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TANGENT CONES AND LIPSCHITZ STRATIFICATIONS

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1. Introduction and statement of the results

This paper is closely related to [1].

In [3] B. Teissier gave an algebraic characterisation of Whitney's conditions. It would be interesting to have also an algebraic characterisation of stratifications satisfying the estimates of Proposition 1.1 in [1].

We shall do here the first step in this direction, i.e. for a given (germ at 0 of an) analytic set $X \subset C^n$ we shall give an algebraic description of all analytic sets Y such that, for some constant C,

$$(*) \qquad |P_{q_1} - P_{q_2}| \leqslant C|q_1 - q_2|/\operatorname{dist}(\{q_1, q_2\}, Y)$$

for all $q_1, q_2 \in X_{reg}$, where P_q: $T_q C^n \to T_q X$ is the orthogonal projection. It relates the inequality $(*)$ to singular parts of tangent cones to X at points of X_{sing}.

To get an idea of how our characterisation looks like, consider for a moment a hypersurface X given by one equation $F = 0$, F without multiple factors. Let $p \in X_{sing}$; we have the notion of the tangent cone $C_p(X) \subset T_p C^n$ to X at p ([4]). It is given by $G_p(\xi) = 0$, where G_p is the homogeneous part of $F(p+\xi)$ and $\xi = (\xi_1, \ldots, \xi_n)$. Assume that G_p has no multiple factors for all p. Let $C'_p(X)$ be the singular part of $C_p(X)$. Then, as we shall prove, a necessary condition for Y to satisfy $(*)$ is

$$(1) \qquad C'_p(X) \subset C_p(Y) \quad \text{for all } p \in X_{sing},$$

and this is the only condition for tangent cones to Y.

To treat the general case we need a definition, closely related to Zariski's equisingularity.

Consider the space C^n with a distinguished hyperplane H, given by $\{x_1 = 0\}$. We shall say that a linear projection π: $C^n \to C^d$ (where C^d is given by $x_{d+1} = 0, \ldots, x_n = 0$) is parallel to H if its kernel is spanned by vectors in H.

Let $X \subset C^n$ be a hypersurface, given by a reduced equation $F = 0$, such that $\dim X \cap H < n-1$. A point $p \in X \cap H$ will be called a *Z-point of X* if, for a generic projection π: $C^n \to C^{n-1}$ parallel to H, the discriminant of F with respect to π is $\neq 0$ for all $x_1 \neq 0$, in a neighbourhood of $\pi(p)$. (The kernel of any projection π, parallel to H, contains a unique vector of the form $a = (0, a_2, \ldots, a_{n-1}, 1)$; the space of all such projections can be thus identified with C^{n-2} and genericity means "outside of an algebraic set". If a projection π induces a finite map $X \to C^{n-1}$ and we choose the x_n-axis so that $\pi(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1})$, then F is equivalent, in a neighbourhood of p, to a distinguished polynomial with respect to x_n; the discriminant of this polynomial we call the *discriminant of F with respect to* π).

A point in $X \cap H$ which is not a Z-point will be called an *NZ-point*; thus we defined two subsets of $X \cap H$: $Z(X)$ and $NZ(X)$.

Let $p \in Z(X)$; we choose the x_n-axis to be the kernel of the generic projection π. Then X can be described in a neighbourhood of p in terms of Puiseux series:

$$x_n = p_n + \varphi_\alpha(x_1^{1/r}, x_2, \ldots, x_{n-1}), \qquad \alpha = 1, \ldots, k,$$

where p_n is the x_n-th coordinate of p and the analytic functions $\varphi_\alpha(t, x_2, \ldots, x_{n-1})$ satisfy, for every r-th root of unity ε,

$$(2) \qquad \varphi_\alpha(t, x_2, \ldots, x_{n-1}) - \varphi_\beta(\varepsilon t, x_2, \ldots, x_{n-1})$$

is either identically 0 or $\neq 0$ for all $x_1 \neq 0$.

Now suppose that $X \subset C^n$ is of pure dimension d and $\dim X \cap H < d$. A point $p \in X \cap H$ will be called a *Z-point of X* if for a generic projection π: $C^n \to C^{d+1}$, parallel to H, the point $\pi(p)$ is a Z-point of the hypersurface $\pi(X)$; the distinguished hyperplane of C^{d+1} is of course $\{x_1 = 0\}$.

It is easy to prove that p is a Z-point of X if and only if for generic axes $x_{d+1}, \ldots, x_n$ in H the coordinates of points of X in a neighbourhood of p satisfy equations of the form

$$(3) \qquad x_i = \psi_{i,\alpha}(x_1^{1/r}, x_2, \ldots, x_d), \qquad \alpha = 1, \ldots, \alpha_i,\ i = d+1, \ldots, n,$$

where, for every i, the analytic functions $\varphi_\alpha(t, x_2, \ldots, x_d) = \psi_{i,\alpha}(t, x_2, \ldots, x_d)$ satisfy (2).

Now we return to our problem. Let $X \subset C^n$ be of pure dimension d and $p \in X$. Let $M \to C^n$ be the σ-process centered at p and M_0 its exceptional fiber. We cover M by open sets $M_1, \ldots, M_n$ such that $(M_i, M_0 \cap M_i) \approx (C^n, \{x_1 = 0\})$. Let $\tilde{X}_p$, $\tilde{Y}_p$ be the strict transforms of X and Y; put

$$NZ(\tilde{X}_p) = \bigcup_i NZ(\tilde{X}_p \cap M_i).$$

Then a necessary condition for Y to satisfy (*) is

$$(4) \qquad NZ(\tilde{X}_p) \subset \tilde{Y}_p \cap M_0 \quad \text{for all } p \in X_{\text{sing}}.$$

This is a condition for the tangent cones $C_p(Y)$, for $\tilde{Y}_p \cap M_0$ can be identified with $C_p(Y)$ in the following way. $M_0 \approx CP^{n-1}$, so $\tilde{Y}_p \cap M_0$ is a projective variety given, say, by homogenous equations $G_i(\xi) = 0$. These equations define also a subset of C^n: $\{p+\xi: G_i(\xi) = 0\}$; this set is $C_p(Y)$.

It is easy to check that (4) generalises (1); under the assumptions of (1) $NZ(\tilde{X}_p)$ is identified with $C_p(X)$ as $\tilde{Y}_p \cap M_0$ is identified with $C_p(Y)$.

Now $C_p(Y)$ can be considered as the "first non-trivial jet" of Y at p; this suggests that (4) should be considered as the first of a sequence of conditions for Y, of the same nature.

We need some notation. C^n with subscripts C_0^n, C_1^n, ... will all be copies of C^n with the distinguished hyperplanes H_0, H_1, ..., given by $x_1 = 0$.

For every i, $1 \leqslant i \leqslant n$, and $p \in C^n$ we put

$$\sigma_p^i\colon C_0^n \to C^n, \quad \sigma_p^i(x_1, \ldots, x_n) = p+(x_1 x_i, \ldots, x_{i-1} x_i, x_i, x_i x_{i+1}, \ldots, x_i x_n)$$

(σ-process). For every $a \in N$ let

$$\psi^a\colon C_1^n \to C_0^n, \quad \psi^a(x_1, \ldots, x_n) = (x_1^a, x_2, \ldots, x_n).$$

For $p \in C_j^n$ ($j \geqslant 1$) let σ_p be again the σ-process

$$C_{j+1}^n \ni (x_1, \ldots, x_n) \mapsto p+(x_1, x_1 x_2, \ldots, x_1 x_n) \in C_j^n.$$

For an equidimensional $X \subset C^n$ we put

$$X_0^i(p) = \text{strict transform of } X \text{ under } \sigma_p^i,$$
$$X_0^{\wedge i}(p) = NZ(X_0^i(p)) \subset H_0,$$
$$X_0^i = \{(p, p_0)\colon p \in X_{\rm sing}, p_0 \in X_0^i(p)\} \subset X_{\rm sing} \times C_0^n,$$
$$X_0^{\wedge i} = \{(p, p_0)\colon p \in X_{\rm sing}, p_0 \in X_0^{\wedge i}(p)\} \subset X_{\rm sing} \times H_0.$$

For every $Y \subset C^n$ we define

$$Y_0^i(p) = \text{strict transform of } Y \text{ under } \sigma_p^i,$$
$$Y_0^i = \{(p, p_0)\colon p \in X_{\rm sing}, p_0 \in Y_0^i(p)\} \subset X_{\rm sing} \times C_0^n,$$
$$Y_0^{\vee i} = \{(p, p_0)\colon p \in X_{\rm sing}, p_0 \in Y_0^i(p) \cap H_0\} \subset X_{\rm sing} \times H_0.$$

For every $a \in N$ let

$$X_1^{ia}(p) = (\psi^a)^{-1}(X_0^i(p)),$$
$$Y_1^{ia}(p) = (\psi^a)^{-1}(Y_0^i(p)),$$
$$X_1^{ia} = \{(p, p_1)\colon p \in X_{\rm sing}, p_1 \in X_1^{ia}(p)\},$$
$$Y_1^{ia} = \{(p, p_1)\colon p \in X_{\rm sing}, p_1 \in Y_1^{ia}(p)\},$$
$$X_1^{\wedge ia}(p) = X_0^{\wedge i}(p),$$
$$Y_1^{\vee ia}(p) = Y_0^{\vee ia}(p);$$

the two last sets are considered as subsets of H_1; H_1 is just another copy of H_0.

