|)^{-M-1}$ for all $x \in \mathbb{R}^2$, for some $M > 0$. Then for $0 < \varepsilon < M$ there exists $c_M < \infty$ such that*

$$|f * d\sigma_t(x)| \le c_{M,\varepsilon} \min(1,t)(1 + ||x| - t|)^{-M+\varepsilon}$$

for all $x \in \mathbb{R}^2$ and $t > 0$.

Proof. For $t \geq 1$, this is Lemma 2.3 in [EF] (stated there for f radial, but clearly the result follows for f having the stated radial majorant). For $t < 1$, if $|y| = t$ we have $|f(x-y)| \leq c(1+|x|)^{-M-2}$, so

$$|f * d\sigma_t(x)| \leq \int\limits_{\{|y|=t\}} |f(x-y)| d\sigma_t(y) \leq c(1+|x|)^{-M-2} \int d\sigma_t,$$

giving the result. ■

The next lemma gives an estimate that will be used repeatedly. Recall the definition of $D_{\mu,\xi}$ (before the statement in §1 of Theorem 7.2) and let

$$\Delta_\varrho = \frac{\partial^2}{\partial \varrho^2} + \frac{1}{\varrho}\frac{\partial}{\partial \varrho}.$$

LEMMA 2.2. *Suppose* $\mu \in \mathbb{N}_0$, $L, K \in \mathbb{Z}$, $L \geq 1$, *and* $K \geq 0$. *Given* $f : [0,\infty) \times 2^{-\mu}\mathbb{Z} \to \mathbb{C}$, *define* $F : \mathbb{R}^2 \times 2^{-\mu}\mathbb{Z} \to \mathbb{C}$ *by* $F(x,\xi) = f(|x|,\xi)$. *Suppose* $\partial_x^\beta F(x,\xi)$ *exists and is continuous on* $\mathbb{R}^2$, *for each* $\xi \in 2^{-\mu}\mathbb{Z}$, *and*

$$\int\limits_{\mathbb{R}^2} \sum_{\xi \in 2^{-\mu}\mathbb{Z}} |\partial_x^\beta F(x,\xi)|\, dx < \infty, \tag{2.10}$$

for all $\beta = (\beta_1, \beta_2)$ *such that* $\beta_1 + \beta_2 \leq 2L$. *Define*

$$B_\mu = \max_{\substack{l=0,L \\ k=0,K}} 2^{-\mu} \int\limits_0^\infty \sum_{\xi \in 2^{-\mu}\mathbb{Z}} |(D_{\mu,\xi})^{(k)} (\Delta_\varrho)^{(l)} f(\varrho,\xi)| \varrho\, d\varrho.$$

Then there exists c *independent of* μ *such that*

$$\Big| \sum_{\xi \in 2^{-\mu}\mathbb{Z}} \int\limits_0^\infty f(\varrho,\xi) e^{i2^\mu \xi\theta} J_0(2^\mu r\varrho) J_0(2^\mu \varrho s) \varrho\, d\varrho \Big| \tag{2.11}$$
$$\leq c 2^\mu B_\mu (1 + 2^\mu |e^{i\theta} - 1|)^{-K} (1 + 2^\mu |r-s|)^{-2L+2} / \max(1, 2^\mu r, 2^\mu s)$$

for each $r, s > 0$ *and* $\theta \in \mathbb{R}$.

Proof. Let $g_\theta(x) = \sum_{\xi \in 2^{-\mu}\mathbb{Z}} F(x,\xi) e^{i2^\mu \xi\theta}$ for $\theta \in \mathbb{R}$. Then g_θ is a radial L^1 function on $\mathbb{R}^2$. By (2.2),

$$g_\theta(x) J_0(2^\mu r|x|) = (2\pi 2^\mu r)^{-1} \mathcal{F}^{-1}(\mathcal{F} g_\theta * d\sigma_{2^\mu r})(x). \tag{2.12}$$

Let $\gamma(x) = g_\theta(x) J_0(2^\mu r|x|)$ and let $|(*)|$ denote the left side of (2.11). By (2.3) and (2.12), we have, for any $\varphi \in \mathbb{R}$,

$$(*) = (2\pi)^{-1} (\mathcal{F}\gamma)(2^\mu s e^{i\varphi}) = (2\pi)^{-2} \Big(\mathcal{F} g_\theta * \frac{1}{2^\mu r} d\sigma_{2^\mu r} \Big) (2^\mu s e^{i\varphi}). \tag{2.13}$$

We want to apply Lemma 2.1 to $\mathcal{F} g_\theta$. We have $|\mathcal{F} g_\theta(y)| \leq \|g_\theta\|_{L^1(\mathbb{R}^2)}$, for any y, and our assumption (2.10) allows us to integrate by parts to obtain $|\mathcal{F} g_\theta(y)| \leqq |y|^{-2L} \|\Delta^{(L)} g_\theta\|_{L^1}$, where Δ is the usual Laplacian in $\mathbb{R}^2$. Let

$$c_g = \max(\|g_\theta\|_{L^1}, \|\Delta^{(L)} g_\theta\|_{L^1}).$$

Then $|\mathcal{F} g_\theta(y)| \leq c \cdot c_g (1+|y|)^{-2L}$, so by (2.13) and Lemma 2.1,

$$|(*)| \leq c \cdot c_g (1 + 2^\mu |s-r|)^{-2L+2} / \max(1, 2^\mu r).$$

Also, we can interchange s and r in this estimate by symmetry. Hence it suffices to prove

$$c_g \le c2^\mu B_\mu (1 + 2^\mu |e^{i\theta} - 1|)^{-K}. \tag{2.14}$$

To prove this, first note that for any function $h(x,\xi)$,

$$\begin{aligned}\sum_{\xi\in 2^{-\mu}\mathbb{Z}} h(x,\xi)e^{i2^\mu\xi\theta}2^\mu(e^{i\theta}-1) &= \sum_{\xi\in 2^{-\mu}\mathbb{Z}} h(x,\xi)2^\mu[e^{i2^\mu(\xi+2^{-\mu})\theta} - e^{i2^\mu\xi\theta}]\\ &= \sum_{\xi\in 2^{-\mu}\mathbb{Z}} 2^\mu[h(x,\xi-2^{-\mu}) - h(x,\xi)]e^{i2^\mu\xi\theta}\\ &= -\sum_{\xi\in 2^{-\mu}\mathbb{Z}} D_{\mu,\xi}h(x,\xi)e^{i2^\mu\xi\theta},\end{aligned}$$

by a change of summation index. For $l = 0$ or L and $k = 0$ or K, we apply this repeatedly to obtain

$$\begin{aligned}2^{\mu k}(e^{i\theta}-1)^k\Delta^{(l)}g_\theta(x) &= \sum_{\xi\in 2^{-\mu}\mathbb{Z}} \Delta^{(l)}_{(x)}F(x,\xi)e^{i2^\mu\xi\theta}(2^\mu(e^{i\theta}-1))^k\\ &= \sum_{\xi\in 2^{-\mu}\mathbb{Z}} (-1)^k(D_{\mu,\xi})^{(k)}\Delta^{(l)}_{(x)}F(x,\xi)e^{i2^\mu\xi\theta}.\end{aligned}$$

Thus

$$\|\Delta^{(l)}g_\theta\|_{L^1(\mathbb{R}^2)} \le (2^\mu|e^{i\theta}-1|)^{-k}\int_{\mathbb{R}^2}\sum_{\xi\in 2^{-\mu}\mathbb{Z}} |D^{(k)}_{\mu,\xi}\Delta^{(l)}_{(x)}F(x,\xi)|\,dx. \tag{2.15}$$

However, for $|x| = \varrho$, $F(x,\xi) = f(\varrho,\xi)$ is radial, so $\Delta_{(x)}F(x,\xi) = \Delta_\varrho f(\varrho,\xi)$ by writing the Laplacian in polar coordinates $\Delta = \Delta_\varrho + \frac{1}{\varrho^2}\frac{\partial^2}{\partial\theta^2}$. Taking $k = 0$ if $|e^{i\theta}-1| \le 2^{-\mu}$ and $k = K$ otherwise, (2.15) yields (2.14). ■

We now give the discussion of the product Fourier–Hankel transform that was promised in the introduction. Recall the definition of $\widehat{f}(\varrho,n)$ (i.e. (1.1)), the inner product on $L^2([0,\infty)\times\mathbb{Z})$ (i.e. (1.2)), and the definition (1.3) of $\check{f}$.

LEMMA 2.3. (A) *Suppose* $f \in L^2(\mathbb{R}^2)$. *Then* $\widehat{f} \in L^2([0,\infty)\times\mathbb{Z})$,

$$\|\widehat{f}\|_{L^2} = (2\pi)^{-1/2}\|f\|_{L^2}, \tag{2.16}$$

and $f = (\widehat{f})^\vee$ *in the* L^2 *sense.*

(B) *Suppose* $g \in L^2([0,\infty)\times\mathbb{Z})$. *Then* $\check{g} \in L^2(\mathbb{R}^2)$,

$$\|\check{g}\|_{L^2} = (2\pi)^{1/2}\|g\|_{L^2}, \tag{2.17}$$

and $g = (\check{g})^\wedge$ *in the* L^2 *sense.*

Proof. First note that by (2.5) and Plancherel's formula ($\|\mathcal{F}f\|_{L^2} = 2\pi\|f\|_{L^2}$ in $\mathbb{R}^2$), we have, for any $g : [0,\infty) \to \mathbb{C}$,

$$\int_0^\infty |g(r)|^2 r\,dr = \int_0^\infty |Hg(r)|^2 r\,dr, \tag{2.18}$$

by considering the radial extension to $\mathbb{R}^2$ of g. If $f \in L^2(\mathbb{R}^2)$, let

$$f_n(r) = \int_0^{2\pi} f(re^{i\theta})e^{-in\theta}\,d\theta/2\pi.$$

Note that $f_n \in L^2([0,\infty), r\,dr)$, so we can define

$$\widehat{f}(r,n) = (Hf_n)(r). \tag{2.19}$$

By Parseval's theorem, $\int_0^{2\pi} |f(re^{i\theta})|^2\,d\theta = 2\pi \sum_{n\in\mathbb{Z}} |f_n(r)|^2$. Hence by (2.18),

$$\|f\|_{L^2}^2 = 2\pi \sum_{n\in\mathbb{Z}} \int_0^\infty |f_n(r)|^2 r\,dr = 2\pi \sum_{n\in\mathbb{Z}} \int_0^\infty |Hf_n(r)|^2 r\,dr,$$

so (2.16) follows from (2.19).

Similarly, if $g \in L^2([0,\infty)\times\mathbb{Z})$, let $g_n(r) = g(r,n)$. By Parseval's theorem,

$$\int_0^{2\pi} |\check{g}(re^{i\theta})|^2\,d\theta = 2\pi \sum_{n\in\mathbb{Z}} |Hg_n(r)|^2,$$

so

$$\|\check{g}\|_{L^2}^2 = 2\pi \sum_{n\in\mathbb{Z}} \int_0^\infty |Hg_n(r)|^2 r\,dr = 2\pi \|g\|_{L^2}^2.$$

Returning to $f \in L^2(\mathbb{R}^2)$ and $f_n(r)$ as above, we have $f(re^{i\theta}) = \sum_{n\in\mathbb{Z}} f_n(r)e^{in\theta}$ in the $L^2([-\pi,\pi))$ sense for a.e. $r > 0$. Then by (2.6), (2.19), and the definition of the inverse FHT,

$$f(re^{i\theta}) = \sum_{n\in\mathbb{Z}} H(Hf_n)(r)e^{in\theta} = (\widehat{f})^\vee(re^{i\theta}), \tag{2.20}$$

with convergence in L^2 of the symmetric partial sums of the middle term.

Finally, for $g \in L^2([0,\infty)\times\mathbb{Z})$, apply (2.20) to $\check{g} \in L^2(\mathbb{R}^2)$ to get $\check{g} = ((\check{g})^\wedge)^\vee$. This implies $g = (\check{g})^\wedge$ by (2.17). ■

We now consider the kernel of an FHT multiplier. If $f \in L^2(\mathbb{R}^2)$ and $m(\varrho,n) \in L^\infty([0,\infty)\times\mathbb{Z})$, then by Lemma 2.3, $T_m f = (m\widehat{f})^\vee \in L^2(\mathbb{R}^2)$ and

$$\|T_m f\|_{L^2} \le \|m\|_{L^\infty}\|f\|_{L^2}. \tag{2.21}$$

LEMMA 2.4. *Suppose* $m \in L^\infty([0,\infty)\times\mathbb{Z})$, $\sum_{n\in\mathbb{Z}}\int_0^\infty |m(\varrho,n)|\varrho\,d\varrho < \infty$, *and* $f \in L^1\cap L^2(\mathbb{R}^2)$. *Then for* $r > 0$ *and* $\theta \in \mathbb{R}$,

$$T_m f(re^{i\theta}) = \int_0^{2\pi}\int_0^\infty L_m(re^{i\theta}, se^{i\varphi}) f(se^{i\varphi}) s\,ds\,d\varphi, \tag{2.22}$$

where

$$L_m(re^{i\theta}, se^{i\varphi}) = (2\pi)^{-1} \sum_{n\in\mathbb{Z}} \int_0^\infty m(\varrho,n) J_0(s\varrho) J_0(r\varrho)\varrho\,d\varrho\, e^{in(\theta-\varphi)}. \tag{2.23}$$

Proof. By definition,

$$\begin{aligned} T_m f(re^{i\theta}) &= (m\widehat{f})^\vee(re^{i\theta}) \\ &= \sum_{n\in\mathbb{Z}} \int_0^\infty m(\varrho,n) \int_0^\infty \int_0^{2\pi} f(se^{i\varphi}) J_0(s\varrho) e^{-in\varphi} \frac{d\varphi}{2\pi} s\,ds\, J_0(r\varrho)\varrho\,d\varrho\, e^{in\theta}, \end{aligned}$$

and our assumptions justify the application of Fubini's theorem needed to complete the proof. ■

We are concerned in particular with the operators

$$R_\mu f = (\widehat{\varphi}_\mu \widehat{f})^\vee \quad \text{and} \quad S_\mu f = (\overline{\widehat{\varphi}}_\mu \widehat{\psi}_\mu \widehat{f})^\vee, \tag{2.24}$$

for $\mu \in \mathbb{N}_0$ and $\widehat{\varphi}_\mu, \widehat{\psi}_\mu$ as defined in (1.6)–(1.8). These multipliers have compact support, so Lemma 2.4 applies. The corresponding kernels are

$$K_\mu(re^{i\theta}, se^{i\varphi}) = (2\pi)^{-1} \sum_{n\in\mathbb{Z}} \int_0^\infty \widehat{\varphi}_\mu(\varrho, n) J_0(s\varrho) J_0(r\varrho)\varrho\, d\varrho\, e^{in(\theta-\varphi)} \tag{2.25}$$

and

$$G_\mu(re^{i\theta}, se^{i\varphi}) = (2\pi)^{-1} \sum_{n\in\mathbb{Z}} \int_0^\infty \overline{\widehat{\varphi}}_\mu(\varrho, n)\widehat{\psi}_\mu(\varrho, n) J_0(s\varrho) J_0(r\varrho)\varrho\, d\varrho\, e^{in(\theta-\varphi)}. \tag{2.26}$$

We use Lemma 2.2 to estimate the kernels.

Lemma 2.5. *For $M > 0$, there exists $c_M < \infty$ such that for each $\mu \in \mathbb{Z}$, $r, s > 0$, and $\theta, \varphi \in \mathbb{R}$,*

$$\begin{aligned} &|K_\mu(re^{i\theta}, se^{i\varphi})|, |G_\mu(re^{i\theta}, se^{i\varphi})| \\ &\qquad \le c_M 2^{3\mu}(1 + 2^\mu|e^{i\theta} - e^{i\varphi}|)^{-M}(1 + 2^\mu|r - s|)^{-M}/\max(1, 2^\mu r, 2^\mu s). \end{aligned} \tag{2.27}$$

Proof. First take $\mu \ge 1$. By (1.7),

$$\begin{aligned} K_\mu(re^{i\theta}, se^{i\varphi}) &= (2\pi)^{-1} \sum_{n\in\mathbb{Z}} \int_0^\infty \mathcal{F}\varphi(2^{-\mu}\varrho, 2^{-\mu}n) J_0(s\varrho) J_0(r\varrho)\varrho\, d\varrho\, e^{in(\theta-\varphi)} \\ &= (2\pi)^{-1} 2^{2\mu} \sum_{\xi\in 2^{-\mu}\mathbb{Z}} \int_0^\infty \mathcal{F}\varphi(\varrho, \xi) e^{i2^\mu(\theta-\varphi)\xi} J_0(2^\mu s\varrho) J_0(2^\mu r\varrho)\varrho\, d\varrho. \end{aligned}$$

Since $\varphi \in \mathcal{S}(\mathbb{R}^2)$, $\mathcal{F}\varphi(\varrho, \xi)$ is defined for all $(\varrho, \xi) \in \mathbb{R}^2$ and $\mathcal{F}\varphi$ belongs to $\mathcal{S}(\mathbb{R}^2)$. For $x \in \mathbb{R}^2$ and $\xi \in \mathbb{R}$, define $F(x, \xi) = \mathcal{F}\varphi(|x|, \xi)$. Since φ is radial, we have $F \in \mathcal{S}(\mathbb{R}^3)$. Let B_μ be as defined in Lemma 2.2, except with f replaced by $\mathcal{F}\varphi$. If $|x| = \varrho$, then $\Delta_\varrho^{(l)}\mathcal{F}\varphi(\varrho, \xi) = \Delta_x^{(l)}F(x, \xi)$, where $\Delta_x = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$ as usual, since F is radial in x. So

$$B_\mu = (2\pi)^{-1} \max_{\substack{l=0,L \\ k=0,K}} 2^{-\mu} \int_{\mathbb{R}^2} \sum_{\xi\in 2^{-\mu}\mathbb{Z}} |D_{\mu,\xi}^{(k)} \Delta_x^{(l)} F(x, \xi)|\, dx.$$

By the mean value theorem,

$$D_{\mu,\xi}\Delta_x^{(l)}F(x, \xi) = \frac{\partial}{\partial \xi}\Delta_x^{(l)}F(x, \xi - \theta 2^{-\mu})$$

for some $\theta \in (0, 1)$. By a repetition of this we obtain

$$|D_{\mu,\xi}^{(k)}\Delta_x^{(l)}F(x, \xi)| \le c_{k,l}$$

since $F \in \mathcal{S}(\mathbb{R}^3)$. However, $\operatorname{supp} F(x, \xi) \subseteq B(0, 1)$, since the same is true for $\mathcal{F}\varphi$. Therefore the sum over ξ contains at most $c2^\mu$ non-zero terms. Hence (2.10) holds and $B_\mu \le c_{L,K}$. This is true for arbitrary L and K, so (2.27) follows from Lemma 2.2, for $\mu \ge 1$ and K_μ. For $\mu = 0$, replace $\widehat{\varphi}$ with $\widehat{\Phi}$ in this argument, and similarly for G_μ consider $\overline{\widehat{\varphi}}\psi$ and $\overline{\widehat{\Phi}}\widehat{\Psi}$. ∎

We note, since we will use this later, that estimates (2.27) and (2.29) below hold, by the same proof, for the kernel of any FHT multiplier operator with multiplier $\mathcal{F}h(2^{-\mu}\varrho, 2^{-\mu}n)$, for any radial h with $\mathcal{F}h \in \mathcal{D}(\mathbb{R}^2)$.

As an application of this, note that by the definitions (1.13) and (2.25),

$$\varphi_{\mu km}(re^{i\theta}) = \sqrt{\pi}b_k 2^{-3\mu/2}K_\mu(re^{i\theta}, 2^{-\mu}j_k e^{i2^{-\mu}\pi m}). \tag{2.28}$$

Using Lemma 2.5 and noting that the same estimates apply to $\psi_{\mu km}$, we obtain (1.15). This gives, for arbitrarily large M,

$$\|\varphi_{\mu km}\|_{L^2}^2 \le c_M b_k^2 j_k^{-2} 2^{3\mu} \int_0^{2\pi} (1 + 2^\mu|e^{i\theta} - e^{i\pi 2^{-\mu}m}|)^{-M}\, d\theta \int_{\mathbb{R}} (1 + 2^\mu|r - 2^{-\mu}j_k|)^{-M} r\, dr.$$

The first integral is bounded by $2^{-\mu}$, the second by $2^{-2\mu}j_k$ (see e.g. Lemma 3.1 in [EF]). Then (2.9) yields (1.16).

In addition, we will need estimates on the derivatives of K_μ.

