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MICROLOCAL SOLVABILITY OF THE CAUCHY PROBLEM AND BOUNDARY REGULARITY

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In this paper we announce some results concerning the one-sided Cauchy problem

(1) $$p(x, t, D_x, D_t)u = 0, \quad t > 0,$$

(2) $$D_t^j u|_{t=0} = f_j, \quad j = 0, \ldots, m-1,$$

for a class of linear partial differential operators with analytic coefficients. Our main result is that the obstructions to solve this problem are essentially of microlocal nature (cf. Theorem 5 and the related result on microregularity from Theorem 18). When microlocalisation is here with respect to the analytic wave front set, this result is essentially known. For constant coefficients it is in fact proved in O. Liess [13] and for analytic coefficients it can be proved starting from a result of P. Schapira [19] (also cf. J. Sjöstrand [20] and L. Hörmander [8]). We shall therefore add an additional assumption on the operator under consideration and gain additional precision in the results in that microlocality and microregularity will be obtained with respect to some wave front set which is better adapted to the problem under study. (The wave front set localizes on much smaller sets, in general, than the standard wave front set.) Technically these assumptions depend on some weight function on $\boldsymbol{R}^n$ and for a particular choice of this function, we recover the set-up from P. Schapira's paper [19] with another proof (which is, admittedly, much longer. The arguments could, however, be simplified for that choice of the weight function at many places). For all other choices of the weight function our additional assumption implies that the principal part of the operator degenerates on a nontrivial set and that the lower order terms satisfy a condition of Levi type. Typical examples of equations which one then obtains are quasi-elliptic equations, the Schrödinger equation, products of such equations perturbed by low order terms and the image of such equations under linear changes of coordinates.

To be more precise, we shall assume that $p(x, t, D_x, D_t)$ has the form

$$p(x, t, D_x, D_t) = D_t^m + \sum_{\substack{j<m \\ |\alpha|+j\leqslant m}} q_{j\alpha}(x, t) D_x^\alpha D_t^j \tag{3}$$

for some real-analytic coefficients $q_{j\alpha}(x, t)$ defined for (x, t) near $0 \in \boldsymbol{R}_x^n \times \boldsymbol{R}_t$. The main assumption is now that we are given some (globally) Lipschitz-continuous function φ: $\boldsymbol{R}^n \to \boldsymbol{R}$ (by this we mean that $|\varphi(\xi) - \varphi(\eta)| \leqslant C|\xi - \eta|$, $\forall \xi$, $\forall \eta \in \boldsymbol{R}^n$, for some $C > 0$) such that

$$\Big|D_{x,t}^\gamma D_\xi^\beta \sum_\alpha q_{j\alpha}(x, t)\xi^\alpha\Big| \leqslant c^{|\gamma|+1}\gamma!\,\varphi(\xi)^{m-j-|\beta|}, \tag{4}$$

$$\forall j,\ \forall \gamma,\ \forall \beta,\ \forall \xi \in \boldsymbol{R}^n, \quad \text{if } |(x, t)| \leqslant \varepsilon,$$

for some $c > 0$ and $\varepsilon > 0$.

For technical reasons we shall always assume that $\varphi(\xi) \geqslant |\xi|^\delta$, for some positive δ, at infinity, and $\varphi(\xi) > 0$, $\forall \xi \in \boldsymbol{R}^n$.

To give an example, assume that we are given some rational numbers $M_i \geqslant 1$, $i = 1, \ldots, n$, and suppose

$$p(x, t, D_x, D_t) = D_t^m + \sum_{\substack{j<m \\ \alpha_1 M_1 + \ldots + \alpha_n M_n + j \leqslant m}} q_{j\alpha}(x, t) D_x^\alpha D_t^j \tag{5}$$

for some real-analytic functions $q_{j\alpha}$. (4) is then valid with $\varphi = \sum_j (1 + |\xi_j|)^{1/M_j}$. Also note that (4) is always valid with $\varphi = 1 + |\xi|$.

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We now return to (1) and (2). In order to give a meaning to (2) we must at least assume that u is a germ of an extendable distribution defined for $t > 0$ in a neighborhood of $0 \in \boldsymbol{R}^{n+1}$. We may, and shall, then as well assume that u is a germ of a distribution defined in a full neighborhood of $0 \in \boldsymbol{R}^{n+1}$. Likewise, the f_j will be germs of distributions defined near $0 \in \boldsymbol{R}^n$. We shall henceforth denote the space of germs of distributions in n or $n+1$ variables defined near $0 \in \boldsymbol{R}^n$ or $0 \in \boldsymbol{R}^{n+1}$ by $\mathscr{D}'$. (The precise meaning of $\mathscr{D}'$ must be clear from the context.)