By induction on k we shall define subsets

$$X_k^{ia} \subset X_{\text{sing}} \times H_1 \times \ldots \times H_{k-1} \times Ł_k^n,$$
$$X_k^{\wedge ia} \subset X_{\text{sing}} \times H_1 \times \ldots \times H_{k-1} \times H_k;$$

they will be in the form

$$X_k^{ia} = \{(p, p_1, \ldots, p_k): (p, p_1, \ldots, p_{k-1}) \in X_{k-1}^{\wedge ia}, \\ (p, p_1, \ldots, p_k) \in X_k^{ia}(p, p_1, \ldots, p_{k-1})\},$$
$$X_k^{\wedge ia} = \{(p, p_1, \ldots, p_k): (p, p_1, \ldots, p_{k-1}) \in X_{k-1}^{\wedge ia}, \\ (p, p_1, \ldots, p_k) \in X_k^{\wedge ia}(p, p_1, \ldots, p_{k-1})\}.$$

It suffices to define $X_k^{ia}(p, p_1, \ldots, p_{k-1})$ and $X_k^{\wedge ia}(p, p_1, \ldots, p_{k-1})$; they are given by

$$X_k^{ia}(p, p_1, \ldots, p_{k-1}) \text{ is the strict transform of} \\ X_{k-1}^{ia}(p, p_1, \ldots, p_{k-2}) \text{ under } \sigma_{p_{k-1}}: C_k^n \to C_{k-1}^n,$$
$$X_k^{\wedge ia}(p, p_1, \ldots, p_{k-1}) = NZ\big(X_k^{ia}(p, p_1, \ldots, p_{k-1})\big).$$

For every $Y \subset C^n$ and $k \geqslant 1$ we define

$$Y_k^{ia} = \{(p, p_1, \ldots, p_k): (p, p_1, \ldots, p_{k-1}) \in X_{k-1}^{\wedge ia}, p_k \in Y_k^{ia}(p, p_1, \ldots, p_{k-1})\},$$
$$Y_k^{\vee ia} = \{(p, p_1, \ldots, p_k): (p, p_1, \ldots, p_{k-1}) \in X_{k-1}^{\wedge ia}, p_k \in Y_k^{\vee ia}(p, p_1, \ldots, p_{k-1})\},$$

where $Y_k^{ia}(p, p_1, \ldots, p_{k-1})$ is the strict transform of $Y_{k-1}^{ia}(p, p_1, \ldots, p_{k-2})$ under $\sigma_{p_{k-1}}$,

$$Y_k^{\vee ia}(p, p_1, \ldots, p_{k-1}) = Y_k^{ia}(p, p_1, \ldots, p_{k-1}) \cap H_k.$$

Our characterisation of sets Y satisfying (*) is as follows.

PROPOSITION. *Y satisfies (*) if and only if $X_k^{\wedge ia} \subset Y_k^{\vee ia}$ for all i, k, a.*

We shall give two applications of this proposition. We shall work only with (germs of) algebraic sets.

Let K, N be given positive integers. Let $g_{i\alpha}$ be variables, $i = 1, \ldots, K$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \ldots + \alpha_n \leqslant N$. We consider $g_{i\alpha}$ as coordinates in an affine space G_{KN}. Any point $g = (g_{i\alpha}) \in G_{KN}$ gives K polynomials $g_i(x) = \sum_\alpha g_{i\alpha} x^\alpha$, and so we can define

$$Y_g = \{x \in C^n: g_i(x) = 0 \text{ for all } i = 1, \ldots, K\}.$$

Recall that the family of constructible sets in C^n is the Boolean algebra of subsets of C^n generated by all algebraic sets.

COROLLARY 1. *Let $X \subset C^n$ be an algebraic set of pure dimension d. Then, for every K, N, the sets*

$$G_{KN}(X) = \{g \in G_{KN}\colon\ Y_g \text{ satisfies } (*) \text{ in some neighbourhood of } 0\},$$

$$G'_{KN}(X) = \{g \in G_{KN}(X)\colon\ \dim Y_g < d\}$$

are constructible (and, by [1], non-empty for sufficiently big K, N).

$G_{KN}(X)$, $G'_{KN}(X)$ are not always algebraic.

EXAMPLE. Let X be a surface having 0 as an isolated singular point such that $C'_0(X) \neq \{0\}$. By [1], there exists a curve Y satisfying (*). By Łojasiewicz's inequality

$$|P_q - P_{q'}| \leqslant C|q-q'|/\text{dist}(\{q, q'\}, 0)^k \quad \text{for all } q, q' \in X \setminus \{0\},$$

for some k, and therefore, as is easy to see, there exists an integer p such that any curve Y satisfies (*) provided that

$$\text{dist}(x, Y') \leqslant |x|^p \quad \text{for all } x \in Y, \text{ sufficiently close to } 0.$$

Enlarging Y if necessary, we can assume that it is given by $P_1 = 0, \ldots, P_{n-1} = 0$ and the ideal generated by the P_i's and the $(n-1)\times(n-1)$-minors of the matrix $(\partial P_i/\partial x_j)$ contains (for some m) $\mathfrak{m}^m$, where $\mathfrak{m} = (x_1, \ldots, x_n)$. We take a coordinate system such that the x_n-axis is not contained in $C'_0(X)$. Let Y_ε be given by

$$x_1^N = \varepsilon P_1(x), \ldots, x_{n-1}^N = \varepsilon P_{n-1}(x).$$

Then, if N is big enough, we have, for every $\varepsilon \neq 0$,

$$\text{dist}(x, Y_\varepsilon) \leqslant |x|^p \quad \text{for } x \in Y, |x| < \delta_\varepsilon,$$

where $\delta_\varepsilon > 0$, and so every Y_ε satisfies (*) (for $\varepsilon \neq 0$). However Y_0 doesn't satisfy (*), since $C_0(Y) \not\supset C'_0(X)$.

Now let $X, Y \subset C^n$ be two algebraic sets, X of pure dimension d. Let

$$L(X, Y) = \{p \in X_{\text{sing}}\colon (*) \text{ is satisfied in a neighbourhood of } p, \text{ with a constant } C \text{ depending on } p\},$$

$$NL(X, Y) = X_{\text{sing}} \setminus L(X, Y).$$

COROLLARY 2. *$NL(X, Y)$ is algebraic.*

In the sequel the letter C will denote different constants.

2. Preliminaries

We consider two copies C_x^n, C_y^n of C^n with coordinates $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ respectively. Let $H_x = \{x_1 = 0\} \subset C_x^n$, $H_y = \{y_1 = 0\} \subset C_y^n$. We shall list some obvious properties of the maps σ, ψ: $C_y^n \to C_x^n$ given by

$$\sigma(y_1, \ldots, y_n) = (y_1, y_1 y_2, \ldots, y_1 y_n) \ (\sigma\text{-process}),$$

$$\psi(y_1, \ldots, y_n) = (y_1^a, y_2, \ldots, y_n),$$

where a is a given positive integer. Let f be either σ or ψ.