LEMMA 2.6. *Suppose $M > 0$, and $k, n \in \mathbb{N}_0$. Then there exists $c_{M,k,n} < \infty$ such that for all $\mu \in \mathbb{N}_0$, $r, s > 0$, and $\theta, \varphi \in \mathbb{R}$,*

$$\begin{aligned} \left| \frac{\partial^k}{\partial r^k}\frac{\partial^n}{\partial \theta^n} K_\mu(re^{i\theta}, se^{i\varphi})\right| \\ \le c_{M,k,n} 2^{\mu(n+k+3)}(1 + 2^\mu|e^{i\theta} - e^{i\varphi}|)^{-M}(1 + 2^\mu|r - s|)^{-M}/\max(1, 2^\mu s). \end{aligned} \tag{2.29}$$

Proof. This proof is a variation on the proofs of Lemmas 2.2 and 2.5. Suppose $\mu \ge 1$ first, and apply the formula for K_μ at the start of Lemma 2.5 to obtain

$$\frac{\partial^n}{\partial\theta^n}K_\mu(re^{i\theta}, se^{i\varphi}) = (2\pi)^{-1}2^{2\mu} \sum_{\xi \in 2^{-\mu}\mathbb{Z}} (i2^\mu\xi)^n \int_0^\infty \mathcal{F}\varphi(\varrho, \xi) e^{i2^\mu(\theta-\varphi)\xi} J_0(2^\mu s\varrho) J_0(2^\mu r\varrho)\varrho\, d\varrho.$$

For $x \in \mathbb{R}^2$, let $F(x, \xi) = (i2^\mu\xi)^n \mathcal{F}\varphi(|x|, \xi)$ and set

$$g_{\theta-\varphi}(x) = \sum_{\xi \in 2^{-\mu}\mathbb{Z}} F(x, \xi) e^{i2^\mu\xi(\theta-\varphi)}.$$

Since $\operatorname{supp}\mathcal{F}\varphi \subseteq B(0,1)$, $\varphi \in \mathcal{S}(\mathbb{R}^2)$, and φ is radial, we have $\operatorname{supp} F \subseteq B(0,1)$, $F \in \mathcal{S}(\mathbb{R}^3)$, $g_{\theta-\varphi} \in \mathcal{S}(\mathbb{R}^2)$, and $g_{\theta-\varphi}$ is radial. Let

$$\gamma(x) = g_{\theta-\varphi}(x) J_0(2^\mu s|x|),$$

which is also radial. So by (2.3),

$$\frac{\partial^n}{\partial\theta^n}K_\mu(re^{i\theta}, se^{i\varphi}) = (2\pi)^{-2}2^{2\mu}(\mathcal{F}\gamma)(2^\mu r\mathbf{1}),$$

for $\mathbf{1} = (1, 0) \in \mathbb{R}^2$. Hence

$$\begin{aligned} (*) \equiv \frac{\partial^k}{\partial r^k}\frac{\partial^n}{\partial\theta^n}K_\mu(re^{i\theta}, se^{i\varphi}) &= (2\pi)^{-2}2^{2\mu}\frac{\partial^k}{\partial x_1^k}[\mathcal{F}\gamma(2^\mu x)]|_{x=(r,0)} \\ &= (2\pi)^{-2}2^{2\mu}(-i2^\mu)^k[\mathcal{F}(x_1^k\gamma(x))](2^\mu r\mathbf{1}). \end{aligned}$$

Let $h(x) = x_1^k g_{\theta-\varphi}(x)$ and $c_{h,L} = \max(\|h\|_{L^1}, \|\Delta^{(L)}h\|_{L^1})$. Then as in (2.12)–(2.13),

$$\begin{aligned} |(*)| &\le c2^{\mu(2+k)}(2^\mu s)^{-1}|(\mathcal{F}h * d\sigma_{2^\mu s})(2^\mu r\mathbf{1})| \\ &\le c \cdot c_{h,L}2^{\mu(2+k)}(1 + 2^\mu|s - r|)^{-2L+2}/\max(1, 2^\mu s), \end{aligned} \tag{2.30}$$

using Lemma 2.1. Similarly to (2.15), then for $m = 0$ or $m = K$,

$$\|\Delta^{(l)} h\|_{L^1} \le (2^\mu |e^{i(\theta-\varphi)} - 1|)^{-m} \int_{\mathbb{R}^2} \sum_{\xi \in 2^{-\mu}\mathbb{Z}} |D^{(m)}_{\mu,\xi} \Delta^{(l)}_{(x)} (x_1^k F(x,\xi))| \, dx.$$

Recall $F(x,\xi) = (i2^\mu \xi)^n \mathcal{F}\varphi(|x|,\xi)$. Since $x_1^k \xi^n \mathcal{F}\varphi(|x|,\xi)$ is C^∞ on $\mathbb{R}^3$ and supported in $B(0,1)$, we obtain, for each ξ, as in the proof of Lemma 2.5,

$$\int_{\mathbb{R}^2} |D^{(m)}_{\mu,\xi} \Delta^{(l)}_{(x)} (x_1^k F(x,\xi))| \, dx \le c_{m,l} 2^{\mu n}.$$

Since there are at most $c2^\mu$ nonzero terms in the sum on ξ, by taking $l = 0$ and $l = L$, we obtain

$$c_{h,L} \le c_{L,K} 2^{\mu(n+1)} (1 + 2^\mu |e^{i\theta} - e^{i\varphi}|)^{-K}.$$

Since L and K are arbitrary, the result follows from (2.30) for $\mu \ge 1$. If $\mu = 0$, replace φ by Φ in the above argument. ■

Using (2.28), Lemma 2.6 implies (1.19) for $\varphi_{\mu k m}$, and $\psi_{\mu k m}$ clearly satisfies the same estimates.

Recall the assumptions in (1.5) on $\varphi, \Phi \in \mathcal{S}(\mathbb{R}^2)$. With (1.6) and (1.7), these imply that there exist constants $c_1, c_2 > 0$ such that

$$c_1 \le \sum_{\mu=0}^{\infty} |\widehat{\varphi}_\mu(\varrho,n)|^2 \le c_2$$

for all $(\varrho,n) \in [0,\infty) \times \mathbb{N}$. Hence by Lemma 2.3,

$$\begin{aligned}
\|f\|^2_{L^2(\mathbb{R}^2)} &= 2\pi \sum_{n\in\mathbb{Z}} \int_0^\infty |\widehat{f}(\varrho,n)|^2 \varrho \, d\varrho \approx \sum_{n\in\mathbb{Z}} \int_0^\infty \sum_{\mu=0}^\infty |\widehat{\varphi}_\mu(\varrho,n)|^2 |\widehat{f}(\varrho,n)|^2 \varrho \, d\varrho \\
&= \sum_{\mu=0}^\infty \|\widehat{\varphi}_\mu \widehat{f}\|^2_{L^2([0,\infty)\times\mathbb{Z})} = (2\pi)^{-1} \sum_{\mu=0}^\infty \|R_\mu f\|^2_{L^2} \\
&= (2\pi)^{-1} \int_{\mathbb{R}^2} \sum_{\mu=0}^\infty |R_\mu f|^2 = (2\pi)^{-1} \Big\| \Big(\sum_{\mu=0}^\infty |R_\mu f|^2 \Big)^{1/2} \Big\|^2_{L^2}.
\end{aligned}$$

Notice by our definitions that

$$\|f\|_{A_p^{\alpha q}} = \Big\| \Big(\sum_{\mu=0}^\infty (2^{\mu\alpha} |R_\mu f|)^q \Big)^{1/q} \Big\|_{L^p}. \tag{2.31}$$

So we have proved that

$$L^2 \approx A_2^{02}, \tag{2.32}$$

i.e. the norms are equivalent. In §8, we extend this to obtain $L^p \approx A_p^{02}$, $1 < p < \infty$.

Define operators $\widetilde{R}_\mu$ and T_μ on $L^2(\mathbb{R}^2)$ by

$$\widetilde{R}_\mu f = (\overline{\widehat{\varphi}}_\mu \widehat{f})^\vee \quad \text{and} \quad T_\mu f = (\widehat{\psi}_\mu \widehat{f})^\vee. \tag{2.33}$$

Then $S_\mu = \widetilde{R}_\mu T_\mu$. By (1.11), for $f \in L^2$ we have

$$f = \sum_{\mu=0}^\infty S_\mu f = \sum_{\mu=0}^\infty \widetilde{R}_\mu T_\mu f, \tag{2.34}$$

with convergence in L^2 norm, by Lemma 2.3. This is our analogue of the Calderón formula (see e.g. [FJW] for a discussion of the classical Calderón formula).

We will extend (2.34) to L^p, $1 < p < \infty$. First we must define our operators on L^p. Lemma 2.5 implies that

$$\sup_{se^{i\varphi}\in\mathbb{R}^2} \int_0^{2\pi}\int_0^{\infty} |K_\mu(re^{i\theta}, se^{i\varphi})| r\,dr\,d\theta < \infty \tag{2.35}$$

and

$$\sup_{re^{i\theta}\in\mathbb{R}^2} \int_0^{2\pi}\int_0^{\infty} |K_\mu(re^{i\theta}, se^{i\varphi})| s\,ds\,d\varphi < \infty, \tag{2.36}$$

and similarly for the kernels of S_μ, $\widetilde{R}_\mu$ and T_μ. By a standard result (e.g. [Fo], p. 1), the operator

$$f \to \int_0^{2\pi}\int_0^{\infty} K_\mu(\cdot, se^{i\varphi}) f(se^{i\varphi}) s\,ds\,d\varphi \tag{2.37}$$

is bounded on $L^p(\mathbb{R}^2)$ for $1 \le p \le \infty$. By Lemma 2.4, it coincides with $R_\mu f$ on $L^1 \cap L^2$, hence on all of L^2. Thus (2.37) gives the continuous extension of R_μ to L^p, $1 \le p \le \infty$, which we still denote R_μ. Similarly we extend S_μ, T_μ, and $\widetilde{R}_\mu$ to L^p, $1 \le p \le \infty$.

To proceed much further, we need an appropriate maximal function which controls our operators S_μ, etc. To do this, we adapt the usual Hardy–Littlewood maximal operator M to the polar grid. Let $\mathcal{M}$ denote the collection of all sets in $\mathbb{R}^2$ that are either an *admissible annular sector* (or wedge if $r_0 = 0$) of the form

$$I = \{re^{i\theta} : 0 \le r_0 < r < r_1,\, \theta_0 < \theta < \theta_1,\, r_1 - r_0 = \theta_1 - \theta_0 \le 2\pi,\, \theta_0 \in \mathbb{R}\}$$

or an *admissible annulus* (or ball if $r_0 = 0$) of the form

$$I = \{re^{i\theta} : 0 \le r_0 < r < r_1,\ \theta \in \mathbb{R},\ r_1 \ge r_0 + 2\pi\}.$$

If $x \in \mathbb{R}^2 \setminus \{0\}$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$, we define

$$M^{\mathrm{pol}} f(x) = \sup_{\{I\in\mathcal{M}: x\in I\}} \frac{1}{|I|}\int_I |f(y)|\,dy. \tag{2.38}$$

We also have an analogue of the strong maximal function M_s (see e.g. [CF] for a discussion of M_s). Let $\mathcal{N}$ denote the collection of all open *annular sectors*, i.e., all sets of the form

$$I = \{re^{i\theta} \in \mathbb{R}^2 : 0 \le r_0 < r < r_1,\ \theta_0 < \theta < \theta_1,\ \theta_0, \theta_1 \in \mathbb{R}\}.$$

For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ and $x \in \mathbb{R}^2 \setminus \{0\}$, define $M_{\mathrm{s}}^{\mathrm{pol}}$ by

$$M_s^{\mathrm{pol}} f(x) = \sup_{\{I\in\mathcal{N}: x\in I\}} \frac{1}{|I|}\int_I |f(y)| dy.$$

We call $M_{\mathrm{s}}^{\mathrm{pol}}$ the strong polar maximal function. Obviously

$$M^{\mathrm{pol}} f(x) \le M_{\mathrm{s}}^{\mathrm{pol}} f(x). \tag{2.39}$$

The next lemma is analogous to the standard domination of a convolution integral by the usual maximal operator.

LEMMA 2.7. *Let* $\mu \in \mathbb{N}_0$, $\lambda_1 > 0$ *and* $\lambda_2 > 1$. *Suppose* $D_\mu : \mathbb{R}^2 \to \mathbb{C}$ *satisfies*

$$|D_\mu(re^{i\theta}, se^{i\theta})| \le 2^{3\mu}(1 + 2^\mu|e^{i\theta} - e^{i\varphi}|)^{-\lambda_1}(1 + 2^\mu|r - s|)^{-\lambda_2}/\max(1, 2^\mu r, 2^\mu s) \tag{2.40}$$

for $r, s > 0$ *and* $\theta, \varphi \in \mathbb{R}$. *Define* $H_\mu : L^p(\mathbb{R}^2) \to L^p(\mathbb{R}^2)$, $1 \le p \le \infty$, *by*

$$H_\mu f(re^{i\theta}) = \int_0^{2\pi}\int_0^{\infty} D_\mu(re^{i\theta}, se^{i\varphi})f(se^{i\varphi})s\,ds\,d\varphi$$

for $r > 0$ (H_μ *is bounded on* L^p *by the analogue of* (2.35)–(2.36) *following from* (2.40).) *Then there exists* $c = c_{\lambda_1,\lambda_2} < \infty$, *independent of* μ, *such that*

(A) *if* $\lambda_1, \lambda_2 > 2$, *then* $|H_\mu f(re^{i\theta})| \le cM_{\rm s}^{\rm pol} f(re^{i\theta})$ *for all* $r > 0$ *and* $\theta \in \mathbb{R}$, *and*
(B) *if* $\lambda_1 > 4$ *and* $\lambda_2 > 3$, *then* $|H_\mu f(re^{i\theta})| \le cM^{\rm pol} f(re^{i\theta})$ *for all* $r > 0$ *and* $\theta \in \mathbb{R}$.

Proof. Recall the definition (1.18) of the annular sectors $\{R_{\mu km}\}_{k\in\mathbb{N}, m\in A_\mu}$. Suppose $re^{i\theta} \in R_{\mu k_0 m_0}$ for some k_0, m_0. By the rotational properties of (2.40), $M^{\rm pol}$, and $M_s^{\rm pol}$, we can (and do) assume $m_0 = 0$ (else we need to re-index $\{R_{\mu km}\}$ to run over $m - m_0 \in A_\mu$). By (2.40), if $se^{i\varphi} \in R_{\mu km}$, we have

$$|D_\mu(re^{i\theta}, se^{i\varphi})| \le c2^{3\mu}(1 + |m|)^{-\lambda_1}(1 + |j_k - j_{k_0}|)^{-\lambda_2}/\max(j_k, j_{k_0}), \tag{2.41}$$

since $|1 - e^{i\varphi}| \approx |\varphi|$ for $-\pi \le \varphi \le \pi$.

Let $R^*_{\mu km}$ denote the annular sector (i.e., element of $\mathcal{N}$) of smallest measure containing $R_{\mu k_0 0}$ and $R_{\mu km}$. Then $R^*_{\mu km}$ has inner radius $2^{-\mu}\min(j_{k-1}, j_{k_0-1})$, outer radius $2^{-\mu}\max(j_k, j_{k_0})$, and angular variation $2^{-\mu}\pi(1 + |m|)$. Hence

$$|R^*_{\mu km}| \le c2^{-3\mu}(1 + |j_k - j_{k_0}|)\max(j_k, j_{k_0})(1 + |m|).$$

By (2.41), then,

$$\begin{aligned}\iint_{R_{\mu km}} &|D_\mu(re^{i\theta}, se^{i\varphi})f(se^{i\varphi})|s\,ds\,d\varphi\\ &\le c\frac{2^{3\mu}(1 + |m|)^{-\lambda_1}(1 + |j_k - j_{k_0}|)^{-\lambda_2}}{\max(j_k, j_{k_0})}|R^*_{\mu km}|\frac{1}{|R^*_{\mu km}|}\iint_{R^*_{\mu km}} |f(se^{i\varphi})|s\,ds\,d\varphi\\ &\le c(1 + |m|)^{1-\lambda_1}(1 + |j_k - j_{k_0}|)^{1-\lambda_2}M_{\rm s}^{\rm pol} f(re^{i\theta}).\end{aligned}$$

Summing over $m \in A_\mu$ and $k \in \mathbb{N}$ (using (2.7)) yields (A).

For (B), let $R^{**}_{\mu km}$ be the admissible annular sector of smallest measure containing $R_{\mu k_0 0}$ and $R_{\mu km}$. If $\max(j_k - j_{k_0-1}, j_{k_0} - j_{k-1}) \ge \pi(1+|m|)$, then $R^{**}_{\mu km}$ has inner and outer radius as for $R^*_{\mu km}$, and angular variation $\le c2^{-\mu}(1 + |j_k - j_{k_0}|)$. So $|R^{**}_{\mu km}| \le c2^{-3\mu}(1 + |j_k - j_{k_0}|)^2\max(j_k, j_{k_0})$. If $\max(j_k - j_{k_0-1}, j_{k_0} - j_{k-1}) \le \pi(1 + |m|) \le \max(j_k, j_{k_0})$, then $R^{**}_{\mu km}$ has angular variation $\alpha \equiv 2^{-\mu}\pi(1 + |m|)$, outer radius $\beta \equiv 2^{-\mu}\max(j_k, j_{k_0})$, and inner radius $\beta - \alpha$. Using $\beta^2 - (\beta - \alpha)^2 \le 2\alpha\beta$, we obtain $|R^{**}_{\mu km}| \le c2^{-3\mu}(1 + |m|)^2\max(j_k, j_{k_0})$ in this case. In the remaining case, $R^{**}_{\mu km}$ is a wedge with inner radius 0, and outer radius and angular variation both $\pi(1 + |m|)$, so $|R^{**}_{\mu km}| \le c2^{-3\mu}(1 + |m|)^3$. In all cases we obtain

$$|R^{**}_{\mu km}| \le c2^{-3\mu}(1 + |m|)^3(1 + |j_k - j_{k_0}|)^2\max(j_k, j_{k_0}).$$

Now the same argument as for (A) gives (B). ∎

In the $F_p^{\alpha q}$ theory, the Fefferman–Stein vector-valued maximal inequality ([FS]) plays a critical role (see e.g. [FJ3]). We require a similar control of M^{pol} and $M_{\mathrm{s}}^{\mathrm{pol}}$. The simplest way to obtain this control is to use a weighted vector-valued maximal inequality for doubling measures. A (measurable) function $\omega(x) \geq 0$ on $\mathbb{R}^n$ is called a *doubling weight* if there exists $c > 0$ such that for all $x \in \mathbb{R}^n$ and $r > 0$, $\omega(B(x,2r)) \leq c\omega(B(x,r))$, where $\omega(E) = \int_E \omega\, dx$, for E measurable. The corresponding weighted maximal operator M_ω is defined by

$$M_\omega f(x) = \sup_{Q:x\in Q} \frac{1}{\omega(Q)} \int_Q |f(y)|\omega(y)\, dy, \tag{2.42}$$

for $f \in L^1_{\mathrm{loc}}(\omega)$ and the sup is over all cubes Q in $\mathbb{R}^n$ containing x. Then (see [JT] or adapt the proof in [S2], §2) if $1 < p < \infty$ and $1 < q \leq \infty$,

$$\Big\| \Big(\sum_{i=1}^{\infty} |M_\omega f_i|^q \Big)^{1/q} \Big\|_{L^p(\omega\, dx)} \leq c_{p,q,\omega} \Big\| \Big(\sum_{i=1}^{\infty} |f_i|^q \Big)^{1/q} \Big\|_{L^p(\omega\, dx)}. \tag{2.43}$$

LEMMA 2.8. *Suppose* $1 < p < \infty$, *and* $1 < q \leq \infty$. *Then there exists* $c_{p,q} < \infty$ *such that for any locally integrable sequence* $\{f_i\}_{i\in\mathbb{N}}$ *on* $\mathbb{R}^2$,

$$\begin{aligned} \Big\| \Big(\sum_{i=1}^{\infty} |M^{\mathrm{pol}} f_i|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^2)} &\leq \Big\| \Big(\sum_{i=1}^{\infty} |M_{\mathrm{s}}^{\mathrm{pol}} f_i|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^2)} \\ &\leq c_{p,q} \Big\| \Big(\sum_{i=1}^{\infty} |f_i|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^2)}. \end{aligned} \tag{2.44}$$

Proof. Of course the first inequality follows from (2.39) and is just stated for emphasis. For each i, define $f_i^* : \mathbb{R}^2 \to \mathbb{C}$ as follows. If $x < 0$ or $|\theta| > 3\pi$, let $f_i^*(x,\theta) = 0$. Otherwise let $f_i^*(x,\theta) = f_i(xe^{i\theta})$. Let $\omega(x) = |x|$ for $x \in \mathbb{R}$; then ω is a doubling weight on $\mathbb{R}$. Define M_ω by (2.42), and let M_θ be the usual maximal function on $\mathbb{R}$ (i.e. (2.42) with $\omega(x) = 1$ for all x). Let I be an annular sector containing $re^{i\theta}$, $-\pi < \theta \leq \pi$. Then I can be written (not uniquely, but this is not necessary) as $I = \{re^{i\theta} : 0 \leq r_0 < r < r_1,\ -3\pi \leq \theta_0 < \theta < \theta_1 \leq 3\pi,\ \theta_1 - \theta_0 \leq 2\pi\}$. Let $I^* = (r_0, r_1) \times (\theta_0, \theta_1)$ be the corresponding true rectangle in $\mathbb{R}^2$. Then $|I| = \omega((r_0,r_1))(\theta_1 - \theta_0)$. So

$$\begin{aligned} \frac{1}{|I|} \iint_I |f_i(se^{i\varphi})| s\, ds\, d\varphi &= \frac{1}{|I|} \iint_{I^*} |f_i^*(x,\varphi)| \omega(x)\, dx\, d\varphi \\ &= \frac{1}{\theta_1 - \theta_0} \int_{\theta_0}^{\theta_1} \frac{1}{\omega((r_0,r_1))} \int_{r_0}^{r_1} |f_i^*(x,\varphi)|\omega(x)\, dx\, d\varphi \\ &\leq M_\theta^{(2)} M_\omega^{(1)} f_i^*(r,\theta), \end{aligned}$$

where $M_\omega^{(1)}$ acts in the first variable, $M_\theta^{(2)}$ in the second. Therefore for $r > 0$, $-\pi < \theta < \pi$,

$$M_{\mathrm{s}}^{\mathrm{pol}} f_i(re^{i\theta}) \leq M_\theta^{(2)} M_\omega^{(1)} f_i^*(r,\theta).$$

Hence the pth power of the middle term of (2.44) is dominated by

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \Big(\sum_{i=1}^{\infty} |M_\theta^{(2)} M_\omega^{(1)} f_i^*(r,\theta)|^q\Big)^{p/q} d\theta\, r\, dr \le c_{p,q}^p c_{p,q,\omega}^p \int_{\mathbb{R}}\int_{\mathbb{R}} \Big(\sum_{i=1}^{\infty} |f_i^*(r,\theta)|^q\Big)^{p/q} r\, dr\, d\theta$$
$$\le 3\, c_{p,q}^p c_{p,q,\omega}^p \Big\| \Big(\sum_{i=1}^{\infty} |f_i|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^2)}^p,$$

by an iteration of (2.43). ■

Of course we now obtain L^p boundedness

$$\|M^{\mathrm{pol}} f\|_{L^p(\mathbb{R}^2)} \le \|M_{\mathrm{s}}^{\mathrm{pol}} f\|_{L^p(\mathbb{R}^2)} \le c_p \|f\|_{L^p(\mathbb{R}^2)}, \quad 1 < p \le \infty, \tag{2.45}$$

the case $p = \infty$ being trivial. By Lemmas 2.5 and 2.7 we have

$$|R_\mu f|, |S_\mu f|, |\widetilde{R}_\mu f|, |T_\mu f| \le c M^{\mathrm{pol}} f \le c M_{\mathrm{s}}^{\mathrm{pol}} f, \tag{2.46}$$

since the proof of Lemma 2.5 applies to the kernel of $\widetilde{R}_\mu$ and T_μ as well. Although in some cases (e.g., Lemma 2.7), $M_{\mathrm{s}}^{\mathrm{pol}}$ gives a sharper result, M^{pol} is still useful because it is of weak type (1,1) (see §8).