Our first concern is then to see how much regularity on the f_j is required if we want to find a solution for the problem (1) and (2).

In the case of operators of the form (5), the answer can be formulated in terms of anisotropic Gevrey classes. Let us in fact denote by G^M the set of germs f of $\mathscr{C}^\infty$ functions defined near $0 \in \boldsymbol{R}^n$ such that for some $\varepsilon > 0$ and $c > 0$ (which may depend on f)

$$|D_x^\alpha f(x)| \leqslant c^{|\alpha|+1}(\alpha_1!)^{M_1}\ldots(\alpha_n!)^{M_n}, \quad \forall \alpha,\ \forall x,\ |x| \leqslant \varepsilon.$$

It is then (e.g.) a result of Persson [17] that (1) and (2) have a solution if $f_j \in G^M$, $j = 0, \ldots, m-1$. Moreover, this solution solves in fact the two-sided Cauchy problem

(6) $$p(x, t, D_x, D_t)u = 0 \quad \text{near } 0 \in \mathbf{R}^{n+1},$$

(7) $$D_t^j u|_{t=0} = f_j, \quad j = 0, \ldots, m-1,$$

and it is well known (in view of the regularity results for quasi-elliptic equations due to Cavalucci [1]) that this result cannot be improved if no assumption on the type of the operator is made.

To state a similar result for the case of a general φ, we introduce:

DEFINITION 1 (Liess–Rodino [14]). Consider $x^0 \in \mathbf{R}^n$ and let f be a germ of a distribution defined near x^0. We say that f is of *class* G_φ near x^0 if there is a neighborhood X of x^0, $c > 0$, and a bounded sequence of distributions $f_j \in \mathscr{E}'(\mathbf{R}^n)$ such that

(a) $f = f_1$ in the sense of germs,

(b) $f_j = f_k$ on X, $\forall j$, $\forall k$,

(c) $|\hat{f}_j(\xi)| \leqslant c(cj/\varphi(\xi))^j$, $j = 1, 2, \ldots$, $\forall \xi \in \mathbf{R}^n$.

$\mathscr{E}'(\mathbf{R}^n)$ is here the space of distributions with compact support defined on $\mathbf{R}^n$. (More generally, $\mathscr{E}'(U) = \{u \in \mathscr{E}'(\mathbf{R}^n);\ \operatorname{supp} u \subset U\}$.) When $v \in \mathscr{E}'(\mathbf{R}^n)$ we denote by $\hat{v}$ the Fourier–Borel transform $\hat{v}(\zeta) = v(\exp(-i\langle x, \zeta\rangle))$, $\zeta \in \mathbf{C}^n$, of v. (Of course this definition is modelled on Hörmander's definition of the analytic wave front set. Cf. Hörmander [6].)

We denote by G_φ the set of germs of distributions defined near $x^0 = 0$ which are of class G_φ there. Of course, when $\varphi = \sum (1 + |\xi_j|)^{1/M_j}$ we just have $G_\varphi = G^M$.

Our first result in this paper is now:

THEOREM 2. *Assume that* $p(x, t, D_x, D_t)$ *satisfies* (4) *and let* $f_j \in G_\varphi$, $j = 0, \ldots, m-1$, *be given. Then we can find a germ of a* $\mathscr{C}^\infty$ *function defined near* $0 \in \mathbf{R}^{n+1}$ *for which* (6) *and* (7) *are valid.*

Note that this is just the natural formulation of the Cauchy–Kowalewska theorem in G_φ classes.

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To state our next result, we shall, once more, place ourselves at first in the quasihomogeneous case from (5). Thus assume that $M_i \geqslant 1$, $i = 1, \ldots, n$, are given and choose $x^0 \in \mathbf{R}^n$, $\xi^0 \in \dot{\mathbf{R}}^n$ ($\dot{\mathbf{R}}^n = \mathbf{R}^n \setminus \{0\}$).

DEFINITION 3. Let f be a germ of a distribution defined near x^0. We shall say that $(x^0, \xi^0) \notin \mathrm{WF}^M f$ if we can find an open M-quasicone $\Gamma \subset \dot{\mathbf{R}}^n$ which contains ξ^0 (a set $A \subset \dot{\mathbf{R}}^n$ is called an *M-quasicone* if *a*

$=(a_1, \ldots, a_n) \in A$ implies $(t^{M_1} a_1, \ldots, t^{M_n} a_n) \in A$ for all $t > 0$), $c > 0$, and a bounded sequence $f_j \in \mathscr{E}'(\boldsymbol{R}^n)$ such that (a) and (b) from Definition 1 are valid and

$$|\hat{f}_j(\xi)| \leqslant c\,(cj/\varphi(\xi))^j, \quad j = 1, 2, \ldots, \xi \in \Gamma.$$

Here $\varphi = \sum (1+|\xi_j|)^{1/M_j}$. (For similar definitions cf. Lascar [9], Liess–Rodino [14], Rodino [18], Zanghirati [21].)