For $q \in C_x^n$ (resp. $\tilde{q} \in C_y^n$) let $H_{x,q}$ (resp. $H_{y,\tilde{q}}$) be the hyperplane parallel to H_x (resp. H_y) passing through q (resp. $\tilde{q}$).

Let $\tilde{q}_1, \tilde{q}_2 \in C_y^n$ and $q_1 = f(\tilde{q}_1)$, $q_2 = f(\tilde{q}_2)$. Let

$$\tilde{\Pi}_i \subset T_{\tilde{q}_i} C_y^n \cap H_{y,\tilde{q}_i}, \quad i = 1, 2,$$

be two linear subspaces and let $\Pi_i \subset T_{q_i} C_x^n$ be their images under df. Since C_x^n, C_y^n are linear spaces, we can speak about the angles $\measuredangle(\tilde{\Pi}_1, \tilde{\Pi}_2)$, $\measuredangle(\Pi_1, \Pi_2)$;

$$\measuredangle(\tilde{\Pi}_1, \tilde{\Pi}_2) = \measuredangle(\Pi_1, \Pi_2). \tag{5}$$

This is obvious, since

$$\sum_j a_j(\partial/\partial y_j)|_{\tilde{q}_i} \in \tilde{\Pi}_i \iff \sum_j a_j(\partial/\partial x_j)|_{q_i} \in \Pi_i \quad \text{for } i = 1, 2.$$

Let $X \subset C_x^n$ (resp. $\tilde{X} \subset C_y^n$) be analytic sets. For $q \in X_{\text{reg}}$ (resp. $\tilde{q} \in \tilde{X}_{\text{reg}}$) we define

$$T_q^0 X = T_q X \cap H_{x,q} \quad (\text{resp. } T_{\tilde{q}}^0 \tilde{X} = T_{\tilde{q}} \tilde{X} \cap H_{y,\tilde{q}}). \tag{6}$$

Let

$$P_q^0\colon T_q C_x^n \to T_q^0 X, \quad P_{\tilde{q}}^0\colon T_{\tilde{q}} C_y^n \to T_{\tilde{q}}^0 \tilde{X} \tag{7}$$

be the orthogonal projections and

$$P_q^{0\perp} = I - P_q^0, \quad P_{\tilde{q}}^{0\perp} = I - P_{\tilde{q}}^0. \tag{8}$$

Let $X \subset C_x^n$ be a given analytic set and take for $\tilde{X}$ the strict transform of X under σ if $f = \sigma$ and $\psi^{-1}(X)$ if $f = \psi$. Let $\tilde{q}_1, \tilde{q}_2 \in \tilde{X}_{\text{reg}} \setminus H_y$, $q_i = f(\tilde{q}_i)$ for $i = 1, 2$. Then (5) gives

$$|P_{\tilde{q}_1}^{0\perp} P_{\tilde{q}_2}^0| = |P_{q_1}^{0\perp} P_{q_2}^0|. \tag{9}$$

For every $q \in C_x^n$ (resp. $\tilde{q} \in C_y^n$) we put

$$d^0(q, X) = \operatorname{dist}(q, X \cap H_{x,q}) \quad (\text{resp. } d^0(\tilde{q}, \tilde{X}) = \operatorname{dist}(\tilde{q}, \tilde{X} \cap H_{y,\tilde{q}})). \tag{10}$$

Let $\tilde{q} \in C_y^n$ and $q = f(\tilde{q}) \in C_x^n$; then we have

$$d^0(q, X) = \begin{cases} d^0(\tilde{q}, \tilde{X}), & \text{if } f = \psi, \\ q_1 d^0(\tilde{q}, \tilde{X}), & \text{if } f = \sigma, \end{cases} \tag{11}$$

where q_1 is the x_1-coordinate of q.

If $q(t)$ is a germ of a curve at $0 \in C^n_x$ such that

$$v = \lim_{t \to 0} \dot{q}(t)/|\dot{q}(t)| \in C_0(X),$$

but $\sphericalangle(v, H_x) > \varepsilon$, then there exists a C, depending on ε, such that

$$d^0(q(t), X) \leqslant C \operatorname{dist}(q(t), X). \tag{12}$$

We shall also use two trivial observations from elementary geometry. Let $H = H^{n-1} \subset C^n$ be a hyperplane and Π_1^0, Π_2^0 two linear subspace of H, of the same dimension. Let v_1, v_2 be two unit vectors in C^n such that

$$\sphericalangle(H, v_i) \geqslant \beta > 0 \quad \text{for } i = 1, 2, |v_1 - v_2| \leqslant \alpha.$$

Put $\Pi_i = \Pi_i^0 \oplus C v_i$ and let P_i (resp. P_i^0) be the orthogonal projection of C^n onto Π_i (resp. Π_i^0), $P_i^\perp = I - P_i$, $P_i^{0\perp} = I - P_i^0$. Then

$$(13) \qquad \begin{aligned} |P_2^{0\perp} P_1^0| &\leqslant |P_2^\perp P_1|(1 + (1/\sin\beta)), \\ |P_2^\perp P_1| &\leqslant |P_2^{0\perp} P_1^0|(1 + (1/\sin\beta)) + (\alpha/\sin\beta). \end{aligned}$$

Let π: $C^n \to C^d$ be the standard projection, $\pi(x_1, \ldots, x_n) = (x_1, \ldots, x_d)$ and let T, T' be two d-dimensional planes in C^n such that $\sphericalangle(T, \ker\pi) > \varepsilon$, $\sphericalangle(T', \ker\pi) > \varepsilon$. If w_k, w_k' are liftings of $\partial/\partial x_k$ to T, T' respectively ($k \leqslant d$), then, for some C, C', depending on ε,

$$C \max \sphericalangle(w_k, w_k') \leqslant \sphericalangle(T, T') \leqslant C' \max \sphericalangle(w_k, w_k'). \tag{14}$$

3. A characterisation of Z-points

Let $X \subset C^n$ be an analytic set of pure dimension d and $H = \{x_1 = 0\} \subset C^n$. $T_q^0 X$, P_q^0, $P_q^{0\perp}$ are defined (for $q \in X_{\text{reg}}$) as in (7) and (8).

LEMMA 1. *A point $p \in X \cap H$ is a Z-point of X if and only if there exists a C, depending on p, such that*

$$|P_{q_1}^{0\perp} P_{q_2}^0| \leqslant C|q_1 - q_2| \tag{15}$$

for all q_1, $q_2 \in X_{\text{reg}} \setminus H$, having the same x_1-coordinate, lying in a suitable neighbourhood of p.

Proof of the "if" part. Let p be a Z-point. We choose axes $x_{d+1}, \ldots, x_n$ in H so that the coordinates of points of X satisfy (2) and (3). Then if $q \in X$ has coordinates

$$(x_1, \ldots, x_d, \psi_{d+1,\alpha_{d+1}}(x_1^{1/r}, \ldots, x_d), \ldots, \psi_{n,\alpha_n}(x_1^{1/r}, \ldots, x_d)),$$

then $T_q^0 X$ is spanned by

$$w_k(q) = (\partial/\partial x_k) + \sum_{i=d+1}^{n} [(\partial/\partial x_k)\psi_{i,\alpha_i}] \, \partial/\partial x_i,$$

$1 < k \leqslant d$. It is easy to prove (the details are in [1]) that $|w_k(q)| \leqslant C$, $|w_k(q) - w_k(q')| \leqslant C|q-q'|$ for all k and for every $q' \in X_{\mathrm{reg}} \setminus H$ having the same x_1-coordinate as q. This, together with (14), implies (15). □

For the "only if" part we need a lemma.

