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ASYMPTOTIC METHODS APPLIED TO PROBLEMS OF DIFFUSION, CRACK PROPAGATION AND CRACK TIP STRESS ANALYSIS

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1. Introduction

The theme of these lectures will be the application of a combination of mathematical techniques to certain physical problems. Often it is in precisely the region where one method fails that another is most successful, hence if a combination of techniques is used each method can serve as a check on the other as well as possibly extending the range of the solution. Moreover, it is sometimes possible to get an answer in some limit of a physical parameter that is relatively insensitive to the constitutive model under consideration. Some examples of results of this kind will be given.

The two principle mathematical techniques to be discussed are singular perturbation theory (matched asymptotic expansions) and transform methods leading to functional equations. The solution of the complex functional equations arising from many physical problems defies exact analysis and various approximate methods for their solution must be considered. We will not discuss these methods in detail here but we will quote some results calculated by these methods in order to compare with our asymptotic results.

In each case the physical motivation for the mathematical problem will be introduced prior to a description of the mathematical treatment. In Section 2 some problems of solid-solid phase transformations are considered. First certain moving boundary problems involving non-linear diffusion due to a concentration dependent diffusion coefficient are treated by asymptotic methods (Section 2.1). Then is Section 2.2 the steady motion of a step under volume diffusion control conditions is studied. A comparison of these results with a complete numerical treatment of a functional equation shows good

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agreement. In Section 3 recent numerical calculations of stresses at line crack tips in non-local elastic media are reexamined. On the basis of careful analysis, doubt is cast on these numerical results. In this section both asymptotic methods and Fourier transform rechniques are used. Finally in Section 4 dynamic crack propagation in a viscoelastic strip is considered by matched asymptotic expansions and comparison made with more complete Fourier transform and numerical integration results.

2. Moving boundary problems arising in the theory of the growth kinetics of metallurgical phase transformation

There are many problems in metallurgy in which diffusion is the rate controlling process in a transformation from one metallic phase to another. For example, there are many experimental observations of growing precipitate particles, dendrites and plates and a knowledge of the growth rates of these transformations is essential in order to control and understand the processes involved. There are additional complications due to the effect of favored crystallographic orientations, surface energies, interface kinetics etc. Furthermore, in some alloy systems, thermodynamic arguments plus experimental data have been used to predict the dependence of diffusion coefficients on composition. In these systems the growth of precipitate particles is determined by diffusion in a medium in which the diffusion coefficient depends on composition, i.e., the diffusion equation takes the form

$$\frac{\partial c}{\partial t} = \nabla \cdot (D(C) \nabla C). \tag{2.1}$$

In the next two subsections we consider situations where particular particle shapes are involved in the transformation process. In Section 2.1 we formulate the moving boundary problem for particles of ellipsoidal shape an in Section 2.2 the question of lateral growth due to step motion is considered.

2.1. Asymptotic results for the effect of composition-dependent diffusion on the growth rates of precipitate particles. In this section our object is to derive, via perturbation methods, quite general formulae for the growth rate of particles under certain conditions without restricting the diffusion coefficient $D(c)$ to have any specific form. We give here a rather abbreviated treatment (see Atkinson (1974) for a more complete account).

The following assumptions are made in the model.

(i) The precipitate particle, which is assumed to grow from zero size, is assumed to be of a constant composition X_1, matter is provided (or taken away in the case of negative precipitation) by diffusion to the particle from the matrix phase. In the matrix phase the composition profile is determined as the appropriate solution of equation (2.1) where C is the composition of the diffusing species.

(ii) The value of the composition in the matrix at the particlematrix interface is assumed to be constant C_0. The diffusion coefficient at this interface is written $D(C_0) = D_0$.

(iii) The value of C at infinity (in the matrix phase), or at the initial stages in time is assumed to be a constant C_∞.

(iv) The growth of the particle will be determined by the flux of matter through the particle matrix interface, e.g.,

$$(X_1 - C_0)\frac{dR_0}{dt} = D_0\left(\frac{\partial C}{\partial R}\right)_{R=R_0}$$

in the case when R_0 is the radius of a spherical or cylindrical particle. In general for an isoconcentrate surface the gradient of C at the surface is parallel to the normal $N = (N_1, N_2, N_3)$ to the surface. Condition (2.2) then has the form $(X_1 - C_0)\, N\cdot\dfrac{dN}{dt} = D_0 N\cdot \nabla C$ at the interface. It is fairly easy to see that the above conditions will all be satisfied if the particle is assumed to be spherical and then C can be seen to depend only on the similarity variable $R/t^{1/2}$. Thus expanding spheres satisfy the above conditions, the rate of the spheres expansion being determined from the flux equation subject to (2.1) and the boundary conditions. What is not so obvious, however, is that solutions can also be obtained for families of expanding ellipsoids as shown here.

2.1.1. Ellipsoidal growth. Consider the following change to the variable ω where

$$\frac{x^2}{\omega + a_1} + \frac{y^2}{\omega + a_2} + \frac{z^2}{\omega + a_3} = 4tD_0. \tag{2.2}$$

The moving boundary is assumed to be the expanding ellipse $\omega = \omega_0$. Making the change of variable (2.2) in equation (2.1) and assuming that C depends only on ω reduces (2.1) to the ordinary differential equation

$$\frac{dC}{d\omega} + \frac{d}{d\omega}\left(F\frac{dc}{d\omega}\right) + \frac{F}{2}\frac{dc}{d\omega}\left(\frac{1}{a_1+\omega} + \frac{1}{a_2+\omega} + \frac{1}{a_3+\omega}\right) = 0 \tag{2.3}$$

where we have written

$$D(c) = D_0 F(c), \qquad F(c_0) = 1. \tag{2.4}$$

Conditions (ii) and (iii) above are consistent with the boundary conditions

$$\begin{aligned} C &= C_0 \quad \text{when } \omega = \omega_0 \text{ (the moving boundary)},\\ C &= C_\infty \quad \text{when } \omega \to \infty. \end{aligned} \tag{2.5}$$

Thus equation (2.3) subject to boundary conditions (2.5) defines the ellipsoid growth problem; the unknown ω_0 which determines the moving boundary is to be found from the flux condition at the interface which can be written as

$$\frac{1}{(X_1 - C_0)} \left(\frac{dc}{d\omega}\right)_{\omega=\omega_0} = 1. \tag{2.6}$$

Explicit solutions of the non-linear equation (2.4) are not available in general so numerical methods must be employed. However, for limiting values of ω_0 some general results can be obtained.

2.1.2. Fast growth ($\omega_0 \gg 1$). This situation might be expected to occur when the composition of the particles is very close to that in the matrix phase at infinity, and the composition in the matrix at the matrix-particle interface is either very rich, or very depleted, in solute so that very steep concentration gradients are set up in the matrix, i.e., $(C_\infty - X_1)/(C_0 - C_\infty) \ll 1$.

To attempt a solution of this problem ($\omega_0 \gg 1$), put $\omega = \omega_0 + u$, where $\omega_0 = 1/\varepsilon$ and $\varepsilon \ll 1$ is to be a small parameter in the following analysis.

(a) A sphere, cylinder or plane. First, we treat the situation when the ellipsoid reduces to either a sphere, a cylinder or a plane; then (2.2) gives $r^2 = 4\omega D_0 t$ where r denotes distance from a plane, radius of a cylinder or of a sphere. In terms of the new variable u, (2.3) becomes

$$\frac{dC}{du} + \frac{d}{du}\left(F \frac{dC}{du}\right) + \frac{\lambda F}{2} \frac{dC}{du}\left(\frac{\varepsilon}{1+\varepsilon u}\right) = 0, \tag{2.7}$$

where $\lambda = 1, 2, 3$ signifies a plane, cylinder or sphere, respectively.

The boundary conditions (2.5) become,

$$\begin{aligned} C &= C_0 \quad \text{when} \quad u = 0, \\ C &= C_\infty \quad \text{when} \quad u \to \infty. \end{aligned} \tag{2.8}$$

It is convenient to replace this latter boundary condition by

$$\frac{dC}{du} = 0 \quad \text{as} \quad C \to C_\infty. \tag{2.9}$$

With this condition a first integral of equation (2.7) can be written

$$F \frac{dC}{du} = - \int_{C_\infty}^{C} \{1 + \tfrac{1}{2}\lambda F \varepsilon (1 + \varepsilon u)^{-1}\}\, dC, \tag{2.10}$$

where u in the integrand is of course an unknown function of C. We look for an inverse coordinate expansion of the solution u as a function of C and write

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots \tag{2.11}$$

With this in mind, rewrite (2.10) as

$$F^{-1}\frac{du}{dC} = -\left\{\int_{C_\infty}^{C}[1+\tfrac{1}{2}\lambda F\varepsilon(1+\varepsilon u)^{-1}]\,dC\right\}^{-1}, \tag{2.12}$$

$$\frac{1}{F}\frac{du}{dC} = -(C-C_\infty)^{-1}\left\{1+\frac{\lambda\varepsilon}{2(C-C_\infty)}\left[\int_{C_\infty}^{C}\frac{F\,dC}{(1+\varepsilon u)}\right]\right\}^{-1}. \tag{2.13}$$

Substituting from (2.11) gives the following sequence of problems:

$$F^{-1}\frac{du_0}{dC} = -(C-C_\infty)^{-1}, \tag{2.14}$$

$$F^{-1}\frac{du_1}{dC} = \tfrac{1}{2}\lambda(C-C_\infty)^{-2}\int_{C_\infty}^{C}F\,dC, \tag{2.15}$$

(2.16)

$$F^{-1}\frac{du_2}{dC} = -\tfrac{1}{2}\lambda(C-C_\infty)^{-2}\int_{C_\infty}^{C}Fu_0\,dC-\tfrac{1}{4}\lambda^2(C-C_\infty)^{-3}\left(\int_{C_\infty}^{C}F\,dC\right)^2,$$

$$F^{-1}\frac{du_3}{dC} = \tfrac{1}{2}\lambda(C-C_\infty)^{-2}\int_{C_\infty}^{C}F(u_0^2-u_1)\,dC+ \tag{2.17}$$

$$+\tfrac{1}{2}\lambda^2(C-C_\infty)^{-3}\left(\int_{C_\infty}^{C}F\,dC\right)\left(\int_{C_\infty}^{C}u_0F\,dC\right)+\tfrac{1}{3}\lambda^3(C-C_\infty)^{-4}\left(\int_{C_\infty}^{C}F\,dC\right)^3.$$

The boundary conditions $C = C_0$ at $u = 0$ could then be interpreted as $C = C_0$ at $u_0 = 0$, $u_1 = 0$, $u_2 = 0$, etc. Under these conditions the solution of (2.14) is

$$u_0 = \int_{C}^{C_0}(C-C_\infty)^{-1}F\,dC, \tag{2.18}$$

and the solution of (2.15) is

$$u_1 = -\tfrac{1}{2}\lambda\int_{C}^{C_0}(C_1-C_\infty)^{-2}F(C_1)\,dC_1\int_{C_\infty}^{C_1}F(C_2)\,dC_2. \tag{2.19}$$

Our procedure so far has been entirely formal; we have not stated under what conditions this inverse expansion may be valid or what restrictions on $F(C)$ are necessary in order that the above solution would be valid. Our interest is in determining ω_0 (i.e. $1/\varepsilon$) from the condition (2.6) which becomes in the new variables

$$\left(\frac{du}{dC}\right)_{C=C_0} = (X_1 - C_0)^{-1}, \tag{2.20}$$

and equations (2.14) to (2.17) can be used directly in (2.20) providing the expansion procedure is valid. Assuming that it is, substitution from (2.14) to (2.17) in (2.20) gives (recalling that $F(C_0) = 1$)

$$\begin{aligned}(X_1 - C_0)^{-1} = -(C_0 - C_\infty)^{-1} + \tfrac{1}{2}\varepsilon\lambda(C_0 - C_\infty)^{-2} \int_{C_\infty}^{C_0} F(C)\,dC - \\ -\varepsilon^2(C_0 - C_\infty)^{-1}\Big[(\tfrac{1}{2}\lambda)^2(C_0 - C_\infty)^{-2}\Big\{\int_{C_\infty}^{C_0} F(C)\,dC\Big\}^2 + \\ + \tfrac{1}{2}\lambda(C_0 - C_\infty)^{-1}\int_{C_\infty}^{C_0} F(C)u_0\,dC\Big] + O(\varepsilon^3).\end{aligned}$$

This equation can be rearranged as

$$\begin{aligned}(X_1 - C_\infty)/(X_1 - C_0) = \tfrac{1}{2}\lambda\varepsilon(C_0 - C_\infty)^{-1}\int_{C_\infty}^{C_0} F(C)\,dC - \\ -\varepsilon^2\Big[(\tfrac{1}{2}\lambda)^2(C_0 - C_\infty)^{-2}\Big\{\int_{C_\infty}^{C_0} F(C)\,dC\Big\}^2 + \\ + \tfrac{1}{2}\lambda(C_0 - C_\infty)^{-1}\int_{C_\infty}^{C_0} u_0 F(C)\,dC\Big] + O(\varepsilon^3),\end{aligned} \tag{2.21}$$

with u_0 given by equation (2.18). Thus, as a first approximation, (2.21) gives

$$\omega_0 = \varepsilon^{-1} \approx \tfrac{1}{2}\lambda(X_1 - C_0)(X_1 - C_\infty)^{-1}(C_0 - C_\infty)^{-1}\int_{C_\infty}^{C_0} F(C)\,dC. \tag{2.22}$$

A number of authors have approximated the effect of a composition-dependent diffusion coefficient by using the results of a constant diffusion-coefficient analysis and replacing the diffusion coefficient by an average one. The above result suggests that, for very fast growth rates ($\omega_0 \gg 1$), the best average to use is the simple one $D_{av} = \int_{C_\infty}^{C_0} D(C)\,dC/(C_0 - C_\infty)$. Moreover, equation (2.21) suggests that such a procedure gives an upper bound to the growth rate.