We can now discuss the convergence of our Calderón formula (2.34) when $f \in L^p$, $1 < p < \infty$. For $N \in \mathbb{N}_0$, let

$$H_N f = \sum_{\mu=0}^{N} S_\mu f = \Big(\sum_{\mu=0}^{N} \overline{\widehat{\varphi}}_\mu \widehat{\psi}_\mu \widehat{f}\Big)^{\vee}.$$

Define $h \in \mathcal{S}(\mathbb{R}^2)$ by $\mathcal{F}h = \overline{\mathcal{F}\Phi}\mathcal{F}\Psi$ for Φ, Ψ as in §1. By (1.7), (1.8), and (1.11),

$$\mathcal{F}h(\varrho, n) = (\overline{\widehat{\varphi}}_0 \widehat{\psi}_0)(\varrho, n) = 1 - \sum_{\mu=1}^{\infty} (\overline{\widehat{\varphi}}_\mu \widehat{\psi}_\mu)(\varrho, n) = 1 - \sum_{\mu=1}^{\infty} (\overline{\mathcal{F}\varphi}\mathcal{F}\psi)(2^{-\mu}\varrho, 2^{-\mu} n).$$

Similarly, then,

$$\sum_{\mu=0}^{N} (\overline{\widehat{\varphi}}_\mu \widehat{\psi}_\mu)(\varrho, n) = 1 - \sum_{\mu=N+1}^{\infty} (\overline{\widehat{\varphi}}_\mu \widehat{\psi}_\mu)(\varrho, n)$$
$$= 1 - \sum_{\mu=N+1}^{\infty} (\overline{\mathcal{F}\varphi}\mathcal{F}\psi)(2^{-\mu}\varrho, 2^{-\mu} n) = \mathcal{F}h(2^{-N}\varrho, 2^{-N} n).$$

So the kernel D_N of H_N is as in (2.23) with $m(\varrho, n) = \mathcal{F}h(2^{-N}\varrho, 2^{-N} n)$. The proof of Lemma 2.5 applies to give the same bounds on D_μ as for K_μ in (2.27). Hence by Lemma 2.7,

$$|D_N f| \le c M^{\mathrm{pol}} f, \tag{2.47}$$

with c independent of N. We need one more basic fact about H_N to show that it behaves like a standard approximate identity.

LEMMA 2.9. *For $N \in \mathbb{N}_0$, $(D_N 1)(re^{i\theta}) = 1$ for all $r > 0$ and $\theta \in \mathbb{R}$.*

P r o o f. Note that the kernel estimates for D_N imply the analogue of (2.36), so $D_N 1$ is defined pointwise on $\mathbb{R}^2 \setminus \{0\}$. By the dominated convergence theorem,

$$\begin{aligned}(D_N 1)(re^{i\theta})\\ &= \lim_{\varepsilon\to 0}\int_0^{2\pi}\int_0^{\infty} D_N(re^{i\theta}, se^{i\theta})e^{-\varepsilon s^2}s\,ds\,d\varphi\\ &= \lim_{\varepsilon\to 0}(2\pi)^{-1}\int_0^{2\pi}\int_0^{\infty}\sum_{n\in\mathbb{Z}}\int_0^{\infty}\mathcal{F}h(2^{-N}\varrho, 2^{-N}n)J_0(s\varrho)J_0(r\varrho)\varrho\,d\varrho\,e^{in(\theta-\varphi)}e^{-\varepsilon s^2}s\,ds\,d\varphi\\ &= \lim_{\varepsilon\to 0}\int_0^{\infty}\mathcal{F}h(2^{-N}\varrho, 0)J_0(r\varrho)\int_0^{\infty}e^{-\varepsilon s^2}J_0(s\varrho)s\,ds\,\varrho\,d\varrho,\end{aligned}$$

by integrating out φ and using Fubini's theorem. Define f_ε and $g_N : \mathbb{R}^2\to\mathbb{C}$ by $f_\varepsilon(x) = e^{-\varepsilon|x|^2}$ and $g_N(x) = \mathcal{F}h(2^{-N}|x|, 0)$. By (2.3),

$$\int_0^{\infty} e^{-\varepsilon s^2}J_0(s\varrho)s\,ds = (2\pi)^{-1}\mathcal{F}f_\varepsilon(\varrho e^{i\varphi}), \quad \text{for any } \varphi\in\mathbb{R}.$$

Since h is radial, (2.5) and the above equality imply

$$(D_N 1)(re^{i\theta}) = \lim_{\varepsilon\to 0}\mathcal{F}^{-1}(g_N\mathcal{F}f_\varepsilon)(re^{i\theta}) = \lim_{\varepsilon\to 0}(\mathcal{F}^{-1}g_N * f_\varepsilon)(re^{i\theta}) = \int_{\mathbb{R}^2}\mathcal{F}^{-1}g_N,$$

by the dominated convergence theorem. But

$$\int_{\mathbb{R}^2}\mathcal{F}^{-1}g_N = g_N(0) = \mathcal{F}h(0) = 1,$$

by (1.9)–(1.11) and the definition of h. ■

The same proof shows that for all $r > 0$ and $\theta\in\mathbb{R}$, $R_0 1(re^{i\theta}) = \mathcal{F}\Phi(0)$ and, for $\mu\in\mathbb{N}$, $R_\mu 1(re^{i\theta}) = 0$. By the same argument, or by the symmetry in (2.25), we have

$$\int_0^{2\pi}\int_0^{\infty} K_\mu(re^{i\theta}, se^{i\varphi})r\,dr\,d\theta \equiv 0$$

for $\mu\in\mathbb{N}$, and the integral is identically $\mathcal{F}\Phi(0)$ if $\mu = 0$. By (2.28), this implies (1.20) for $\varphi_{\mu km}$, and $\psi_{\mu km}$ has the same property for the same reasons. For $\mu = 0$, (2.28) implies that

$$\int_{\mathbb{R}^2}\varphi_{0km} = \sqrt{\pi}\,b_k\mathcal{F}\Phi(0),$$

and similarly for ψ. In fact, by (1.9)–(1.10), the same methods show that a stronger orthogonality property holds:

$$\int_0^{2\pi}\int_0^{\infty}\varphi_{\mu km}(re^{i\theta})e^{in\theta}r\,dr\,d\theta = 0$$

unless $2^{\mu-2}\le |n|\le 2^{\mu}$ in the case $\mu\in\mathbb{N}$, and unless $n = 0$ if $\mu = 0$ (and similarly for ψ). The case $\mu = 0$ shows that φ_{0km} and ψ_{0km} are radial, which can be readily seen by (1.9), (1.13), and (1.14).

Finally, we come to the main convergence result concerning (2.34).

LEMMA 2.10. *Suppose* $1 < p < \infty$ *and* $f \in L^p(\mathbb{R}^2)$. *Then*

$$H_N f = \sum_{\mu=0}^{N} S_\mu f = \sum_{\mu=0}^{N} \widetilde{R}_\mu T_\mu f = \Big(\sum_{\mu=0}^{N} \overline{\widehat{\varphi}}_\mu \widehat{\psi}_\mu \widehat{f} \Big)^\vee$$

converges to f *a.e. and in* L^p *as* $N \to \infty$.

Proof. By Lemma 2.9, we have

$$(H_N f - f)(re^{i\theta}) = \int_0^{2\pi} \int_0^{\infty} D_N(re^{i\theta}, se^{i\varphi})(f(se^{i\varphi}) - f(re^{i\theta}))s\,ds\,d\varphi. \tag{2.48}$$

Fix $re^{i\theta} \in \mathbb{R}^2 \setminus \{0\}$ and μ large enough that $r > 2^{-\mu}$, and define sets

$$B_\mu = \{se^{i\varphi} \in \mathbb{R}^2 : r - 2^{-\mu} < s < r + 2^{-\mu},\ \theta - 2^{-\mu} < \varphi < \theta + 2^{-\mu}\}.$$

Taking $N > \mu$ and $M > 1$ and applying (2.27) for D_N gives

$$\begin{aligned}\iint\limits_{\mathbb{R}^2 \setminus B_\mu} &|D_N(re^{i\theta}, se^{i\varphi})|s\,ds\,d\varphi \\ &\le c \int\limits_{|s-r| \ge 2^{-\mu}} \int\limits_{|\theta-\varphi| \ge 2^{-\mu}} 2^{2N}(1 + 2^N|e^{i\theta} - e^{i\varphi}|)^{-M}(1 + 2^N|r-s|)^{-M}\,d\varphi\,ds \\ &\le c_M 2^{-2(N-\mu)(M-1)},\end{aligned}$$

which goes to 0 as $N \to \infty$. From this and the analogue of (2.36), and (2.48), it follows easily that if $f \in C_0(\mathbb{R}^2)$ (i.e. continuous with compact support), then $H_N f$ converges to f a.e. as $N \to \infty$.

Now let $f \in L^p(\mathbb{R}^2)$. For $\delta > 0$, pick $f_0 \in C_0(\mathbb{R}^2)$ such that $\|f - f_0\|_{L^p} < \delta$. Then a.e. we have

$$\begin{aligned}G &\equiv \limsup_{N\to\infty} H_N f - \liminf_{N\to\infty} H_N f = \limsup_{N\to\infty} H_N(f - f_0) - \liminf_{N\to\infty} H_N(f - f_0) \\ &\le 2cM^{\mathrm{pol}}(f - f_0),\end{aligned}$$

by (2.47). Hence by (2.45),

$$\|G\|_{L^p} \le 2cc_p\|f - f_0\|_{L^p} \le 2cc_p\delta.$$

Since $\delta > 0$ is arbitrary, $G = 0$ a.e., and $H_N f \to f$ a.e. Then by (2.47) again, $|f - H_N f|^p \le 2^p(|f|^p + |M^{\mathrm{pol}} f|^p) \in L^1$, so the dominated convergence theorem implies $\|f - H_N f\|_{L^p} \to 0$ as $N \to \infty$. ∎

We can extend the a.e. convergence in Lemma 2.10 to $p = 1$ by the usual argument ([S1], §1.5) once we have (Lemma 8.1) that M^{pol} is of weak type $(1, 1)$.

3. The sampling theorem and polar wavelet identity

We are now in position to prove our main results. We begin with the FHT sampling formula (Theorem 3.1) and the polar wavelet identity (Theorem 3.2), both stated in the introduction.

Proof of Theorem 3.1. Let B, N, f and g be as stated. As in the Shannon sampling theorem, the key step is to expand $\widehat{f}$ in terms of an orthonormal basis. Equip $L^2([0,B]\times\{-N,\dots,N-1\})$ with the inner product

$$\langle F,G\rangle = \sum_{n=-N}^{N-1}\int_0^B F(r,n)\overline{G(r,n)}r\,dr.$$

For $k\in\mathbb{N}$, $m,n\in\{-N,\dots,N-1\}$, and $r\in[0,B]$, let

$$a_{k,m}(r,n) = b_k(2N)^{-1/2}B^{-1}J_0(j_k r/B)e^{-i\pi mn/N}.$$

Then $\{a_{k,m}\}_{k,m}$ is a complete orthonormal basis for $L^2([0,B]\times\{-N,\dots,N-1\})$, since $\{(2N)^{-1/2}e^{-i\pi mn/N}\}_{m=-N}^{N-1}$ and $\{b_kB^{-1}J_0(j_kr/B)\}_{k\in\mathbb{N}}$ are complete orthonormal bases for $\ell^2(\{-N,\dots,N-1\})$ and $L^2([0,B], r\,dr)$ respectively (using a dilation and the fact stated after (2.8)). By Lemma 2.3,

$$\begin{aligned}\langle \widehat{f}, a_{k,m}\rangle &= \sum_{n=-N}^{N-1}\int_0^B \widehat{f}(\varrho,n)b_kB^{-1}J_0(j_k\varrho/B)(2N)^{-1/2}e^{i\pi mn/N}\varrho\,d\varrho\\ &= b_kB^{-1}(2N)^{-1/2}(\widehat{f})^{\vee}(j_ke^{i\pi m/N}/B)\\ &= b_kB^{-1}(2N)^{-1/2}f(j_ke^{i\pi m/N}/B)\end{aligned}$$

since f is (B,N)-bandlimited and, as noted in the introduction, we identify f with its continuous representative $(\widehat{f})^{\vee}$. Hence for $(\varrho,n)\in[0,B]\times\{-N,\dots,N-1\}$,

$$\begin{aligned}\widehat{f}(\varrho,n) &= \sum_{k\in\mathbb{N}}\sum_{m=-N}^{N-1}\langle\widehat{f},a_{k,m}\rangle a_{k,m}(\varrho,n)\\ &= \sum_{k\in\mathbb{N}}\sum_{m=-N}^{N-1} b_k^2B^{-2}(2N)^{-1}f(j_ke^{i\pi m/N}/B)J_0(j_k\varrho/B)e^{-i\pi mn/N}.\end{aligned}\tag{3.1}$$

Since $\widehat{g}$ is (B,N)-bandlimited, we can use (3.1) to substitute for $\widehat{f}$ in $(\widehat{f}\widehat{g})^{\vee}$ and obtain (1.4).

To be more precise about convergence, let

$$\widehat{f}_M(\varrho,n) = \sum_{k=1}^{M}\sum_{m=-N}^{N-1} b_k^2B^{-2}(2N)^{-1}f(j_ke^{i\pi m/N}/B)J_0(j_k\varrho/B)e^{-i\pi mn/N}.$$

Then by our derivation,

$$\|\widehat{f}-\widehat{f}_M\|_{L^2}\underset{M\to\infty}{\longrightarrow}0.\tag{3.2}$$

The partial sum over $k=1,\dots,M$ instead of $k\in\mathbb{N}$ on the right side of (1.4) is just $(\widehat{f}_M\widehat{g})^{\vee}(re^{i\theta})$, and

$$\begin{aligned}|(\widehat{f}\widehat{g})^{\vee}-(\widehat{f}_M\widehat{g})^{\vee}| &\le \|J_0\|_{L^\infty}\sum_{n\in\mathbb{N}}\int_0^\infty |(\widehat{f}-\widehat{f}_M)(\varrho,n)||\widehat{g}(\varrho,n)|\varrho\,d\varrho\\ &\le \|J_0\|_{L^\infty}\|\widehat{f}-\widehat{f}_M\|_{L^2}\|\widehat{g}\|_{L^2}\underset{M\to\infty}{\longrightarrow}0.\end{aligned}$$

So $(\widehat{f}_M\widehat{g})^{\vee}$ converges uniformly to $(\widehat{f}\widehat{g})^{\vee}$ on $\mathbb{R}^2\setminus\{0\}$, as stated in Theorem 3.1. If $g\in L^1(\mathbb{R}^2)$ then $\widehat{g}\in L^\infty$ and a similar argument gives the L^2 convergence. ∎

P r o o f o f T h e o r e m 3.2. We write

$$f = \sum_{\mu=0}^{\infty} \widetilde{R}_\mu T_\mu f = \sum_{\mu=0}^{\infty} (\widehat{f}\,\overline{\widehat{\varphi}_\mu}\widehat{\psi}_\mu)^\vee, \tag{3.3}$$

with convergence in L^2 by Lemma 2.3, and a.e. by Lemma 2.10. Lemma 2.10 also gives convergence in L^p if $f \in L^p$, $1 < p < \infty$. For each μ, $\widehat{f}\overline{\widehat{\varphi}_\mu}$ and $\widehat{\psi}_\mu$ are $(2^\mu, 2^\mu)$-bandlimited, $\widehat{f}\overline{\widehat{\varphi}_\mu} \in L^2$, and $\widehat{\psi}_\mu \in L^1 \cap L^2$. So by Theorem 3.1,

$$\begin{aligned}(\widehat{f}\,\overline{\widehat{\varphi}_\mu}\widehat{\psi}_\mu)^\vee(re^{i\theta}) &= 2^{-3\mu-1} \sum_{k\in\mathbb{N}} \sum_{m\in A_\mu} b_k^2 (\widehat{f}\,\overline{\widehat{\varphi}_\mu})^\vee(2^{-\mu} j_k e^{i\pi 2^{-\mu} m}) \\ &\quad \times \sum_{n\in A_\mu} \int_0^{2^\mu} \widehat{\psi}_\mu(\varrho, n) J_0(2^{-\mu} j_k \varrho) J_0(r\varrho)\varrho\, d\varrho\, e^{in(\theta - \pi 2^{-\mu} m)} \\ &= \sqrt{\pi}2^{-3\mu/2} \sum_{k\in\mathbb{N}} \sum_{m\in A_\mu} b_k (\widehat{f}\overline{\widehat{\varphi}_\mu})^\vee(2^{-\mu} j_k e^{i\pi 2^{-\mu} m}) \psi_{\mu k m}(re^{i\theta}),\end{aligned} \tag{3.4}$$

with the convergence stated in the theorem. By polarizing (2.16) (or a simple direct proof) and using the observation $\varphi_{\mu k m} = b_k 2^{-3\mu/2}(2\sqrt{\pi})^{-1}(\widehat{\varphi}_\mu g_{\mu k m})^\vee$ stated prior to (1.20), we obtain

$$\begin{aligned}\langle f, \varphi_{\mu k m}\rangle &= 2\pi \langle \widehat{f}, \widehat{\varphi}_{\mu k m}\rangle \\ &= 2\pi \sum_{n\in\mathbb{Z}} \int_0^{\infty} \widehat{f}(\varrho, n) b_k 2^{-3\mu/2}(2\sqrt{\pi})^{-1} \overline{\widehat{\varphi}}_\mu(\varrho, n) J_0(2^{-\mu} j_k \varrho)\varrho\, d\varrho\, e^{i\pi 2^{-\mu} m n} \\ &= \sqrt{\pi}2^{-3\mu/2} b_k (\widehat{f}\overline{\widehat{\varphi}_\mu})^\vee(2^{-\mu} j_k e^{i\pi 2^{-\mu} m}).\end{aligned} \tag{3.5}$$

Substituting this in (3.4) and the result in (3.3) yields (1.17). ∎

4. Boundedness of almost diagonal matrices on $a_p^{\alpha q}$

Here we will prove Theorem 4.2, that an almost diagonal matrix is bounded on the sequence space $a_p^{\alpha q}$. Recall from the introduction the definitions of $a_p^{\alpha q}$, of the action of a matrix on $a_p^{\alpha q}$, and of an almost diagonal matrix. Theorem 4.2 is of interest in its own right, but in addition it will be used in the proof of the norm equivalences (Theorem 6.3) for the polar wavelet identity and in our FHT multiplier result (Theorem 7.2). We follow the same general strategy as in [FJ3]. First we need a technical lemma.