LEMMA 2. *Let $\Gamma \subset C^n$ be a germ of a curve at p, singular at p, and let $l \subset C^n$ be a line. Assume that the orthogonal projection π_l: $C^n \to l$ induces an isomorphism $T_q \Gamma \to T_{\pi_l(q)} l$ for every $q \neq p$ and the norm of its inverse is $\leqslant C$, C independent of q. Then there exist sequences of points q_ν, $q'_\nu \in \Gamma_{\mathrm{reg}}$ such that $q'_\nu \neq q_\nu$, q_ν, $q'_\nu \to p$, and*

$$\sphericalangle(T_{q_\nu}\Gamma, T_{q'_\nu}\Gamma)/|q_\nu - q'_\nu| \to \infty .$$

Proof. We can assume that $p = 0$ and l is the x_1-axis. Consider the case $n = 2$. Assume first that Γ contains a component Γ_0, singular at the origin. Then Γ_0 can be described by

$$x_1 = t^r, \quad x_2 = \lambda(t), \quad r \geqslant 2, \lambda \text{ analytic}.$$

For every r-th root of unity $\varepsilon \neq 1$ we have $\lambda(\varepsilon t) \neq \lambda(t)$ for all $t \neq 0$, so

$$\lambda(\varepsilon t) - \lambda(t) = t^k u(t), \quad u(0) \neq 0,$$

and thus $\lambda'(\varepsilon t) - \lambda'(t)$ is of order t^{k-1}. It is easy to calculate that if q corresponds to $t \neq 0$ and q' to εt, then $|q-q'|$ is of order $|t|^k$, while $\sphericalangle(T_q\Gamma, T_{q'}\Gamma)$ is of order $|t|^{k-r}$.

In the other case Γ contains two nonsingular components Γ_1, Γ_2, intersecting at 0, given by

$$\Gamma_1\colon x_2 = \lambda(x_1), \quad \Gamma_2\colon x_2 = \mu(x_1), \quad \lambda, \mu \text{ analytic}.$$

Clearly $\lambda(0) = \mu(0) = 0$ and $\lambda(x_1) \neq \mu(x_1)$ for $x_1 \neq 0$. We take q lying on Γ_1 and q' on Γ_2 and repeat the reasoning.

If $n > 2$, we take a linear projection π: $C^n \to P$ on a plane P containing l such that $\pi|\Gamma$ is proper and, for some C,

$$|q-q'| \leqslant C|\pi(q) - \pi(q')| \quad \text{for all } q, q \in \Gamma_{\mathrm{reg}}, \text{ close to } 0.$$

Necessarily $\pi(0)$ is a singular point of $\pi(\Gamma)$. Let $\tilde{q}_\nu$, $\tilde{q}'_\nu \in \pi(\Gamma)_{\mathrm{reg}}$ satisfy the conclusion of the lemma for $\pi(\Gamma)$ and we take for q_ν, q'_ν any points in Γ_{reg} projecting into $\tilde{q}_\nu$, $\tilde{q}'_\nu$.

Returning to the proof of Lemma 1 we take a point $p \in X \cap H$ such that (15) is satisfied in a neighbourhood of p. Let p_t be a germ of an analytic map $(C, 0) \to (C^n, p)$ such that $p_t \in X_{\mathrm{reg}} \setminus H$ for $t \neq 0$. There exists an open and dense set D of projections $C^n \to C^d$, parallel to H, such that for every $\pi \in D$ we have: π: $X \to C^d$ is proper and for some $C = C(\pi)$ and every $t \neq 0$, $d\pi(p_t)$: $T^0_{p_t} X \to T^0_{\pi(p_t)} C^d$ is an isomorphism and the norm of its inverse is bounded by C.

Fix a $\pi \in D$; after a translation and a coordinate change in H we can assume that $p = 0$, $\pi(x_1, \ldots, x_n) = (x_1, \ldots, x_d)$.

1° There exists a $C' = C'(\pi)$ such that for every $q \in X_{\mathrm{reg}} \setminus H$, sufficiently close to p,

$$d\pi(q)\colon\ T_q^0 X \to T_{\pi(q)}^0 \mathbf{C}^d$$

is an isomorphism and the norm of its inverse is bounded by C'. For otherwise, using the curve selection lemma for real semianalytic sets, we prove that there exists a real-analytic map $q(r)$ such that $q(r) \in X_{\mathrm{reg}} \setminus H$ for $r \neq 0$, $q(0) = 0$ and the norm of the inverse to $d\pi(q(r))$ tends to ∞ ($r \in \mathbf{R}$). Let $q(t)$ ($t \in \mathbf{C}$) be the complexification of $q(r)$. We reparametrise $q(t)$ and $p(t)$ so that the x_1-coordinates of $q(t)$ and $p(t)$ coincide and we get a contradiction with (15).

2° $X_{\mathrm{sing}} \subset H$. For suppose it is not so. Then $\dim \overline{X_{\mathrm{sing}} \setminus H} = d-1$. Let $W = \pi(\overline{X_{\mathrm{sing}} \setminus H})$; clearly $\dim W = d-1$. A general line l in $\mathbf{C}^d$, parallel to $\{x_1 = 0\}$, intersects W transversally; its lifting Γ to X has singular points. Applying to Γ Lemma 2 we get sequences of points q_ν, $q'_\nu \in \Gamma_{\mathrm{reg}}$ such that $\sphericalangle(T_{q_\nu}\Gamma, T_{q'_\nu}\Gamma)/|q_\nu - q'_\nu| \to \infty$. Now, using (14), we get a contradiction with (15).

3° Let $x' = (x_2, \ldots, x_d)$, $y = (x_{d+1}, \ldots, x_n)$. It follows from 2° that there exists finitely many $\mathbf{C}^{n-d}$-valued analytic functions $\varphi_\alpha(t, x')$, $\varphi_\alpha = (\varphi_{\alpha,d+1}, \ldots, \varphi_{\alpha,n})$ and an integer r such that if $(x_1, x', y) \in X$, then, for some α,

$$y = \varphi_\alpha(x_1^{1/r}, x').$$

For any α and any r-th root of unity ε we put

$$\psi(t, x') = \varphi_\alpha(\varepsilon t, x') - \varphi_\alpha(t, x') = \big(\psi_{d+1}(t, x'), \ldots, \psi_n(t, x')\big).$$

For these ψ that don't vanish identically let (for a generic x') $\mathrm{ord}_t \psi_i = s_i$ (at $t = 0$); after a permutation of $x_{d+1}, \ldots, x_n$ we can assume for simplicity that $s_{d+1} = \ldots = s_{d+k} = s$ and $s_i > s$ for $i > d+k$. Thus

$$\psi_i = t^{s_i} \tilde{\psi}_i(t, x'), \quad \tilde{\psi}_i(0, x') \quad \text{not identically } 0.$$

We shall show that at least one of the $\tilde{\psi}_i$ for $i \leqslant d+k$ is $\neq 0$ at $x' = 0 = \pi(p)$.

Suppose that $\tilde{\psi}_i(0, 0) = 0$ for all $i \leqslant d+k$. Then

$$\psi_i(t, x') = t^s \psi_i^*(x') + O(t^{s+1}), \quad \psi_i^*(0) = 0, \quad i \leqslant d+k,$$

$$\psi_i(t, x') = O(t^{s+1}), \quad i > d+k.$$

For t, x' unprecised for the moment let

$$q = \big(t^r, x', \varphi_\alpha(t, x')\big), \quad q' = \big(t^r, x', \varphi_\alpha(\varepsilon t, x')\big) \in X. \tag{16}$$

Then $|q-q'| = |\psi(t, x')|$ and it follows from 1°, (14) that

$$\measuredangle(T_q^0 X, T_{q'}^0 X) \text{ is of order } \max_{2\leqslant j\leqslant d} |\partial\psi(t, x')/\partial x_j|,$$

so (15) implies that

$$|\partial_j \psi(t, x)| \leqslant C|\psi(t, x)| \quad \text{for } 2\leqslant j\leqslant d, \tag{17}$$

where $\partial_j = \partial/\partial x_j$. For $2\leqslant j\leqslant d$ we have

$$\partial_j \psi_i(t, x') = t^s \partial_j \psi_i^*(x') + O(t^{s+1}), \quad i\leqslant d+k,$$
$$\partial_j \psi_i(t, x') = O(t^{s+1}), \quad i > d+k.$$

Let $\psi_i^0(x')$ be the homogenous part of ψ_i^*, of degree, say, $m_i (i\leqslant d+k)$; then

$$\psi_i^*(x') = \psi_i^0(x') + O(|x'|^{m_i+1}).$$

Let Ω be an open cone in the x'-space, with vertex at 0, disjoint with all the cones $\psi_i^0(x') = 0$. Let $m = \min m_i$; again we can assume that $m_i = m$ for $i\leqslant d+l$, $m_i > m$ for $i > d+l$, where l is some number $\leqslant k$. After a linear change among $x_2, \ldots, x_d$ we can assume that for all $i\leqslant d+l$

$$|\partial_2 \psi_i^0(x')| \geqslant C|x'|^{m-1} \quad \text{for } x'\in\Omega;$$

clearly for all x'

$$|\psi_i^*(x')| \leqslant C|x'|^m.$$

Now take in (16) $x'\in\Omega$, $|t| = |x'|^m$. Then

$$|\psi_i(t, x')| \leqslant |t|^s |\psi_i^*(x')| + |x'|^{m(s+1)} \leqslant C|x'|^{m(s+1)}, \quad i\leqslant d+k,$$
$$|\psi_i(t, x')| \leqslant C|t|^{s+1} \leqslant C|x'|^{m(s+1)}, \quad i > d+k,$$

so

$$|\psi(t, x')| \leqslant C|x'|^{m(s+1)}.$$

But for $i\leqslant d+l$

$$|\partial_2 \psi_i(t, x')| \geqslant |t|^s |\partial_2 \psi_i^*(x')| - O(|t|^{s+1})$$
$$\geqslant C|t|^s|x'|^{m-1} - C|t|^s|x'|^m - C|x'|^{m(s+1)} \geqslant C|x'|^{m(s+1)-1}$$

and we have a contradiction with (17).