Using equation (2.21) to give a correction to the result (2.22) we have

$$\omega_0 = \varepsilon^{-1} \approx \tfrac{1}{2}\lambda(X_1 - C_0)(X_1 - C_\infty)^{-1}(C_0 - C_\infty)^{-1}\int_{C_\infty}^{C_0} F(C)\,dC - \tag{2.23}$$

$$-(X_1-C_\infty)(X_1-C_0)^{-1}\left[\tfrac{1}{2}\lambda(C_0-C_\infty)^{-1}\int_{C_\infty}^{C_0} F(C)dC+\right.$$

$$\left.+\int_{C_\infty}^{C_0}(C_1-C_\infty)^{-1}F(C_1)dC_1\int_{C_\infty}^{C_1}F(C)dC\left\{\int_{C_\infty}^{C_0}F(C)dC\right\}^{-1}\right],$$

where those terms after the first can be considered as the error made in using the formula (2.22).

A check on the above results can be made when $F(C) = C^\beta/C_0^\beta$, $\beta > 0$, using results for this case derived in [8]. Using a different notation from that, but applying the same method, we find

$$\frac{C}{F(C)}\frac{d\omega}{dC} = -1+a_1 Y_1-a_2 Y_1^2+O(Y_1^3), \tag{2.24}$$

where

$$a_1 = \lambda(1+\beta)^{-1}, \qquad a_2 = 2\lambda(1+\beta)^{-1}(1+2\beta)^{-1}+\lambda^2(1+\beta)^{-2}, \tag{2.25}$$

and the boundary condition $C = C_\infty = 0$ at infinity is satisfied.

We have not yet satisfied the boundary condition $C = C_0$ at $\omega = \omega_0$. Substituting this in (2.24) and putting $\omega = \omega_0+u$ gives

$$\frac{C}{F(C)}\frac{du}{dC} = -1+\tfrac{1}{2}F(C)a_1\varepsilon-\tfrac{1}{4}\{F(C)\}^2\varepsilon^2 a_2+O(\varepsilon^3) \tag{2.26}$$

at $C = C_0$, and this agrees with (2.11), together with (2.14) to (2.16), when $C = C_0$.

(b) The ellipsoid

$$\frac{x^2}{a_1+\omega}+\frac{y^2}{a_2+\omega}+\frac{z^2}{a_3+\omega} = 4D_0 t.$$

Results for this case can be obtained in a similar way to that for the sphere, cylinder or plane. If $a_i/\omega_0 \ll 1$ then the corresponding result to (2.23) is

$$\omega_0 = 1/\varepsilon \approx \tfrac{3}{2}(X_1-C_0)(X_1-C_\infty)^{-1}(C_0-C_\infty)^{-1}\int_{C_\infty}^{C_0} F\,dC- \tag{2.27}$$

$$-(X_1-C_\infty)(X_1-C_0)^{-1}\left\{\tfrac{1}{3}(a_1+a_2+a_3)+\right.$$

$$+\int_{C_\infty}^{C_0}(C_1-C_\infty)^{-1}F(C_1)dC_1\int_{C_\infty}^{C_1}F\,dC\left(\int_{C_\infty}^{C_0}F\,dC\right)^{-1}+$$

$$\left.+\tfrac{3}{2}(C_0-C_\infty)^{-1}\int_{C_\infty}^{C_0}F\,dC\right\}+\ldots$$

The above result applies also to spheroidal particles whose equation is written

(2.28) $$\frac{x^2+y^2}{a+\omega}+\frac{z^2}{\bar{a}+\omega} = 4D_0 t$$

with aspect ratio $A = \{(\bar{a}+\omega_0)/(a+\omega_0)\}^{1/2}$. The two parameters a and $\bar{a}$ are superfluous for spheroids so we define oblate spheroids when $\bar{a} = 0$, $a \geqslant 0$ and prolate spheroids with $a = 0$, $\bar{a} \geqslant 0$. Putting $a_1 = a_2 = a$, $a_3 = \bar{a}$ in (2.27) gives the result for this case.

2.1.3. Slow growth ($\omega_0 \ll 1$). We expect slow growth when there are not very steep concentration gradients in the matrix phase, so the rate of flux of material into the particle of new phase is correspondingly small, i.e., $|C_0 - C_\infty| \ll |C_0 - X_1|$. Equations (2.3) to (2.6) still apply but it is convenient to change variables by writing

(2.29) $$\phi(C) = \int_C^{C_0} F(C)dC \Big/ \int_{C_\infty}^{C_0} F(C)dC$$

so that

(2.30) $$\frac{d\phi}{d\omega} = -\frac{F(C)}{A_0}\frac{dC}{d\omega},$$

where $A_0 = \int_{C_\infty}^{C_0} F(C)dC$. With this change of variable (2.3) becomes

(2.31) $$f(\phi)\frac{d\phi}{d\omega}+\frac{d^2\phi}{d\omega^2}+\frac{1}{2}\frac{d\phi}{d\omega}\left(\frac{1}{a_1+\omega}+\frac{1}{a_2+\omega}+\frac{1}{a_3+\omega}\right) = 0,$$

where

(2.32) $$f(\phi) = \{F(C(\phi))\}^{-1},$$

together with the boundary conditions

(2.33) $$\phi = \begin{cases} 0 & \text{when } \omega = \omega_0, \\ 1 & \text{when } \omega = \infty. \end{cases}$$

We look for ω_0 as the solution of the flux equation (2.6) which, when written in terms of ϕ, is

(2.34) $$A_0\left(\frac{d\phi}{d\omega}\right)_{\omega=\omega_0} = (C_0 - X_1)$$

since $F(C_0) = 1$.

We begin by considering the situation when the particle shape is that of a sphere, a cylinder or a plane.

(a) A sphere, a cylinder or a plane. In this case equation (2.31) becomes

$$f(\phi)\frac{d\phi}{d\omega}+\frac{d^2\phi}{d\omega^2}+\frac{\lambda}{2\omega}\frac{d\phi}{d\omega}=0, \tag{2.35}$$

where $\lambda = 1, 2, 3$ represents the plane, cylinder or sphere, respectively.

When ω_0 is much less than unity we take $\omega_0 = \varepsilon$ to be the small parameter in our problem; further, since $1/\omega$ occurs in equation (2.29), a singular-perturbation approach to the problem is suggested. Putting $\omega = \alpha\omega_1$ in (2.35) reduces it to

$$\alpha f(\phi)\frac{d\phi}{d\omega_1}+\frac{d^2\phi}{d\omega_1^2}+\frac{\lambda}{2\omega_1}\frac{d\phi}{d\omega_1}=0 \tag{2.36}$$

so if $\alpha \gg 1$, this reduces to

$$\frac{d\phi}{d\omega_1}=0 \tag{2.37}$$

and if $\alpha \ll 1$, to

$$\frac{d^2\phi}{d\omega_1^2}+\frac{\lambda}{2\omega_1}\frac{d\phi}{d\omega_1}=0, \tag{2.38}$$

while α of order unity gives equation (2.35).

Although the ranges of validity of (2.37) and (2.38) do not overlap, singular-perturbation methods can be used to get asymptotic solutions to the problem. These methods have been used in Lagerstrom and Casten (1972) to treat a differential equation which has a similar character to equation (2.35) (their example has ϕ in place of $f(\phi)$ in equation (2.35) and $(N-1)$ in place of $\lambda/2$). Intuitively one expects that close to the particle its shape would be important, whereas at infinity with boundary conditions (2.33), to a first approximation, the presence of the particle would be unnoticed. This suggests that we scale the variables for the inner problem by writing $\omega = \varepsilon u$.

Inner problem. With $\omega = \varepsilon u$, equation (2.35) becomes

$$\varepsilon f(\phi)\frac{d\phi}{du}+\frac{d^2\phi}{du^2}+\frac{\lambda}{2u}\frac{d\phi}{du}=0 \tag{2.39}$$

with inner boundary conditions $\phi = 0$ when $u = 1$, the other boundary condition to be determined by matching. To a first approximation we have

$$\phi = \begin{cases} -B(u^{1-\lambda/2}-1), & \lambda \neq 2, \\ B\log u, & \lambda = 2, \end{cases} \tag{2.40}$$

with solution

$$\frac{d^2\phi}{du^2}+\frac{\lambda}{2u}\frac{d\phi}{du}=0 \tag{2.41}$$

where B is a constant to be determined by matching.

For $\lambda = 3$ (the sphere) this solution is finite at $u = \infty$, for $\lambda = 2$ it tends to infinity like $\log u$, whereas for $\lambda = 1$ (the plane) it goes to infinity like $u^{1/2}$ as $u \to \infty$. Thus for $\lambda = 1$, matching is not appropriate and the problem appears to be a regular perturbation problem (see [13]) for which the first approximation would be the stationary-interface problem, equation (2.35), with $\lambda = 1$ and $\omega_0 = 0$ which involves the full non-linear equation so no simplification seems possible. The problem of getting sharp bounds for $dC/d\omega$ at $\omega = 0$ for this equation seems surprisingly difficult (see [14], [15]). We consider first the sphere.

2.4.1. The sphere ($\lambda = 3$). Write the inner solution as

$$\phi_i \sim g_0+\alpha g_1+\dots. \tag{2.42}$$

and (2.40) gives

$$g_0 = B(1-u^{-1/2}) \tag{2.43}$$

as the first approximation to the inner solution. We want this to match with the outer solution as u tends to infinity so we take $B = 1$. We write the outer solution as

$$\phi \sim 1+\beta\phi_1+\beta^2\phi_2+\dots, \tag{2.44}$$

where both α and β are to be determined.

The equations governing ϕ_1, ϕ_2, etc. can be obtained by expanding $f(\phi)$ in a Taylor series and equating to zero terms involving β, β^2, etc. For the general case equation (2.31) gives

$$f_1\frac{d\phi_1}{d\omega}+\frac{d^2\phi_1}{d\omega^2}+\frac{1}{2}\frac{d\phi_1}{d\omega}\left\{\frac{1}{a_1+\omega}+\frac{1}{a_2+\omega}+\frac{1}{a_3+\omega}\right\}=0, \tag{2.45}$$

$$f_1\frac{d\phi_2}{d\omega}+\frac{d^2\phi_2}{d\omega^2}+\frac{1}{2}\frac{d\phi_2}{d\omega}\left\{\frac{1}{a_1+\omega}+\frac{1}{a_2+\omega}+\frac{1}{a_3+\omega}\right\}=-f_1'\phi_1\frac{d\phi_1}{d\omega}, \tag{2.46}$$

etc., where

$$f_1 \equiv f(1) \equiv \{F(C_\infty)\}^{-1}, \tag{2.47}$$

$$f_1' = \left(\frac{df}{d\phi}\right)_{\phi=1} = A_0F'(C_\infty)\{F(C_\infty)\}^{-3}, \tag{2.48}$$

together with the boundary conditions $\phi_1 = 0 = \phi_2$, etc. at $\omega = \infty$.

W assume that $f_1 \neq 0$ in the following analysis. The solutions of (2.45) and (2.46) which apply to the general ellipsoid are

$$\phi_1 = A_1 \int_{\omega}^{\infty} e^{-f_1\omega} \{(a_1+\omega)(a_2+\omega)(a_3+\omega)\}^{-1/2} d\omega, \tag{2.49}$$

$$\begin{aligned}\phi_2 = {} & A_2 \int_{\omega}^{\infty} e^{-f_1\omega} \{(a_1+\omega)(a_2+\omega)(a_3+\omega)\}^{-1/2} d\omega + \\ & + f_1' A_1^2 \int_{\omega}^{\infty} e^{-f_1\omega_2} \{(a_1+\omega_2)(a_2+\omega_2)(a_3+\omega_2)\}^{-1/2} d\omega_2 \times \\ & \times \int_{\omega_2}^{\infty} e^{-f_1\omega_1} (\omega_1-\omega_2) \{(a_1+\omega_1)(a_2+\omega_1)(a_3+\omega_1)\}^{-1/2} d\omega_1,\end{aligned} \tag{2.50}$$

where A_1 and A_2 are constants to be determined by matching. In the case of the sphere the above relations reduce to

$$\phi_1 = A_1 \int_{\omega}^{\infty} e^{-f_1\omega} \omega^{-3/2} d\omega, \tag{2.51}$$

$$\begin{aligned}\phi_2 = {} & A_2 \int_{\omega}^{\infty} e^{-f_1\omega} \omega^{-3/2} d\omega + \\ & + f_1' A_1^2 \int_{\omega}^{\infty} e^{-f_1\omega_2} \omega^{-3/2} d\omega_2 \int_{\omega_2}^{\infty} e^{-f_1\omega_1} \omega_1^{-3/2} (\omega_1-\omega_2) d\omega_1;\end{aligned} \tag{2.52}$$

β and A_1 can be determined by matching with (2.43), i.e., $g_0 = 1-u^{-1/2}$. Expanding (2.51) for small ω gives

(2.53)
$$\phi_1 = A_1 [2\omega^{-1/2} e^{-f_1\omega} - 2\pi^{1/2} f_1^{1/2} \{1 - 2\pi^{-1/2} (f_1\omega)^{1/2} (1 - \tfrac{1}{3} f_1 \omega + \ldots)\}].$$

Written in outer coordinates, g_0 becomes

$$g_0 = 1 - \varepsilon^{1/2} \omega^{-1/2} \tag{2.54}$$

so matching with $\phi = 1 + \beta\phi_1$ is possible, provided $\beta = \varepsilon^{1/2}$and $A_1 = -1/2$. To proceed to higher-order approximations for the inner solution we look for the solution of

$$\frac{d^2\phi}{du^2} + \frac{3}{2u}\frac{d\phi}{du} = -\varepsilon f(g_0) \frac{dg_0}{du}, \tag{2.55}$$

which should give a solution to (2.39) correct to order ε.