LEMMA 4.1. *Suppose* $0 < a \le 1$, $\lambda_1 > 1 + 1/a$, $\lambda_2 > 2/a$, $\mu, \nu \in \mathbb{Z}$, $s = \{s_{\mu k m}\}_{k\in\mathbb{N}, m\in A_\mu}$ *(recall that* $A_\mu = \{-2^\mu, \ldots, 2^\mu - 1\}$*) is a sequence of complex numbers*, $r > 0$, *and* $\theta \in \mathbb{R}$. *Let*

$$C = \{k \in \mathbb{N} : k > 2^{\mu-\nu}\} \times A_\mu$$

and

$$D = \{k \in \mathbb{N} : k \le 2^{\mu-\nu}\} \times A_\mu.$$

Then there exists $c < \infty$, c *depending only on* a, λ_1, *and* λ_2, *such that*

$$(4.1)\quad \sum_{(k,m)\in C} |s_{\mu km}|(1+\min(2^\mu,2^\nu)|e^{i\theta}-e^{i\pi 2^{-\mu}m}|)^{-\lambda_1}(1+\min(2^\mu,2^\nu)|r-2^{-\mu}j_k|)^{-\lambda_2} \le c2^{(\mu-\nu)_+ 2/a}\Big(M_{\rm s}^{\rm pol}\Big(\sum_{(k,m)\in C}|s_{\mu km}|^a\chi_{R_{\mu km}}\Big)(re^{i\theta})\Big)^{1/a},$$

and

$$(4.2)\quad \sum_{(k,m)\in D} |s_{\mu km}|(1+\min(2^\mu,2^\nu)|e^{i\theta}-e^{i\pi 2^{-\mu}m}|)^{-\lambda_1}(1+\min(2^\mu,2^\nu)|r-2^{-\mu}j_k|)^{-\lambda_2} \le c2^{(\mu-\nu)_+ 3/a}\Big(M_{\rm s}^{\rm pol}\Big(\sum_{(k,m)\in D}|s_{\mu km}|^a\chi_{R_{\mu km}}\Big)(re^{i\theta})\Big)^{1/a},$$

where $x_+=\max(x,0)$.

Proof. If $\mu\le\nu$, the statement is independent of ν (and (4.2) is void), so it is sufficient to prove the case $\mu\ge\nu$, as follows. Pick $l_0\in\mathbb{N}$ and $n_0\in A_\nu$ such that $re^{i\theta}\in A_{\nu l_0 n_0}$. We consider (4.1) first. For $n\in A_\mu$, let

$$C_{2,n}=\{(k,m)\in\mathbb{N}\times A_\mu : 2^{\mu-\nu}<k,\ 2^{-\mu}j_{k-1}<2^{-\nu}j_2,\ \text{and}\ 2^{-\nu}n\le 2^{-\mu}m<2^{-\nu}(n+1)\}$$

and, for $l\in\mathbb{N}$, $l\ge 3$, and $n\in A_\mu$,

$$C_{l,n}=\{(k,m)\in\mathbb{N}\times A_\mu : 2^{\mu-\nu}<k,\ 2^{-\nu}j_{l-1}\le 2^{-\mu}j_{k-1}<2^{-\nu}j_l\ \text{and}\ 2^{-\nu}n\le 2^{-\mu}m<2^{-\nu}(n+1)\}.$$

(We have used this definition of $C_{2,n}$ just so we could formulate the theorem in terms of the set C without reference to j_k and j_1.) Note that if $(k,m)\in C_{l,n}$, then $|e^{i\theta}-e^{i\pi 2^{-\nu}n_0}|\le c2^{-\nu}$, $|e^{i\pi 2^{-\mu}m}-e^{i\pi 2^{-\nu}n}|\le c2^{-\nu}$, $|r-2^{-\nu}j_{l_0}|\le c2^{-\nu}$, and $|2^{-\mu}j_k-2^{-\nu}j_l|\le c2^{-\nu}$. Hence

$$(4.3)\qquad 1+2^\nu|r-2^{-\mu}j_k|\approx 1+2^\nu|2^{-\nu}j_{l_0}-2^{-\nu}j_l|=1+|j_{l_0}-j_l|$$

and

$$(4.4)\qquad 1+2^\nu|e^{i\theta}-e^{i\pi 2^{-\mu}m}|\approx 1+2^\nu|e^{i\pi 2^{-\nu}n_0}-e^{i\pi 2^{-\nu}n}|\equiv b_\nu(n-n_0),$$

where we set $b_\nu(t)=1+2^\nu|e^{i\pi 2^{-\nu}t}-1|$ (note that for $|t|\le 2^\nu$, $b_\nu(t)\approx 1+|t|$, and $b_\nu(t)$ is $2^{\nu+1}$-periodic).

Note that if $(k,m)\in C_{l,n}$, then $k\ge 2$ so $|R_{\mu km}|\approx 2^{-3\mu}j_k\ge c2^{-2\mu}2^{-\nu}j_{l-1}\ge c2^{-2\mu}2^{-\nu}j_l$. Hence

$$(4.5)\qquad B_{l,n}\equiv \min_{(k,m)\in C_{l,n}}|R_{\mu km}|\ge c\cdot 2^{-2\mu}2^{-\nu}j_l.$$

Let $R^*_{\nu ln}$ be the smallest annular sector containing $R_{\nu l_0 n_0}$ and $\bigcup_{(k,m)\in C_{l,n}}R_{\mu km}$. The angular spread of $R^*_{\nu ln}$ is equivalent to $2^{-\nu}b_\nu(n-n_0)$ (to see this, take $n_0=0$, so the spread is equivalent to $2^{-\nu}(1+|n-n_0|)$; the general case is the periodized form of this). For $l>2$, the inner radius of $R^*_{\nu ln}$ its greater than or equal to $2^{-\nu}\min(j_{l-1},j_{l_0-1})$, and the outer radius is less than or equal to $2^{-\nu}\max(j_{l_0},j_l+c)$ for some absolute constant c. For $l=2$, the outer radius is less than or equal to $2^{-\nu}\max(j_{l_0},c)\le c2^{-\nu}(1+j_{l_0})\approx c2^{-\nu}(1+|j_{l_0}-j_2|)$, and we bound the inner radius below just by 0. In all cases we obtain

$$(4.6)\qquad |R^*_{\nu ln}|\le c2^{-3\nu}(1+|j_l-j_{l_0}|)\max(j_l,j_{l_0})b_\nu(n-n_0).$$

By (4.3)–(4.4) we obtain

$$
\begin{aligned}
(*)_{l,n} &\equiv \sum_{(k,m)\in C_{l,n}} |s_{\mu k m}|(1+2^{\nu}|e^{i\theta}-e^{i\pi 2^{-\mu}m}|)^{-\lambda_1}(1+2^{\nu}|r-2^{-\mu}j_k|)^{-\lambda_2} \\
&\le c(b_v(n-n_0))^{-\lambda_1}(1+|j_{l_0}-j_l|)^{-\lambda_2}\sum_{(k,m)\in C_{l,n}}|s_{\mu k m}| \\
&\le c(b_v(n-n_0))^{-\lambda_1}(1+|j_{l_0}-j_l|)^{-\lambda_2}\Big(\sum_{(k,m)\in C_{l,n}}|s_{\mu k m}|^a\Big)^{1/a},
\end{aligned}
$$

since $a\le 1$. Now by (4.5)–(4.6),

$$
\begin{aligned}
\sum_{(k,m)\in C_{l,n}}|s_{\mu k m}|^a &\le B_{l,n}^{-1}\int \sum_{(k,m)\in C_{l,n}}|s_{\mu k m}|^a\chi_{R_{\mu k m}} \\
&\le |R^*_{\nu l n}|B_{l,n}^{-1}M_{\mathrm{s}}^{\mathrm{pol}}\Big(\sum_{(k,m)\in C_{l,n}}|s_{\mu k m}|^a\chi_{R_{\mu k m}}\Big)(re^{i\theta}) \\
&\le c2^{2(\mu-\nu)}\max(1,j_{l_0}/j_l)b_v(n-n_0)(1+|j_l-j_{l_0}|) \\
&\quad\times M_{\mathrm{s}}^{\mathrm{pol}}\Big(\sum_{(k,m)\in C}|s_{\mu k m}|^a\chi_{R_{\mu k m}}\Big)(re^{i\theta}).
\end{aligned}
$$

So altogether

$$
\begin{aligned}
(4.7)\quad (*)_{l,n} &\le c2^{2(\mu-\nu)/a}\max(1,j_{l_0}/j_l)^{1/a}b_v(n-n_0)^{-\lambda_1+1/a} \\
&\quad\times(1+|j_{l_0}-j_l|)^{-\lambda_2+1/a}\Big(M_{\mathrm{s}}^{\mathrm{pol}}\Big(\sum_{(k,m)\in C}|s_{\mu k m}|^a\chi_{R_{\mu k m}}\Big)(re^{i\theta})\Big)^{1/a}.
\end{aligned}
$$

Since $\lambda_1>1+1/a$,

$$
\sum_{n\in A_\mu}(b_v(n-n_0))^{-\lambda_1+1/a}\approx\sum_{n\in A_\mu}(1+|n|)^{-\lambda_1+1/a}\le c.
$$

Let $[l_0/2]$ denote the greatest integer in $l_0/2$. Then

$$
\sum_{l=[l_0/2]+1}^{\infty}\max(1,j_{l_0}/j_l)^{1/a}(1+|j_{l_0}-j_l|)^{-\lambda_2+1/a}\le c,
$$

since $j_l\approx\pi l$ by (2.7) and $\lambda_2>2/a\ge 1+1/a$. For $l\le[l_0/2]$, $1+|j_{l_0}-j_l|\approx j_{l_0}$, so

$$
\sum_{l=1}^{[l_0/2]}(j_{l_0}/j_l)^{1/a}(1+|j_{l_0}-j_l|)^{-\lambda_2+1/a}\le cj_{l_0}^{-\lambda_2+2/a}\log j_{l_0}\le c,
$$

since $\lambda_2>2/a$. Hence summing (4.7) over l,n implies (4.1).

For (4.2) the argument is similar. Let

$$
D_n=\{(k,m)\in\mathbb{N}\times A_\mu : k\le 2^{\mu-\nu}\text{ and }2^{-\nu}n\le 2^{-\mu}m<2^{-\nu}(n+1)\}
$$

for $n\in A_\mu$. For $(k,m)\in D_n$, we have $2^{-\mu}j_k\approx 2^{-\mu}k\le 2^{-\nu}\approx 2^{-\nu}j_1$, which implies $|2^{-\mu}j_k-2^{-\nu}j_1|\le c2^{-\nu}$, so (4.3) holds with j_l replaced by j_1, and (4.4) holds as stated. However, we can only obtain $\min_{(k,m)\in D_n}|R_{\mu k m}|\ge c2^{-3\mu}$, since $k=1$ is possible. If $R^*_{\nu l n}$

is as above except with D_n in place of $C_{l,n}$, then $|R^*_{\nu l n}| \le c2^{-3\nu} j_{l_0}^2 b_v(n-n_0)$. Now denote by $(*)_n$ the sum on the left side of (4.2) with D replaced by D_n. Then

$$(*)_n \le c(b_v(n-n_0))^{-\lambda_1+1/a} j_{l_0}^{-\lambda_2+2/a} 2^{3(\mu-\nu)/a} \Big(M_s^{\mathrm{pol}} \Big(\sum_{(k,m)\in D} |s_{\mu k m}|^a \chi_{R_{\mu k m}} \Big)(re^{i\theta}) \Big)^{1/a}.$$

Since $j_{l_0}^{-\lambda_2+2/a} \le c$, summing on n as above gives (4.2). ■

P r o o f o f T h e o r e m 4.2. Suppose $re^{i\theta} \in R_{\nu l n}$. Then $|r - 2^{-v} j_l| \le c2^{-\nu}$ and $|e^{i\theta} - e^{i\pi 2^{-\nu} n}| \le c \cdot 2^{-\nu}$. Hence, with χ_j as in the statement of the theorem,

$$\begin{aligned} |(B_s)_{\nu l n}| &\le \sum_{\mu=0}^{\infty} \sum_{(k,m)\in \mathbb{N}\times A_\mu} \omega_{\nu l n;\mu k m}(\varepsilon)|s_{\mu k m}| \\ &\le |R_{\nu l n}|^{1/2} \sum_{\mu=0}^{\infty} 2^{(\mu-\nu)\alpha} \min(2^{(\nu-\mu)(J+\varepsilon)}, 2^{(\mu-\nu)\varepsilon}) \\ &\quad \times \sum_{k,m} (1 + (2^{(\nu-\mu)J/2} - 1)\chi_{\mu-\nu}(k))|s_{\mu k m}| \cdot |R_{\mu k m}|^{-1/2} \\ &\quad \times (1 + \min(2^\mu, 2^\nu)|e^{i\theta} - e^{i\pi 2^{-\mu} n}|)^{-J/2-1-\varepsilon} (1 + \min(2^\mu, 2^\nu)|r - 2^{-\mu} j_k|)^{-J-\varepsilon}. \end{aligned}$$

There exists $a > 0$ such that $a < \min(1, p, q)$ and

$$\varepsilon' = \min\left(\frac{J}{2} - \frac{1}{a} + \varepsilon, J - \frac{2}{a} + \varepsilon, \frac{3J}{2} - \frac{3}{a} + \varepsilon \right) > 0.$$

Apply Lemma 4.1 with this a, $\lambda_1 = \frac{J}{2} + 1 + \varepsilon \ge 1 + \varepsilon' + 1/a$ and $\lambda_2 = J + \varepsilon \ge \varepsilon' + 2/a$. We obtain

$$|(Bs)_{\nu l n}| \le c|R_{\nu l n}|^{1/2} \sum_{\mu=0}^{\infty} 2^{(\mu-\nu)\alpha} 2^{-|\mu-\nu|\varepsilon'} \Big(M_{\mathrm{s}}^{\mathrm{pol}} \Big(\sum_{k,m} |s_{\mu k m}| \widetilde{\chi}_{\mu k m} \Big)^a (re^{i\theta}) \Big)^{1/a},$$

since the $\{R_{\mu k m}\}_{k,m}$ are disjoint for each μ. Hence

$$\begin{aligned} &\Big(\sum_{\nu=0}^{\infty} (2^{\nu\alpha} |(Bs)_{\nu l n}| \widetilde{\chi}_{\nu l n}(re^{i\theta}))^q \Big)^{1/q} \\ &\qquad \le c \Big(\sum_{\nu=0}^{\infty} \Big(\sum_{\mu=0}^{\infty} 2^{-|\mu-\nu|\varepsilon'} \Big(M_{\mathrm{s}}^{\mathrm{pol}} \Big(\sum_{k,m} 2^{\mu\alpha} |s_{\mu k m}| \widetilde{\chi}_{\mu k m} \Big)^a (re^{i\theta}) \Big)^{1/a} \Big)^q \Big)^{1/q} \\ &\qquad \le c \Big(\sum_{\mu=0}^{\infty} \Big(M_{\mathrm{s}}^{\mathrm{pol}} \Big(\sum_{k,m} 2^{\mu\alpha} |s_{\mu k m}| \widetilde{\chi}_{\mu k m} \Big)^a (re^{i\theta}) \Big)^{q/a} \Big)^{1/q}, \end{aligned}$$

by $\|a * b\|_{\ell^q} \le \|a\|_{\ell^1} \|b\|_{\ell^q}$ if $q \ge 1$ and by $\|a * b\|_{\ell^q} \le \||a|^q * |b|^q\|_{\ell^1}^{1/q} \le \|a\|_{\ell^q} \|b\|_{\ell^q}$ if $q < 1$. Taking the L^p norm, writing $p/q = (a/q) \cdot (p/a)$, noting p/a, $q/a > 1$, and applying (2.44) yields the boundedness of B on $a_p^{\alpha q}$. ■

We remark that Theorem 4.2 can be slightly sharpened by a duality argument. We omit this result, since it is more technical and not needed later.

5. Peetre's maximal inequality

A maximal inequality due to Peetre [P] plays an important role in the theory of $F_p^{\alpha q}$ spaces. The main point is to use facts about functions of exponential type to control the $**$-maximal function, which in turn gives a strong pointwise control. Here we adapt to our polar situation the presentation in Triebel ([T1], pp. 16–17) of Peetre's result.

Our analogue of exponential type is defined using the FHT. For $\nu \in \mathbb{N}_0$, let

$$E_\nu = \{f \in L^2(\mathbb{R}^2) : \widehat{f}(\varrho, n) = 0 \text{ for } (\varrho^2 + n^2)^{1/2} > 2^\nu\}.$$

LEMMA 5.1. *Suppose $\nu \in \mathbb{N}_0$ and $f \in E_\nu$. Then $f \in L^\infty$ and*

$$\|f\|_{L^\infty} \le c2^{3\nu/2}\|f\|_{L^2}. \tag{5.1}$$

Proof. By (1.10)–(1.11), we have

$$f = \Big(\sum_{\mu=0}^{\nu+2} \overline{\widehat{\varphi}}_\mu \widehat{\psi}_\mu \widehat{f}\Big)^\vee = H_{\nu+2} f,$$

for H_N defined as above, after Lemma 2.8. As noted there, the kernel $D_{\nu+2}$ of $H_{\nu+2}$ satisfies (as in (2.27))

$$|D_{\nu+2}(re^{i\theta}, se^{i\varphi})| \le c_M 2^{3\nu}(1 + 2^{\nu+2}|e^{i\theta} - e^{i\varphi}|)^{-M}(1 + 2^{\nu+2}|r - s|)^{-M}/\max(1, 2^\nu r, 2^\nu s).$$

So, taking M sufficiently large,

$$\begin{aligned}|f(re^{i\theta})| &\le \int_0^{2\pi}\int_0^\infty |D_{\nu+2}(re^{i\theta}, se^{i\varphi})||f(se^{i\varphi})|s\,ds\,d\varphi \\ &\le c\|f\|_{L^2}\Big[\int_0^{2\pi}\int_0^\infty 2^{5\nu}(1 + 2^{\nu+2}|e^{i\theta} - e^{i\varphi}|)^{-2M}(1 + 2^{\nu+2}|r - s|)^{-2M} ds\,d\varphi\Big]^{1/2} \\ &\le c2^{3\nu/2}\|f\|_{L^2}. \ \blacksquare\end{aligned}$$

Recall from (2.24) the operator R_μ with FHT multiplier $\widehat{\varphi}_\mu$ and kernel K_μ satisfying (2.27) and (2.29). For $\nu \in \mathbb{N}_0$, $\lambda_1, \lambda_2 > 0$, $r > 0$, and $\theta \in \mathbb{R}$, we define our analogue R_ν^{**} of Peetre's $**$-maximal function by

$$R_\nu^{**} f(re^{i\theta}) = \sup_{s>0,\varphi\in\mathbb{R}} (1 + 2^\nu|r - s|)^{-\lambda_1}(1 + 2^\nu|e^{i\theta} - e^{i\varphi}|)^{-\lambda_2}|R_\nu f(se^{i\varphi})|$$

(we suppress the dependence on λ_1, λ_2). Regarding $R_\nu f(re^{i\theta})$ as a function of the real variables r and θ, define

$$\nabla_{r,\theta} R_\nu f(re^{i\theta}) = \Big(\frac{\partial R_\nu f}{\partial r}(re^{i\theta}), \frac{\partial R_\nu f}{\partial \theta}(re^{i\theta})\Big),$$

and set

$$\nabla_\nu^{**} R_\nu f(re^{i\theta}) = \sup_{s>0,\varphi\in\mathbb{R}} (1 + 2^\nu|r - s|)^{-\lambda_1}(1 + 2^\nu|e^{i\theta} - e^{i\varphi}|)^{-\lambda_2}|\nabla_{r,\theta} R_\nu f(se^{i\varphi})|.$$

As in the case of exponential type functions, we can control the derivatives by the function in the following sense.

LEMMA 5.2. *Suppose $\nu \in \mathbb{N}_0$ and $f \in L^2(\mathbb{R}^2)$. Then*

$$\nabla_\nu^{**} R_\nu f(re^{i\theta}) \le c 2^\nu R_\nu^{**} f(re^{i\theta}), \tag{5.2}$$

for $r > 0$ and $\theta \in \mathbb{R}$, with c independent of $\nu, f, r,$ and θ.

