COLLOQUIUM MATHEMATICUM

VOL. XXXII 1974 FASC. 1

ACCUMULATION POINTS OF NOWHERE DENSE SETS

BY

W. KULPA AND A. SZYMAŃSKI (KATOWICE)

The following question, originated from Hewitt's paper [2] (see also Elkin [1]), remains open: does there exist a regular topology which is maximal among topologies without isolated points? (**P 914**)

Related to this question is another one: is it true that every point of a regular space without isolated points is an accumulation point of a nowhere dense subset? (**P 915**)

As it is easy to observe, the affirmative answer to the second question implies the negative answer to the first one. We shall give a partial positive answer to the second question in the case where the space is compact Hausdorff. The answer in the case of minimal Hausdorff spaces is unknown; when the space is H-closed, the question is answered negatively by examples of maximal H-closed topologies.

THEOREM. *If X is a dense-in-itself compact Hausdorff space, then, for every point $x \in X$, there exists a closed nowhere dense set F such that* $x \in \operatorname{cl}[F \setminus \{x\}]$.

Proof. Let $x \in X$. Denote by R_x a family of all regular open sets $U \subset X$ such that $x \in \operatorname{cl} U$. The relation of inclusion yields a partial order on the family R_x. Note that if $U \neq V$, $U, V \in R_x$ and $U \subset V$, then $V \setminus \operatorname{cl} U \neq \emptyset$.

(I) Assume that there exists a chain $R' \subset R_x$ such that

$$\{x\} = \bigcap \{\operatorname{cl} U: \ U \in R'\}.$$

There exists a chain $R = \{U_\xi: \xi < a\}$ which is well-ordered and cofinal in R'. It is clear that $\{x\} = \bigcap \{\operatorname{cl} U: \ U \in R\}$ and that a is a limit ordinal, since x is not isolated. Choose a set $E = \{x_\xi: \xi < a\}$ of points $x_\xi \neq x$ of X such that

$$(1) \qquad x_\xi \in U_\xi \setminus \operatorname{cl} U_{\xi+1} \quad \text{for every } \xi < a.$$

Every one-point set $\{x_\xi\} \subset E$ is nowhere dense in X and open in E, x_ξ being non-isolated in X. Hence, by Theorem 3 of [3], § 8, III, it follows that E is nowhere dense in X. From (1) and the fact that X is compact Hausdorff and $\{x\} = \bigcap \{\mathrm{cl}\, U\colon\ U \in R\}$ we have $x \in \mathrm{cl}(\mathrm{cl}\, E \setminus \{x\})$. Thus the conclusion of the Theorem holds in case (I) for $F = \mathrm{cl}\, E$.

(II) Let R be a chain in R_x. According to the Kuratowski-Zorn Lemma we can assume that R is maximal in R_x. It is clear that the set

$$F = \bigcap \{\mathrm{cl}\, U\colon\ U \in R\} \setminus \mathrm{int} \bigcap \{\mathrm{cl}\, U\colon\ U \in R\}$$

is closed and nowhere dense. From the maximality of the chain R it follows that

$$x \notin \mathrm{cl}\big(\mathrm{int} \bigcap \{\mathrm{cl}\, U\colon\ U \in R\}\big). \tag{2}$$

This implies that $x \in F$. We show that $x \in \mathrm{cl}(F \setminus \{x\})$.

Suppose that x is isolated in F. Then (2) implies that x is isolated in $\bigcap \{\mathrm{cl}\, U\colon\ U \in R\}$. Thus, X being regular, there exists a regular open neighbourhood G of x such that

$$\mathrm{cl}\, G \cap \bigcap \{\mathrm{cl}\, U\colon\ U \in R\} = \{x\}.$$

Put $R_G = \{U \cap G\colon\ U \in R\}$. The sets $U \cap G$, $U \in R$, are regular open. We have $\{x\} = \bigcap \{\mathrm{cl}\, U\colon\ U \in R_G\}$, but this is case (I). Thus the proof is completed.

REFERENCES

[1] А. Г. Елкин, *О разложимости пространств*, Доклады Академии наук СССР 186 (1969), p. 9-12.

[2] E. Hewitt, *A problem on set-theoretic topology*, Duke Mathematical Journal 10 (1943), p. 309-333.

[3] K. Kuratowski, *Topology I*, New York and London 1966.

Reçu par la Rédaction le 24. 4. 1973