The solution of (2.55) with $\phi = 0$ at $u = 1$ can be written

$$\phi = B_1(1-u^{-1/2}) - \varepsilon \int_1^u u_1^{-3/2} du_1 \int_1^{u_1} u_2^{3/2} f(g_0)(dg_0/du_2) du_2,$$

or after changing the order of integration,

$$(2.56)\qquad \phi = B_1(1-u^{-1/2})-\varepsilon\int_1^u f(g_0)(u_2^{-1/2}-u^{-1/2})\,du_2,$$

where B_1 is an arbitrary constant, $g_0 = 1-u_2^{-1/2}$, and $f(g_0)$ is a function defined implicitly in terms of g_0 (or ϕ) through equations (2.32) and (2.29). The constant B_1 is to be determined by matching the above solution (equation (2.56)), expanded for large u, with the outer expansion $\phi \sim 1+\varepsilon^{1/2}\phi_1+\varepsilon\phi_2$ expanded for small ω. To get the expansion of ϕ in (2.56) for large u and for a general f we can expand $f(g_0)$ by Taylor series giving

$$(2.57)\qquad f(g_0) = f(1)-u_2^{-1/2}f'(1)+\tfrac{1}{2}u_2^{-1}f''(1)\ldots,$$

and (2.56) becomes

$$(2.58)\qquad \phi_{\text{inner}} \sim B_1(1-u^{-1/2})-\varepsilon\{u^{1/2}f(1)-f'(1)\log u- -\tfrac{1}{2}f_1''u^{-1/2}\log u+\text{constant}+(\text{constant})\,u^{-1/2}\ldots\}.$$

Using the Taylor expansion (2.57) to a finite number of terms will not give a closed form expression for the constants in equation (2.58). We could, using the full Taylor expansion, obtain an expression for those constants in terms of a series, however by rewriting the second integral in (2.56) in terms of g_0 and integrating by parts we get the following expression:

$$(2.59)\qquad \phi_{\text{inner}} = B_1(1-u^{-1/2})-\varepsilon\{u^{1/2}f(g_0)-2f(0)+f'(g_0)+ +u^{-1/2}(f(0)-f'(0))-f'(g_0)\log u-\tfrac{1}{2}u^{-1/2}f''(g_0)\log u- -2\int_0^{1-u^{-1/2}} f''(g_0)\log(1-g_0)\,dg_0-u^{-1/2}\int_0^{1-u^{-1/2}} f'''(g_0)\log(1-g_0)\,dg_0\}.$$

This is true to order ε for all u with $g_0 = 1-u^{-1/2}$.

If $f(g_0)$ is such that f and its derivatives have no worse than integrable singularities in the range $0 \leqslant g_0 \leqslant 1$, then the integrals in equation (2.59) above are finite as $u \to \infty$ (i.e. $g_0 \to 1$ in the upper limit), and so the first of the two integrals in (2.59) can be used to evaluate the constant times ε term in the expansion of (2.59) for u large. The result for large u is

$$(2.60)\qquad \phi_{\text{inner}} \approx B_1(1-u^{-1/2})-\varepsilon\{u^{1/2}f(1)-2f(0)- -2\int_0^1 f''(g_0)\log(1-g_0)\,dg_0-f'(1)\log u+O(u^{-1/2})\},$$

where in $O(u^{-1/2})$ we have included terms like $u^{-1/2}\log u$. It should be recalled that we have no explicit expression for $f(g_0)$ as it is determined implicitly through equations (2.29) and (2.32). We also have $f(0) = \{F(C_0)\}^{-1} = 1$.

Expanding the outer solution $1+\varepsilon^{1/2}\phi_1+\varepsilon\phi_2$ for small ω, we get

$$(2.61)\qquad \phi_{\text{outer}} \approx 1-\tfrac{1}{2}\varepsilon^{1/2}\left(2\omega^{-1/2}-2\sqrt{\pi}f_1^{1/2}+2f_1\,\omega^{1/2}+O(\omega^{3/2})+\right.$$
$$+\varepsilon\left(A_2\{2\omega^{-1/2}-2\sqrt{\pi}f_1^{1/2}+2f_1\,\omega^{1/2}+O(\omega^{3/2})\}\right)+$$
$$\left.+f_1'\{\tfrac{1}{2}\pi^{1/2}(f_1\,\omega)^{-1/2}+\log(2f_1\,\omega)-1+\gamma-\tfrac{1}{2}\pi^{1/2}(f_1\,\omega)^{1/2}+O(f_1\,\omega)\ldots\}\right),$$

where f_1 and f_1' are the values of $f(\phi)$ and $f'(\phi)$ evaluated at $\phi = 1$ and γ is the Euler constant $\gamma = 0.57721566$.

Writing the inner expansion (2.60) in terms of ω gives

$$(2.62)\qquad \phi_{\text{inner}} \approx B_1(1-\varepsilon^{1/2}\omega^{-1/2})-\varepsilon\{\omega^{1/2}\varepsilon^{-1/2}f_1-f_1'\log(\omega/\varepsilon)-$$
$$-2-2\int_0^1 f''(g_0)\log(1-g_0)\,dg_0\}+O(\varepsilon^{1/2}/\omega^{1/2}).$$

To match (2.61) and (2.62) to order ε we choose

$$(2.63)\qquad A_2 = -\tfrac{1}{4}\pi^{1/2}f_1'f_1^{-1/2}-\tfrac{1}{2}\pi^{1/2}f_1^{1/2}$$

and

$$(2.64)\qquad B_1 = 1+\pi^{1/2}f_1^{1/2}\varepsilon^{1/2}+f_1'\varepsilon\log\varepsilon+\varepsilon\{2+2\int_0^1 f''(g_0)\log(1-g_0)\,dg_0+$$
$$+f_1'(\log(2f_1)+\gamma+\pi/2-1)+\pi f_1\}.$$

From (2.56), we have $(d\phi/du)_{u=1} = \frac{1}{2}B_1$ and so equation (2.34) gives

$$(2.65)\qquad \tfrac{1}{2}\varepsilon^{-1}A_0B_1 = (C_0-X_1)$$

as the equation which determines $\varepsilon = \omega_0$.

The result (2.60) can be checked in special cases when an explicit expression for $f(g_0)$ can be obtained from equations (2.29) and (2.32).

EXAMPLE. When $F(C) = e^{(c-c_0)\alpha_1}$, and α_1 is a constant, then

$$(2.66)\qquad f(\phi) = \{1-H\phi(C)\}^{-1},$$

where $H = 1-e^{-\alpha_1(C_\infty-C_0)}$. Equation (2.56) for the inner solution can then be integrated to give

$$(2.67)\qquad \phi = B_1(1-u^{-1/2})-\varepsilon\{u^{1/2}(1-H)^{-1}+2(2H-1)(1-H)^{-2}+$$
$$+(1-3H)u^{-1/2}(1-H)^{-2}-2H(1-H)^{-2}\log|(1-H)u^{1/2}+H|$$
$$-2H^2u^{-1/2}(1-H)^{-3}\log|(1-H)u^{1/2}+H|\}$$

and expanding for large u gives

$$\phi \approx B_1(1-u^{-1/2})-\varepsilon[u^{1/2}(1-H)^{-1}+2(2H-1)(1-H)^{-2}- -2H(1-H)^{-2}\log|(1-H)u^{1/2}|+O(u^{1/2}\log u)] \tag{2.68}$$

$$\approx B_1(1-u^{-1/2})-\varepsilon[u^{1/2}f(1)-f'(1)\log u+ +2(2H-1)(1-H)^{-2}-2H(1-H)^{-2}\log|1-H|+O(u^{-1/2}\log u)],$$

since the results $f(1)=(1-H)^{-1}$, $f'(1)=H(1-H)^{-2}$ follow directly from (2.65).

The other two terms in the square brackets check with

$$-2-2\int_0^1 f''(g_0)\log(1-g_0)\,dg_0$$

on integration.

2.2. Step motion during lateral growth in solid-solid phase transformations. In this section we consider situations where volume diffusion in the parent phase is the predominant contribution to the growth of steps during lateral growth in solid-solid phase transformations. It is assumed that the ledged interphase boundaries have faces whose structure is sufficiently coherent to render them immobile in the direction normal to these faces; all motion thus takes place by the formation and lateral movement of the ledges. The specific situation considered here is that of a single step, such as shown in Figure 1, representing a precipitate growing

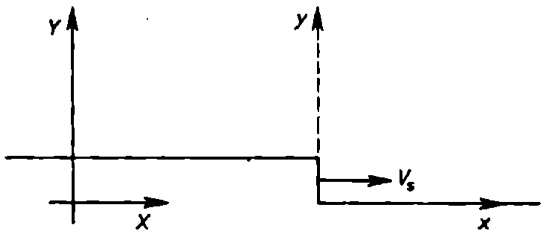

Fig. 1. The steady motion of a single precipitate step growing into the parent phase

into the parent phase by means of the ledge mechanism. The equilibrium solute concentration at the interface is indicated by C_e while the actual value at the step face is denoted by C_m. It is assumed that the forward step velocity V_s is constant and that the solute distribution is independent of time when written in terms of a coordinate system moving with the step.

The governing equations can then be written

$$\nabla^2\Gamma(x,y)+2p\frac{\partial\Gamma}{\partial x}=0 \tag{2.69}$$

where

(2.70) $$x = \frac{X - v_s t}{h}, \qquad y = \frac{Y}{h},$$

(2.71) $$2p = V_s h/D$$

and

(2.72) $$\Gamma(x, y) = (c(x, y) - C_\infty)(C_m^0 - C_\infty^m)^{-1},$$

D is the diffusion coefficient for solute in the matrix.

The dimensionless velocity parameter p is the peclet number, h is the height of the step in the stationary (X, Y) coordinate system, C_m^0 is the concentration at the base of the step and C_∞ the concentration at infinity.

Since it is assumed that the step progresses without change of shape the following boundary conditions are imposed on equation (2.69)

(2.73) $$\begin{aligned} \frac{\partial \Gamma}{\partial y} &= 0 \quad \text{on} \quad y = 1,\ x < 0, \\ \frac{\partial \Gamma}{\partial y} &= 0 \quad \text{on} \quad y = 0,\ x > 0, \\ \frac{\partial \Gamma}{\partial x} &= g \quad \text{on} \quad x = 0,\ 0 \leqslant y < 1; \end{aligned}$$

g is constant and Γ tends to zero as $(x^2 + y^2)^{1/2} \to \infty$ in the matrix $y > 0$, $x > 0$; $y > 1$, $x < 0$.

Once the problem specified by equations (2.69), (2.72) and (2.73) is solved, the condition

(2.74) $$\Gamma(0, 0) = 1$$

can be used to determine g as a function of p. Note that condition (2.74) is a consequence of $C(0, 0) = C_m^0$ the concentration at the base of the step.

The dependence of the steady growth velocity V_s on volume diffusion can be found by performing a flux balance across the step face, i.e.,

(2.75) $$V_s = D\left(\frac{\partial C}{\partial x}\right)_{\text{step}} (C_p - C_m)^{-1}$$

where D is the diffusion coefficient for solute in the matrix and C_p is the constant solute concentration of the precipitate. Assuming that $C_p \gg C_m$ and that $C_m \approx C_e$ the flux equation (2.75) can be seen to be consistant with boundary condition $(2.73)_3$ and leads to the equation

(2.76) $$\Omega_0 = 2p\alpha(p)$$

for the growth velocity V_s (recall definition (2.71)) in terms of the dimensionless normalised supersaturation

$$\Omega_0 = \frac{C_\infty - C_e}{C_p - C_e} \tag{2.77}$$

and

$$\alpha(p) = -g^{-1} \tag{2.78}$$

which follows from the solution of the boundary value problem specified by equations (2.69) to (2.74).

This problem will be treated here by the method of matched asymptotic expansions assumed to be applicable for $p \ll 1$. The results of the asymptotic method will be compared with results obtained by numerical solutions of a functional equation. A more complete account of these two methods can be found in Atkinson [6].