4° After a linear change in the $x_{d+1}, \ldots, x_n$-coordinates we can assume that for all α, ε, and i, either ψ_i is identically 0 or $\psi_i(0, 0) \neq 0$. This shows that the set of $\pi\in D$ for which the latter condition holds, is open and dense. But this is equivalent to the definition of a Z-point, so $p\in Z(X)$.

4. Proof of the proposition

Let $X \subset C^n$ be analytic, $\dim X = d$.

LEMMA 3. *Let Γ be a germ at $p \in X$ of a curve such that $\Gamma \setminus \{p\} \subset X_{reg}$. Then the C^∞-function*

$$\Gamma_{reg} \ni q \mapsto P_q \in C^{n^2}$$

(*where, of course,* P_q: $C^n \to T_q X$) *satisfies, for some* C,

$$|DP_q| \leqslant C/|q-p|$$

(*even* $|DP_q| \leqslant C/|q-p|^\alpha$ *for some* $\alpha < 1$, *but we don't need that*).

Proof. Assume that $p = 0$. Let π: $C^n \to C^d$ be a projection such that for all $q \in \Gamma \setminus \{0\}$ $d\pi(q)$: $T_q X \to T_{\pi(q)} C^d$ is an isomorphism and the norm of its inverse is bounded by some C. After a coordinate change in C^d we can assume that Γ is given by $x_i = \lambda_i(x^{1/r})$, $i = 2, \ldots, n$, λ_i analytic, $\operatorname{ord} \lambda_i \geqslant r$. Let $w_1(q), \ldots, w_d(q)$ be liftings of $\partial/\partial x_1, \ldots, \partial/\partial x_d$ to vectors in $T_q X$, $q \in \Gamma_{reg}$; then w_j are analytic in $x_1^{1/s}$ for some s; since they are bounded, $\operatorname{ord} w_j \geqslant s$. Thus $|Dw_j| \leqslant C/|x_1|^{1-(1/s)}$.

Now let Γ be again a germ of a curve at p ($= 0$ for simplicity), $\Gamma \setminus \{p\} \subset X_{reg}$; let Γ be given by $x_i = \lambda_i(x_1^{1/r})$, $\operatorname{ord} \lambda_i \geqslant r$.

LEMMA 4. *For every $q \in \Gamma \setminus \{0\}$ and any number $a \in C$, $|a| \neq 0$ and small enough, there exists a $q' \in \Gamma \setminus \{0\}$ having a as its x_1-coordinate such that*

$$|P_q^\perp P_{q'}| \leqslant C|q-q'|/\min(|q_1|, |a|),$$

where q_1 is the x_1-coordinate of q and C is independent of q, a.

Proof. In the x_1-axis we join q_1 and a by an arc L of length $\leqslant 2\pi|q_1 - a|$ such that for every $t \in L$

$$|t| \geqslant \min(|q_1|, |a|).$$

We lift L to a real curve in Γ_{reg}, starting at q. If q' is its end, then, by Lemma 3,

$$|P_q^\perp P_{q'}| \leqslant C|q_1 - a|/\min(|q_1|, |a|) \leqslant C|q-q'|/\min(|q_1|, |a|). \qquad \square$$

Lemma 4, together with (13), implies the following lemma.

LEMMA 5. *Let Γ_1, Γ_2 be germs of curves at $p \in X$, $\Gamma_i \setminus \{p\} \subset X_{reg}$, such that the angles between the tangent vectors to Γ_i and $\partial/\partial x_1$ are $< (\pi/2) - \alpha$, $\alpha > 0$. Let $H = \{x_1 = 0\}$ and let $Y \subset C^n$ be any analytic set. Then the following conditions are equivalent*:

1° $|P_{q_1}^\perp P_{q_2}| \leqslant C|q_1 - q_2|/\operatorname{dist}(\{q_1, q_2\}, Y)$ *for all* $q_i \in \Gamma_i \setminus \{p\}$;

2° $|P_{q_1}^{0\perp} P_{q_2}^0| \leqslant C|q_1 - q_2|/\operatorname{dist}(\{q_1, q_2\}, Y)$ *for all* $q_i \in \Gamma_i \setminus \{p\}$, $i = 1, 2$, *such that q_1, q_2 have the same x_1-coordinate.*

We can now prove the proposition. We shall use the same symbols P^0, $P^{0\perp}$ for projections onto $T^0 X_k^{ia}$, $T^{0\perp} X_k^{ia}$ for various i, a, k.

1° Assume that $Y_k^{\vee ia} \not\supset X_k^{\wedge ia}$ for some k, i, a; we shall show that $(*)$ doesn't hold. We assume that the smallest k with the latter property is > 1 (the case $k = 1$, or, which is the same, $k = 0$, we leave to the reader). Thus, for every $p \in X_{\text{sing}}$ we have $C_p'(X) \subset C_p(Y)$.

Let $\tilde{p} = (p, p_1, \ldots, p_{k-1}) \in NZ(X_{k-1}^{ia}) \setminus Y_{k-1}^{\vee ia}$. By Lemma 1 and the curve selection lemma there exist germs of analytic maps $\tilde{q}_1(t)$, $\tilde{q}_2(t)$ such that $\tilde{q}_1(0) = \tilde{q}_2(0) = \tilde{p}$, $\tilde{q}_1(t)$, $\tilde{q}_2(t) \in X_{k-1,\text{reg}}^{ia}$ for $t \neq 0$, the x_1-coordinates of $\tilde{q}_1(t)$, $\tilde{q}_2(t)$ coincide, and

$$|P^{0\perp}_{\tilde{q}_1(t)} P^0_{\tilde{q}_2(t)}| / |\tilde{q}_1(t) - \tilde{q}_2(t)| \quad \text{is unbounded.}$$

Let $q_1(t)$, $q_2(t)$ be the images of $\tilde{q}_1(t)$, $\tilde{q}_2(t)$ under $\sigma_p^i \psi^a \sigma_{p_1} \ldots \sigma_{p_{k-1}}$; then, by (9), (11)

$$|P^{0\perp}_{q_1(t)} P^0_{q_2(t)}| \, d^0(\{q_1(t), q_2(t)\}, Y) / |q_1(t) - q_2(t)| \quad \text{is unbounded.}$$

Clearly, $q_1(t)$, $q_2(t)$ are tangent at p to $C_p'(X) \subset C_p(Y)$ and $q_1(t)$ and $q_2(t)$ have the same x_i-coordinates; further

$$\lim_{t \to 0} (\dot{q}_j(t) / |\dot{q}_j(t)|) \notin \{x_i = 0\} \qquad \text{for } j = 1, 2.$$

Now (12) and (13) imply that

$$|P^{\perp}_{q_1(t)} P_{q_2(t)}| \, \text{dist}(\{q_1(t), q_2(t)\}, Y) / |q_1(t) - q_2(t)| \quad \text{is unbounded,}$$

so $(*)$ is not satisfied.