Proof. We write the proof of $\partial/\partial r$; the proof for $\partial/\partial\theta$ is exactly the same since all that is used are the analogues of (2.27) and (2.29) for the kernel D_ν of H_ν (defined in §2), which follow by the same argument. Recall that the FHT multiplier of $H_{\nu+2}$ is 1 on $\operatorname{supp}(R_\nu f)^\wedge$, so $R_\nu = H_{\nu+2} R_\nu$. Using the trivial inequality $1 + a + b \le (1+a)(1+b)$ for $a, b > 0$,

$$\begin{aligned}
&(1 + 2^\nu|r - s|)^{-\lambda_1}(1 + 2^\nu|e^{i\theta} - e^{i\varphi}|)^{-\lambda_2}\left|\frac{\partial R_\nu f}{\partial r}(se^{i\varphi})\right| \\
&\quad \le \int_0^{2\pi}\int_0^\infty \left|\frac{\partial D_{\nu+2}}{\partial r}(se^{i\varphi}, te^{i\beta})\right| \frac{(1 + 2^\nu|s - t|)^{\lambda_1}}{(1 + 2^\nu|r - t|)^{\lambda_1}} \frac{(1 + 2^\nu|e^{i\varphi} - e^{i\beta}|)^{\lambda_2}}{(1 + 2^\nu|e^{i\theta} - e^{i\beta}|)^{\lambda_2}} |R_\nu f(te^{i\beta})| t\, dt\, d\beta \\
&\quad \le c_M R_\nu^{**} f(re^{i\theta}) \int_0^{2\pi}\int_0^\infty 2^{3\nu}(1 + 2^\nu|s - t|)^{-M+\lambda_1}(1 + 2^\nu|e^{i\varphi} - e^{i\beta}|)^{-M+\lambda_2}\, dt\, d\beta \\
&\quad \le c 2^\nu R_\nu^{**} f(re^{i\theta}),
\end{aligned}$$

by taking M sufficiently large. Taking the sup on the left side yields (5.2). ■

Now we come to the key estimate.

LEMMA 5.3. *Suppose $f \in L^2$, $\nu \in \mathbb{N}_0$, $a > 0$, $\lambda_1 > 2/a$, $\lambda_2 > 1/a$, $r > 0$, and $\theta \in \mathbb{R}$. Then*

$$R_\nu^{**} f(re^{i\theta}) \le c(M_{\mathrm{s}}^{\mathrm{pol}}(|R_\nu f|^a)(re^{i\theta}))^{1/a}, \tag{5.3}$$

where c depends on a but not on $\nu, f, r,$ or θ.

Proof. Fix $r, s > 0$ and $\theta, \varphi \in \mathbb{R}$. For $\delta \in (0, 1]$ to be fixed later, let

$$F_\delta = \{te^{i\beta} \in \mathbb{R}^2 : s < t < s + 2^{-\nu}\delta \text{ and } \varphi < \beta < \varphi + 2^{-\nu}\delta\}.$$

Note that

$$|F_\delta| = 2^{-2\nu}\delta^2(s + 2^{-\nu-1}\delta) \approx 2^{-2\nu}\delta^2 \max(s, 2^{-\nu}\delta). \tag{5.4}$$

For $te^{i\beta} \in F_\delta$, the mean value theorem implies

$$|R_\nu f(se^{i\varphi})| \le |R_\nu f(te^{i\beta})| + c2^{-\nu}\delta \sup_{F_\delta} |\nabla_{r,\theta} R_\nu f|.$$

Taking the ath power of both sides and averaging over $te^{i\beta} \in F_\delta$ gives

$$|R_\nu f(se^{i\varphi})|^a \le c|F_\delta|^{-1} \iint_{F_\delta} |R_\nu f(te^{i\beta})|^a t\, dt\, d\beta + c(2^{-\nu}\delta)^a \sup_{F_\delta} |\nabla_{r,\theta} R_\nu f|^a. \tag{5.5}$$

By definition

$$\left|\frac{\partial R_\nu f}{\partial r}(te^{i\beta})\right| \le (1 + 2^\nu|t - r|)^{\lambda_1}(1 + 2^\nu|e^{i\beta} - e^{i\theta}|)^{\lambda_2} \nabla_\nu^{**} R_\nu f(re^{i\theta}).$$

However, for $te^{i\beta} \in F_\delta$,

$$(1 + 2^\nu|t - r|)^{\lambda_1} \le (1 + 2^\nu|t - s|)^{\lambda_1}(1 + 2^\nu|s - r|)^{\lambda_1} \le c(1 + 2^\nu|s - r|)^{\lambda_1},$$

and similarly for the factors involving θ, β and φ. Also $\partial R_\nu f/\partial\theta$ satisfies the same estimates. Using Lemma 5.2 then gives

$$\sup_{F_\delta} |\nabla_{r,\theta} R_\nu f| \le c2^\nu (1+2^\nu|s-r|)^{\lambda_1}(1+2^\nu|e^{i\varphi}-e^{i\theta}|)^{\lambda_2} R_\nu^{**} f(re^{i\theta}). \tag{5.6}$$

To consider the other term on the right of (5.5), let I_δ be the smallest annular sector containing F_δ and $re^{i\theta}$. The inner radius of I_δ is $\min(r,s)$, the outer radius is $\max(r, s+2^{-\nu}\delta) \le \max(r,s)+2^{-\nu}\delta$, and the angular spread is bounded above by $c(|e^{i\theta}-e^{i\varphi}|+2^{-\nu}\delta)$. Hence

$$|I_\delta| \le c2^{-2\nu}(1+2^\nu|r-s|)(1+2^\nu|e^{i\theta}-e^{i\varphi}|)(r+s+2^{-\nu}\delta), \tag{5.7}$$

since $\delta \le 1$. If $r \le 2s$, then

$$r+s+2^{-\nu}\delta \le 3s+2^{-\nu}\delta \le 3\max(s, 2^{-\nu}\delta),$$

while if $r > 2s$, then $r \approx |r-s|$, so

$$r+s+2^{-\nu}\delta \le c2^{-\nu}(\delta+2^\nu|r-s|) \le \frac{c}{\delta}(1+2^\nu|r-s|)\max(s,2^{-\nu}\delta).$$

Hence by (5.4) and (5.7),

$$|I_\delta|\cdot|F_\delta|^{-1} \le c\delta^{-3}(1+2^\nu|r-s|)^2(1+2^\nu|e^{i\theta}-e^{i\varphi}|).$$

Therefore

$$\begin{aligned}|F_\delta|^{-1}\iint_{F_\delta}|R_\nu f(te^{i\beta})|^a t\,dtd\beta &\le |F_\delta|^{-1}\iint_{I_\delta}|R_\nu f(te^{i\beta})|^a t\,dt\,d\beta\\ &\le |I_\delta|\cdot|F_\delta|^{-1} M_{\mathrm{s}}^{\mathrm{pol}}(|R_\nu f|^a)(re^{i\theta})\\ &\le c\delta^{-3}(1+2^\nu|r-s|)^2(1+2^\nu|e^{i\theta}-e^{i\varphi}|)M_{\mathrm{s}}^{\mathrm{pol}}(|R_\nu f|^a)(re^{i\theta}).\end{aligned}$$

Substituting this and (5.6) in (5.5) and using $\lambda_1 > 2/a$, $\lambda_2 > 1/a$ gives

$$\begin{aligned}(1+2^\nu|r-s|)^{-\lambda_1}&(1+2^\nu|e^{i\theta}-e^{i\varphi}|)^{-\lambda_2}|R_\nu f(se^{i\varphi})|\\ &\le c\delta^{-3/a}(M_{\mathrm{s}}^{\mathrm{pol}}(|R_\nu f|^a)(re^{i\theta}))^{1/a} + c'\delta R_\nu^{**} f(re^{i\theta}).\end{aligned}$$

Taking the sup over $s>0$, $\varphi\in\mathbb{R}$ on the left side gives

$$R_\nu^{**} f(re^{i\theta}) \le c\delta^{-3/a}(M_{\mathrm{s}}^{\mathrm{pol}}(|R_\nu f|^a)(re^{i\theta}))^{1/a} + c'\delta R_\nu^{**} f(re^{i\theta}). \tag{5.8}$$

Since $f\in L^2$, we have $R_\nu f\in E_\nu$. So by Lemma 5.1, $R_\nu f\in L^\infty$. Hence $R_\nu^{**} f\in L^\infty$. Pick δ so that $c'\delta = 1/2$ and subtract in (5.8) to obtain (5.3). ∎

This and the polar vector-valued inequality now give us the maximal characterization of $A_p^{\alpha q}$ that we need.

THEOREM 5.4. *Suppose $\alpha\in\mathbb{R}$ and $0<p,q<\infty$. Define R_μ^{**} for $\mu\in\mathbb{N}_0$ with $\lambda_1 > 2/\min(p,q)$ and $\lambda_2 > 1/\min(p,q)$. Then for $f\in L^2(\mathbb{R}^2)$,*

$$\|f\|_{A_p^{\alpha q}} \approx \Big\|\Big(\sum_{\mu=0}^\infty (2^{\mu\alpha}|R_\mu^{**} f|)^q\Big)^{1/q}\Big\|_{L^p}, \tag{5.9}$$

with constants depending only on α, p, q, λ_1 and λ_2.

Proof. One direction is trivial since $R_\mu f(re^{i\theta}) \le R_\mu^{**} f(re^{i\theta})$. For the other, our assumptions allow us to select $a>0$ such that $a<\min(p,q)$ but $\lambda_1 > 2/a$ and $\lambda_2 > 1/a$.

Then Lemma 5.3 gives that the right side of (5.9) is dominated by

$$\Big\|\Big(\sum_{\mu=0}^{\infty}(M_{\mathrm{s}}^{\mathrm{pol}}((2^{\mu\alpha}|R_\mu f|)^a)(re^{i\theta}))^{q/a}\Big)^{a/q}\Big\|_{L^{p/a}}^{1/a}.$$

Since $p/a, q/a > 1$, applying (2.44) allows us to remove $M_{\mathrm{s}}^{\mathrm{pol}}$ and obtain the other inequality in (5.9). ∎

This result is critical for the main norm-equivalence results (Theorems 6.3 and 6.4). In addition, it gives an easy way to see that $A_p^{\alpha q}$ is independent of the choice of φ and Φ satisfying the conditions in the introduction (cf. Peetre [P]).

THEOREM 5.5. *For $i = 1, 2$, suppose $\varphi^{(i)}$ and $\Phi^{(i)}$ satisfy the conditions in* (1.5), *and define corresponding $\{\varphi_\mu^{(i)}\}_{\mu=0}^{\infty}$ as in* (1.6)–(1.7). *Define operators $\{R_\mu^{(i)}\}_{\mu\in\mathbb{N}_0}$ acting on $f \in L^2(\mathbb{R}^2)$ by $R_\mu^{(i)} = (\widehat{\varphi}_\mu^{(i)}\widehat{f})^\vee$. For $\alpha \in \mathbb{R}$, $0 < p, q < \infty$ and $f \in L^2$, let*

$$\|f\|_{A_p^{\alpha q}(i)} = \Big\|\Big(\sum_{\mu=0}^{\infty}|2^{\mu\alpha}R_\mu^{(i)}f|^q\Big)^{1/q}\Big\|_{L^p}.$$

Then

$$\|f\|_{A_p^{\alpha q}(1)} \approx \|f\|_{A_p^{\alpha q}(2)}, \tag{5.10}$$

with constants depending on $\alpha, q, p, \varphi^{(1)}, \Phi^{(1)}, \varphi^{(2)}$, and $\Phi^{(2)}$, but not f.

Proof. By assumption, there exist $\psi^{(2)}$ and $\Psi^{(2)}$ as specified in (1.5) such that with $\{\psi_\mu^{(2)}\}_{\mu=0}^{\infty}$ defined as in (1.6), (1.8), we then have the analogue of (1.11). Taking the conjugate of this and defining $\widetilde{T}_\mu^{(2)} f = (\overline{\widehat{\psi}_\mu^{(2)}}\widehat{f})^\vee$, we obtain

$$f = \sum_{\nu=0}^{\infty} R_\nu^{(2)}\widetilde{T}_\nu^{(2)} f.$$

So (by L^2 continuity and convergence, e.g. (2.34)),

$$R_\mu^{(1)} f = \sum_{\nu=\mu-1}^{\mu+1} R_\mu^{(1)}\widetilde{T}_\nu^{(2)}R_\nu^{(2)} f, \tag{5.11}$$

since multiplier operators commute and $R_\mu^{(1)}R_\nu^{(2)} = 0$ if $|\mu-\nu| > 1$ by (1.9)–(1.10). Setting $\nu = \mu + j$, $j \in \{-1, 0, 1\}$ (or $j \in \{0, 1\}$ if $\mu = 0$), $R_\mu^{(1)}\widetilde{T}_\nu^{(2)}$ has multiplier

$$(\widehat{\varphi}_\mu^{(1)}\overline{\widehat{\psi}_\nu^{(2)}})(\varrho, n) = \mathcal{F}\varphi^{(1)}(2^{-\mu}\varrho, 2^{-\mu}n)\overline{\mathcal{F}\psi^{(2)}}(2^{-\mu-j}\varrho, 2^{-\mu-j}n) = \mathcal{F}h_j(2^{-\mu}\varrho, 2^{-\mu}n)$$

for h_j defined by $\mathcal{F}h_j(\xi) = \mathcal{F}\varphi^{(1)}(\xi)\overline{\mathcal{F}\psi^{(2)}}(2^{-j}\xi)$, by (1.6)–(1.8), if $\mu \neq 0$ and $\nu \neq 0$. If $\mu = 0$, substitute $\Phi^{(1)}$ for $\varphi^{(1)}$ and if $\nu = 0$, substitute $\Psi^{(1)}$ for $\psi^{(1)}$. Note that h_j is radial and $\mathcal{F}h_j \in \mathcal{D}(\mathbb{R}^2)$. So by the remark after Lemma 2.5, the kernel $H_{j,\mu}$ of $R_\mu^{(1)}\widetilde{T}_\nu^{(2)}$ satisfies the analogue of (2.27). Hence

$$|R_\mu^{(1)}\widetilde{T}_{\mu+j}^{(2)}R_{\mu+j}^{(2)}f(re^{i\theta})| \le \int_0^{2\pi}\int_0^{\infty}|H_{j,\mu}(re^{i\theta}, se^{i\varphi})|\cdot|R_{\mu+j}^{(2)}f(se^{i\varphi})|s\,ds\,d\varphi$$

$$\leq R_{\mu+j}^{(2)**} f(re^{i\theta}) \int_0^{2\pi} \int_0^{\infty} |H_{j,\mu}(re^{i\theta}, se^{i\varphi})|(1+2^{\mu+j}|r-s|)^{\lambda_1}(1+2^{\mu+j}|e^{i\theta}-e^{i\varphi}|)^{\lambda_2} s\, ds\, d\varphi$$
$$\leq cR_{\mu+j}^{(2)**} f(re^{i\theta}),$$

by (2.27), where we take λ_1 and λ_2 as in Theorem 5.4 and pick M large enough in (2.27). Here the constant c is independent of μ. So by (5.11) we have

$$|R_\mu^{(1)} f| \leq c \sum_{j=-1}^{+1} R_{\mu+j}^{(2)**} f.$$

Multiplying by $2^{\mu\alpha}$, taking the $L^p(\ell^q)$ norm, and applying Theorem 5.4 gives one direction in (5.10). By symmetry we have the other direction. ∎

6. Norm characterizations

We are now ready to obtain the precise characterization of the size of the coefficients $\langle f, \varphi_{\mu km}\rangle$ in the polar wavelet identity (1.17) in terms of the function space behavior of f with respect to the $A_p^{\alpha q}$ scale of spaces. It is useful to introduce our version S_φ of the φ-transform (cf. [FJ1–3]), which is defined initially on L^2 but will eventually be extended to all of $A_p^{\alpha q}$. For $f \in L^2$, let $S_\varphi f$ be the sequence $\{(S_\varphi f)_{\mu km}\}_{\mu\in\mathbb{N}_0, k\in\mathbb{N}, m\in A_\mu}$, where

$$(S_\varphi f)_{\mu km} = \langle f, \varphi_{\mu km}\rangle,$$

which makes sense by (1.16).

Lemma 6.1. *Suppose $f \in L^2(\mathbb{R}^2)$, $\alpha \in \mathbb{R}$ and $0 < p, q < \infty$. Then*

$$\|S_\varphi f\|_{a_p^{\alpha q}} \leq c\|f\|_{A_p^{\alpha q}}, \tag{6.1}$$

where c does not depend on f.

Proof. Let $\widetilde{R}_\mu f = (\overline{\widehat{\varphi}}_\mu \widehat{f})^\vee$ as before. By (3.5),

$$\langle f, \varphi_{\mu km}\rangle = \sqrt{\pi}\, 2^{-3\mu/2} b_k \widetilde{R}_\mu f(2^{-\mu} j_k e^{i\pi 2^{-\mu} m}).$$

By (2.9), $2^{-3\mu/2} b_k \approx (2^{-3\mu} j_k)^{1/2} \approx |R_{\mu km}|^{1/2}$. Also, for $re^{i\theta} \in R_{\mu km}$, we have $|r - 2^{-\mu} j_k| \leq c2^{-\mu}$ and $|e^{i\theta} - e^{i\pi 2^{-\mu} m}| \leq c2^{-\mu}$. So

$$|\widetilde{R}_\mu f(2^{-\mu} j_k e^{i\pi 2^{-\mu} m})| \leq (1+2^\mu|r-2^{-\mu} j_k|)^{\lambda_1}(1+2^\mu|e^{i\theta}-e^{i\pi 2^{-\mu} m}|)^{\lambda_2} \widetilde{R}_\mu^{**} f(re^{i\theta})$$
$$\leq c\widetilde{R}_\mu^{**} f(re^{i\theta}),$$

where we take λ_1, λ_2 as in Theorem 5.4. Hence

$$\sum_{k=1}^{\infty} \sum_{m\in A_\mu} |\langle f, \varphi_{\mu km}\rangle| \widetilde{\chi}_{\mu km} \leq c\widetilde{R}_\mu^{**} f.$$

Since the $\{R_{\mu km}\}_{k\in\mathbb{N}, m\in A_\mu}$ are disjoint for each fixed μ, we obtain

$$\|S_\varphi f\|_{a_p^{\alpha q}} \leq c\Big\| \Big(\sum_{\mu=0}^{\infty} |2^{\mu\alpha} \widetilde{R}_\mu^{**} f|^q \Big)^{1/q} \Big\|_{L^p} \leq c\|f\|_{A_p^{\alpha q}},$$

using Theorem 5.4 and the observation that the conditions on φ and Φ in (1.5) are invariant under complex conjugation, so that by Theorem 5.5, replacing R_μ by $\widetilde{R}_\mu$ gives an equivalent definition of the $A_p^{\alpha q}$ quasi-norm. ■

The inverse φ-transform T_ψ is defined formally on a sequence $s=\{s_{\mu km}\}_{\mu\in\mathbb{N}_0,k\in\mathbb{N},m\in A_\mu}$ by

$$T_\psi s = \sum_{\mu=0}^{\infty}\sum_{k=1}^{\infty}\sum_{m\in A_\mu} s_{\mu km}\psi_{\mu km}.$$

Initially we define T_ψ on sequences s with only finitely many nonzero terms. The next lemma allows us to extend T_ψ to $\ell^2 = \{s : (\sum_{\mu,k,m}|s_{\mu km}|^2)^{1/2} = \|s\|_{\ell^2} < \infty\}$, and eventually T_ψ will be extended to all of $a_p^{\alpha q}$.

LEMMA 6.2. *Suppose $\alpha\in\mathbb{R}$, $0<p,q<\infty$, and $s\in\ell^2$. For $N\in\mathbb{N}$, let $s^N=\{s^N_{\mu km}\}$ where $s^N_{\mu km} = s_{\mu km}$ if $0\le\mu\le N$, $1\le k\le N$, and $m\in A_\mu$, and $s^N_{\mu km}=0$ otherwise. Then $\lim_{N\to\infty}T_\psi s^N = \lim_{N\to\infty}\sum_{\mu=0}^{N}\sum_{k=1}^{N}\sum_{m\in A_\mu}s_{\mu km}\psi_{\mu km}$ exists in $L^2(\mathbb{R}^2)$. If we denote this limit by*

$$T_\psi s = \sum_{\mu=0}^{\infty}\sum_{k=1}^{\infty}\sum_{m\in A_\mu} s_{\mu km}\psi_{\mu km}, \tag{6.2}$$

then

$$\|T_\psi s\|_{A_p^{\alpha q}} \le c\|s\|_{a_p^{\alpha q}}, \tag{6.3}$$

with c independent of s.