2.2.1. Analysis. Making the change of variable

$$\Gamma = e^{-px} U \tag{2.79}$$

in equation (2.69) reduces it to

$$\nabla^2 U - p^2 U = 0 \tag{2.80}$$

and the boundary conditions (2.73) becomes

$$\frac{\partial U}{\partial y} = 0, \qquad y = 0,\ x > 0, \quad y = 1,\ x < 0; \tag{2.81}$$

$$\frac{\partial U}{\partial x} - pU = g, \qquad x = 0,\ 0 \leqslant y \leqslant 1. \tag{2.82}$$

Matched asymptotic expansions valid for $p \ll 1$. Viewed on a length scale large compared to the size of the step it seems plausible to assume that the diffusion field is not affected by the fine detail of the step geometry (the outersolution) whereas close to the step the geometry becomes all important (the innersolution). At some intermediate length scale in favorable cases these two solutions will overlap and the method of matched asymptotic expansions can in principle be used to provide a complete solution.

The "outer" problem. We define "outer" coordinates (x_0, y_0) as

$$x_0 = px, \qquad y_0 = py$$

then the differential equation (2.80)

$$\frac{\partial^2 u^{(0)}}{\partial x_0^2} + \frac{\partial^2 u^{(0)}}{\partial y_0^2} - u^{(0)} = 0 \tag{2.80'}$$

and the boundary conditions (2.81) and (2.82) become

(2.81′) $$\frac{\partial u^{(0)}}{\partial y_0} = 0, \qquad y_0 = 0,\ x_0 > 0,$$
$$y_0 = p,\ x_0 < 0$$

and

(2.82′) $$\frac{\partial u^{(0)}}{\partial x_0} - u^{(0)} = \frac{g}{p}, \qquad x_0 = 0,\ 0 \leqslant y_0 \leqslant p.$$

Letting p tend to zero in these equations the boundary conditions (2.82′) is lost, to zero order, and the first approximation to the outer solution is thus an eigensolution which could be written in general as

(2.83) $$u_1^{(0)} = \sum_0^\infty b_n K_n(r_0) \cos n\theta$$

where

$$r_0^2 = x_0^2 + y_0^2 \qquad \text{and} \qquad x_0 = r_0 \cos \theta.$$

Clearly this solution satisfies the differential equation (2.80′) and the boundary conditions (2.81′) with $p = 0$. Subsequently we will need to rewrite $u(r_0, \theta)$ in terms of the inner coordinates r ($r_0 = pr$) and expand in terms of p for fixed r, i.e., the "inner" limit of the outer expansion. Note that

$$K_0(r_0) = \sum_{s=0}^\infty \frac{1}{s!}\left\{\frac{\Gamma'(s+1)}{\{\Gamma(s+1)\}^2}\left(\frac{r_0}{2}\right)^{2s} - \frac{1}{\Gamma(s+1)}\left(\frac{r_0}{2}\right)^{2s} \ln\left(\frac{r_0}{2}\right)\right\}$$

and

$$K_n(r_0) \sim (r_0)^{-N} = p^{-N} r^{-N} \qquad \text{as} \quad r_0 \to 0 \text{ if } N > 0.$$

This suggests that only the eigensolution $b_0 K_0(r_0)$ is relevant to our first "outer" approximation and the unknown coefficient b_0 is to be determined by matching this solution with an appropriate inner solution.

The "inner" problem. Taking (x, y) as "inner" coordinates, the equations governing the inner problem are equations (2.80) to (2.82), i.e.,

(2.80″) $$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - p^2 U = 0$$

with

(2.81″) $$\frac{\partial U}{\partial y} = 0, \qquad y = 0,\ x > 0,$$
$$y = 1,\ x < 0;$$

(2.82″) $$\frac{\partial U}{\partial x} - pU = g, \qquad x = 0,\ 0 \leqslant y \leqslant 1.$$

Formally letting $p \to 0$ we have that the zero order inner approximation must satisfy the equations

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0,$$

$$(2.84) \qquad \frac{\partial U}{\partial y} = 0, \qquad y = 0,\ x > 0,$$

$$y = 1,\ x < 0;$$

$$\frac{\partial U}{\partial x} = g, \qquad x = 0,\ 0 \leqslant y \leqslant 1.$$

Equations (2.84) lack conditions on U as $(x^2+y^2)^{1/2} \to \infty$, these conditions are to be determined by matching with "outer" solutions. The solution of (2.84) can be written

$$(2.85) \qquad U = gx + A + U_0$$

where U_0 satisfies equations (2.84) and

$$(2.86) \qquad \frac{\partial U_0}{\partial x} = 0, \qquad x = 0,\ 0 \leqslant y \leqslant 1.$$

To find the eigensolution U_0 we use conformal mapping. The region exterior to the step in mapped onto the upper half of the complex t plane by the transformation

$$(2.87) \qquad z = \frac{1}{\pi}\left\{(t^2-1)^{1/2} + \ln\left(t + (t^2-1)^{1/2}\right)\right\}$$

where $z = x+iy$, $t = t_1 + it_2$ (t_1, t_2) real.

The step corners $z = 0$ and $z = i$ map into $t = 1$ and $t = -1$ respectively, the logarithm has its principal value and the cut from $t = -1$ to $+1$ lies in $\operatorname{Im} t < 0$. The problem for U_0 satisfying equations (2.84) and (2.86) is satisfied by $U_0 = \operatorname{Re} F(t)$ where $F(t)$ is analytic in $\operatorname{Im} t > 0$ continuous in $\operatorname{Im} t \geqslant 0$ and such that $\operatorname{Im} F(t) = 0$ on $\operatorname{Im} t = 0$.

The most general such $F(t)$ depends upon conditions prescribed as $|t| \to \infty$ in the upper half of the t plane. However, the "outer" solutions discussed earlier have algebraic-logarithmic singularities as $x_0 \to 0$ and the matching procedure would then demand such behavior as $|t| \to \infty$. Hence one can deduce via Liouville's theorem that $F(t)$ should be a polynomial in t.

Hence we take

$$(2.88) \qquad U_0 = B_0 \operatorname{Re} t$$

where the real constant B_0 is to be determined by matching.

Thus the zero order inner approximation is written

$$U^{(i)} = gx + A + B_0 \operatorname{Re} t \tag{2.89}$$

where z and t are connected by the transformation (2.87). To match with the outer solution we require the expansion $U^{(i)}$ for $(x^2+y^2)^{1/2}$ large, hence note that from (2.87) with $|z|$ large we can write

$$t = \pi z - \ln \pi z + \frac{0.5 + \ln 2}{\pi z} - \ln 2 + \frac{\ln \pi z}{\pi z} + O(z^{-2}). \tag{2.90}$$

The matching principle. This principle which follows Van Dyke [16] says that $U^{(0)}$ $(r_0 \to 0)$ is equivalent to $U^{(i)}$ $(r \to \infty)$ where $r_0 = (x_0^2+y_0^2)^{1/2}$, $r = (x^2+y^2)^{1/2}$. More precisely write $U^{O(N)}$ to denote the outer expansion of $U^{(0)}$ (r_0, p) up to terms of order p^N. Then rewriting $U^{O(N)}(r_0, p)$ in terms of the other variable $r = r_0/p$ and expanding to order p^M with r fixed gives a result denoted $U^{O(N,M)}$. Similarly the Mth order inner expansion $U^{i(M)}(r, p)$ is expanded outwards by writing in terms of $r_0 = pr$ and retaining terms of order p^N with r_0 fixed to get $U^{i(M,N)}$. The matching principle then states that

$$U^{O(N,M)} = U^{i(M,N)}. \tag{2.91}$$

Using the matching procedure and grouping terms of $\log p$ and p^0 together (similarly p^N and $p^N \log p$, etc.)

$$U^{O(0,0)} = b_0(\Gamma'(1) - \ln \tfrac{1}{2} pr) \tag{2.92}$$

from equation (2.83) with $r_0 = pr$. $\Gamma'(1) = -C = -0.5772$ (Eulers constant). Also from (2.89) and (2.90)

$$U^{i(0,0)} = \frac{gx_0}{p} + A + B_0\left(\frac{\pi x_0}{p} - \ln \frac{\pi r_0}{p} - \ln 2\right). \tag{2.93}$$

Note that although including the $K_1(r_0)$ eigensolution in $U_1^{(0)}$ would have led to a $1/p$ term in $U^{O(0,0)}$ there is no way in which it could have been matched to this order.

Matching the above two equations is indeed possible and leads to

$$g + B_0\pi = 0, \qquad b_0 = B_0,$$

$$+b_0(-C + \ln 2) = A + B_0\left(\ln \frac{p}{\pi} - \ln 2\right)$$

hence

$$B_0 = b_0 = -\frac{g}{\pi}$$

and

$$A = -\frac{g}{\pi}\left(2 \ln 2 - C - \ln \frac{p}{\pi}\right). \tag{2.94}$$

Now from (2.89) we can deduce the concentration at the base of the step

$$U^{(i)}(0, 0) = A + B_0 \tag{2.95}$$

since $t = 1$ maps into $z = 0$ in the mapping (2.87).

Thus the zero order application of the matching procedure leads to the approximation

$$U^{(i)}(0, 0) = -\frac{g}{\pi}\left(1 + 2 \ln 2 - C - \ln \frac{p}{\pi}\right). \tag{2.96}$$

Setting $U^{(i)}(0, 0)$ equal to one gives $\alpha(p) = -1/g$ as a function of p. In Figure 2 this is plotted and compared with the results of a numerical

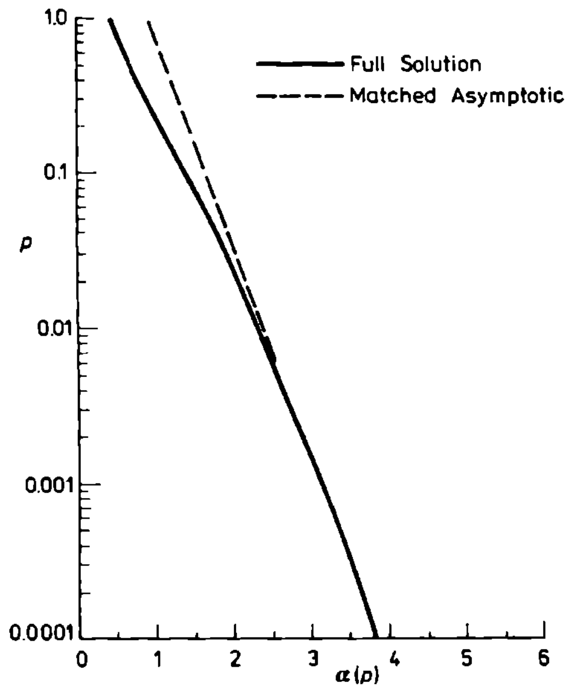

Fig. 2. Peclet number p versus $\alpha(p)$. Comparison of matched asymptotics with full numerical solution of complex functional equation

solution of a functional equation derived in [6]. As the figure demonstrates the results of the asymptotic method are in excellent agreement with the numerical ones for $p \leqslant 0.01$.

3. Line crack problems in non-local theories of elasticity

3.1. Introduction. In a recent series of papers Eringen and co-workers ([10]-[12]) have applied a non-local theory of elasticity to the problem of fracture from a line crack. A major conclusion of their work is that the crack tip stresses are finite for the non-local moduli they consider, in contrast to the well-known inverse square root singularity of linear elastic fracture mechanics. Another aspect of their work is an approximation scheme in which they compute the stress field for their non-local continuum by substituting the classical elastic crack face displacements into the non-local stress-strain constitutive equation. Both these aspects have been questioned by Atkinson ([3], [4]). In the first paper a detailed analytical treatment of the model problem considered in [12] is made and in the second one generalisations to plane strain, anti-plane strain and non-local moduli with a delta function component are made. The conclusions of the papers of Atkinson ([3], [4]) are that the approximation scheme used in [12] and [10], [11] has a non-uniform character and that consequently the finite crack tip stress result is not substantiated. Moreover, for the problems in which direct numerical calculations were made in [12] non-existence results can be proven. Furthermore, for non-local moduli with a delta function component solutions with *singular* crack tip stress fields are constructed in Atkinson [4] and the behaviour of the solution has been confirmed by numerical analysis. Here we consider the simple model problems treated in Eringen *et al.* [12] and Atkinson [3] and refer the reader to Atkinson [4] and Eringen [10], [11] for the full plane non-local elastic situation.

3.2. A model one-dimensional problem. In [12] the following one-dimensional problem is proposed, defined by the equations

$$t_{yy} = (\lambda + 2\mu) \int_{-\infty}^{\infty} \alpha(|x^1 - x|) \frac{\partial v(x^1, y)}{\partial y} dx^1 \tag{3.1}$$

with

$$\gamma^2 v_{xx} + v_{yy} = 0, \qquad \gamma^2 = \mu/(\lambda + 2\mu) \tag{3.2}$$

(μ and λ are constants), together with the boundary condition:

$$\begin{aligned} & t_{yy}(x, 0) = -t_0(x), && \text{given for} \quad |x| < l, \\ & v(x, 0) = 0, && |x| \geqslant l, \\ & v \to 0 && \text{as} \quad y \to +\infty. \end{aligned} \tag{3.3}$$

In equation (3.2) the subscripts denotes partial differentiation. In [12] it is shown that the problem specified by equations (3.1) to (3.3) has similar characteristics to the more complicated plane strain crack situation.