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We also need

DEFINITION 4. Let $p(x, t, D_x, D_t)$ be as in (5) and consider some germs of distributions $f_0, \ldots, f_{m-1}$ defined near $0 \in \boldsymbol{R}^n$. We say that the one-sided Cauchy problem is *solvable in microgerms at* $(0, \xi^0)$ if there is a germ of a distribution u_{ξ^0} defined near $0 \in \boldsymbol{R}^{n+1}$ such that

(8) $$p(x, t, D_x, D_t)\, u_{\xi^0} = 0 \quad \text{for } t > 0,$$

(9) $$(0, \xi^0) \notin \mathrm{WF}^M(D_t^j u_{\xi^0}|_{t=0} - f_j), \quad j = 0, \ldots, m-1.$$

Our main result in this paper is now the following theorem, and its variant in G_φ classes from Theorems 9 and 12 below:

THEOREM 5. *Suppose that there are given* $f_0, \ldots, f_{m-1}$ *in* $\mathscr{D}'$ *and assume that the Cauchy problem is solvable in microgerms at* $(0, \xi^0)$ *for any* $\xi^0 \in \dot{\boldsymbol{R}}^n$. *Then we can find a solution* $u \in \mathscr{D}'$ *for* (1) *and* (2).

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In the case of a general φ it seems difficult to associate some wave front set directly with vectors $\xi^0 \in \dot{\boldsymbol{R}}^n$ in a natural way. In fact, as has been first observed in Hörmander [7], wave front sets are associated rather with the points at infinity of a suitable compactification of $\boldsymbol{R}^n$ related to φ, than with the points from $\dot{\boldsymbol{R}}^n$. We avoid this difficulty altogether by introducing

DEFINITION 6. (Liess–Rodino [14]). Let f be a germ of a distribution defined near $x^0 \in \boldsymbol{R}^n$ and consider $\Gamma \subset \boldsymbol{R}^n$. We shall write $(0, \Gamma) \cap \mathrm{WF}_\varphi f = \emptyset$ if there are a neighborhood X of x^0, $c_1 > 0$, $c_2 > 0$, and a bounded sequence $f_j \in \mathscr{E}'(\boldsymbol{R}^n)$ such that (a) and (b) from Definition 1 are valid and

$$|\hat{f}_j(\xi)| \leqslant c_1\,(c_1 j/\varphi(\xi))^j, \quad \forall j, \quad \text{if dist}\,(\xi, \Gamma) \leqslant c_2\,\varphi(\xi).$$

To simplify the notation, we shall henceforth denote $\{\xi \in \boldsymbol{R}^n;\ \mathrm{dist}(\xi, \Gamma) \leqslant c\varphi(\xi)\}$ by $\Gamma_{c\varphi}$. If Γ' contains some set of the form $\Gamma_{c\varphi}$ then we say that Γ' is a *φ-neigborhood* of Γ and write $\Gamma \subset_\varphi \Gamma'$.

DEFINITION 7. Let $f_0, \ldots, f_{m-1}$ be given germs of distributions defined near $0 \in \boldsymbol{R}^n$ and consider $\Gamma \subset \boldsymbol{R}^n$. We say that the one-sided Cauchy problem is *solvable in microgerms at* $(0, \Gamma)$ if there is a germ of a distribution u_Γ defined near $0 \in \boldsymbol{R}^{n+1}$ for which

$$p(x, t, D_x, D_t)u_\Gamma = 0, \quad t > 0, \tag{10}$$

$$(0, \Gamma) \cap \mathrm{WF}_\varphi(f_j - D_t^j u_\Gamma|_{t=0}) = \emptyset, \quad j = 0, \ldots, m-1. \tag{11}$$

Remark 8. If $p(x, t, D_x, D_t)$ has the form (5) and if the Cauchy problem is solvable in microgerms at $(0, \xi^0)$ (in G^M), then there is an open M-quasicone Γ, $\xi^0 \in \Gamma$, such that the Cauchy problem is solvable in microgerms at $(0, \Gamma)$.

We now have

THEOREM 9. *Consider* $\Gamma^1, \ldots, \Gamma^s$, *some sets in* $\dot{\boldsymbol{R}}^n$ *such that* $\bigcup \Gamma^k = \dot{\boldsymbol{R}}^n$, *and let* $f_0, \ldots, f_{m-1}$ *be given. Assume that the Cauchy problem is solvable in microgerms at* $(0, \Gamma^k)$ *for any* k. *Then we can find a solution* u *for the problem* (1), (2).