Proof. Suppose first that $s=\{s_{\mu km}\}_{\mu km}$ has only finitely many nonzero terms. Then for $\nu\in\mathbb{N}_0$,

$$R_\nu(T_\psi s)(re^{i\theta}) = \sum_{\mu=\nu-1}^{\nu+1}\sum_{k=1}^{\infty}\sum_{m\in A_\mu} s_{\mu km}R_\nu(\psi_{\mu km})(re^{i\theta}), \tag{6.4}$$

since $R_\nu\psi_{\mu km}=0$ if $|\mu-\nu|>1$ by (1.10). By the observation that

$$\widehat{\psi}_{\mu km} = b_k 2^{-3\mu/2}(2\sqrt{\pi})^{-1}\widehat{\psi}_\mu g_{\mu km}$$

stated just prior to (1.20), we have

$$\begin{aligned} R_\nu(\psi_{\mu km})(re^{i\theta}) &= (\widehat{\varphi}_\nu\widehat{\psi}_{\mu km})^\vee(re^{i\theta}) \\ &= (2\sqrt{\pi})^{-1}b_k 2^{-3\mu/2} \\ &\quad\times\sum_{n\in\mathbb{Z}}\int_0^\infty \widehat{\varphi}_\nu(\varrho,n)\widehat{\psi}_\mu(\varrho,n)J_0(2^{-\mu}j_k\varrho)e^{-i\pi 2^{-\mu}mn}J_0(\varrho r)\varrho\,d\varrho\,e^{in\theta}. \end{aligned}$$

Let $\nu=\mu+j$, $j\in\{-1,0,1\}$, and let $\gamma_j\in\mathcal{S}(\mathbb{R}^2)$ be defined so that $\mathcal{F}\gamma_j(\xi) = \mathcal{F}\varphi(2^{-j}\xi)\mathcal{F}\psi(\xi)$ if $\mu,\nu\ge 1$, and similarly with φ replaced by Φ if $\nu=0$ and ψ replaced by Ψ if $\mu=0$. By (1.6)–(1.8),

$$\widehat{\varphi}_\nu(\varrho,n)\widehat{\psi}_\mu(\varrho,n) = \mathcal{F}\gamma_j(2^{-\mu}\varrho, 2^{-\mu}n).$$

Changing variables gives

$$R_\nu(\psi_{\mu km})(re^{i\theta}) = (2\sqrt{\pi})^{-1} b_k 2^{\mu/2} \sum_{\xi\in 2^{-\mu}\mathbb{Z}} \int_0^\infty \mathcal{F}\gamma_j(\varrho,\xi) e^{i2^\mu \xi(\theta - 2^{-\mu}\pi m)} J_0(2^\mu r\varrho) J_0(j_k\varrho)\varrho\, d\varrho.$$

Let B_μ be as in Lemma 2.2, but with $f(\varrho,\xi)$ replaced by $\mathcal{F}\gamma_j(\varrho,\xi)$. The argument in the proof of Lemma 2.5 shows that for any L and K, $B_\mu \le c_{L,K,\gamma_j}$. Hence by Lemma 2.2,

$$|R_\nu(\psi_{\mu km})(re^{i\theta})| \\ \le c_M b_k 2^{3\mu/2} (1 + 2^\mu |e^{i\theta} - e^{i\pi 2^{-\mu} m}|)^{-M} (1 + 2^\mu |r - 2^{-\mu} j_k|)^{-M} / \max(j_k, 2^\mu r).$$

Fix $M > 2/\min(1,p,q)$, and define, for $\nu,\mu\in\mathbb{N}_0$, $k,l\in\mathbb{N}$, $n\in A_\nu$, and $m\in A_\mu$,

$$b_{\nu ln;\mu km} = b_k j_l^{1/2} (1 + 2^\mu |e^{i\pi 2^{-\nu} n} - e^{i\pi 2^{-\mu} m}|)^{-M} (1 + 2^\mu |2^{-\nu} j_l - 2^{-\mu} j_k|)^{-M} / j_k$$

if $|\mu-\nu| \le 1$ and $b_{\nu ln;\mu km} = 0$ if $|\mu - \nu| \ge 1$. Let

$$B = \{b_{\nu ln;\mu km}\}.$$

By (2.9) and (1.22), B is almost diagonal, hence bounded, on $a_p^{\alpha q}$. For $re^{i\theta}\in R_{\nu ln}$, we have $|r - 2^{-\nu} j_l| \le c2^{-\nu}$ and $|e^{i\theta} - e^{i\pi 2^{-\nu} n}| \le c2^{-\nu}$, so

$$|R_\nu(\psi_{\mu km})(re^{i\theta})| \le c 2^{3\mu/2} j_l^{-1/2} b_{\nu ln;\mu km} \le c|R_{\nu ln}|^{-1/2} b_{\nu ln;\mu km}$$

if $|u - v| \le 1$. So by (6.4), for a.e. $re^{i\theta}\in\mathbb{R}^2$,

$$|R_\nu(T_\psi s)(re^{i\theta})| \le c \sum_{\mu=\nu-1}^{\nu+1} \sum_{k=1}^\infty \sum_{m\in A_\mu} |s_{\mu km}| |R_\nu(\psi_{\mu km})(re^{i\theta})|$$

$$\le c \sum_{l=1}^\infty \sum_{n\in A_\nu} \sum_{\mu,k,m} |s_{\mu km}| b_{\nu ln;\mu km} \widetilde{\chi}_{\nu ln} = c \sum_{l=1}^\infty \sum_{n\in A_\nu} (B|s|)_{\nu ln} \widetilde{\chi}_{\nu ln},$$

where $|s|$ is the sequence with $|s|_{\mu km} = |s_{\mu km}|$. Since the $\{R_{\nu ln}\}_{l,n}$ are disjoint for each ν, this gives

$$\|T_\psi s\|_{A_p^{\alpha q}} \le c\|B|s|\|_{a_p^{\alpha q}} \le c\||s|\|_{a_p^{\alpha q}} = c\|s\|_{a_p^{\alpha q}},$$

by Theorem 4.2.

Now suppose $s\in\ell^2$. By definition, $\|s\|_{a_2^{02}} = \|s\|_{\ell^2}$ and as we noted in (2.32), $L^2 \approx A_2^{02}$. Clearly $\{s^N\}_{N=1}^\infty$ is Cauchy in ℓ^2, so by (6.3) for finite sequences, $\{T_\psi s^N\}_{N=1}^\infty$ is Cauchy in L^2. So $T_\psi s = \lim_{N\to\infty} T_\psi s^N$ exists in $L^2(\mathbb{R}^2)$. By the continuity of R_ν on L^2, (6.4) still holds, and the previous argument carries over verbatim to yield (6.3) in general. ∎

Proof of Theorem 6.3. For α, p, q, f, and $s = \{s_{\mu km}\}$ as in the statement of the theorem (see §1), we have

$$\|s\|_{a_p^{\alpha q}} \le c\|f\|_{A_p^{\alpha q}}$$

immediately by Lemma 6.1. Also since $f\in L^2$, taking $\alpha = 0$ and $q = p = 2$ gives that $s\in\ell^2$. By Theorem 3.2, $f = T_\psi s$, so (6.3) gives the other inequality. ∎

It is worth noting that for $f = \sum_{\mu,k,m} s_{\mu km}\psi_{\mu km}$, we do not in general have $\|f\|_{A_p^{\alpha q}} \approx \|s\|_{a_p^{\alpha q}}$, due to the non-orthogonality of the expansion in Theorem 3.2. In fact, since $\psi_{\nu ln} = \sum_{\mu,k,m} \langle \psi_{\nu ln}, \varphi_{\mu km}\rangle \psi_{\mu km}$ we see that 0 can be represented nontrivially as

$\sum_{\mu,k,m} s_{\mu km}\psi_{\mu km}$. Theorem 6.3 only gives equivalence of norms for the particular representation where $s_{\mu km} = \langle f, \varphi_{\mu km}\rangle$. However, Lemma 6.2 carries more information since it guarantees that one of the estimates, namely

$$\Big\| \sum_{\mu,k,m} s_{\mu km}\psi_{\mu km}\Big\|_{A_p^{\alpha q}} \le c\|s\|_{a_p^{\alpha q}}$$

holds for any sequence $s \in \ell^2$ (and even this restriction on s will eventually be dropped).

Recall that $\mathcal{A}_p^{\alpha q} = \{f \in L^2(\mathbb{R}^2) : \|f\|_{A_p^{\alpha q}} < \infty\}$ and $A_p^{\alpha q}$ is the completion of $\mathcal{A}_p^{\alpha q}$ with respect to the $A_p^{\alpha q}$ quasi-norm. It is now a formal matter to extend S_φ to $A_p^{\alpha q}$, T_ψ to $a_p^{\alpha q}$, and obtain Theorem 6.4. However, it is worthwhile to be precise about the resulting definitions.

First, suppose $f \in A_p^{\alpha q}$. Pick $\{f_n\}_{n=1}^\infty$ a sequence of elements of $\mathcal{A}_p^{\alpha q}$ such that $\lim_{n\to\infty} \|f_n - f\|_{A_p^{\alpha q}} = 0$. Then $\{f_n\}_{n=1}^\infty$ is Cauchy in $\mathcal{A}_p^{\alpha q}$, so by Lemma 6.1, $\{S_\varphi f_n\}_{n=1}^\infty$ is Cauchy in $a_p^{\alpha q}$. But $a_p^{\alpha q}$ is complete, so define $S_\varphi f$ to be the limit in $a_p^{\alpha q}$ of $\{S_\varphi f_n\}_{n=1}^\infty$. Now for any sequence $s = \{s_{\mu km}\}$,

$$|s_{\mu km}| \le 2^{-\mu\alpha}|R_{\mu km}|^{1/2-1/p}\|s\|_{a_p^{\alpha q}},$$

so

$$\langle f, \varphi_{\mu km}\rangle \equiv \lim_{n\to\infty} \langle f_n, \varphi_{\mu km}\rangle$$

exists in $\mathbb{C}$ for each μ, k, and m. It is easy to see that this value is independent of the choice of $\{f_n\}_{n=1}^\infty$, hence agrees with the usual meaning of $\langle f, \varphi_{\mu km}\rangle$ if $f \in \mathcal{A}_p^{\alpha q}$, and that with the above definition of $S_\varphi f$ we still have

$$S_\varphi f = \{\langle f, \varphi_{\mu km}\rangle\}_{\mu km}.$$

Also $S_\varphi : A_p^{\alpha q} \to a_p^{\alpha q}$ is bounded.

Similarly we can extend T_ψ. For $s \in a_p^{\alpha q}$, define $\{s^N\}_{N=1}^\infty$ as in Lemma 6.2. By the dominated convergence theorem,

$$\lim_{N\to\infty} \|s - s^N\|_{a_p^{\alpha q}} = 0.$$

So by Lemma 6.2, $\{T_\psi s^N\}_{N=1}^\infty$ is Cauchy in $A_p^{\alpha q}$. We define

$$T_\psi s = \lim_{N\to\infty} T_\psi s^N = \lim_{N\to\infty} \sum_{\mu=0}^{N}\sum_{k=1}^{N}\sum_{m\in A_\mu} s_{\mu km}\psi_{\mu km}$$

and we denote this limit in $A_p^{\alpha q}$ as $\sum_{\mu,k,m} s_{\mu km}\psi_{\mu km}$. Then $T_\psi : a_p^{\alpha q} \to A_p^{\alpha q}$ is bounded and agrees with the former definition on $a_p^{\alpha q} \cap \ell^2$. Finally, $T_\psi \circ S_\varphi : A_p^{\alpha q} \to A_p^{\alpha q}$ is bounded and agrees with identity on the dense subspace $\mathcal{A}_p^{\alpha q}$ of $A_p^{\alpha q}$, hence $T_\psi \circ S_\varphi$ is the identity on $A_p^{\alpha q}$.

Proof of Theorem 6.4. Let $s_{\mu km} = \langle f, \varphi_{\mu km}\rangle$ and $s = \{s_{\mu km}\}_{\mu,k,m}$, for f as given in the theorem. Then f_N as given there is just $T_\psi s^N$. But by the above,

$$\|T_\psi(s^N - s)\|_{A_p^{\alpha q}} \le c\|s^N - s\|_{a_p^{\alpha q}} \underset{N\to\infty}{\longrightarrow} 0.$$

So (1.17) holds in the required sense. Also

$$\|f\|_{A_p^{\alpha q}} = \|T_\psi S_\varphi f\|_{A_p^{\alpha q}} \le c\|S_\varphi f\|_{a_p^{\alpha q}} \le c\|f\|_{A_p^{\alpha q}},$$

so (1.21) holds. ∎

7. FHT multiplier and potential operators

Theorem 6.4 can be used to study linear operators on $A_p^{\alpha q}$. This is analogous to the classical case of $F_p^{\alpha q}$, as in e.g. [FJ3]. The following is trivial to obtain formally, but requires some care to make precise. Recall that $\mathcal{A}_p^{\alpha q} = L^2 \cap A_p^{\alpha q}$.

LEMMA 7.1. *Suppose $\alpha \in \mathbb{R}$, $0 < p, q < \infty$, and T is a linear operator which is bounded on $L^2(\mathbb{R}^2)$. Define a matrix $B = \{b_{\nu ln;\mu km}\}$ for $\mu, \nu \in \mathbb{N}_0$, $l, k \in \mathbb{N}$, $n \in A_\nu$ and $m \in A_\mu$ by*

$$b_{\nu ln;\mu km} = \langle T\psi_{\mu km}, \varphi_{\nu ln}\rangle.$$

Suppose B is bounded on $a_p^{\alpha q}$. Then for $f \in \mathcal{A}_p^{\alpha q}$, we have

$$Tf = T_\psi B S_\varphi f,$$

for S_φ and T_ψ defined as in §6. Hence T extends to be a bounded operator on $A_p^{\alpha q}$.

Proof. Let $f \in \mathcal{A}_p^{\alpha q}$ and set $s = S_\varphi f$, i.e. $s_{\mu km} = \langle f, \varphi_{\mu km}\rangle$. For $N \in \mathbb{N}$, define s^N, the truncation of s, as in Lemma 6.2. Let

$$f_N = \sum_{\mu=0}^{N}\sum_{k=1}^{N}\sum_{m\in A_\mu} s_{\mu km}\psi_{\mu km} = T_\psi s^N.$$

Then $Tf = \lim_{N\to\infty} Tf_N$ in L^2. For each N,

$$Tf_N = \sum_{\mu=0}^{N}\sum_{k=1}^{N}\sum_{m\in A_\mu} s_{\mu km} T\psi_{\mu km}.$$

But by Theorem 3.2,

$$T\psi_{\mu km} = \sum_{\nu=0}^{\infty}\sum_{l=1}^{\infty}\sum_{n\in A_\nu} \langle T\psi_{\mu km}, \varphi_{\nu ln}\rangle \psi_{\nu ln},$$

with $\sum_{\nu=0}^{\infty}\sum_{l=1}^{\infty}$ interpreted as the L^2 limit as $M \to \infty$ of $\sum_{\nu=0}^{M}\sum_{l=1}^{M}$. So we can interchange the $\lim_{M\to\infty}$ with the finite sum to get

$$Tf_N = \sum_{\nu=0}^{\infty}\sum_{l=1}^{\infty}\sum_{n\in A_\nu} (Bs^N)_{\nu ln}\psi_{\nu ln} = T_\psi B s^N.$$

So

$$\|Tf_N - T_\psi Bs\|_{A_p^{\alpha q}} = \|T_\psi B(s^N - s)\|_{A_p^{\alpha q}} \le c\|B(s^N - s)\|_{a_p^{\alpha q}} \le c\|s^N - s\|_{a_p^{\alpha q}},$$

by the boundedness of T_ψ and B. By the dominated convergence theorem, we have $\lim_{N\to\infty}\|s^N - s\|_{a_p^{\alpha q}} = 0$. Hence Tf_N converges to $T_\psi Bs$ in $A_p^{\alpha q}$. But B is also ℓ^2-bounded. (In fact, by L^2 convergence and the L^2 boundedness of T, we have, for $s \in \ell^2$,

$$\langle T(T_\psi s), \varphi_{\nu ln}\rangle = \Big\langle \sum_{\mu,k,m} s_{\mu km} T\psi_{\mu km}, \varphi_{\nu ln}\Big\rangle = \sum_{\mu,k,m} s_{\mu km} b_{\nu ln;\mu km},$$

or $B = S_\varphi T T_\psi$.) So by the same argument Tf_N converges to $T_\psi Bs$ in L^2, so $Tf = T_\psi Bs$, as required. Then the boundedness of S_φ and T_ψ imply that T is bounded on $\mathcal{A}_p^{\alpha q}$. ∎

With this, Lemma 2.2, and Theorem 4.2, we can now prove our FHT multiplier theorem, i.e. Theorem 7.2 stated in §1.

Proof of Theorem 7.2. Corresponding to $T = T_m$ is the matrix $B = \{b_{\nu ln;\mu km}\}$, where

$$b_{\nu ln;\mu km} = \langle T\psi_{\mu km}, \varphi_{\nu ln}\rangle = 2\pi\langle m\widehat{\psi}_{\mu km}, \widehat{\varphi}_{\nu ln}\rangle,$$

by polarizing (2.16). This is zero if $|\mu-\nu| > 1$ by (1.9)–(1.10). For $|\mu-\nu| \le 1$, let $\nu = \mu + j$, $j \in \{-1,0,1\}$, and define $\gamma_j \in \mathcal{S}(\mathbb{R}^2)$ by $\gamma_j(x) = \mathcal{F}\psi(x)\overline{\mathcal{F}\varphi}(2^{-j}x)$, for $x \in \mathbb{R}^2$, if $\mu, \nu \ge 1$, and similarly with ψ replaced by Ψ if $\mu = 0$ and φ replaced by Φ if $\nu = 0$. Then by the characterization of $\widehat{\varphi}_{\mu km}$ and $\widehat{\psi}_{\mu km}$ before (1.20) and by (1.6)–(1.8),

$$\begin{aligned} b_{\nu ln;\mu km} &= 2\pi \sum_{t=0}^{\infty} \int_0^{\infty} m(\varrho,t) b_k 2^{-3\mu/2}(2\sqrt{\pi})^{-1}\widehat{\psi}_\mu(\varrho,t) J_0(2^{-\mu}j_k\varrho) e^{-i\pi 2^{-\mu}mt} \\ &\quad\times b_l 2^{-3\nu/2}(2\sqrt{\pi})^{-1}\overline{\widehat{\varphi}}_\nu(\varrho,t) J_0(2^{-\nu}j_l\varrho)e^{i\pi 2^{-\nu}nt}\varrho\,d\varrho \\ &= \frac{1}{2} b_k b_l 2^{\mu/2} 2^{-3\nu/2} \sum_{\xi\in 2^{-\mu}\mathbb{Z}} \int_0^{\infty} m(2^\mu\varrho, 2^\mu\xi)\gamma_j(\varrho,\xi) e^{i2^\mu\xi(2^{-\mu}\pi m - 2^{-\nu}\pi n)} \\ &\quad\times J_0(j_k\varrho)J_0(2^{-j}j_l\varrho)\varrho\,d\varrho. \end{aligned}$$

For $\mu \in \mathbb{N}_0$, and L, K as in the statement of the theorem, let

$$B_\mu = \max_{\substack{l=0,L/2\\k=0,K}} 2^{-\mu} \int_0^{\infty} \sum_{\xi\in 2^{-\mu}\mathbb{Z}} |((D_{\mu,\xi})^{(k)}(\Delta_\varrho)^l)(m(2^\mu\varrho,2^\mu\xi)\gamma_j(\varrho,\xi))|\varrho\,d\varrho.$$

We claim that $\sup_{\mu\ge 0} B_\mu < \infty$. Assuming this for the moment, note that for $x \in \mathbb{R}^2$, $\gamma_j^*(x,\xi) \equiv \gamma_j(|x|,\xi)$ is C^∞ and compactly supported as a function of x, and by assumption $m^*(x,\xi) = m(|x|,\xi) \in C^L(\mathbb{R}^2)$, for each ξ. Hence we can apply Lemma 2.2 to obtain

$$\begin{aligned} &|b_{\nu ln;\mu km}| \\ &\quad\le c b_k b_l 2^{3\mu/2} 2^{-3\nu/2} j_k^{-1}(1 + 2^\mu|e^{i2^{-\mu}\pi m} - e^{i2^{-\nu}\pi n}|)^{-K}(1 + 2^\mu|2^{-\mu}j_k - 2^{-\nu}j_l|)^{-L+2}. \end{aligned}$$

Since $|\mu-\nu| \le 1$, $b_k \approx j_k^{1/2}$ (see (2.9)), $|R_{\mu km}| \approx 2^{-3\mu}j_k$, and similarly for $|R_{\nu ln}|$, we have by the definition (1.22) that B is almost diagonal on $a_p^{\alpha q}$. So by Theorem 4.2, B is $a_p^{\alpha q}$ bounded. Therefore by Lemma 7.1, m is an FHT multiplier on $A_p^{\alpha q}$.