Taking the Fourier transform of (3.2) and using $(3.3)_3$ gives the transform of the displacement in the half-space $y \geqslant 0$ as

$$\bar{v}(k, y) = A(k) \exp(-\gamma |k| y) \tag{3.4}$$

where

$$\bar{v}(k, y) = \int_{-\infty}^{\infty} v(x, y) e^{ikx} dx \tag{3.5}$$

and $|k|$ is defined so as to have positive real part in the complex k plane.

Also, taking the Fourier transform of (3.1) and using (3.4) gives

$$\bar{t}_{yy} = -\gamma |k| (\lambda + 2\mu) A(k) \int_{-\infty}^{\infty} \alpha(|x_1|) e^{-ikx_1} dx_1. \tag{3.6}$$

In [12] the problem is reduced to a Fredholm integral equation for the unknown function $A(k)$ and arguments are given, together with some numerical work, to support "strongly" the approximation of replacing $A(k)$ in (3.6) by the corresponding result for the classical elastic problem, i.e., the problem with $\alpha(|x_1|) \equiv \delta(|x_1|)$. This approximation amounts to using the displacement from the classical elastic problem, $v_c(x^1, y)$ say, in (3.1) to compute the stress field.

We first investigate the consequences of this approximation in a fairly straightforward example.

3.2.1. The semi-infinite problem. This problem which mimics a semi-infinite crack problem has boundary conditions

$$\begin{aligned} &t_{yy}(x, 0) = -e^{\lambda x} && \text{for} \quad x < 0, \\ &v(x, 0) = 0, && x > 0, \\ &v \to 0 && \text{as} \quad y \to +\infty. \end{aligned} \tag{3.7}$$

In (3.7), $1/\lambda$ plays the role of a characteristic length. Writing

$$\begin{aligned} &t_{yy}(x, 0) = t_+(x) && \text{for} \quad x > 0, \\ &v(x, 0) = v_-(x) && \text{for} \quad x < 0 \end{aligned} \tag{3.8}$$

with both $t_+(x)$ and $v_-(x)$ as yet unknown functions of x, the tranforms of (3.7) and (3.8) together with equations (3.4) and (3.6) lead to the functional equation

$$\bar{t}_{yy}(k, 0) = \frac{-1}{(\lambda + ik)} + \bar{t}_+(k) = -\gamma_0 |k| \hat{\alpha}(k) \bar{v}_-(k) \tag{3.9}$$

where

$$\begin{aligned} &\gamma_0 \equiv \gamma(\lambda + 2\mu) = \{\mu(\lambda + 2\mu)\}^{1/2}, \\ &\hat{\alpha}(k) \equiv \int_{-\infty}^{\infty} \alpha(|x_1|) e^{-ikx_1} dx_1. \end{aligned} \tag{3.10}$$

If $\hat{\alpha}(k) = 1$, (the classical elastic case), the functional equation (3.9), which holds on the line Im $k = 0$, can be solved by writing $|k| = k_+^{1/2} k_-^{1/2}$ where $k_+^{1/2}$ has a branch cut from $-i0$ to $-i\infty$ and $k_-^{1/2}$ a cut from $+i0$ to $+i\infty$. These branches are chosen so that $|k|$ has positive real part when viewed as a function of k in the complex k plane. Using this factonsation (3.9) can be rearranged as

$$(3.11) \qquad J = \frac{\bar{t}_+(k)}{k_+^{1/2}} + \frac{i}{(k-i\lambda)}\left(\frac{1}{k_+^{1/2}} - \frac{1}{(i\lambda)_+^{1/2}}\right)$$

$$= -\gamma_0 k_-^{1/2} \bar{v}_{c-}(k) - \frac{i}{(i\lambda)_+^{1/2}(k-i\lambda)}.$$

Using analytic continuation, a generalized form of Liouville's theorem and edge conditions on the crack tip, it can be shown that J defined by (3.11) is in fact zero. Thus the solution of (3.9) with $\hat{\alpha}(k) = 1$ is

$$(3.12) \qquad \bar{v}_{c-}(k) = \frac{-i}{\gamma_0 (i\lambda)_+^{1/2} k_-^{1/2}(k-i\lambda)}.$$

We use the notation $\bar{v}_c$ to indicate that (3.12) is derived from the approximation $\hat{\alpha}(k) = 1$. In the spirit of the approximations used in Eringen *et al.* (1977) we substitute from (3.12) into (3.9) and take this as the result for the stress $\bar{t}_{yy}$ in the model non-local problem defined in (3.1). The result is

$$(3.13) \qquad \bar{t}_{yy} = \frac{i\hat{\alpha}(k) k_+^{1/2}}{(i\lambda)_+^{1/2}(k-i\lambda)}.$$

Clearly if $\hat{\alpha}(k) \neq 1$ then (3.9) is not satisfied exactly, what is required is a measure of how the accuracy of the approximation (3.13) varies with the parameters defining $\hat{\alpha}(k)$. To investigate this we consider three forms for $\alpha(|x|)$.

$$(3.14)\ (\text{i}) \qquad \alpha(|x|) = \frac{\beta}{2} e^{-\beta|x|} \qquad \text{then} \qquad \hat{\alpha}(k) = \frac{\beta^2}{(\beta^2 + k^2)},$$

and $\hat{\alpha}(k)$ is analytic in $-\beta < \text{Im } k < \beta$. The dimensionless constant β/λ is assumed to be much greater than unity. For this example $\bar{t}_{yy}$ given by (3.13) can be split into the sum of plus and minus functions by inspection. The result is

$$(3.15) \qquad \bar{t}_{yy}(k, 0) = \bar{t}_-(k, 0) + \bar{t}_+(k, 0)$$

where

$$(3.16) \quad \bar{t}_+(k, 0) = \frac{i\beta^2 \{(i\lambda)_+^{1/2} - k_+^{1/2}\}}{(i\lambda)_+^{1/2}(k-i\lambda)(\lambda^2-\beta^2)} + $$

$$+ \frac{i\beta\{(i\beta)_+^{1/2} - k_+^{1/2}\}}{2(i\lambda)_+^{1/2}(\beta-\lambda)(k-i\beta)} - \frac{i\beta k_+^{1/2}}{2(\lambda+\beta)(k+i\beta)(i\lambda)_+^{1/2}}$$

and

$$\bar{t}_-(k,0) = \frac{-i\beta^2}{(k-i\lambda)(\lambda^2-\beta^2)} - \frac{i\beta}{2}\left(\frac{\beta}{\lambda}\right)_+^{1/2} \frac{1}{(\beta-\lambda)(k-i\beta)}.$$

Inverting the expression for $\bar{t}_-(k, 0)$ gives for the stress on $y = 0$, $x < 0$ the result

$$t_{yy}(x,0) = -\frac{e^{\lambda x}\beta^2}{(\beta^2-\lambda^2)} + \frac{\beta}{2}\left(\frac{\beta}{\lambda}\right)^{1/2} \frac{e^{\beta x}}{(\beta-\lambda)}.$$

Rearranging this as

$$\text{(3.17)} \qquad P_c(x) = t_{yy}(x,0) + e^{\lambda x} = \frac{-\lambda^2 e^{\lambda x}}{(\beta^2-\lambda^2)} + \frac{\beta}{2}\left(\frac{\beta}{\lambda}\right)^{1/2} \frac{e^{\beta x}}{(\beta-\lambda)},$$

it is easy to see that for $\beta/\lambda \gg 1$ the boundary condition (3.7) seems to be more nearly satisfied as β increases, since $x < 0$. For fixed $x < 0$, the right hand side of (3.17) tends to zero as $\beta \to \infty$. However, if we write $\beta x = X$, then (3.17) becomes

$$\text{(3.18)} \qquad P_c(x) = \frac{-\lambda^2 e^{\lambda X/\beta}}{(\beta^2-\lambda^2)} + \frac{\beta}{2}\left(\frac{\beta}{\lambda}\right)^{1/2} \frac{e^{X}}{(\beta-\lambda)}$$

and clearly $P_c(x)$ does not tend to zero, as $\beta \to \infty$, for $x < 0$ uniformly in x. In fact

$$\text{(3.19)} \qquad P_c(x) \to \tfrac{1}{2}\left(\frac{\beta}{\lambda}\right)^{1/2} e^{X}$$

as $\beta \to \infty$, $X \leqslant 0$.

In [12], [10] various plots are given of $P_c(x)$. Superficially, it looks as if the boundary condition (3.7), which is $P_c(x) = 0$, $x < 0$, is satisfied more accurately as a parameter, analogous to β, increases. However, we contend that the same non-uniform behaviour described above is present in their numerical results.

$$\text{(3.20) (ii)} \qquad \alpha(|x|) = \begin{cases} \dfrac{1}{a}\left(1 - \dfrac{|x|}{a}\right), & |x| \leqslant a, \\ 0, & |x| > a. \end{cases}$$

This expression is used in [12], a is a lattice parameter and is given the value 2.48 A° in the case of steel. Thus with a of order 10^{-8} cms, the ratio of any macroscopic crack length to a will be much greater than unity. In this case

$$\text{(3.21)} \qquad \hat{\alpha}(k) = \frac{\sin^2 ka}{k^2 a^2}$$

which is analytic everywhere. The expression for $\bar{t}_{yy}(k, 0)$ given in (3.13) is thus analytic in, $0 < \text{Im } k < \lambda$. The Fourier inversion theorem then gives

$$t_{yy} = \frac{1}{2\pi} \int_{-\infty + id}^{\infty + id} \bar{t}_{yy}(k, 0) e^{-ikx} dk \tag{3.22}$$

with $0 < d < \lambda$. From (3.13)

$$\bar{t}_{yy} = \frac{ik_+^{1/2}}{(i\lambda)_+^{1/2}(k - i\lambda)} \frac{\sin^2 ka}{k^2 a^2}$$

and $\sin^2 ka = \frac{1}{2} - \frac{1}{4} e^{2ika} - \frac{1}{4} e^{-2ika}$, hence substituting into the above integral gives

$$t_{yy} = -\frac{\sin h^2(\lambda^2 a^2)}{(\lambda^2 a^2)} e^{\lambda x}, \qquad \text{for} \quad x \leqslant -2a. \tag{3.23}$$

This result is obtained by closing the contour in the upper half-plane and picking up the pole at $k = i\lambda$. The condition $x \leqslant -2a$ is necessary in order that the exponential terms decay on a large semi-circular contour in this upper half-plane.

From (3.23) it is easily seen that as $a\lambda \to 0$, $t_{yy} \to -e^{\lambda x}$ uniformly in x provided $x \leqslant -2a$. The boundary condition is thus satisfied uniformly in the region $x \leqslant -2a$, it remains to investigate what happens in $-2a < x < 0$. To do this we take the complex integral (3.22) along the real axis ($d = 0$) and obtain

$$t_{yy} = \frac{1}{\pi} \int_0^\infty \frac{r^{1/2}}{\lambda^{1/2}} \left(\frac{\sin^2 ra}{r^2 a^2} \right) \frac{\{r \sin(\frac{1}{4}\pi + rx) - \lambda \cos(\frac{1}{4}\pi + rx)\} dr}{(r^2 + \lambda^2)}. \tag{3.24}$$

This integral is valid for all x and is a continuous function of x with a constant value at $x = 0$. As $a \to 0$ the contribution to the integral from the integrand with the factor $\lambda \cos(\frac{1}{4}\pi + rx)$ will be finite. This is easily seen, since $\dfrac{\sin^2 ra}{r^2 a^2} \leqslant 1$, the contribution to this part of the integral is of magnitude $< \dfrac{1}{\pi} \displaystyle\int_0^\infty \frac{\lambda^{1/2} r^{1/2}}{(r^2 + \lambda^2)} dr$ which is finite for $\lambda > 0$. The rest of the integrand is, however, much more sensitive to the limit $a \to 0$. To see this, write

$$r = \frac{R}{a}, \qquad x = aX$$

to give

$$(3.25) \qquad t_{yy} \approx \frac{1}{\pi(a\lambda)^{1/2}} \int_0^\infty \frac{R^{3/2}\sin^2 R}{R^2} \frac{\sin(\frac{1}{4}\pi + RX)\,dR}{(R^2 + a^2\lambda^2)}.$$

We use the $\approx$ sign to indicate that we are neglecting the contribution from the second term of the integrand of (3.24) since this is finite as $a \to 0$ as shown above.