Remark 10. In view of Remark 8 it is clear that Theorem 5 is a consequence of Theorem 9.

Remark 11. Definitions 4 and 7 both refer to the one-sided Cauchy problem. We obtain related definitions for the two-sided Cauchy problem (6), (7) if we just drop the condition "$t > 0$" in (8), respectively (10). In this way we arrive at natural variants of Theorems 5 and 9 which are also true (and in fact easier to prove. One can also obtain them by using the uniqueness of the solutions.)

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In Theorems 5 and 9 we have studied the solvability of (1), (2) in distributions. One may ask if u is a $\mathscr{C}^\infty$ function if the solutions for the corresponding microlocal problems (8), (9), respectively (10), (11), are $\mathscr{C}^\infty$ functions. This is indeed the case:

THEOREM 12. *Consider* $\Gamma^1, \ldots, \Gamma^s \subset \boldsymbol{R}^n$ *with* $\bigcup \Gamma^k = \boldsymbol{R}^n$ *and let* $f_0, \ldots, f_{m-1}$ *be given. Assume that for every* k, $1 \leqslant k \leqslant s$, *there is a* $\mathscr{C}^\infty$ *function* u_{Γ^k} *which satisfies* (10) *and* (11). *Then we can find a germ of a* $\mathscr{C}^\infty$ *function defined near* $0 \in \boldsymbol{R}^{n+1}$ *for which* (1) *and* (2) *are valid.*

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So far we have only analyzed solvability questions for (1) and (2). These questions are naturally related to questions of microlocal uniqueness (or regularity) for the corresponding solutions. To state the relevant results, we must first introduce a natural notion of boundary wave front set. As is customary we shall define such boundary wave front sets only for a subclass of distributions. Here we shall assume that u is $\mathscr{C}^\infty$ in the t-variable. We shall in fact denote by $\mathscr{F}$ the space of germs of distributions u defined near $0 \in \boldsymbol{R}^{n+1}$ with the following property: $\exists \varepsilon > 0$, $\forall b \in \boldsymbol{R}$, $\exists b' \in \boldsymbol{R}$, $\exists c > 0$ such that $|u(v)| \leqslant c$ if $v \in \mathscr{C}_0^\infty(\boldsymbol{R}^{n+1})$ satisfies

$$|\hat{v}(\lambda)| \leqslant \exp\big(\varepsilon |\operatorname{Im} \zeta| + \varepsilon \operatorname{Im} \tau^+ + b \ln(1+|\lambda|) + b' \ln(1+|\zeta|)\big).$$

Here $\lambda = (\zeta, \tau)$, $\zeta \in \boldsymbol{C}^n$, $\tau \in \boldsymbol{C}$, are the Fourier-dual variables to $z = (x, t)$. Furthermore, when $a \in \boldsymbol{R}$, then we denote by a^+ its positive part. Finally, we should mention perhaps that in the above we have identified (as we shall also do later on) u with some suitable distribution defining it.

If $u \in \mathscr{D}'$ satisfies $p(x, t, D_x, D_t)u = 0$ for $t > 0$, then it follows from Theorem 4.3.1 in Hörmander [5] that we can find $u' \in \mathscr{F}$ such that $u = u'$ for $t > 0$.

To justify the notion of boundary wave front set which we introduce later on, we recall:

Proposition 13 (cf. Liess–Rodino [14]). *Consider $f \in \mathscr{D}'(\boldsymbol{R}^n)$, $\Gamma \subset \boldsymbol{R}^n$, $x^0 \in \boldsymbol{R}^n$. Then the following conditions are equivalent:*

(i) $(x^0, \Gamma) \cap \mathrm{WF}_\varphi f = \emptyset$.

(ii) *There are $d > 0$, $\varepsilon > 0$, $c > 0$, $c' > 0$ and $b \in \boldsymbol{R}$ such that $|v(f)| \leqslant c$ whenever $v \in \mathscr{C}_0^\infty(\boldsymbol{R}^n)$ satisfies*

$$|\hat{v}(\zeta)| \leqslant \exp\big(d\varphi(-\operatorname{Re} \zeta) + \langle x^0, \operatorname{Im} \zeta\rangle + \varepsilon |\operatorname{Im} \zeta| + b \ln(1+|\zeta|)\big) \quad \text{if } \zeta \in \boldsymbol{C}^n,\ \operatorname{Re} \zeta \in -\Gamma_{c'\varphi}, \tag{12}$$

$$|\hat{v}(\zeta)| \leqslant \exp\big(\langle x^0, \operatorname{Im} \zeta\rangle + \varepsilon |\operatorname{Im} \zeta| + b \ln(1+|\zeta|)\big) \quad \text{if } \zeta \in \boldsymbol{C}^n,\ \operatorname{Re} \zeta \notin -\Gamma_{c'\varphi}. \tag{13}$$

Moreover, when $f \in \mathscr{C}^\infty(\boldsymbol{R}^n)$, then (i) *and* (ii) *are also equivalent to*

(ii)′ *There are $d > 0$, $\varepsilon > 0$, $c' > 0$, and for every b some $c > 0$ such that $|v(f)| \leqslant c$ whenever $v \in \mathscr{E}'(\boldsymbol{R}^n)$ satisfies* (12) *and* (13).