2° Assume that $X_k^{\wedge ia} \subset Y_k^{\vee ia}$ for all i, a, k, but $(*)$ is not satisfied. Then, by the curve selection lemma, there exist germs of analytic maps $q_1(t)$, $q_2(t)$ such that $q_1(0) = q_2(0) = p \in X_{\text{sing}}$, $q_1(t)$, $q_2(t) \in X_{\text{reg}}$ for $t \neq 0$ and

$$|P^{\perp}_{q_1(t)} P_{q_2(t)}| \, \text{dist}(\{q_1(t), q_2(t)\}, Y) / |q_1(t) - q_2(t)| \quad \text{is unbounded ([1]).}$$

Select one of the coordinate axes, x_i, such that, for some $\alpha > 0$,

$$\sphericalangle\left(\lim_{t \to 0} (\dot{q}_j(t) / |\dot{q}_j(t)|), \{x_i = 0\}\right) \leqslant \tfrac{1}{2}\pi - \alpha.$$

By Lemma 5 we can assume that $q_1(t)$, $q_2(t)$ have the same x_i-coordinate. Let $q_1^*(t)$, $q_2^*(t)$ be the liftings of $q_1(t)$, $q_2(t)$ via σ_p^i; then

$$(18) \qquad |P^{0\perp}_{q_1^*(t)} P^0_{q_2^*(t)}| \, d^0(\{q_1^*(t), q_2^*(t)\}, Y_0^i) / |q_1^*(t) - q_2^*(t)| \quad \text{is unbounded.}$$

We choose a so that $(\psi^a)^{-1}(q_j^*(t))$ are sums of smooth curves $\Gamma_{j\beta}$ given by

$$\Gamma_{j\beta}\colon \; x_s = \varphi_{j\beta,s}(x_1), \qquad s = 2, \ldots, n.$$

By (9), (11), (18) remains unchanged if we pass from $q_j^*(t)$ to $(\psi^a)^{-1}(q_j^*(t))$ and from X_0^i, Y_0^i to X_1^{ia}, Y_1^{ia}; further it remains unchanged after liftings via the σ-processes. But after a finite number of such liftings the strict transforms of $\Gamma_{j\beta}$ become disjoint and we get a contradiction.

5. Constructibility of $X_k^{\wedge ia}$ and $Y_k^{\vee ia}$

First we introduce some notation. If

$$F(t) = b_0 + b_1 t + \ldots + b_{n-1} t^{n-1} + t^n,$$

we define ([1])

$$\Delta_i^F = \sum_{\alpha_1, \ldots, \alpha_i}^* \prod_{r,s}^* (t_r - t_s),$$

where $\sum^*$ denotes the summation over all α_k such that $\alpha_k \neq \alpha_j$ for $k \neq j$, and $\prod^*$ the product over all r, s such that $r \neq s$ and $r, s \neq \alpha_j$ for all j. The t_r are of course all the roots of F. We consider Δ_i^F as polynomials in $b_0, \ldots, b_{n-1}$. Thus Δ_0^F is the discriminant of F and F has less than $n-k$ distinct roots if and only if $\Delta_i^F = 0$ for all $i \leqslant k$.

If $F(t) = a_0 + a_1 t + \ldots + a_n t^n$, then we put $G = b_0 + b_l t + \ldots + b_{n-1} t^{n-1} + t^n$, where $b_i = a_i / a_n$, and

$$\Delta_i^F (a_0, \ldots, a_n) = a_n^{k(i)} \Delta_i^G (b_0, \ldots, b_{n-1}),$$

where $k(i)$ is the smallest number such that Δ_i^F is a polynomial.

LEMMA 6. *Let $S \subset C^p$ be algebraic and $X \subset S \times C^n$ algebraic. Let π: $S \times C^n \to S$ be the standard projection, $H = \{x_1 = 0\} \subset C^n$; assume that all the fibers $X_s = \pi^{-1}(s) \cap X \subset \{s\} \times C^n \approx C^n$ are of pure dimension d and $\dim X_s \cap H < d$. Then there exists an algebraic set $S_0 \subsetneq S$ and an algebraic set $Z \subset S \times C^n$ such that $Z_s = NZ(X_s)$ for all $s \in S \setminus S_0$, where $Z_s = Z \cap \pi^{-1}(s)$.*

Proof. Assume first that $d = n-1$. There exists a polynomial $F(s, x)$ ($s \in C^p$, $x \in C^n$) and an algebraic set $S_1 \subsetneq S$ such that

$$X_s = \{x\colon F(s, x) = 0\} \quad \text{for all } s \in S \setminus S_1.$$

For every $\xi \in H$ consider $F(s, x + \lambda\xi)$ as a polynomial in one variable λ with s, x, ξ as parameters; let $\Delta_i^F(s, x, \xi)$ be its generalised discriminants. Let

$$C_i = \{(s, \xi)\colon \Delta_i^F(s, x, \xi) = 0 \text{ for all } x \in H\}.$$

Let j be the smallest number such that $(S \setminus S_1) \times H \not\subset C_j$; put $C = C_j$, $\Delta = \Delta_j^F$. If $\Delta(s, x, \xi) = \sum \Delta_\alpha(s, \xi) x^\alpha$ ($x \in H$), then C is given by $\Delta_\alpha(s, \xi) = 0$ for all α. Let $\Delta_\alpha(s, \xi) = \sum \Delta_{\alpha\beta}(s) \xi^\alpha$; put $S_0 = S_1 \cup \{\Delta_{\alpha\beta}(s) = 0 \text{ for all } \alpha, \beta\}$. It is easy to see that $NZ(X_s)$ is given (for $s \in S \setminus S_0$) by $A_\gamma(s, x) = 0$ for all γ, where

$$\Delta(s, \xi, x) = \sum A_\gamma(s, x) \xi^\gamma.$$

Now suppose that d is arbitrary. Let $\Pi \subset H \times \ldots \times H(n-d-1$ times$) = H^{n-d-1}$ be the set of all $a = (a_{d+2}, \ldots, a_n)$ such that $\partial/\partial x_2, \ldots, \partial/\partial x_{d+1}, a_{d+2}, \ldots, a_n$ are linearly independent. Every $a \in \Pi$ determines a projection $\Pi(a)$: $C^n \to C^{d+1}$. There is a Zariski open set $\Omega \subset S \times \Pi$ and an algebraic set $\mathfrak{X} \subset S \times H^{n-d-1} \times C^{d+1}$ such that if $(s, a) \in \Omega$, then $\mathfrak{X}_{(s,a)} = \pi(a)(X_s)$ and $\dim \mathfrak{X}_{(s,a)} = d$. Choose a polynomial $P(s, x)$ such that the complement of Ω in $S \times H^{n-d-1}$ is contained in $\{P = 0\}$. By the codimension 1 – case there exists an algebraic set $W \subset S \times H^{n-d-1}$, given by $Q_i(s, a) = 0$, and an algebraic set $\mathfrak{Z} \subset S \times H^{n-d-1} \times C^{d+1}$ such that $\Omega \not\subset \mathfrak{Z}$ and $\mathfrak{Z}_{(s,a)} = NZ(\mathfrak{X}_{(s,a)})$ for $(s, a) \in \Omega \setminus W$. Let $S_0 = \{s\colon P(s, a) = 0,\ Q_i(s, a) = 0$ for all i and for all $a \in H^{n-d-1}\} \subsetneq S$. Let $G_j(s, a, z) = 0$ be the equations of $\mathfrak{Z}$ (where $z \in C^{d+1}$). For $s \in S \setminus S_0$ we have: $x \in Z(X_s)$ if and only if $\pi(a)x \notin \mathfrak{Z}_{(s,a)}$ for an open set of a's, so $NZ(X_s)$ is given by $G_j(s, a, \pi(a)x) = 0$ for all $a \in H^{n-d-1}$.

Corollary 3. *Let $S \subset C^p$, $X \subset S \times C^n$ be algebraic such that all the fibers X_s are equidimensional. Then*

$$NZ(X) = \bigcup_{s \in S} (\{s\} \times NZ(X_s))$$

is constructible.