When $X = 0$, expression (3.25) shows that when $(a\lambda) \to 0$ the stress at the origin behaves like

$$(3.26) \qquad t_{yy} \approx \frac{1}{\pi(a\lambda)^{1/2}} \int_0^\infty \left(\frac{\sin^2 R}{R^2}\right) \frac{dR}{(2R)^{1/2}}.$$

Further this result is irrespective of whether $X \to 0$ with $X > 0$ or with $X < 0$. Thus the same characteristics as shown in example (i) are present in this case, i.e., that as the region of the discrepancy in the boundary condition gets smaller the magnitude gets larger tending to infinity as $a \to 0$. In [12] it is asserted that the increase in magnitude of t_{yy} at $x = 0$ over the boundary value is the stress concentration. Our contention is that the stress is in fact continuous in this approximation and the value (3.26) merely a property of the approximation *not* of the original boundary value problem.

$$(3.27)\ (\text{iii}) \qquad \alpha(|x|) = \alpha_0 \exp\left\{-\left(\frac{\beta}{a}\right)^2 x^2\right\} \qquad \text{with} \quad \alpha_0 = \frac{\beta}{a\sqrt{\pi}}.$$

Typical values of constants in this expression are given in [10] as $a = 2.48$ A° and $\beta = 1.65$ for steel. Now $\bar{\alpha}(k) = \exp\{-k^2 a^2/(4\beta^2)\}$ and is an entire function of k. Hence $\bar{t}_{yy}(k, 0)$ with this expression for $\alpha(k)$ is analytic in $0 < \operatorname{Im} k < \lambda$ so (3.22) applies. Evaluating this integral along the real axis leads to

$$(3.28)$$

$$t_{yy} = \frac{1}{\pi} \int_0^\infty \frac{r^{1/2}}{\lambda^{1/2}} \frac{\exp\{-r^2 a^2/(4\beta^2)\}\{r\cos(\frac{1}{4}\pi - rx) - \lambda\sin(\frac{1}{4}\pi - rx)\}\,dr}{(r^2 + \lambda^2)}.$$

The second term in the integrand gives a finite contribution to t_{yy} as $a \to 0$. To investigate the contribution from the first term write

$$r = 2\beta R/a \qquad \text{and} \qquad x = aX/2\beta$$

to get

(3.29) $$t_{yy} \approx \frac{1}{\pi}\int_0^\infty \left(\frac{2\beta}{a\lambda}\right)^{1/2} \frac{R^{1/2}\exp(-R^2)\cos(\frac{1}{4}\pi - RX)\,dR}{R^2+\left(\frac{a\lambda}{2\beta}\right)^2}.$$

Hence, this integral shows that t_{yy} goes to infinity like $(2\beta/(a\lambda))^{1/2}$ as $(a\lambda/\beta) \to 0$ with $X \to +0$ or -0.

Thus in each of the examples we have considered, the approximation of using the displacement of the classical elastic problem in order to calculate the stress for the non-local problem has been shown to be non-uniform and hence unsatisfactory. In the next section we show that the same kind of behavior is present in the finite crack problem.

3.2.2. The finite crack, model problem (specified displacement). To demonstrate the effect of the approximation suggested in [10] and [12] we consider the problem specified by equations (3.1) and (3.2) together with the boundary conditions:

(3.30) $$\begin{aligned} &v(x,0) = 0, && |x| \geqslant l,\\ &v(x,0) = (l^2-x^2)^{1/2}, && |x| \leqslant l,\\ &v \to 0 && \text{as } y \to +\infty. \end{aligned}$$

For reasonable crack lengths we expect a small parameter ε to appear in the problem because of the magnitude of the non-local moduli, for example a (the lattice parameter) is given in Angstroms (10^{-8} cms) in examples (ii) and (iii). Hence we define

(3.31) $$\varepsilon_1 = \frac{1}{\beta l}, \qquad \varepsilon_2 = \frac{a}{l}, \qquad \varepsilon_3 = \frac{a}{\beta l}$$

where $\varepsilon_i \ll 1$ and $i = 1, 2$ or 3 refers to examples (i), (ii) and (iii) of the section (3.1).

To investigate the behavior near the crack tip $x = l$ we write

(3.32) $$\begin{aligned} x &= l+\varepsilon lX, & y &= \varepsilon lY,\\ x' &= l+\varepsilon lX', & y' &= \varepsilon lY',\\ v &= (\varepsilon l)^{1/2}V, & t_{yy} &= (\varepsilon l)^{-1/2}T, \end{aligned}$$

where ε without a subscript refers to either of (3.31) whichever is appropriate. In these new coordinates (3.1) and (3.2) become

(3.33) $$T(X, Y) = (\lambda+2\mu)\int_{-\infty}^{\infty} \alpha_i(|X^1-X|)\frac{\partial V(X^1, Y)}{\partial Y}\,dX^1$$

and

(3.34) $$\gamma^2 V_{,XX} + V_{,YY} = 0$$

where

(3.35) (i) $$\alpha_1(|X|) = \tfrac{1}{2}\exp(-|X|),$$

(ii) $$\alpha_2(|X|) = \begin{cases}(1-|X|), & |X| \leqslant 1,\\ 0, & |X| \geqslant 1,\end{cases}$$

(iii) $$\alpha_3(|X|) = \frac{1}{\sqrt{\pi}}\exp(-X^2).$$

The boundary conditions (3.30) become

(3.36) $$\begin{aligned} V(X, 0) &= 0, && X > 0,\ X < -\frac{2}{\varepsilon},\\ V(X, 0) &= (-X)^{1/2}(2l+\varepsilon l X)^{1/2}, && -\frac{2}{\varepsilon} < X < 0.\end{aligned}$$

As $\varepsilon \to 0$ these boundary conditions become

(3.37) $$\begin{aligned} V(X, 0) &= 0, && X > 0,\\ V(X, 0) &= +(2l)^{1/2}(-X)^{1/2}, && -\infty < X < 0.\end{aligned}$$

This problem now has similar characteristics to the semi-infinite one discussed in section (3.1) and it is straightforward to derive the corresponding results

(3.38) $$\bar{T}(k, 0) = -\gamma_0|k|\hat{\alpha}_i(k)\bar{V}_-(k)$$

and from (3.37)

(3.39) $$\bar{V}_-(k) = -l^{1/2}(\tfrac{1}{2}\pi)^{1/2}e^{\pi i/4}k_-^{-3/2}$$

with

$$\hat{\alpha}_1(k) = \frac{1}{(1+k^2)}, \qquad \hat{\alpha}_2(k) = \frac{\sin^2 k}{k^2}$$

and

$$\hat{\alpha}_3(k) = \exp(-k^2/4).$$

The calculation of $T(X, 0)$ from (3.38) is similar to the evaluation of t_{yy} in section (3.1). For example (i), $\bar{T}$ can be split by inspection into plus and minus functions as

(3.40) $$\bar{T}_+ = \frac{\gamma_0 l^{1/2}}{2}\left(\frac{\pi}{2}\right)^{1/2} e^{-i\pi/4}\left\{\frac{-k_+^{-1/2}}{(k+i)} + \frac{1}{(k-i)}\left(k_+^{-1/2} - (i)^{-1/2}\right)\right\}$$

and

$$\bar{T}_- = \frac{\gamma_0}{2} l^{1/2} \left(\frac{\pi}{2}\right)^{1/2} \frac{e^{-\pi i/2}}{(k-i)}$$

since $(i)^{-1/2} = e^{-i\pi/4}$. Inverting $\bar{T}_-$ gives

$$T = \frac{\gamma_0 l^{1/2}}{2} \left(\frac{\pi}{2}\right)^{1/2} e^X \quad \text{for} \quad X < 0. \tag{3.41}$$

Recalling that $X = (x-l)/(\varepsilon l)$ and that $t_{yy} = (\varepsilon l)^{-1/2} T$ we see that t_{yy} tends to zero, the stress free crack boundary condition, except when $x - l = O(\varepsilon)$, i.e., except within distances of order ε from the crack tip. Within such distances the crack boundary stress goes to infinity like $\varepsilon^{-1/2}$, as $\varepsilon \to 0$.

Thus for the finite crack problem the correct boundary condition cannot be satisfied uniformly by the approximation of using the elastic crack displacement. Similar results can be obtained for examples (ii) and (iii) following the analysis of Section 3.1.

3.3. Miscellaneous results. The results of Section 3.2 demonstrate (in our opinion) the inadequacy of the approximation suggested in [10] and [12], however the question remains as to what is the precise nature of the solution to the problem originally formulated in Section 3.2 with boundary conditions (3.3). Note, that the case with $t_0(x)$, a constant, was treated numerically in [12] and results displayed which apparently justified the approximation scheme subsequently used and that we have criticized in Section 3.2.

To investigate this further, we consider solutions of (3.2) in terms of a continuous distribution of virtual screw dislocations within the crack $y = 0$, $-l < x < l$. Thus we can write

$$\frac{\partial v(x,y)}{\partial y} = \gamma \int_{-l}^{l} \frac{(x-\xi) f(\xi)\, d\xi}{(x-\xi)^2 + \gamma^2 y^2} \tag{3.42}$$

where

$$v(x,0) = \int_{-l}^{x} f(\xi)\, d\xi. \tag{3.43}$$

We have presupposed here that the crack displacements will be finite by assuming $f(\xi)$ is integrable. It is possible that there may be solutions to the non-local problem in which $v(x, 0)$ is *not* finite particularly as $x \to \pm l$. However, assuming $f(\xi)$ is integrable we can substitute for (3.42) into (3.43) to get

$$t_{yy} = (\lambda + 2\mu)\gamma \int_{-\infty}^{\infty} \alpha(|x^1 - x|)\, dx^1 \int_{-l}^{l} \frac{(x^1-\xi) f(\xi)}{(x^1-\xi)^2 + \gamma^2 y^2}\, d\xi \tag{3.44}$$

$$= \gamma_0 \int_{-l}^{l} f(\xi)\,d\xi \int_{-\infty}^{x} \frac{\alpha(|x^1 - x|)(x^1 - \xi)}{(x^1 - \xi)^2 + \gamma^2 y^2}\,dx^1 \tag{3.45}$$

the last equation following by interchanging the order of integration.

Taking the limit y tending to zero (3.45) can be written

$$t_{yy} = \gamma_0 \int_{-l}^{l} f(\xi)\,d\xi \int_{-\infty}^{\infty} \frac{\alpha(|x_0|)\,dx_0}{x_0 + (x - \xi)}. \tag{3.46}$$

The inner integral being a Cauchy principal value (Hilbert transform). For convenience rewrite (3.46) as

$$t_{yy} = \gamma_0 \int_{-l}^{l} f(\xi)\,d\xi K(x - \xi) \tag{3.47}$$

where

$$K(x - \xi) = \int_{-\infty}^{\infty} \frac{\alpha(|x_0|)\,dx_0}{x_0 + x - \xi}. \tag{3.48}$$

For $\alpha(|x_0|)$ given in our previous three examples we have

(i) $$\alpha(|x|) = \tfrac{1}{2}\beta \exp(-\beta|x|)$$

then

$$K(x - \xi) = \tfrac{1}{2}\beta \operatorname{sgn}(-x + \xi) \otimes \tag{3.49}$$
$$\otimes \{\exp(\beta|\xi - x|) E_i(-\beta|\xi - x|) - \exp(-\beta|\xi - x|) \bar{E}_i(\beta|\xi - x|)\}$$

(see tables of the Hilbert transform, Erdélyi, Magnus, Oberhettinger (1934)).

(ii) $$\alpha(|x|) = \begin{cases} \dfrac{1}{a}\left(1 - \dfrac{|x|}{a}\right), & |x| \leqslant a, \\ 0, & |x| \geqslant a \end{cases}$$

then

$$K(x - \xi) = \frac{1}{a^2} \{(a + x - \xi) \log|x + a - \xi| - \tag{3.50}$$
$$- (a - x + \xi) \log|x - a - \xi| - 2(x - \xi) \log|x - \xi|\}.$$

(iii) $$\alpha(|x|) = \frac{\beta}{a\sqrt{\pi}} \exp\left\{-\left(\frac{\beta}{a}\right)^2 x^2\right\}$$

then

$$K(x-\xi) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\exp(-x_0^2)\,dx_0}{ax_0/\beta + x - \xi}. \tag{3.51}$$

For interest we consider a fourth example

(iv) $$\alpha(|x|) = \frac{1}{\pi} \frac{a}{(x^2+a^2)}$$

then

$$K(x-\xi) = \frac{x-\xi}{(x-\xi)^2+a^2}. \tag{3.52}$$

The functions $\alpha(|x|)$ given above can be ordered in terms of how quickly they tend to zero as $|x| \to \infty$. Such an ordering is (ii), (iii), (i), (iv). The boundary condition $t_{yy} = 1$ for $|x| \leqslant l$, $y = 0$ leads to the integral equation

$$1 = \gamma_0 \int_{-l}^{l} K(x-\xi) f(\xi)\,d\xi, \qquad |x| \leqslant l. \tag{3.53}$$

We first consider the kernel (3.52) since it is perhaps the easiest to deal with. When $a = 0$, $K(x-\xi) = 1/(x-\xi)$ and (3.53) becomes the usual Cauchy integral equation associated with the classical elastic problem.

3.3.1. Non-existence theorem. We sketch here a proof that no solutions of the integral equation (3.53) can exist for integrable $f(\xi)$. The proof is illustrated by example (iv) above, although note that in [3] other examples are briefly considered. The importance of this non-existence theorem is that it suggests that numerical solutions obtained in [12] are untenable as solutions to the precise mathematical problem posed. Of course, arguments are often put forward to suggest that a discrete approximation may be a better physical model than a continuous one but such was not the original purpose of the non-local models considered here.