In particular, $f \in \mathscr{C}^\infty(\boldsymbol{R}^n)$ defines an element in G_φ precisely if we can find $d > 0$, $\varepsilon > 0$, and for every b some $c > 0$ such that $|v(f)| \leqslant c$ whenever $v \in \mathscr{E}'(\boldsymbol{R}^n)$ satisfies

$$|\hat{v}(\zeta)| \leqslant \exp\big(d\varphi(-\operatorname{Re} \zeta) + \varepsilon |\operatorname{Im} \zeta| + b \ln(1+|\zeta|)\big), \quad \forall \zeta \in \boldsymbol{C}^n.$$

Definition 14. Consider $u \in \mathscr{F}$ and $\Gamma \subset \boldsymbol{R}^n$. We say that $(0, \Gamma)$ is not in the boundary wave front set $\mathrm{WF}_\varphi^{\mathrm{b}}$ of u ("0" is here the one from $\boldsymbol{R}^{n+1}$), and write $(0, \Gamma) \cap \mathrm{WF}_\varphi^{\mathrm{b}} u = \emptyset$ if:

$$\exists d > 0, \quad \exists \varepsilon > 0, \quad \exists c', \quad \forall b \in \boldsymbol{R}, \quad \exists b' \in \boldsymbol{R}, \quad \exists c,$$

such that $|u(v)| \leqslant c$ for any $v \in \mathscr{C}_0^\infty(R^{n+1})$ which satisfies

$$|\hat{v}(\lambda)| \leqslant \exp\big(d\varphi(-\mathrm{Re}\,\zeta)+\varepsilon|\mathrm{Im}\,\zeta|+\varepsilon\,\mathrm{Im}\,\tau^+ + b\ln(1+|\lambda|) + b'\ln(1+|\zeta|)\big) \quad \text{if } \mathrm{Re}\,\zeta \in -\Gamma_{c'\varphi},$$

$$|\hat{v}(\lambda)| \leqslant \exp\big(\varepsilon|\mathrm{Im}\,\zeta|+\varepsilon\,\mathrm{Im}\,\tau^+ + b\ln(1+|\lambda|)+b'\ln(1+|\zeta|)\big) \quad \text{if } \mathrm{Re}\,\zeta \notin -\Gamma_{c'\varphi}.$$

Remark 15. Various notions of boundary wave front sets have been introduced in the literature, using a variety of definitions and serving different purposes (cf. e.g. J. Chazarain [2], R. Melrose–J. Sjöstrand [16]). Any notion of boundary regularity should be such that boundary regularity for u implies at least regularity of the traces $D_t^j u|_{t=0}$. In the present situation this comes to:

If $(0, \Gamma) \cap \mathrm{WF}_\varphi^{\mathrm{b}} u = \emptyset$, *then* $(0, \Gamma) \cap \mathrm{WF}_\varphi D_t^j u|_{t=0} = \emptyset$, $\forall j$.

This is in fact an easy consequence of Proposition 13. In particular, $(0, \Gamma) \cap \mathrm{WF}_\varphi^{\mathrm{b}} u = \emptyset$ implies that $(0, \Gamma) \cap \mathrm{WF}_{\mathrm{a}}^{\mathrm{b}} u = \emptyset$ in the sense of J. Sjöstrand [20] if $\varphi = 1+|\xi|$ and $p(x, t, D_x, D_t)u = 0$ (the latter condition is necessary since $\mathrm{WF}_{\mathrm{a}}^{\mathrm{b}}$ is only defined for solutions of equations of the type $p(x, t, D_x, D_t)u = 0$). The converse is also true in view of Theorem 18 below.

Remark 16. If $\chi_j \in \mathscr{C}_0^\infty(R^n)$ is a sequence of functions such that

$$|D^{\alpha+\beta}\chi_j(x)| \leqslant c_\alpha c^{|\beta|+1} j^{|\beta|} \quad \text{for } |\beta| \leqslant j,$$

then it follows from Proposition 13 (and its proof) that we have

$$|g_j(\xi, t)| \leqslant c'\big(c'j/\varphi(\xi)\big)^j \quad \text{for } \xi \in \Gamma_{c''\varphi} \tag{14}$$

and for $g_j(\xi, t) = \int \chi_j(x)u(x, t)\exp(-i\langle x, \xi\rangle)\,dx$ if

(a) $0 \leqslant t \leqslant \varepsilon$ and ε is small,

(b) $(0, \Gamma) \cap \mathrm{WF}_\varphi^{\mathrm{b}} u = \emptyset$,

(c) the supports of the χ_j are close to zero.