Lemma 7. *Let $S \subset C^p$, X, $Y \subset S \times C^n$ be algebraic. Then there exist algebraic sets $S_0 \subsetneq S$ and $Z \subset S \times C^n$ such that $Z_s = \overline{X_s \setminus Y_s}$ for all $s \in S \setminus S_0$.*

Proof. Take any non-zero $f \in I(X)$, $g \in I(Y)$. After a linear change of coordinates in C^n we can assume that $f = a(s)x_n^k + \ldots$, $g = b(s)x_n^l + \ldots$, where $\ldots$ denote terms of lower degree with respect to x_n. Let $S_1 = \{s \in S\colon a(s) = 0,\ b(s) = 0\} \subsetneq S$. Put $X_s' = \pi_0(X_s)$, where π_0: $C^n \to C^{n-1}$ is the projection parallel to the x_n-axis. Note that π_0: $X_s \to X_s'$ is proper for $s \notin S_1$ and therefore X_s' are algebraic for $s \notin S_1$. There exist an algebraic set $X^* \subset S \times C^{n-1}$ such that $X_s' = X_s^*$ for $s \notin S_1$. There exists polynomials $\delta(s, x')$ and $\varphi(s, x)$ (where $x' = (x_1, \ldots, x_{n-1})$) such that: 1° δ does not vanish identically on X^*, 2° for every $(s, x') \in X^* \setminus \{\delta = 0\}$ we have

$$(s, x', x_n) \in X \setminus Y \;\Leftrightarrow\; \varphi(s, x', x_n) = 0.$$

Put

$$\tilde{X}_s = [X_s \cap \pi_0^{-1}(\{\delta_s = 0\})] \cup [X_s \cap \{\varphi_s = 0\}],$$

where $\delta_s(x') = \delta(s, x')$ and $\varphi_s(x) = \varphi(s, x)$. Clearly $\overline{X_s \setminus Y_s} = \overline{\tilde{X}_s \setminus Y_s}$ and $\tilde{X} = \bigcup_s [\{s\} \times \tilde{X}_s]$ is algebraic. If $\tilde{X}_s \subsetneq X_s$ for some $s \notin S_1$, we can repeat the argument with X replaced by $\tilde{X}$. If $X_s = \tilde{X}_s$ for all $s \notin S_1$, then

$$\overline{X_s \setminus Y_s} = \overline{[X_s \cap \pi_0^{-1}(\{\delta_s = 0\})] \setminus Y_s} \cup \overline{[X_s \cap \pi_0^{-1}(X_s^* \setminus \{\delta_s = 0\})]}$$

and the conclusion of the lemma can be assumed to hold for $\overline{[X_s \cap \pi_0^{-1}(\{\delta_s = 0\})] \setminus Y_s}$ and $\overline{X_s^* \setminus \{\delta_s = 0\}}$.

COROLLARY 4. $\bigcup_{s\in S}(\{s\}\times\overline{X_s\setminus Y_s})$ *is constructible.*

By induction on k we get now:

COROLLARY 5. *Let* $S\subset C^p$, $X, Y\subset S\times C^n$ *be algebraic and all the fibres* X_s *are equidimensional. Then, for all* i, a, k $\bigcup_s[\{s\}\times(X_s)_k^{\wedge ia}]$, $\bigcup_s[\{s\}\times(Y_s)_k^{\vee ia}]$ *are constructible ; in particular (putting* $S=$ *point)* $X_k^{\wedge ia}$, $Y_k^{\vee ia}$ *are constructible.*

6. Proofs of the corollaries

Proof of Corollary 1. We observe first that $G_{KN}(X)$ is semialgebraic (if we consider C^n and G_{KN} as real vector spaces). In fact, the function

$$X_{\text{reg}}\times X_{\text{reg}}\ni(q_1, q_2)\mapsto|P_{q_1}-P_{q_2}|\in R$$

is semialgebraic (i.e. its graph is semialgebraic) and similarly the distance function $(q_1, q_2)\mapsto|q_1-q_2|$. We rewrite the definition of $G_{KN}(X)$:

$$G_{KN}(X)=\{g\colon \exists\varepsilon>0,\ C>0\ \forall q_1, q_2\in X_{\text{reg}}, |q_1|<\varepsilon, |q_2|<\varepsilon, \exists x\ g_1(x)=0, \ldots, g_K(x)=0, |P_{q_1}-P_{q_2}|\leqslant C|q_1-q_2|/\min(|q_1-x|, |q_2-x|)\};$$

now our claim follows directly from Tarski's theorem (e.g. [2]).

Similarly the sets

$$K_k^{ia}(X)=\{g\colon X_k^{\wedge ia}\subset(Y_g)_k^{\vee ia}\}$$

are constructible since their definition can be rewritten as

$$\{g\colon \forall(p, p_1, \ldots, p_k)\ (p, p_1, \ldots, p_k)\in X_k^{\wedge ia}\ \Rightarrow\ (p, p_1, \ldots, p_k)\in(Y_g)_k^{\vee ia}\}$$

and, because of Corollary 5, the results of [2] can be again used. So the semialgebraic set $G_{KN}(X)$ is a countable intersection $\bigcap_{i,a,k} K_k^{ia}(X)$ of constructible sets; this is possible, as is easy to see, only if the intersection stabilises. Thus, for sufficiently big a, k we have

$$G_{KN}(X)=\bigcap_{i,b\leqslant a,l\leqslant k} K_l^{ib}(X).$$

To prove that $G'_{KN}(X)$ is constructible, we have only to observe that $\{g\in G_{KN}\colon \dim Y_g\leqslant r\}$ is constructible for every r; it is easily proved by induction on n. □

Proof of Corollary 2. We prove as before that $L(X, Y)$ is semialgebraic. Now for every $p\in X_{\text{sing}}$ we put

$$X_k^{\wedge ia}(p)=\{(p_1, \ldots, p_k)\colon (p, p_1, \ldots, p_k)\in X_k^{\wedge ia}\},$$

$$Y_k^{\vee ia}(p)=\{(p_1, \ldots, p_k)\colon (p, p_1, \ldots, p_k)\in Y_k^{\vee ia}\}.$$

Again using [2] we prove that the sets

$$L_k^{ia} = \{p \in X_{\text{sing}}: X_k^{\wedge ia} \subset Y_k^{\vee ia}\}$$

are constructible. By our proposition $L(X, Y)$ is the interior in X_{sing} of $\bigcap_{i,a,k} L_k^{ia}$; this implies, as is easy to see, that $L(X, Y)$ is constructible. It follows that $NL(X, Y)$ is constructible, and, since it is closed, it is algebraic.

7. Examples

We shall use our proposition to give some explicit examples of Lipschitz stratifications of surfaces in C^3.

Let X be the germ at 0 given by

$$y^2 = x^3 + z^2 x^2.$$

We shall describe all curves Y satisfying (*). Of course the only interesting point is the origin. Clearly

$$C_0(X): y = 0.$$

First we find tangents to Y at 0.

a) If we substitute zx for z and zy for y, we get

$$y^2 = x^2 z + x.$$

The only NZ-point is $x = 0$, $y = 0$, $z = 0$; it corresponds to the z-axis so the z-axis must be tangent to a component of Y.

b) If we substitute xz for z and xy for y, we get

$$y^2 = x + x^2 z^2,$$

and this surface has no NZ-points.

Thus we can assume that Y is tangent to the z-axis. Now we take any integer $a \in N$ and put

$$z = t^a, \quad x = t^{a+1} x_1, \quad y = t^{a+1} y_1.$$

The strict transform of X is

$$X_1: \ y_1^2 = t^{a+1} x_1^2 t^{a-1} + x_1;$$

it has only one NZ-point: $x_1 = 0$, $y_1 = 0$, $t = 0$. We have to substitute

$$x_1 = t x_2, \quad y_1 = t y_2.$$

The strict transform of X_1 is

$$X_2: \ y_2^2 = t^{a+2} x_2^2 t^{a-2} + x_2.$$

The only NZ-point is again $x_2 = 0$, $y_2 = 0$, $t = 0$, so we have to substitute

$$x_2 = tx_3, \qquad y_2 = ty_3,$$

etc. After a such steps we get

$$X_a: \quad y_a^2 = t^{2a} x_a^2 (1 + x_a).$$

X_a has two NZ-points:

$$\text{I}: \quad x_a = 0, \ y_a = 0, \ t = 0,$$

$$\text{II}: \quad x_a = -1, \ y_a = 0, \ t = 0.$$

If we blow-up I, i.e. substitute

$$x_a = tx_{a+1}, \qquad y_a = ty_{a+1},$$

we get

$$X_{a+1}: \quad y_{a+1}^2 = t^{2a} x_{a+1}^2 (1 + tx_{a+1})$$

with the only NZ-point $x_{a+1} = 0$, $y_{a+1} = 0$, $t = 0$, and the same situation will appear after any number of blowing-ups. So, remembering that for every k

$$x = t^{a+k}, \qquad y = t^{a+k} y_k, \qquad z = t^a,$$

we see that I corresponds to the z-axis, which must be a component Y_1 of Y.