For complex z, *define*

$$\phi(z) = \gamma_0 \int_{-l}^{l} K(a-\xi) f(\xi)\,d\xi \tag{3.54}$$

where K is defined as in equation (3.52), (3.54) makes sense in the strip $|\text{Im } z| < a$ and is analytic in z. Furthermore, $\phi(z) = 1$ for $z = x$, $|x| \leqslant l$ hence $\phi(z) = 1$ in the whole region $|\text{Im } z| < a$, and in particular for $z = x$ (real) for all x. (Note that to deduce this, the theorem used is that if $\psi(z)$ is analytic in region D, and $\psi(z) = 0$ at a sequence of points $z = z_N$ with $z_N \to z^*$ in D then $\psi(z) = 0$ in D.) However, for real $x > l$, and $|\xi| \leqslant l$

$$|K(x_1 - \xi)| \leqslant \frac{1}{|x-\xi|} \leqslant \frac{1}{x-l}$$

so

$$|\phi(x)| \leqslant \frac{\gamma_0}{x-l} \int_{-l}^{l} f(\xi)\, d\xi .$$

This gives $\phi(x) \to 0$ as $x \to \infty$ contradicting $\phi(x) \equiv 1$. The contradiction proves the theorem that there exists no solution of (3.53) with K specified in (3.52) and $f(\xi)$ to be integrable.

3.3.2. Direct calculations. To further illustrate the kind of results to be expected in such non-local media we present here some explicit results. Consider the problem specified by equation (3.7) (i.e., a semi-infinite "crack" with an exponentially decaying internally applied "stress"). This reduced in equation (3.9) to the functional equation

$$\frac{-1}{(\lambda + ik)} + \bar{t}_+(k) = -|k|\,\hat{\alpha}(k)\, v_-(k) \tag{3.55}$$

(for convenience we have taken $\gamma_0 = 1$).

Consider the non-local modulus with a delta function component

$$\alpha(|x|) = b\delta(|x|) + (1-b)\,\alpha_1(|x|). \tag{3.56}$$

For simplicity we choose $\alpha_1(|x|)$ as in example (i) above. Thus

$$\hat{\alpha}_1(k) = \frac{\beta^2}{(\beta^2 + k^2)}$$

and then

$$\hat{\alpha}(k) = \frac{\beta^2 + bk^2}{(\beta^2 + k^2)} = \frac{(b^{1/2}k + i\beta)(b^{1/2}k - i\beta)}{(k+i\beta)(k-i\beta)} . \tag{3.57}$$

It is then a simple matter to factorise $\hat{\alpha}$ into the product of plus and minus functions (the subscripts + and − denoting regularity in upper and lower halves of the complex k plane). Thus

$$\hat{\alpha}_+(k) = \frac{b^{1/2}k + i\beta}{k + i\beta}; \qquad \hat{\alpha}_-(k) = \frac{b^{1/2}k - i\beta}{k - i\beta} . \tag{3.58}$$

The functional equation (3.55) can now be rearranged as

$$J = \frac{\bar{t}_+(k)}{k_+^{1/2}\hat{\alpha}_+(k)} + \frac{i}{(k-i\lambda)}\left[\frac{1}{k_+^{1/2}\hat{\alpha}_+(k)} - \frac{1}{(i\lambda)_+^{1/2}\hat{\alpha}_+(i\lambda)}\right] \tag{3.59}$$

$$= -k_-^{1/2}\hat{\alpha}_-(k)\bar{v}_-(k) - \frac{i}{(k-i\lambda)(i\lambda)_+^{1/2}\hat{\alpha}_+(i\lambda)}.$$

The expressions $k_+^{1/2}$ and $k_-^{1/2}$ are as defined in Section 3.2.1. Thus the functional equation (3.59) holds on the real k axis, the + region being $\operatorname{Im} k > 0$ the − region being $\operatorname{Im} k < 0$. An application of Liouville's theorem together with edge conditions at the crack tip gives $J = 0$ and thus $\bar{t}_+(k)$ and $\bar{v}_-(k)$ are determined. The behavior of $t_+(x)$ as $x \to 0+$ and $v_-(x)$ as $x \to 0-$ can be determined from the behavior of the transforms $\bar{t}_+$ and $\bar{v}_-$ as $k \to \infty$ in their respective half planes of regularity. These are

$$\lim_{k\to\infty} \bar{t}_+(k) = \frac{+ib^{1/2}}{k_-^{1/2}(i\lambda)_+^{1/2}\hat{\alpha}_+(i\lambda)},$$
$$\lim_{k\to\infty} \bar{v}_-(k) = \frac{-ib^{-1/2}}{k_-^{3/2}(i\lambda)_+^{1/2}\hat{\alpha}_+(i\lambda)}. \tag{3.60}$$

Thus

$$t_+(x) \sim \frac{b^{1/2}}{\pi^{1/2}\lambda^{1/2}}\frac{(\lambda+\beta)}{(\lambda b^{1/2}+\beta)}x^{-1/2} \quad \text{as} \quad x \to 0+,$$
$$v_-(x) \sim \frac{b^{-1/2}\pi^{1/2}}{2\lambda^{1/2}}\frac{(\lambda+\beta)}{(\lambda b^{1/2}+\beta)}(-x)^{1/2} \quad \text{as} \quad x \to 0-. \tag{3.61}$$

As the contribution of the delta function component in (3.56) becomes smaller (i.e. $b \to 0$), when the constitutive equation reduces to that of the type considered in [12] the displacement in $(3.61)_2$ tends to infinity like $b^{-1/2}$. Thus the result of this exact calculation seems to corroborate the non-existence results obtained in (3.31). Note also that for $\beta/\lambda \gg 1$ the expressions in (3.61) tend to their elastic counterparts except that the stress is multiplied by $b^{1/2}$ and the displacement by $b^{-1/2}$. This result has been shown Atkinson [4] to be valid for the full non-local elastic problem and relatively insensitive to the particular form of $\alpha_1(|x|)$. Some numerical evidence in support of this more general result for plane elastic problems is given in [5].

It is of interest to consider the situation when $b = 0$ to begin with. In this case (3.56) reduces to

$$\alpha(|x|) = \alpha_1(|x|) \tag{3.56'}$$

and

$$(3.62) \qquad \hat{\alpha}_+(k) = \frac{\beta}{(k+i\beta)}, \qquad \hat{\alpha}_-(k) = \frac{\beta}{(k-i\beta)}.$$

If t_+ is to be finite as $x \to 0+$ then Liouville's theorem can again be used to show that $J = 0$. However, it then follows from $(3.59)_2$ that v_- is infinite as $x \to 0-$.

4. A crack propagating steadily in a strip

Fixed displacements are applied to the sides of a strip $x_2 = \pm 1$ and a semi-infinite crack propagates on the x_1' axis with uniform velocity v (Fig. 3). We use co-ordinates moving with the crack tip and define $x_1 = x_1' - vt$. On

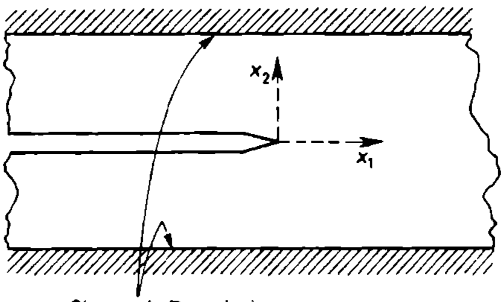

Fig. 3. Moving crack in clamped strip

account of the steady state assumption the stress and displacement field depend only on x_1 and x_2. We assume a two dimensional configuration together with mode 3 deformation for simplicity (the more general plane strain configuration is considered in the cited papers). The boundary conditions of the problem can thus be written

$$(4.1) \qquad \begin{array}{llll} \text{on } x_2 = \pm 1, & u_3 = \pm u_{30} & \text{for all } x_1; \\ \text{on } x_2 = 0, & \sigma_{23} = 0 & \text{for } x_1 < 0 \text{ (stress-free crack)}; \\ \text{on } x_2 = 0, & u_2 = 0 & \text{for } x_1 > 0 \text{ (from symmetry)}, \end{array}$$

u_{30} is a constant.

The fixed grip configuration described above is a reasonable approximation to an experimental configuration although the anti-plane strain assumption (i.e., only one non-zero displacement component u_3) is less realistic. Note, however, that the methods used here can and have been applied to the more realistic plane strain situation. It is also worth recalling the result for an elastic strip; in this case it is relatively easy to show that the energy flow into the crack tip is independent of the crack speed for speeds less than the shear-wave speed in the anti-plane case. A similar result is true in plane strain for speeds less than the Rayleigh wave speed. One way of evaluating the energy flow into the steadily moving crack tip in the elastic case is to simply calculate the energy stored in a thin ligament at $x_1 \to +\infty$, this amount of energy will eventually have been transformed into that at $x_1 \to -\infty$ which is zero. The difference must have gone into the crack tip. Now the displacement in a ligament at $x_1 \to +\infty$ is $u_3 = u_{30} x_2/h$ and the energy stored in this ligament is thus

$$\int_{-h}^{h} \tfrac{1}{2}\mu\left(\frac{u_{30}}{h}\right)^2 dx_2 = \mu \frac{u_{30}^2}{h}. \tag{4.2}$$

Hence

$$G = \mu \frac{u_{30}^2}{h}$$

where μ is the elastic shear modulus, and G is the energy flow into the crack tip. Since this energy could also be calculated by a local work argument at the crack tip it is possible to use (4.2) to deduce the local stress and displacement field at the crack tip. The result is

$$u_3 \sim A_2(-x_1)^{1/2}, \qquad \sigma_{22} \sim A_1 x_1^{-1/2} \tag{4.3}$$

where

$$\begin{aligned} A_1 &= \pi^{-1/2} u_{30}\mu h^{-1/2}\left(1-\frac{v^2}{c_2^2}\right)^{1/4}, \\ A_2 &= 2\pi^{-1/2} u_{30} h^{-1/2}\left(1-\frac{v^2}{c_2^2}\right)^{-1/4} \end{aligned} \tag{4.4}$$

with $c_2^2 = \mu/\varrho$, ϱ being the density of the elastic medium. (See Atkinson [2] for generalisations of this result.)

4.1. The viscoelastic strip. When the strip is made of material which is linearly viscoelastic its constitutive equation can be written

$$\sigma_{i3} = \int_{-\infty}^{t} G(t-\tau)\frac{\partial e_{i3}}{\partial \tau}\, d\tau \tag{4.5}$$

where

$$e_{i3} = \tfrac{1}{2}\frac{\partial u_3}{\partial x_i'}, \qquad i = 1, 2.$$

An alternative form to (4.5) using differential operators would be

$$H_1\left(\frac{d}{dt}\right)\sigma_{13} = P_1\left(\frac{d}{dt}\right)e_{i3} \tag{4.6}$$

where H_1 and P_1 are functions of the operator d/dt. To the above constitutive equation must be added the equation of motion

$$\frac{\partial \sigma_{i3}}{\partial x_i'} = \varrho \frac{\partial^2 u_3}{\partial t^2} \tag{4.7}$$

where ϱ is the density and the x_j' are stationary cartesian co-ordinates. We now change to the moving co-ordinate system

$$x_1 = x_1' - vt, \qquad x_2 = x_2', \qquad x_3 = x_3' \tag{4.8}$$

and apply a Laplace transform over time and a Fourier transform over x, defined by

$$\begin{aligned} \bar{f}(x_1, p) &= \int_0^\infty e^{-pt} f(x_1, t)\, dt, \\ \bar{\bar{f}}(s, p) &= \int_{-\infty}^{\infty} e^{isx_1} \bar{f}(x_1, p)\, dx_1. \end{aligned} \tag{4.9}$$

Then (4.7) becomes

$$\frac{d\bar{\bar{\sigma}}_{23}}{dx_2} - is\bar{\bar{\sigma}}_{13} = \varrho (p + ivs)^2 \bar{\bar{u}}_3 \tag{4.10}$$

while (4.6) becomes

$$H_1(p+ivs)\bar{\bar{\sigma}}_{i3} = P_1(p+ivs)\bar{\bar{e}}_{i3} \tag{4.11}$$

and

$$p\bar{G}(p) \equiv \frac{P_1(p)}{H_1(p)}, \qquad \text{with} \quad \bar{G}(p) = \int_0^\infty e^{-pt} G(t)\, dt. \tag{4.12}$$

The steady state can be achieved by letting p tend to zero in the above equations. Alternatively, if we use the representation (4.6) with d/dt replaced by $-v\partial/\partial x_1$ for the steady state then it is easy to see that (4.6) combined with (4.7) leads to the equation

$$P_1\left(-v\frac{\partial}{\partial x_1}\right)\frac{\partial e_{i3}}{\partial x_1} = \varrho H_1\left(-v\frac{\partial}{\partial x_1}\right)v^2 \frac{\partial^2 u_3}{\partial x_1^2}.$$

It can then be seen that under certain conditions the coefficients of the highest derivatives in this equation will be small, the classical hallmark of a singular perturbation problem. Following this observation an asymptotic method was outlined by Atkinson and Coleman [7] and applied to certain steady moving boundary problems. The key ingredient in the analysis was a dimensionless parameter $\varepsilon = v\tau/L$ where V was the crack speed, τ the relaxation time of the medium and L a length associated with the problem (e.g., half of the strip width). This asymptotic method is presented here in a slightly different way and applied to media where the moduli have small relaxation times. Thus a typical relaxation function $G(t)$ might be written

$$G(t) = G_0 + \sum_{j=2}^{N} G_j \exp(-t/\varepsilon t_j) \tag{4.13}$$

where G_j and t_j are positive constants and ε a small parameter.