Our definition of WF^{b} is, however, a little more than just this property, in that it also refers to the derivatives in t of the functions $g_j(\xi, t)$. More precisely, one can prove:

PROPOSITION 17. *Consider* $u \in \mathscr{F}$. *The following conditions are equivalent*:

(a) $(0, \Gamma) \cap \mathrm{WF}_\varphi^{\mathrm{b}} u = \emptyset$.

(b) *There are* $d > 0$, $\varepsilon > 0$, C *and for every* k *some* c_k *and* b_k *such that*

$$\left|\int v(x)\,D_t^k u(x, t)\,dx\right| \leqslant c_k \quad \textit{if } 0 \leqslant t \leqslant \varepsilon$$

and if $v \in \mathscr{C}_0^\infty(\mathbf{R}^n)$ *satisfies*

$$|\hat{v}(\zeta)| \leqslant \exp(d\varphi(-\operatorname{Re}\zeta)+\varepsilon|\operatorname{Im}\zeta|+b_k\ln(1+|\zeta|)) \quad \text{if } \operatorname{Re}\zeta \in -\Gamma_{C\varphi},$$

$$|\hat{v}(\zeta)| \leqslant \exp(\varepsilon|\operatorname{Im}\zeta|+b_k\ln(1+|\zeta|)) \quad \text{if } \operatorname{Re}\zeta \notin -\Gamma_{C\varphi}.$$

(c) *Let* $\chi_j \in \mathscr{C}_0^\infty(\mathbf{R}^n)$ *be a sequence of functions such that* $\chi_j(x) = 0$ *for* $|x| \geqslant C_1$ *and* $\chi_j(x) = 1$ *in some fixed neighborhood of the origin. Moreover, assume that*

$$|D_x^{\alpha+\beta}\chi_j(x)| \leqslant c_\alpha c^{|\beta|+1} j^{|\beta|} \quad \text{if } |\beta| \leqslant j.$$

Then if C_1 *is small enough, we can find* C_2 *and for every* k *some* c_k *such that*

$$\left|\int \chi_j(x) D_t^k u(x,t) \exp(-i\langle x,\xi\rangle)\,dx\right| \leqslant c_k(c_k j/\varphi(\xi))^j \quad \text{if } \xi \in \Gamma_{C_2\varphi}.$$

Thus our definition is a quantitatively more precise version of (14), but we prefer our own definition since it is closer to what we actually need in the proofs.

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We can now state

THEOREM 18. *Assume that* $u \in \mathscr{F}$ *is a solution of* (1) *and write* $f_j = D_t^j u|_{t=0}$, $j = 0, \ldots, m-1$. *Let also* $\Gamma \subset \mathbf{R}^n$ *be given and assume that* $(0,\Gamma) \cap \mathrm{WF}_\varphi f_j = \emptyset$, $\forall j$. *Then* $(0,\Gamma) \cap \mathrm{WF}_\varphi^{\mathrm{b}} u = \emptyset$.

Remark 19. In the analytic category this theorem gives a result of Schapira [19]. (Cf. also Sjöstrand [20] and Theorem 9.6.9 in Hörmander [8]. For constant coefficients it is also a consequence of the arguments from Liess [13].) In the $\mathscr{C}^\infty$ category a similar result appears in de Gosson [4].

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We mention finally the following result, which may serve as a justification for our notion of boundary wave front set.

PROPOSITION 20. *Assume that* $f \in \mathscr{F}$ *is such that* $(0,\mathbf{R}^n) \cap \mathrm{WF}_\varphi^{\mathrm{b}} f = \emptyset$. *Then there is* $u \in \mathscr{S}'$ *such that* $p(x,t,D_x,D_t)u = f$ *and* $D_t^j u|_{t=0} = 0$, $j = 0, \ldots, m-1$.

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We conclude the paper with a brief discussion of some results for constant coefficient operators which are related to this paper. In fact, in that case one can go much further in the study of the solvability of the Cauchy problem and one also arrives in a natural way at definitions of the type considered in this paper, so this discussion should also serve as a justification for the present approach.