It remains to check that $\sup_{\mu\ge 0} B_\mu < \infty$. If $\mu = \nu = 0$ (so $j = 0$) then $\gamma_0 = \mathcal{F}\Psi\overline{\mathcal{F}\Phi}$ is C^∞, radial, and supported in $\{x \in \mathbb{R}^2 : |x| \le 1\}$. So $\gamma_0(\varrho,n) = 0$ for $n \ne 0$, and the term $(D_{0,\xi})^{(k)}$ can be dropped. Looking at the radial extension in the first variable to $\mathbb{R}^2$ (i.e. for $x \in \mathbb{R}^2$, looking at $m^*(x,\xi)\gamma_j^*(x,\xi)$), and replacing Δ_ϱ by the usual Euclidean Laplacian Δ_x, we get

$$B_0 \approx \max_{l=0,L/2} \int_{\mathbb{R}^2} |(\Delta_x)^l[m^*(x,0)\gamma_0^*(x,0)]|\,dx.$$

By the Leibniz rule and the properties of γ_0, (1.24) implies that $B_0 < \infty$. Now suppose that either $\mu \ne 0$ or $\nu \ne 0$. Then $\gamma_j(\varrho,\xi) = 0$ unless $1/4 \le (\varrho^2+\xi^2)^{1/2} \le 1$. As for B_0, we consider the radial extensions to $\mathbb{R}^2$ in the first argument, and replace the integral by an equivalent integral on $\mathbb{R}^2$. We then use the chain rule and the formula

$$D_{\mu,\xi}(FG)(\varrho,\xi) = F(\varrho,\xi)D_{\mu,\xi}G(\varrho,\xi) + G(\varrho,\xi - 2^{-\mu})D_{\mu,\xi}F(\varrho,\xi)$$

repeatedly, with F the term involving m^* and G the term involving γ_j^*. Then the properties of γ_j show that B_μ is dominated by the left side of (1.25) for all $\mu \in \mathbb{N}$. ■

This result provides the key step in the proof of Theorem 7.3, stated in §1. First we need to be careful about the definition of P^α on $A_p^{\beta q}$, since the FHT multiplier $(1+\varrho^2+n^2)^{-\alpha/2}$ for P^α is not necessarily bounded (i.e. when $\alpha < 0$). However, P^α is defined, by L^2 considerations, on $E = \{f \in L^2 : \operatorname{supp} \widehat{f} \text{ is compact}\}$, and by Theorem 6.4, $E \cap A_p^{\beta q}$ is dense in $A_p^{\beta q}$. We will prove that for any $\alpha, \beta \in \mathbb{R}$, $0 < p, q < \infty$ and $f \in E$,

$$\|P^\alpha f\|_{A_p^{\alpha+\beta,q}} \le c\|f\|_{A_p^{\beta q}} \tag{7.1}$$

with c independent of f. Hence we can extend P^α to a continuous map $P^\alpha : A_p^{\beta q} \to A_p^{\alpha+\beta,q}$.

P r o o f o f T h e o r e m 7.3. Suppose we prove (7.1). Since α and β are arbitrary, we can extend $P^{-\alpha}$ to $P^{-\alpha} : A_p^{\alpha+\beta,q} \to A_p^{\beta q}$ continuously. Then $P^\alpha \circ P^{-\alpha}$ and $P^{-\alpha} \circ P^\alpha$ are the identity on dense subspaces, hence on their domains, by continuity. So Theorem 7.3 will follow from (7.1).

To prove (7.1), let $\eta : \mathbb{R}^2 \to [0,\infty)$ be C^∞, radial, and satisfy $\eta(x) = 1$ for $|x| > 1/2$ and $\eta(x) = 0$ for $|x| < 1/4$. Define $\gamma_\alpha : \mathbb{R}^2 \to [0,\infty)$ by

$$\gamma_\alpha(x) = \left(\frac{1+|x|^2}{|x|^2}\right)^{-\alpha/2} \eta(x).$$

Regarding $(\varrho, n) \in [0,\infty) \times \mathbb{Z}$ as a point in $\mathbb{R}^2$, define the FHT multiplier m_α by

$$m_\alpha(\varrho, n) = \gamma_\alpha(\varrho, n) = \left(\frac{1+\varrho^2+n^2}{\varrho^2+n^2}\right)^{-\alpha/2} \eta(\varrho, n).$$

We claim that m_α satisfies the conditions of Theorem 7.2 (for any K, L) and hence that the associated FHT multiplier operator T_{m_α} is bounded on $A_p^{\beta q}$. First, (1.24) is clear since

$$m_\alpha^*(x,0) = (1+|x|^{-2})^{-\alpha/2} \eta(|x|, 0).$$

Second, for $1/4 \le (|x|^2+\xi^2)^{1/2} \le 1$ and $\mu \ge 1$ we have $\eta(2^\mu|x|, 2^\mu\xi) = 1$, so

$$m_\alpha^*(2^\mu x, 2^\mu \xi) = (1 + 2^{-2\mu}(|x|^2+\xi^2)^{-1})^{-\alpha/2}.$$

Using the chain rule to handle ∂_x^σ and the mean value theorem to handle $(D_{\mu,\xi})^{(k)}$, it is straightforward to check that

$$|((D_{\mu,\xi})^{(k)} \partial_x^\sigma)(m_\alpha^*(2^\mu x, 2^\mu \xi))| \le c_{k,\sigma}$$

for all (x,ξ) with $1/4 \le (|x|^2+\xi^2)^{1/2} \le 1$, and $c_{k,\sigma}$ independent of μ. We see that (1.25) holds, since there are at most $c2^\mu$ nonzero terms in the sum on ξ.

Now the theorem will follow from Theorem 5.5, which says that $A_p^{\alpha q}$ is independent of the choice of test functions φ, ψ, Φ, and Ψ satisfying (1.5). With such functions given, define alternate functions $\varphi^{(1)}, \psi^{(1)}, \Phi^{(1)}$, and $\Psi^{(1)}$ by

$$\mathcal{F}\varphi^{(1)}(\xi) = |\xi|^{-\alpha}\mathcal{F}\varphi(\xi), \quad \mathcal{F}\psi^{(1)}(\xi) = |\xi|^{\alpha}\mathcal{F}\psi(\xi),$$
$$\mathcal{F}\Phi^{(1)}(\xi) = (1+|\xi|^2)^{-\alpha/2}\mathcal{F}\Phi(\xi), \quad \text{and} \quad \mathcal{F}\Psi^{(1)}(\xi) = (1+|\xi|^2)^{\alpha/2}\mathcal{F}\Psi(\xi).$$

Note that these functions are still radial, belong to $\mathcal{S}$ (since $0 \notin \operatorname{supp} \mathcal{F}\varphi, \mathcal{F}\psi$), and satisfy (1.5). Now as in (1.6) and (1.7), define $\{\varphi_\mu^{(1)}\}_{\mu=0}^\infty$ by

$$\widehat{\varphi}_0^{(1)}(\varrho, n) \equiv \mathcal{F}\Phi^{(1)}(\varrho, n) = (1 + \varrho^2 + n^2)^{-\alpha/2}\mathcal{F}\Phi(\varrho, n) = (1 + \varrho^2 + n^2)^{-\alpha/2}\widehat{\varphi}_0(\varrho, n)$$

and, for $\mu \geq 1$,

$$\begin{aligned}\widehat{\varphi}_\mu^{(1)}(\varrho, n) &\equiv \mathcal{F}\varphi^{(1)}(2^{-\mu}\varrho, 2^{-\mu}n) = 2^{\mu\alpha}(\varrho^2 + n^2)^{-\alpha/2}\mathcal{F}\varphi(2^{-\mu}\varrho, 2^{-\mu}n) \\ &= 2^{\mu\alpha}(\varrho^2 + n^2)^{-\alpha/2}\widehat{\varphi}_\mu(\varrho, n).\end{aligned}$$

Hence for $f \in E \cap A_p^{\beta q}$,

$$\begin{aligned}\|P^\alpha f\|_{A_p^{\alpha+\beta,q}} &\approx \|(\widehat{\varphi}_0(\varrho, n)(1 + \varrho^2 + n^2)^{-\alpha/2}\widehat{f})^\vee\|_{L^p} \\ &\quad + \Big\|\Big(\sum_{\mu=1}^\infty |2^{\mu(\alpha+\beta)}(\widehat{\varphi}_\mu(\varrho, n)(1 + \varrho^2 + n^2)^{-\alpha/2}\widehat{f})^\vee|^q\Big)^{1/q}\Big\|_{L^p} \\ &= \|(\widehat{\varphi}_0^{(1)}\widehat{f})^\vee\|_{L^p} + \Big\|\Big(\sum_{\mu=1}^\infty |2^{\mu\beta}(\widehat{\varphi}_\mu^{(1)} m_\alpha \widehat{f})^\vee|^q\Big)^{1/q}\Big\|_{L^p} \\ &\leq c(\|f\|_{A_p^{\beta q}} + \|T_{m_\alpha} f\|_{A_p^{\beta q}}),\end{aligned}$$

by Theorem 5.5. The boundedness of T_{m_α} completes the proof. ■

8. Equivalence of L^p and A_p^{02}, $1 < p < \infty$

This section is devoted to the proof of Theorem 8.8, that $A_p^{02} \approx L^p$ for $1 < p < \infty$. The proof is an adaption to the polar grid of the standard Littlewood–Paley arguments based on vector-valued Calderón–Zygmund theory (as in [S1], §4). We will refer to the literature for the portions of the proofs that follow the usual lines.

We begin with a discussion of the polar maximal operator M^{pol}, defined by (2.38). First we define polar regions $\widetilde{B}(re^{i\theta}, \delta)$ for each $r, \delta > 0$ and $\theta \in \mathbb{R}$. If $\delta \leq \pi$, let

$$\widetilde{B}(re^{i\theta}, \delta) = \{se^{i\varphi} \in \mathbb{R}^2 : \max(0, r - \delta) < s < \max(r + \delta, 2\delta),\ \theta - \delta < \varphi < \theta + \delta\}.$$

If $\delta > \pi$, let

$$\widetilde{B}(re^{i\theta}, \delta) = \{se^{i\varphi} \in \mathbb{R}^2 : \max(0, r - \delta) < s < \max(r + \delta, 2\delta), \varphi \in \mathbb{R}\}.$$

Observe that the class $\mathcal{M}$ of admissible sets, defined in §2, coincides with $\{\widetilde{B}(re^{i\theta}, \delta) : r, \delta > 0, \theta \in \mathbb{R}\}$. Notice that

(8.1) $\quad$ if $\delta' > \delta/2$ and $B(re^{i\theta}, \delta') \cap B(se^{i\varphi}, \delta) \neq \emptyset$, then $B(re^{i\theta}, \delta) \subseteq B(se^{i\varphi}, 5\delta')$.

Also note that $|\widetilde{B}(re^{i\theta}, \delta)| = 4\delta \max(r, \delta) \min(\delta, \pi)$, and hence

$$|\widetilde{B}(re^{i\theta}, 5\delta)| \leq 125|\widetilde{B}(re^{i\theta}, \delta)|. \tag{8.2}$$

Lemma 8.1. *M^{pol} is of weak-type $(1, 1)$; i.e., there exists $c < \infty$ such that for all $f \in L^1(\mathbb{R}^2)$ and $\lambda > 0$,*

$$|\{x \in \mathbb{R}^2 \backslash \{0\} : M^{\mathrm{pol}} f(x) > \lambda\}| \leq \frac{125}{\lambda}\|f\|_{L^1}.$$

P r o o f. We follow the proof in [S1], pp. 6–10, the only possibility for problems coming from the fact that $|\widetilde{B}(re^{i\theta},\delta)|$ depends on r as well as δ. In the covering lemma ([S1], p. 9), we still pick $I_j = \widetilde{B}(x_j,\delta_j)$ inductively to be disjoint from $\bigcup_{l=1}^{j-1} I_l$ and to have δ_j at least $\frac{1}{2}$ the maximum possible among such sets. If $\sum_{j=1}^{\infty}|I_j| < \infty$ then $\lim_{j\to\infty}|I_j| = 0$, so from $4\delta_j^2\min(\delta_j,\pi) \le |I_j|$ we get $\lim_{j\to\infty}\delta_j = 0$. So the same argument applies. ∎

LEMMA 8.2. *Suppose $f \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$. Then almost every $x \in \mathbb{R}^2\backslash\{0\}$ has the property that for any sequence $\{I_j\}_{j=1}^{\infty}$ of admissible sets such that $x \in I_j$ for all j and $\lim_{j\to\infty}|I_j| = 0$,*

$$\lim_{j\to\infty}\frac{1}{|I_j|}\int_{I_j} f(y)\,dy = f(x).$$

P r o o f. This follows from Lemma 8.1 as in [S1], §1.5. ∎

We have a notion of dyadic admissible sets. For $\mu \in \mathbb{Z}$, let $A_\mu = \{-2^\mu,\ldots,2^\mu-1\}$ if $\mu \ge 0$ and $A_\mu = \{0\}$ if $\mu < 0$. For $\mu \in \mathbb{Z}$, $k \in \mathbb{N}$, and $m \in A_\mu$, let

$$I_{\mu km} = \{re^{i\theta} \in \mathbb{R}^2 : (k-1)2^{-\mu}\pi < r < k2^{-\mu}\pi,\ \text{and } 2^{-\mu}\pi m < \theta < 2^{-\mu}\pi(m+1)\}$$

if $\mu \ge 0$, and

$$I_{\mu k0} = \{re^{i\theta} \in \mathbb{R}^2 : (k-1)2^{-\mu}\pi < r < k2^{-\mu}\pi,\ \theta \in \mathbb{R}\}$$

if $\mu < 0$. Then for each $\mu \in \mathbb{Z}$, the sets $\{I_{\mu km}\}_{k,m}$ are disjoint and open with $\bigcup_{k,m}\overline{I}_{\mu km} = \mathbb{R}^2$. We call a set I *dyadic* if and only if $I = I_{\mu km}$ for some μ,k,m. The dyadic sets have the nesting property:

$$\text{(8.3)}\qquad \text{if } I \text{ and } J \text{ are dyadic and } I\cap J \ne \emptyset, \text{ then either } I \subseteq J \text{ or } J \subseteq I.$$

This leads to a version of the Calderón–Zygmund lemma.

LEMMA 8.3. *Suppose $f \in L^1(\mathbb{R}^2)$ and $\lambda > 0$. Then there exist disjoint dyadic sets $\{I_j\}_j$ such that*

$$\text{(8.4)}\qquad |f(x)| \le \lambda \quad \textit{for a.e. } x \in \mathbb{R}^2\backslash\bigcup_j I_j,$$

$$\text{(8.5)}\qquad \sum_j |I_j| \le \|f\|_{L^1}/\lambda,$$

and

$$\text{(8.6)}\qquad \frac{1}{|I_j|}\int_{I_j}|f(x)|\,dx \le 8\lambda \quad \textit{for all } j.$$

P r o o f. Let $\mathcal{O} = \{I \text{ dyadic} : |I|^{-1}\int_I |f(y)|dy > \lambda\}$. Let $\{I_j\}_{j=1}^{\infty}$ be the maximal sets in $\mathcal{O}$, i.e., those not contained in any larger set in $\mathcal{O}$. The disjointness follows by (8.3), (8.4) follows by Lemma 8.2, and (8.5) by the definition of $\mathcal{A}$ and the disjointness, all in the usual manner (see [S1], §3.2). Then (8.6) follows by the maximality as usual. The number 8 arises because $|I_{\mu km}| \ge k2^{-2\mu-1}\pi^3\min(2,2^{-\mu})$ and if $I^*_{\mu km}$ is the unique dyadic set $I_{\mu-1,jl}$ containing $I_{\mu km}$, then $|I^*_{\mu km}| \le 8k2^{-2\mu-1}\pi^3\min(2,2^{-\mu})$. So $|I^*_{\mu km}|/|I_{\mu km}| \le 8$. ∎

We can now set up the notation needed for the proof of Theorem 8.8. Recall that for $\mu \in \mathbb{N}_0$ and $f \in L^p$, $1 < p < \infty$, we have defined operators R_μ and $\widetilde{T}_\mu$ by $R_\mu f = (\widehat{\varphi}_\mu\widehat{f})^\vee$ and $\widetilde{T}_\mu f = (\overline{\widehat{\psi}}_\mu\widehat{f})^\vee$, for $\{\varphi_\mu\}_{u=0}^{\infty}$ and $\{\psi_\mu\}_{\mu=0}^{\infty}$ as in (1.6)–(1.8). The kernel K_μ of R_μ

satisfies the estimates (2.27) and (2.29), and the kernel of $\widetilde{T}_\mu$, which we will call L_μ, satisfies the same estimates. We define a map R on L^p, $1 < p < \infty$, by

$$Rf = \{R_\mu f\}_{\mu=0}^{\infty} \tag{8.7}$$

and a map T on $L^p(\ell^2) = \{\{f_\mu\}_{\mu=0}^\infty : \|\{f_\mu\}_{\mu=0}^\infty\|_{L^p(\ell^2)} < \infty\}$, where $\|\{f_\mu\}_{\mu=0}^\infty\|_{L^p(\ell^2)} = \|(\sum_{\mu=0}^\infty |f_\mu|^2)^{1/2}\|_{L^p(\mathbb{R}^2)}$, by

$$T(\{f_\mu\}_{\mu=0}^\infty) = \sum_{\mu=0}^{\infty} \widetilde{T}_\mu f_\mu. \tag{8.8}$$

It is not immediately clear how to interpret the right side of (8.8), but for $p = 2$ the next lemma resolves this, and Lemmas 8.5 and 8.6 will handle the remaining cases. Note that

$$\|f\|_{A_p^{02}} = \|Rf\|_{L^p(\ell^2)}. \tag{8.9}$$

LEMMA 8.4. (A) $R : L^2(\mathbb{R}^2) \to L^2(\ell^2)$ *is bounded.*
(B) $T : L^2(\ell^2) \to L^2(\mathbb{R}^2)$ *is bounded.*

Proof. (A) follows immediately from (2.32). The proof of (B) is similar. The conditions on ψ and Ψ in (1.5) imply that there exist $c_1, c_2 > 0$ such that

$$c_1 \le \Big(\sum_{\mu=0}^{\infty} |\widehat{\psi}_\mu(\varrho, n)|^2\Big)^{1/2} \le c_2$$

for all $(\varrho, n) \in [0,\infty) \times \mathbb{Z}$. Using this and Lemma 2.3,

$$\begin{aligned}\Big\|\sum_{\mu=0}^{\infty} \widetilde{T}_\mu f_\mu\Big\|_{L^2}^2 &= 2\pi \sum_{n\in\mathbb{Z}} \int_0^\infty \Big|\sum_{\mu=0}^{\infty} \overline{\widehat{\psi}}_\mu(\varrho,n)\widehat{f}_\mu(\varrho,n)\Big|^2 \varrho\, d\varrho \\ &\le 2\pi \sum_{n\in\mathbb{Z}} \int_0^\infty \sum_{\mu=0}^{\infty} |\widehat{\psi}_\mu(\varrho,n)|^2 \sum_{\mu=0}^{\infty} |\widehat{f}_\mu(\varrho,n)|^2 \varrho\, d\varrho \\ &\le c \sum_{n\in\mathbb{Z}} \int_0^\infty \sum_{\mu=0}^{\infty} |\widehat{f}_\mu(\varrho,n)|^2 \varrho d\varrho = c \sum_{\mu=0}^{\infty} \int_{\mathbb{R}^2} |f_\mu|^2 = c\|\{f_\mu\}_\mu\|_{L^2(\ell^2)}. \ \blacksquare\end{aligned}$$

The next step is the difficult part. We could treat both cases at once with a general vector-valued approach (as in [S1]), but for simplicity we prefer to give an explicit argument in the first case and merely indicate the changes needed for the second. Of course we are following the classical Calderón–Zygmund approach ([CZ]).

LEMMA 8.5. (A) *R is of weak-type from L^1 to $L^1(\ell^2)$; i.e., there exists $c < \infty$ so that for all $f \in L^1(\mathbb{R}^2)$ and $\lambda > 0$,*

$$\Big|\Big\{x : \Big(\sum_{\mu=0}^{\infty} |R_\mu f(x)|^2\Big)^{1/2} > \lambda\Big\}\Big| \le \frac{c}{\lambda}\|f\|_{L^1}.$$

(B) *T is of weak-type from $L^1(\ell^2)$ to L^1; i.e., there exists $c < \infty$ so that for all $\{f_\mu\}_{\mu=0}^\infty \in L^1(\ell^2)$ and $\lambda > 0$,*

$$\Big|\Big\{x : \Big|\sum_{\mu\in\mathbb{Z}} \widetilde{T}_\mu f_\mu(x)\Big| > \lambda\Big\}\Big| \le \frac{c}{\lambda}\|\{f_\mu\}_\mu\|_{L^1(\ell^2)}.$$

Proof. (A) Fix $f \in L^1$ and $\lambda > 0$. Apply Lemma 8.3 to obtain disjoint $\{I_j\}_{j=1}^{\infty}$ and properties (8.4)–(8.6). For $x \in I_j$, let $g(x) = f_{I_j} = |I_j|^{-1} \int_{I_j} f(y)\,dy$ and $b(x) = f(x) - f_{I_j}$. For $x \notin \bigcup_j I_j$, let $g(x) = f(x)$ and $b(x) = 0$. So $f = g + b$.