Now we consider II. We substitute

$$x_a = -1 + tx_{a+1}, \qquad y_a = ty_{a+1};$$

we get

$$X_{a+1}: \quad y_{a+1}^2 = t^{2a-1} (-1 + tx_{a+1})^2 x_{a+1};$$

the only NZ-point of X_{a+1} is $x_{a+1} = 0$, $y_{a+1} = 0$, $t = 0$. So we substitute

$$x_{a+1} = tx_{a+2}, \qquad y_{a+1} = ty_{a+2},$$

etc. After $2a$ such steps we get

$$X_{3a}: \quad y_{3a}^2 = (-1 + t^{2a} x_{3a})^2 x_{3a};$$

this surface has no NZ-points, so the procedure stops. Thus Y must contain a curve Y_2 on which

$$x + t^{2a} \equiv 0 \mod t^{4a}, \qquad y \equiv 0 \mod t^{4a}.$$

Such a curve can be of course characterised by

$$Y_2: \quad x + z^2 = \lambda z^4, \ y = \mu z^4, \qquad \lambda, \mu \text{ bounded}.$$

Thus finally any Lipschitz stratification of X is

$$X \supset (z\text{-axis}) \cup Y_2 \supset \{0\}.$$

As a second example we derive a relation between Lipschitz stratifications and polar curves. Let X be a surface in C^3. Assume that the projection $\pi\colon C^3 \to C^2$, parallel to the z-axis, is proper when restricted to X. For every $\xi \in C^2$ we have the projection $\pi(\xi)\colon C^3 \to C^2$, parallel to $(\xi, 1)$. Let $P(\xi)$ be the polar curve determined by $\pi(\xi)$, i.e.

$$P(\xi) = \text{the closure of } \{x \in X_{\text{reg}}\colon d\pi(\xi)\colon T_x X \to T_{\pi(\xi)x} C^2 \text{ is not a linear isomorphisms}\}.$$

There exists an open set Ω in C^2 such that the number of components of $P(\xi)$ for $\xi \in \Omega$ is independent of ξ:

$$P(\xi) = P_1(\xi) \cup \ldots \cup P_\mu(\xi),$$

and the Puiseux expansion of every $P_\alpha(\xi)$ has the form (after introducing $x_1 = x$, $x_2 = y$)

$$P_\alpha(\xi)\colon\quad x_i = \varphi_i^\alpha(z^{1/r}, \xi) = \sum_{j=1}^{j(i,\alpha)-1} a_{ij}^\alpha z^{j/r} + b_i^\alpha(\xi) z^{j(i,\alpha)/r} + o(z^{j(i,\alpha)/r})$$

where φ_i^α are analytic in z, ξ, $a_{ij} = \text{const}$ (independent of ξ) and $b_i^\alpha(\xi) \neq \text{const}$ at least for one i (remark that $j(i, \alpha)$ is finite at least for one i, for every α).

For every α let $j(\alpha) = \min\limits_{i=1,2} j(i, \alpha)$ and

$$Y_\alpha\colon\quad x_i = \sum_{j=1}^{j(\alpha)} a_{ij}^\alpha z^{j/r} + o(z^{j(\alpha)/r}), \qquad i = 1, 2,$$

where, of course, $o(z^{j(\alpha)/r})$ denotes any function going faster to 0 than $z^{j(\alpha)/r}$.

We shall prove that for any choice of the "remainders" $o(z^{j(\alpha)/r})$ the curve Y defined by

$$Y = X_{\text{sing}} \cup Y_1 \cup \ldots \cup Y_\mu$$

satisfies $(*)$.

It is enough to show that for any two curves $q_1(t)$, $q_2(t)$, lying in $X_{\text{reg}} \setminus Y$ for $t \neq 0$, such that

$$\operatorname{ord}|q_1(t) - q_2(t)| > \operatorname{ord} d(q_1(t), Y),$$

$(*)$ holds, with C depending maybe on these curves.

So let $q_1(t)$, $q_2(t)$ be such curves. Then we remark that there exists à number $c > 0$ and an open and non-empty set $\Omega_0 \subset \Omega$ such that for all $\xi \in \Omega_0$

$$d(q_1(t), X_{\text{sing}} \cup P(\xi)) \geqslant c d(q_1(t), Y) \text{ for all } t \text{ sufficiently close to } 0.$$

We change coordinates. Let $\bar{x}_1$ be any axis such that, for some $c' > 0$,

$$\measuredangle (\partial/\partial \bar{x}_1, \dot{q}_1(t)/|\dot{q}_1(t)|) < \tfrac{1}{2}\pi - c'.$$

Take any $\xi_0 \in \Omega_0$ such that $\partial/\partial \bar{x}_1 \notin \ker \pi(\xi_0)$ and let $\ker \pi(\xi_0)$ be the direction of the $\bar{x}_3$-axis. The $\bar{x}_2$-axis we choose arbitrarily. We can suppose that $q_1(0) = q_2(0) = 0$.

We take for H the plane $\bar{x}_1 = 0$ and define $T_q^0 X$, P_q^0 etc. as before. Thus we have to prove that

$$|P^0_{q_1(t)} - P^0_{q_2(t)}| \leqslant C|q_1(t) - q_2(t)|/d(q_1(t), Y).$$

Let us take an integer N such that if

$$\psi(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (\bar{x}_1^N, \bar{x}_2, \bar{x}_3),$$

then $\psi^{-1}(q_i(t))$ have branches which can be described by

$$\tilde{q}_i = \tilde{q}_i(\bar{x}_1): \quad \bar{x}_j = g_j^{(i)}(\bar{x}_1), \quad j = 2, 3;\ i = 1, 2,$$

$g_j^{(i)}$ analytic, and further

$$\operatorname{ord} d(\tilde{q}_1(\bar{x}_1), \psi^{-1}(Y)) = b \in N.$$

Put

$$\bar{x}_j = g_j^{(1)}(\bar{x}_1) + u_j \bar{x}_1^b, \quad j = 2, 3,$$

where u_2, u_3 are new variables. Thus we have maps

$$C^3_{(\bar{x}_1, u_2, u_3)} \xrightarrow{\varphi} C^3 \xrightarrow{\psi} C^3.$$

Let $H' \subset C^3_{(\bar{x}_1, u_2, u_3)}$ be given by $\bar{x}_1 = 0$ and

$$X^* = \overline{(\varphi\psi)^{-1}(X) \setminus H'};$$

let

$$q_i^*: \quad u_j = h_j^{(i)}(\bar{x}_1), \quad j = 2, 3;\ i = 1, 2,$$

be the equations of the curves $\tilde{q}_i$ in the $(\bar{x}_1, u_2, u_3)$-coordinates.

Now if we take a projection $\pi(\xi)$, where $\xi \in \Omega_0$ and $\ker \pi(\xi)$ contains a vector $\alpha(\partial/\partial \bar{x}_2) + \beta(\partial/\partial \bar{x}_3)$, $\alpha \neq 0$ or $\beta \neq 0$, then

$P^*(\xi) \stackrel{\text{def}}{=}$ the polar variety of X^* defined by the linear projection

whose kernel contains $\alpha(\partial/\partial u_2) + \beta(\partial/\partial u_3)$

$= (\varphi\psi)^{-1}(P(\xi)) \subset H'$,

by the choice of b. The set of projections in the $C^3_{(\bar{x}_1, u_2, u_3)}$ – space for which the above formula holds, is of course open in the set of all projections parallel to H' (with one-dimensional kernel). This implies that X^* has no NZ-points. Thus

$$|P^0_{q_1^*(\bar{x}_1)} - P^0_{q_2^*(\bar{x}_1)}| \leqslant C|q_1^*(\bar{x}_1) - q_2^*(\bar{x}_1)|,$$

so

$$|P^0_{q_1(t)} - P^0_{q_2(t)}| \leqslant C|q_1(t) - q_2(t)|/|t|^b. \qquad \square$$

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