For simplicity, we shall look at the two-sided Cauchy problem, although most of the results have analogues for the one-sided problem. Let us in fact consider some constant coefficient linear partial differential operator $p(D)$ which satisfies our assumptions for some Lipschitz-continuous function φ: $\mathbf{R}^n \to \mathbf{R}$. (Recall that any $p(x, t, D_x, D_t)$ of the form (3) satisfies these assumptions for $\varphi = 1+|\xi|$.) In particular, we assume that the operator is of order m. For fixed $\zeta \in \mathbf{C}^n$ we denote by $\tau_1(\zeta), \ldots, \tau_m(\zeta)$ the roots of $\tau \to p(\zeta, \tau) = 0$ labelled so that $|\operatorname{Im} \tau_1(\zeta)| \leqslant \ldots \leqslant |\operatorname{Im} \tau_m(\zeta)|$. (A k-tuple root is written k times.) The functions $|\operatorname{Im} \tau_i(\zeta)|$ are uniquely defined by this prescription, although the functions τ_i are not.

Definition 21. We denote by F_j the set of germs of $\mathscr{C}^\infty$ functions defined in a neighborhood of $0 \in \mathbf{R}^n$ for which $\exists d > 0$, $\exists \varepsilon > 0$, $\forall b \geqslant 0$, $\exists c > 0$ such that $|v(f)| \leqslant c$ for any $v \in \mathscr{C}_0^\infty(\mathbf{R}^n)$ such that

(15) $$|\hat{v}(\zeta)| \leqslant \exp\bigl(d|\operatorname{Im} \tau_{j+1}(-\zeta)| + \varepsilon|\operatorname{Im} \zeta| + b \ln(1+|\zeta|)\bigr).$$

Thus what was a property in Proposition 13 is now a definition. One cannot, however, reduce the study of spaces of the type F_j to that of G_φ classes, since the functions $|\operatorname{Im} \tau_j(\zeta)|$ are not, in general, Lipschitz-continuous. We come back to this question later.

The following proposition should now serve as a justification for the introduction of the functions τ_i and the classes F_j.

Proposition 22. *Let $f \in \mathscr{C}^\infty$ be given. Then $f \in F_0$ if and only if we can find a $\mathscr{C}^\infty$ solution u of $p(D)u = 0$ such that $u|_{t=0} = f$. Moreover, the condition $f_j \in F_0$, $j = 0, \ldots, m-1$, is enough for the solvability of the Cauchy problem*

(16) $$p(D)u = 0,$$

(17) $$D_t^j u|_{t=0} = f_j, \quad j = 0, \ldots, m-1,$$

if and only if there is some constant c such that

$$|\operatorname{Im} \tau_m(\zeta)| \leqslant c\bigl(1+|\operatorname{Im} \zeta| + |\operatorname{Im} \tau_1(\zeta)|\bigr).$$

The first part of this proposition is due to Ehrenpreis [3]. The second is a consequence of the first if we also use Proposition 1 from Liess [10].

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The study of the solvability conditions for (16), (17) can very often be reduced completely to the study of the spaces F_j. (This question has been analyzed in Liess [11].). Theorems like Theorem 12 have then their counterparts in theorems concerning microlocal characterization of the relation "$f \in F_j$". (The latter are much easier to prove.)

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We add some remarks concerning the spaces F_j. Our first concern is to see how much Lipschitzianity there is in the $|\mathrm{Im}\ \tau_j(\zeta)|$. To see this we mention at first:

LEMMA 23. *The functions* $|\mathrm{Im}\ \tau_j(\zeta)|$ *are continuous.*

Lemma 23 is essentially well known. For details cf. e.g. Liess [11], where it is also shown that for suitable $c_1 > 0$, $\delta > 0$,

$$|\zeta-\eta| \leqslant c_1(1+|\zeta|)^{-\delta} \quad \text{implies} \quad \big||\mathrm{Im}\ \tau_j(\zeta)|-|\mathrm{Im}\ \tau_j(\eta)|\big| \leqslant 1$$

(and this is also essentially well known; cf. also Malgrange [15, Chapt. IV]).

To improve the lemma, we introduce the notation

$$X = \Big\{\zeta\in \mathbf{C}^n;\ \prod_{i\neq j}\big(\tau_i(-\zeta)-\tau_j(-\zeta)\big) = 0\Big\}$$

for the discriminant set of $\tau \to p(-\lambda)$, and

$$X_{c\varphi} = \big\{\zeta\in \mathbf{C}^n;\ \mathrm{dist}(\zeta, X) \leqslant c\big(\varphi(-\mathrm{Re}\ \zeta)+|\mathrm{Im}\ \zeta|\big)\big\}$$

if $c > 0$ is given. (This notation is thus in analogy with $\Gamma_{c\varphi}$ for $\Gamma \subset \mathbf{R}^n$.)