If we put $\varepsilon = 0$ in the viscoelastic relaxation functions as a first approximation to the problem, $G(t) \equiv G_0$, we get an elastic strip problem whose solution is well known. However, if a formal perturbation expansion is attempted it is soon seen that the expansion is a singular one. This suggests that the elastic solution formed by putting $\varepsilon = 0$ is valid at distances $\delta \gg \varepsilon$ from the crack tip. The influence of this "outer" solution is transmitted to the "inner" solution through matching conditions near the crack tip where $\varepsilon \ll \delta < 1$, and since both inner and outer approximations are valid in these regions they must be asymptotically equivalent there. Thus the inner limit of the "outer solution" must match with the outer limit of the "inner" solution (cf. Van Dyke (1964) for details of the method of matched asymptotic expansion). One key feature of the present problem is that the zero-order outer solution is just the solution for the elastic strip and the inner limit of this solution (i.e., the solution near the crack tip) has the form given in equations (4.3) and (4.4).

To obtain the zero-order inner solution, we define inner co-ordinates (X_1, X_2) by

$$x_1 = \varepsilon X_1, \qquad x_2 = \varepsilon X_2 \tag{4.14}$$

and write

$$\sigma_{23} = \varepsilon^{-1/2} T_{23}, \qquad u_3 = \varepsilon^{1/2} U_3. \tag{4.15}$$

In this new co-ordinate system the boundary conditions (4.1) become

$$\begin{aligned} &\text{on } X_2 = \pm 1/\varepsilon, \quad U_3 = u_{30}/\varepsilon^{1/2} \quad \text{for all } x_1; \\ &\text{on } X_2 = 0, \quad T_{23} = 0 \quad \text{for } X_1 > 0, \quad U_3 = 0 \quad \text{for } X_1 > 0. \end{aligned} \tag{4.16}$$

Thus as $\varepsilon \to 0$ the boundary condition $(4.16)_1$ is lost and the inner problem becomes that of a stress free semi-infinite crack moving steadily in infinite viscoelastic medium, the conditions at infinity begin determined by matching with the outer solution. Note that from (4.10) and (4.11) the transformed equations in the original (x_1, x_2) co-ordinate system become

$$\frac{d^2 \bar{u}_3}{dx_2^2} - \left[s^2 + \frac{(p+ivs)^2}{c_2^2} \right] \bar{u}_3 = 0 \tag{4.17}$$

where

$$c_2^2 = \frac{P_1(p+ivs)}{\varrho H_1(p+ivs)} \equiv \frac{(p+ivs)\bar{G}(p+ivs)}{2\varrho}. \tag{4.18}$$

Replacing p by zero in (4.17) and (4.18) for the steady state situation, changing to the inner co-ordinate system (X_1, X_2) and replacing the transform variable s by

$$s_1 = \varepsilon s \tag{4.19}$$

reduces (4.17) to the equation

$$\frac{d^2 \bar{U}_3}{dX_2^2} - s_1^2 \left(1 - \frac{v^2}{c_2^2} \right) \bar{U}_3 = 0 \tag{4.20}$$

where

$$\bar{U}_3 = \int_{-\infty}^{\infty} e^{is_1 X_1} U_3(X_1, X_2) dX_1 \tag{4.21}$$

also

$$c_2^2 = \frac{ivs_1 \bar{G}(ivs_1/\varepsilon)}{2\varrho\varepsilon} \equiv \frac{\mu(ivs_1/\varepsilon)}{\varrho} \tag{4.22}$$

with

$$\bar{G}(ivs) = \bar{G}(ivs_1/\varepsilon) = \varepsilon \int_0^{\infty} e^{-ivs_1 t_1} G(\varepsilon t_1) dt_1 \tag{4.23}$$

where the substitution $t = \varepsilon t_1$ has been made in (4.12). Note that with the definition (4.13)

$$G(\varepsilon t_1) = G_0 + \sum_{j=2}^{N} G_j \exp(-t_1/t_j) \tag{4.24}$$

so the expression (4.22) is independent of ε in these new co-ordinates.

The solution of (4.20) with $\bar{U}_3$ bounded as X_2 tends to plus infinity is thus

$$\bar{U}_3 = A \exp(-\gamma X_2) \tag{4.25}$$

where γ is the branch of the square root of

$$\gamma^2 = s_1^2\left(1-\frac{v^2}{c_2^2}\right) \tag{4.26}$$

which has positive real part in the complex s_1 plane.

Transforming the boundary conditions (4.16) on $X_2 = 0$ gives

$$\bar{T}_{23} = \int_0^\infty e^{is_1X_1}\, T_{23}(X_1, 0)\,dX_1 = \bar{T}_+(s_1) \tag{4.27}$$

an unknown plus function (regular in some upper half of the complex s_1 plane)

$$\bar{U}_3 = \int_{-\infty}^0 e^{is_1X_1}\, U_3(X_1, 0)\,dX_1 = \bar{U}_-(s_1) \tag{4.28}$$

regular in some lower half of the complex s_1 plane.

However, the transformed constitutive equation gives

$$\bar{T}_{23} = \left(\frac{ivs_1}{\varepsilon}\right)\frac{d\bar{U}_3}{dX_2} \tag{4.29}$$

μ being defined by (4.22). Thus combining (4.29) with (4.27) and (4.28) gives

$$\bar{T}_+(s_1) = -\mu\gamma\bar{U}_-(s_1) \tag{4.30}$$

where both μ and γ depend on s_1.

To determine the behavior of the complex function $\mu\gamma$ we note that from the definition (4.23), $\mu(ivs_1/\varepsilon)$ and c_2^2 are regular in $\text{Im}\, s_1 < 0$ and c_2^2 is real when s_1 is pure imaginary. It takes its purely real value when $is_1 \to +\infty$ then c_2 is the short time wave speed

$$c_{2\infty}^2 = \left(G_0 + \sum_{j=2}^{N} G_j\right)/2\varrho \tag{4.31}$$

The long time wave speed is when $s_1 = 0$, this is the minimum purely real wave speed

$$c_{20}^2 = G_0/2\varrho. \tag{4.32}$$

Furthermore for $\text{Im}\, s_1 < 0$ it can be shown that c_2^2 is an increasing function of is_1. Thus provided

$$v^2 < c_{20}^2 \tag{4.33}$$

then γ^2 has no zeros in $\text{Im}\, s_1 < 0$. It is then possible to factorise γ as

$$\gamma_+ = s_{1+}^{1/2}, \qquad \gamma_- = s_{1-}^{1/2}\left(1-\frac{v^2}{c_2^2}\right)^{1/2} \tag{4.34}$$

where $s_{1+}^{1/2}$ has a branch cut from 0 to $-i\infty$ in the lower half-plane and $s_{1-}^{1/2}$ a cut from 0 to $+i\infty$ in the upper half plane. Equation (4.30) can be factorised as

$$N(s_1) = \frac{\bar{T}_+(s_1)}{\gamma_+} = -\mu\gamma_- \bar{U}_-(s_1) \tag{4.35}$$

where the minus subscript denotes regularity in $\operatorname{Im} s_1 < 0$ the plus subscript regularity in $\operatorname{Im} s_1 > 0$.

We now need to solve the functional equation (4.35) subject to the matching requirements that the far field should match with (4.3) written in inner co-ordinates. This leads to the requirement that

$$\begin{aligned} U_3 &\sim A_2(-X_1)^{1/2} \quad \text{as} \quad X_1 \to -\infty, \\ T_{23} &\sim A_1 X_1^{-1/2} \quad \text{as} \quad X_1 \to +\infty. \end{aligned} \tag{4.36}$$

These matching conditions will be satisfied if the transforms have the behavior

$$\left.\begin{aligned} \bar{U}_- &\sim -\tfrac{1}{2}\pi^{1/2} s_{1-}^{-3/2} A_2 e^{\pi i/4}, \\ \bar{T}_+ &\sim A_1 \pi^{1/2} e^{\pi i/4} s_{1+}^{-1/2} \end{aligned}\right\} \quad \text{as} \quad s_1 \to 0. \tag{4.37}$$

The function $N(s_1)$ defined by both sides of (4.35) is analytic in the whole s_1 plane except possibly at $s_1 = 0$, and for large s_1 each side of (4.35) is bounded on account of the usual condition that the stress should be no more singular than $r^{-1/2}$ at the crack tip. Matching the stress boundary condition on $\bar{T}_+$ from (4.37) and using Liouville's theorem specifies $N(s_1)$ as

$$N(s_1) = \frac{A_1}{s}\pi^{1/2} e^{\pi i/4}.$$

Thus from (4.35) the transforms $\bar{T}_+$ and $\bar{U}_-$ are determined, and using Tauberian theorems the stress and displacement at the crack tip can be determined. The resulting expressions are on $X_2 = 0$, $|X_1| \ll 1$,

$$T_{23} \sim A_1 X_1^{-1/2}, \qquad U_3 \sim \frac{2A_1(-X_1)^{1/2}}{\varrho c_{2\infty}^2}\left(1 - \frac{v^2}{c_{2\infty}^2}\right)^{-1/2}. \tag{4.38}$$

Referring these expressions to the (x_1, x_2) co-ordinate system and substituting for A_1 from (4.4), noting that in (4.4) the wave speeds etc. are the long time ones (see (4.32)), gives, at the crack tip,

$$\begin{aligned} \sigma_{23} &\sim (x_1\pi)^{-1/2} U_{30}\mu_0 h^{-1/2}\left(1 - \frac{v^2}{c_{20}^2}\right)^{1/4}, \\ U_3 &\sim 2(-x_1)^{1/2}\pi^{-1/2}\frac{c_{20}^2}{c_{2\infty}^2}\frac{(1 - v^2/c_{20}^2)^{1/4}}{(1 - v^2/c_{2\infty}^2)^{1/2}} h^{-1/2} U_{30}. \end{aligned} \tag{4.39}$$

Since $\mu_0 = \varrho c_{20}^2$. A viscoelastic stress intensity factor can be defined as K_{ve} where

(4.40)
$$\sigma_{23} \sim K_{ve}(2\pi x_1)^{-1/2} \qquad \text{as} \quad x_1 \to 0+,$$
$$u_3 \sim \frac{K_{ve}(-2\pi x_1)^{1/2}}{\pi\mu_\infty}\left(1-\frac{v^2}{c_{2\infty}^2}\right)^{-1/2} \qquad \text{as} \quad x \to 0-$$

where μ_∞ and $c_{2\infty}$ are based on the short time modulus,

Then from (4.39)

(4.41)
$$\frac{\mu_\infty}{\mu_0}\frac{K_{ve}}{K_e} = \left(\frac{1-v^2/c_{20}^2}{1-v^2/c_{2\infty}^2}\right)^{1/4}$$

where K_e is the stress intensity factor of the analogous elastic problem based upon the short time modulus μ_∞. In Atkinson and Popelar [9] the full solution of the above problem with no restriction on relaxation times is given in terms of integrals which are evaluated numerically. Figure 4 shows a

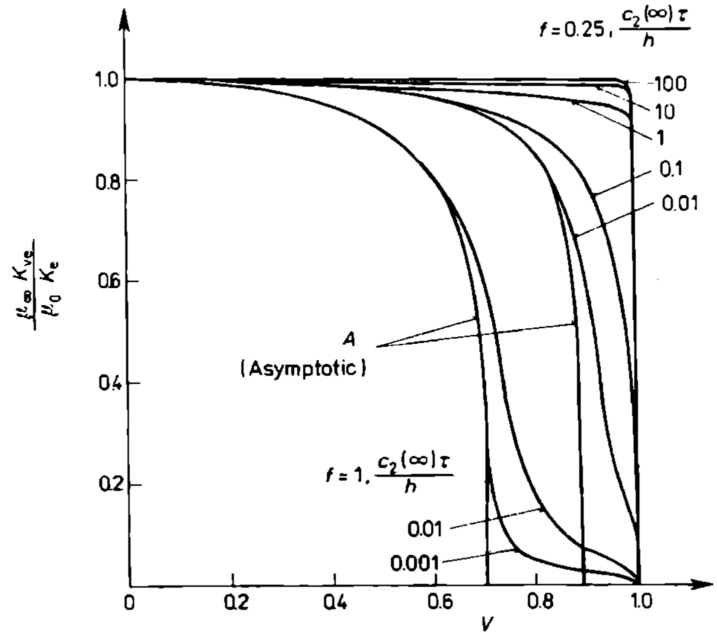

Fig. 4. Normalised stress intensity factor versus normalised crack speed. (Taken from Atkinson and Popelar (1980) standard linear solid.)

comparison between the result (4.4) and the full numerical solution for the standard linear solid described by

$$P_1\left(\frac{d}{dt}\right) = \frac{d}{dt} + \frac{1+f}{\tau}, \qquad H_1\left(\frac{d}{dt}\right) = \mu_\infty\left(\frac{d}{dt}+\frac{1}{\tau}\right).$$

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