As in the standard case (see [S1], p. 31), (8.4)–(8.6) imply

$$\|g\|_{L^2}^2 \le c\lambda \|f\|_{L^1}. \tag{8.10}$$

Then setting $Rg(x) = \{R_\mu g(x)\}_{\mu=0}^{\infty}$, Chebyshev's inequality, Lemma 8.4(A), and (8.10) imply that

$$|\{x : \|Rg(x)\|_{\ell^2} > \lambda/2\}| \le \frac{c}{\lambda}\|f\|_{L^1}.$$

By Minkowski's inequality for ℓ^2, it suffices to obtain a similar estimate for $Rb(x)$. Write $I_j = \widetilde{B}(s_j e^{i\varphi_j}, \delta_j)$ and set $I_j^* = \widetilde{B}(s_j e^{i\varphi_j}, 2\delta_j)$. By (8.5),

$$\Big|\bigcup_j I_j^*\Big| \le \frac{c}{\lambda}\|f\|_{L^1}.$$

Let $V_\lambda = \{x \in \mathbb{R}^2 \backslash \bigcup_j I_j^* : \|Rb(x)\|_{\ell^2} > \lambda/2\}$. So the proof will be complete if we show that

$$|V_\lambda| \le \frac{c}{\lambda}\|f\|_{L^1}. \tag{8.11}$$

Let $b_j = b\chi_{I_j}$. Then $b = \sum_j b_j$ and $\int b_j = 0$ for each j. By Chebyshev's and Minkowski's inequalities we have

$$|V_\lambda| \le \frac{2}{\lambda} \int\limits_{\mathbb{R}^2 \backslash \Sigma_j I_j^*} \|Rb(x)\|_{\ell^2}\,dx \le \sum_j \frac{2}{\lambda} \int\limits_{\mathbb{R}^2 \backslash I_j^*} \|Rb_j(x)\|_{\ell^2}\,dx. \tag{8.12}$$

Since R_μ has kernel K_μ and $\int b_j = 0$, we obtain

$$\begin{aligned}
&\int\limits_{\mathbb{R}^2 \backslash I_j^*} \|Rb_j(x)\|_{\ell^2}\,dx \\
&\quad = \int\limits_{\mathbb{R}^2 \backslash I_j^*} \Big(\sum_{\mu=0}^{\infty} \Big| \int\limits_{I_j} [K_\mu(re^{i\theta}, se^{i\varphi}) - K_\mu(re^{i\theta}, s_j e^{i\varphi_j})] b_j(se^{i\varphi}) s\,ds\,d\varphi \Big|^2\Big)^{1/2} r\,dr\,d\theta \\
&\quad \le \int\limits_{I_j} |b_j(se^{i\varphi})| A_j(se^{i\varphi}) s\,ds\,d\varphi
\end{aligned}$$

for

$$A_j(se^{i\varphi}) = \int\limits_{\mathbb{R}^2 \backslash I_j^*} \Big(\sum_{\mu=0}^{\infty} |K_\mu(re^{i\theta}, se^{i\varphi}) - K_\mu(re^{i\theta}, s_j e^{i\varphi_j})|^2\Big)^{1/2} r\,dr\,d\theta,$$

by Minkowski's inequality and Fubini's theorem.

Let $B_j = \{re^{i\theta} \in \mathbb{R}^2 \backslash I_j^* : |r - s_j| \ge 2\delta_j$ and $\pi \ge |\theta - \varphi_j| \ge 2\delta_j\}$, $C_j = \{re^{i\theta} \in \mathbb{R}^2 \backslash I_j^* : |r - s_j| \ge 2\delta_j$ and $|\theta - \varphi_j| < 2\delta_j\}$, and $D_j = \{re^{i\theta} \in \mathbb{R}^2 \backslash I_j^* : |r - s_j| < 2\delta_j$ and $\pi \ge |\theta - \varphi_j| \ge 2\delta_j\}$. So $\mathbb{R}^2 \backslash I_j^* = B_j \cup C_j \cup D_j$. Fixing j for the moment, note that

$$\begin{aligned}
(*) &\equiv \Big(\sum_{\mu=0}^{\infty} |K_\mu(re^{i\theta}, se^{i\varphi}) - K_\mu(re^{i\theta}, s_j e^{i\varphi_j})|^2\Big)^{1/2} \\
&\le \sum_{\mu=0}^{\infty} |K_\mu(re^{i\theta}, se^{i\varphi}) - K_\mu(re^{i\theta}, s_j e^{i\varphi_j})|.
\end{aligned}$$

If $re^{i\theta} \in B_j$ and $s'e^{i\varphi'} \in I_j$, we have $|r - s'| \approx |r - s_j|$ and $|e^{i\theta} - e^{i\varphi'}| \approx |e^{i\theta} - e^{i\varphi_j}|$. We apply the mean value theorem and note the symmetry of (2.25) and (2.29) under the interchange of $re^{i\theta}$ and $se^{i\varphi}$ to obtain, for $se^{i\varphi} \in I_j$,

$$(*) \le c\delta_j \sum_{\mu=0}^{\infty} 2^{4\mu}(1 + 2^\mu|r - s_j|)^{-4}(1 + 2^\mu|e^{i\theta} - e^{i\varphi_j}|)^{-4}(2^\mu r)^{-1}.$$

Let $k \in \mathbb{Z}$ be such that $2^{-k-1} \le \max(|r - s_j|, |e^{i\theta} - e^{i\varphi_j}|) < 2^{-k}$. If $k \le 0$, we have

$$(*) \le c\delta_j r^{-1}\Big(\sum_{\mu=0}^{\infty} 2^{-\mu}\Big)2^{4k} \le c\delta_j r^{-1}2^{3k} \le c\delta_j r^{-1}|r - s_j|^{-3/2}|e^{i\theta} - e^{i\varphi_j}|^{-3/2}.$$

If $k > 0$,

$$(*) \le cr^{-1}\delta_j\Big(\sum_{\mu=0}^{k} 2^{3\mu} + \sum_{\mu=k+1}^{\infty} 2^{-\mu}2^{4k}\Big) \le cr^{-1}\delta_j 2^{3k} \le c\delta_j r^{-1}|r - s_j|^{-3/2}|e^{i\theta} - e^{i\varphi_j}|^{-3/2}.$$

Hence

$$\int_{B_j} (*) r\,dr\,d\theta \le c\delta_j \int_{|r-s_j|\ge 2\delta_j} |r - s_j|^{-3/2}\,dr \int_{|\theta-\varphi_j|\ge 2\delta_j} |\theta - \varphi_j|^{-3/2}\,d\theta$$
$$\le c\delta_j\delta_j^{-1/2}\delta_j^{-1/2} = c.$$

If $re^{i\theta} \in C_j$ and $s'e^{i\varphi'} \in I_j$, then $|r - s'| \approx |r - s_j|$, so

$$(*) \le c\delta_j \sum_{\mu=0}^{\infty} 2^{4\mu}(1 + 2^\mu|r - s_j|)^{-4}(2^\mu r)^{-1}.$$

Let $k \in \mathbb{Z}$ satisfy $2^{-k-1} \le |r - s_j| < 2^{-k}$. If $k \le 0$,

$$(*) \le c\delta_j r^{-1}2^{4k} \le c\delta_j r^{-1}2^{3k} \le c\delta_j r^{-1}|r - s_j|^{-3}.$$

If $k > 0$,

$$(*) \le c\delta_j r^{-1}\Big(\sum_{\mu=0}^{k} 2^{3\mu} + \sum_{\mu=k+1}^{\infty} 2^{-\mu}2^{4k}\Big) \le c\delta_j r^{-1}2^{3k} \le c\delta_j r^{-1}|r - s_j|^{-3}.$$

So

$$\int_{C_j} (*) r\,dr\,d\theta \le c\delta_j \int_{|r-s_j|\ge 2\delta_j} |r - s_j|^{-3}\,dr \int_{|\theta-\varphi_j|<2\delta_j} d\theta \le c\delta_j\delta_j^{-2}\delta_j = c.$$

If $re^{i\theta} \in D_j$ and $s'e^{i\varphi'} \in I_j$, then $|e^{i\theta} - e^{i\varphi'}| \approx |e^{i\theta} - e^{i\varphi_j}|$, so

$$(*) \le c\delta_j \sum_{\mu=0}^{\infty} 2^{4\mu}(1 + 2^\mu|e^{i\theta} - e^{i\varphi_j}|)^{-4}(2^\mu r)^{-1} \le c\delta_j r^{-1}|e^{i\theta} - e^{i\varphi_j}|^{-3},$$

similarly to the C_j case. So

$$\int_{D_j} (*) r\,dr\,d\theta \le c\delta_j \int_{|r-s_j|<2\delta_j} dr \int_{|\theta-\varphi_j|\ge 2\delta_j} |\theta - \varphi_j|^{-3}\,d\theta \le c\delta_j^2\delta_j^{-2} = c.$$

Therefore $A_j(se^{i\varphi}) \le c$, with c independent of j and $se^{i\varphi} \in I_j$. Using this estimate above gives

$$\int_{\mathbb{R}^2\setminus I_j^*} \|Rb_j(x)\|_{\ell^2}\, dx \le c \int_{I_j} |b_j|\, dx = c \int_{I_j} |f - f_{I_j}| \le c \int_{I_j} |f|.$$

So, by (8.12),

$$|V_\lambda| \le \frac{c}{\lambda} \sum_{j=1}^{\infty} \int_{I_j} |f| \le \frac{c}{\lambda} \|f\|_{L^1},$$

establishing (8.11) and completing the proof of (A).

We only outline the modifications needed to prove (B). We apply the Calderón–Zygmund lemma (Lemma 8.3) to $\bar{f} = (\sum_{\mu=0}^{\infty} |f_\mu|^2)^{1/2} \in L^1$. Then for each μ we let $g_\mu(x) = f_\mu(x)$ if $x \notin \bigcup_j I_j$, $g_\mu(x) = (f_\mu)_{I_j}$ if $x \in I_j$, and set $b_\mu = f_\mu - g_\mu$. Minkowski's inequality shows that

$$\sum_{\mu\in\mathbb{Z}} |(f_\mu)_{I_j}|^2 \le \Big(|I_j|^{-1} \int_{I_j} \Big(\sum_{\mu=0}^{\infty} |f_\mu(x)|^2 \Big)^{1/2} dx \Big)^2 \le c\lambda^2 \tag{8.13}$$

by (8.6) for $\bar{f}$. So the usual methods give

$$\|\{g_\mu\}_\mu\|_{L^2(\ell^2)} \le c\lambda \|\{f_\mu\}\|_{L^1(\ell^2)}.$$

By Lemma 8.4(B) this gives the desired weak-type estimate for $\sum_{\mu=0}^{\infty} \widetilde{T}_\mu g_\mu$.

We let $b_{\mu,j} = b_\mu \chi_{I_j}$ for each μ, j. We then have $\int_{I_j} b_{\mu,j} = 0$. Then $|\bigcup_j I_j^*|$ is controlled as above. Let $V_\lambda = \{x \in \mathbb{R}^2 \setminus \bigcup_j I_j^* : |\sum_{\mu=0}^{\infty} \widetilde{T}_\mu b_\mu(x)| > \lambda/2\}$. Then

$$|V_\lambda| \le \frac{2}{\lambda} \sum_j \int_{\mathbb{R}^2\setminus I_j^*} \Big| \sum_{\mu=0}^{\infty} \widetilde{T}_\mu b_{\mu,j} \Big| \, dx. \tag{8.14}$$

Therefore for $re^{i\theta} \notin I_j^*$ (with $I_j = \widetilde{B}(s_j e^{i\varphi_j}, \delta_j)$, I_j^* similarly with $2\delta_j$)

$$\begin{aligned}
&\sum_{\mu=0}^{\infty} |\widetilde{T}_\mu b_{\mu,j}(re^{i\theta})| \\
&\quad = \sum_{\mu=0}^{\infty} \Big| \int_{I_j} [L_\mu(re^{i\theta}, se^{i\varphi}) - L_\mu(re^{i\theta}, s_j e^{i\varphi_j})] b_{\mu,j}(se^{i\varphi}) s\, ds\, d\varrho \Big| \\
&\quad \le \int_{I_j} \Big(\sum_{\mu=0}^{\infty} |L_\mu(re^{i\theta}, se^{i\varphi}) - L_\mu(re^{i\theta}, s_j e^{i\varphi_j})|^2 \Big)^{1/2} \Big(\sum_{\mu=0}^{\infty} |b_{\mu,j}(se^{i\varphi})|^2 \Big)^{1/2} s\, ds\, d\varphi,
\end{aligned}$$

which can be handled as above, since L_μ satisfies the same estimates as K_μ. Using this estimate in (8.14) and integrating as above gives

$$|V_\lambda| \le \frac{c}{\lambda} \sum_{j=1}^{\infty} \int_{I_j} \Big(\sum_{\mu=0}^{\infty} |b_{\mu,j}|^2 \Big)^{1/2} \le \frac{c}{\lambda} \sum_{j=1}^{\infty} \int_{I_j} \Big(\sum_{\mu=0}^{\infty} |f_\mu|^2 \Big)^{1/2} \le \frac{c}{\lambda} \|\{f_\mu\}_\mu\|_{L^1(\ell^2)},$$

where we have used $\|b_{\mu,j}\|_{\ell^2} \le \|f_\mu\|_{\ell^2} + \|(f_\mu)_{I_j}\|_{\ell^2}$ on I_j, and the first part of (8.13). The estimate for $|V_\lambda|$ completes the proof. ■

The rest follows easily now.

THEOREM 8.6. *Suppose* $1 < p \le 2$. *Then*

$$R : L^p(\mathbb{R}^2) \to L^p(\ell^2) \text{ is bounded}, \tag{8.15}$$

$$T : L^p(\ell^2) \to L^p(\mathbb{R}^2) \text{ is bounded}, \tag{8.16}$$

and

$$A_p^{02} \approx L^p. \tag{8.17}$$

Proof. A straightforward adaptation of the Marcinkiewicz interpolation theorem ([S1], pp. 20–22) to the vector-valued case gives (8.15)–(8.16). Hence if $f \in L^p$ then $\|f\|_{A_p^{02}} \le c\|f\|_{L^p}$ by (8.9) and (8.15). Conversely, if $f \in \mathcal{A}_p^{02} (= L^2 \cap A_p^{02})$ then since $\sum_{\mu=0}^{\infty} \widehat{\varphi}_\mu(\varrho, n)\overline{\widehat{\psi}_\mu(\varrho, n)} \equiv 1$ (the conjugate of (1.11)), we have the Calderón identity $f = \sum_{\mu=0}^{\infty} \widetilde{T}_\mu R_\mu f = T \circ R(f)$. So

$$\|f\|_{L^p} = \|T \circ R(f)\|_{L^p} \le c\|Rf\|_{L^p(\ell^2)} = c\|f\|_{A_p^{02}}$$

by (8.16) and (8.9). So $\|f\|_{L^p} \approx \|f\|_{A_p^{02}}$ on the set $L^2 \cap L^p = L^2 \cap A_p^{02}$, which is dense in both L^p and A_p^{02}. So we can identify L^p with A_p^{02} and obtain (8.17). ■

Duality provides the last step.

THEOREM 8.7. *Suppose* $2 < p < \infty$. *Then* (8.15)–(8.17) *hold.*

Proof. Let p' be the conjugate index to p, i.e., $p^{-1} + p'^{-1} = 1$. Suppose $f \in L^p$. Let

$$B = \{g = \{g_\mu\}_{\mu=0}^{\infty} : \|\{g_\mu\}\|_{L^{p'}(\ell^2)} \le 1\}.$$

By the well-known duality $(L^p(\ell^2))^* \approx L^{p'}(\ell^2)$ (see e.g. [E], 8.20.5), we have

$$\|\{R_\mu f\}_\mu\|_{L^p(\ell^2)} \approx \sup_{g \in B} \Big| \sum_{\mu=0}^{\infty} \langle R_\mu f, g_\mu \rangle \Big|.$$

Letting $\widetilde{R}_\mu$ be defined by (2.33), we have, by a polarization of (2.16),

$$\langle R_\mu f, g_\mu \rangle = 2\pi \sum_{n=0}^{\infty} \int (\widehat{\varphi}_\mu \widehat{f} \overline{\widehat{g}}_\mu)(\varrho, n) \varrho \, d\varrho = \langle f, \widetilde{R}_\mu g_\mu \rangle.$$

But $\overline{\widehat{\varphi}}_\mu$ has all the properties of $\widehat{\psi}_\mu$ (their definitions being indistinguishable), so by the analogue of (8.16),

$$\Big\| \sum_{\mu=0}^{\infty} \widetilde{R}_\mu g_\mu \Big\|_{L^{p'}} \le c\|\{g_\mu\}_\mu\|_{L^{p'}(\ell^2)} \le c.$$

So

$$\|\{R_\mu f\}_\mu\|_{L^p(\ell^2)} \approx \sup_{g \in B} \Big| \Big\langle f, \sum_{\mu=0}^{\infty} \widetilde{R}_\mu g_\mu \Big\rangle \Big| \le c\|f\|_{L^p}$$

by Hölder's inequality. This proves (8.15). For (8.16), the argument is similar but easier since we only need $(L^p)^* \approx L^{p'}$, $\langle \widetilde{T}_\mu f_\mu, g \rangle = \langle f_\mu, T_\mu g \rangle$, the Cauchy–Schwarz inequality, Hölder's inequality, and (8.15) for T_μ in place of $\widetilde{T}_\mu$. Finally, (8.17) follows from (8.15)–(8.16) as in Theorem 8.6. ■

We have therefore proved Theorem 8.8, stated in §1.

9. Conclusion

Here we discuss some problems suggested by this paper. We start with the particular and work toward the general and open-ended.

For the usual Triebel–Lizorkin spaces $F_\varrho^{\alpha q}$, we have $F_p^{02} \approx L^p$, $1 < p < \infty$. So from §8, we have $A_p^{02} \approx F_p^{02}$, $1 < p < \infty$. Does this continue to hold for $0 < p \le 1$, for which it is well known that $F_p^{02} \approx H^p$? Is it true for $q \neq 2$ that $A_p^{0q} \approx F_p^{0q}$? The α index in $A_p^{\alpha q}$ reflects the action of the nonstandard potential operator P^α based on $\Delta_r + \partial^2/\partial\theta^2$; that is, by the discussion of the FHT in §1,

$$P^{-2} f = ((1 + \varrho^2 + n^2)\widehat{f}(\varrho, n))^\vee = (I - \Delta_r - \partial^2/\partial\theta^2) f.$$

For the $F_p^{\alpha q}$ spaces, the α index reflects the action of the usual potential operator based on the standard Laplacian $\Delta_r + r^{-2}\partial^2/\partial\theta^2$, so we do not expect an equivalence between $A_p^{\alpha q}$ and $F_p^{\alpha q}$ to carry over to $\alpha \neq 0$.

A technical issue is raised by comparison with the fact that the $F_p^{\alpha q}$ norm is defined (possibly ∞) for all $f \in \mathcal{S}'$, since it is defined via convolution with a test function. What is the corresponding distribution class for $A_p^{\alpha q}$? Similarly for the φ-transform or Meyer's wavelets, the coefficients $\langle f, \varphi_{\mu k}\rangle$ are defined for all $f \in \mathcal{S}'$ since $\varphi_{\mu k} \in \mathcal{S}$. However, in the polar case, $\varphi_{\mu k m}$ is not continuous at the origin. This is the reason for resorting to the completion of L^2 to define $A_p^{\alpha q}$. Moreover, this is our only reason to avoid defining $A_p^{\alpha q}$ when $q = \infty$, since we do not expect the density of L^2 in this case.

There are many other aspects of the $F_p^{\alpha q}$ theory that could be studied in the $A_p^{\alpha q}$ context. Many of them should be straightforward, such as duality and interpolation. Other aspects may not be so easy to carry over. For example, what is the atomic and molecular theory for $A_p^{\alpha q}$ (cf. [FJ3], [FJ4], and the radial case in [EF], §3). How much operator analysis, as in [FJ3], §10, carries over? What form does the T1 theorm ([DJ]) take in this context?

With Meyer's orthonormal wavelets in mind, we must naturally ask if there is an orthonormal decomposition with the same three basic properties we have noted in §1 for our polar wavelet decomposition. If so, we expect the Littlewood–Paley theory of $A_p^{\alpha q}$ developed here to be the natural function space framework for such a decomposition.

We have only considered $\mathbb{R}^2$ in this paper. What is the analogous theory for $\mathbb{R}^n$, $n > 2$? Presumably the spherical harmonics play the role of the trigonometric system. Still the generalization is by no means obvious. For example, it is trivial to evenly space out sample points on the unit circle $S^1 \subseteq \mathbb{R}^2$, but not so clear how to pick a good sample set in $S^{n-1} \subseteq \mathbb{R}^n$, $n > 2$.

Beyond this, for what sort of manifolds can something like this be carried out? The sampling theory approach avoids a dependence on an underlying group structure, which seems to be necessary for orthonormal wavelet analysis. Some aspects of the Triebel–Lizorkin theory on manifolds are discussed in [T2]. We expect this and the work in [Se-S] to be helpful in any further work in this direction.

We hope to return to some of these questions in the future.

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