PROPOSITION 24. *Let* $c > 0$ *be given. Then there is* $c_1 > 0$ *such that*

$$\big||\mathrm{Im}\ \tau_j(-\zeta)|-|\mathrm{Im}\ \tau_j(-\eta)|\big| \leqslant c_1|\zeta-\eta|$$

for all $\zeta, \eta\in \mathbf{C}^n\setminus X_{c\varphi}$.

Proof (sketch). For some c_2 we have

$$|\tau_j(-\zeta)| \leqslant c_2\big(\varphi(-\mathrm{Re}\ \zeta)+|\mathrm{Im}\ \zeta|\big). \tag{18}$$

As a consequence, it suffices to prove the desired inequality under the additional assumption that

$$|\zeta-\eta| \leqslant c_3\big(\varphi(-\mathrm{Re}\ \zeta)+|\mathrm{Im}\ \zeta|\big).$$

Now, for c_3 suitably small, it follows that

$$\varphi(-\mathrm{Re}\ \zeta)+|\mathrm{Im}\ \zeta| \sim \varphi(-\mathrm{Re}\ \eta)+|\mathrm{Im}\ \eta|$$

then. Shrinking c_3 still further, we have $Y_\zeta \cap X = \emptyset$ for

$$Y_\zeta = \big\{\eta\in \mathbf{C}^n;\ |\eta-\zeta| \leqslant c_3\big(\varphi(-\mathrm{Re}\ \zeta)+|\mathrm{Im}\ \zeta|\big)\big\}. \tag{19}$$

Let us now fix $\eta^0\in Y_\zeta$. We can find unique analytic functions $\mu_1(\eta), \ldots, \mu_m(\eta)$ defined near η^0 with $p(-\eta, \mu_j(\eta)) \equiv 0$ and $\mu_j \neq \mu_k$ if $k \neq j$. In view of Lemma 23 the proof comes to an end if we show that

$$|\mathrm{grad}_\eta\, \mu_k(\eta^0)| \leqslant c_4 \quad \text{if } \eta^0\in Y_\zeta. \tag{20}$$

This is, however, already a consequence of Cauchy's inequalities: if k is fixed, then we can extend μ_k analytically to the whole of Y_ζ so that $p(-\eta, \mu_k(\eta))$

$\equiv 0$, and we also have

$$|\mu_k(\eta)| \leqslant c_2(\varphi(-\operatorname{Re}\eta)+|\operatorname{Im}\eta|) \leqslant c_5(\varphi(-\operatorname{Re}\zeta)+|\operatorname{Im}\zeta|), \qquad \eta \in Y_\zeta,$$

in view of (4) and of the choice of c_3.

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Still for c as in the above we introduce the set

$$Z_c = \{\zeta \in C^n;\ \operatorname{Im}\zeta = 0,\ \zeta \notin X_{c\varphi}\}.$$

We may regard Z_c as a subset of R^n and of course Z_c increases if we shrink c. It is not hard to see in fact that

(21) $$Z_c \subset_\varphi Z_{c'} \quad \text{if } c' < c.$$

Moreover, in view of Proposition 24, $\xi \to |\operatorname{Im}\tau_j(-\xi)|$ is Lipschitz-continuous on Z_c, and (using for example (21)) it is immediate that

(22) $$\big||\operatorname{Im}\tau_j(-\zeta)| - |\operatorname{Im}\tau_j(-\operatorname{Re}\zeta)|\big| \leqslant c_6(1+|\operatorname{Im}\zeta|) \quad \text{for } -\operatorname{Re}\zeta \in Z_c.$$

This leads to the following result:

PROPOSITION 25. *Consider $f \in F_j$ and $c > 0$. Then we can find $\varepsilon > 0$, $c' > 0$, and a bounded sequence $f_j \in \mathscr{E}'(R^n)$ such that*

(23) $$f = f_j \quad \text{for } |x| < \varepsilon,$$

(24) $$|\hat{f}_j(\xi)| \leqslant c'(c'j/|\operatorname{Im}\tau_{j+1}(\xi)|)^j \quad \text{if } \xi \in Z_{c'}.$$

Conversely, let $c'' > c$ and $f \in \mathscr{C}^\infty$ be given and assume that we can find some sequence f_j which is bounded in $\mathscr{E}'(R^n)$ and (23) *and* (24) *are valid. Then there is $\tilde{f} \in F_j$ such that*

$$(0, Z_{c''}) \cap \mathrm{WF}_\varphi(f-\tilde{f}) = \emptyset.$$

This gives a real-variable characterization in terms of G_φ classes of the obstructions to the solvability of the problem (16), (17) which come from Z_c.

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Added in proof (July 1987). Details of the proofs of the results presented here have meanwhile appeared in Comm. Partial Differential Equations 11 (13) (1986), 1379–1437.