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C^*-semigroup bundles and C^*-algebras whose irreducible representations are all finite dimensional

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W R O C Ł A W S K A D R U K A R N I A N A U K O W A

CONTENTS

Abstract. We investigate the structure of C^*-algebras with a finite bound on the dimensions of their irreducible representations, sometimes called "subhomogeneous".

In the first chapter we develop the theory of C^*-semigroup bundles. These are C^*-bundles over semigroups together with a "structure map" which links the semigroup structure of the base space to the bundle. Under suitable conditions we prove the existence of "enough" bounded sections, which are "compatible" with the C^*-semigroup bundle structure. Then we establish a complete duality between a certain class of C^*-semigroup bundles and subhomogeneous C^*-algebras, namely the algebra of compatible sections of such a C^*-semigroup bundle is subhomogeneous and conversely, every subhomogeneous C^*-algebra is isomorphic to the algebra of compatible sections of such a C^*-semigroup bundle. In this way we are able to even represent C^*-algebras with non-Hausdorff spectrum as sections in bundles.

The second chapter is devoted to developing methods for the computation of the functor ΠH^1_R, which classifies certain C^*-bundles with varying finite dimensional fibres. ΠH^1_R is the C^*-bundle analog of Čech-cohomology for bundles with one fibre type. The difficulty here is, that homotopy classes of cocycles of bundle imbeddings have to be computed, while only homotopies that satisfy a corresponding cocycle condition can be considered. We define a functor MH^1_R which describes the multiplicities of the imbeddings of the fibres into the bundle and assignment of multiplicity matrices to cocycles yields a natural transformation: $\Pi H^1_R \to MH^1_R$.

Chapter three finally gives some applications. We calculate ΠH^1_R for C^*-bundles over a two disk for an assignment of different finite dimensional fibres. The result is stated in terms of MH^1_R and quotients of homotopy groups of bundle imbeddings. It provides a new way to describe the group C^*-algebra of an interesting group called *p4gm*, which has been computed by I. Raeburn, and furthermore, our description yields complete invariants — in fact these are given by MH^1_R.

A last example involving bundles over a three ball with 3 different fibres shows the fact that MH^1_R does not always provide complete invariants and at the same time illustrates the limits of our methods.

0. Introduction

Because of the extreme usefulness of the Gelfand duality between commutative C^*-algebras and locally compact Hausdorff spaces – even in the investigation of noncommutative C^*-algebras via the functional calculus normal operators – several attempts have been made to generalize Gelfand duality to the noncommutative case. ([Fe; 61], [Ho; 72], [Va; 66], [Ta; 67]).

The general idea is always to replace the space of maximal ideals (or characters) by more general spaces of ideals or representations of the algebra and then represent the algebra as an algebra of a type of functions on this space.

The reason for the maximal ideal space to contain complete information about the algebra in the commutative case is the fact that there is only the trivial bundle with fibre C and the trivial group $\mathrm{Aut}(C)$ over any given space. Using classical fibre bundle theory, one can easily construct examples of nonisomorphic noncommutative C^*-algebras with equivalent representation theories, because the algebra of continuous sections of a locally trivial bundle over a compact Hausdorff space with a C^*-algebra as fibre and the automorphism group of the fibre as its group is again a C^*-algebra. The generalization of Gelfand duality thus raises two questions:

(1) *Representation.* How can we describe the algebra of functions? and

(2) *Classification.* Given the "representation theory" of an algebra, how many nonisomorphic algebras are there and how can they be distinguished?

In this dissertation we are primarily interested in investigating the structure of a class of C^*-algebras, whose representation theory is relatively close to that of commutative C^*-algebras. It is the class **BD** of C^*-algebras with a finite upper bound on the dimensions of their irreducible representations. These algebras arise as the group C^*-algebras of groups with the same property as characterized by C. C. Moore [Mo; 72], as the C^*-algebras associated to the actions of finite groups on compact Hausdorff spaces or the C^*-algebras generated by n-normal operators [BD; 72]. Even though this is a restricted class of algebras, they can already be very complicated, e.g. in general they do not have a Hausdorff spectrum.

A description of **BD** algebras was given by Vasil'ev [Va; 66], but his description in terms of locally trivial bundles is not very accessible to classification. Similarly, we do not use Hofmann's representation of C^*-algebras as sections in C^*-bundles [Ho; 72], because not much is known yet about their classification. One of the reasons for not using Takesaki's duality here is that his space of all representations is too large for our restricted class of algebras.

In Chapter I then, we treat the problem of representation and we develop the theory of C^*-semigroup bundles. The idea is due to M. J. Dupré [Du; 3], based on the concept of C^*-bundles — a certain type of fibre space — in the sense of Fell [Fe; 61]. The representation used by Fell has the problem that the algebra that is to be represented is not isomorphic to the algebra of all bounded sections of the bundle unless its spectrum is Hausdorff. This problem is overcome by introducing some additional structure leading to the concept of C^*-semigroup bundle. We give a complete duality between **BD** algebras and certain C^*-semigroup bundles, a corrected version of [Du; 3] which does not hold in that generality.

In Chapter II we deal with the problem of classification. Dupre developed the "cohomology" functor ΠH_R^1 [Du; 1], which is the C^*-bundle analog of Čech cohomology in classical fibre bundle theory and it has recently been announced that it is shown to classify those C^*-bundles (in the sense of Fell) that arise in the investigation of **BD** algebras [Du; 2]. Since C^*-algebras will mostly have irreducible representations of various dimensions, the arising C^*-bundles will generally have fibres of different isomorphism type. However, according to [Du; 2], C^*-bundles with varying finite dimensional fibres can be thought of as several locally trivial C^*-bundles "glued" together by a cocycle as given by ΠH_R^1. Consequently, the compact manifolds of imbeddings of one finite dimensional C^*-algebra into another one are of great interest. They are the analog of the group of a classical fibre bundle.

We investigate the topology of these spaces of imbeddings and develop a "multiplicity-cohomology" functor MH_R^1 analogously to the construction of ΠH_R^1, which describes the pure combinatorics of the imbeddings. It gives coarser invariants for the bundles and is related to ΠH_R^1 via a natural transformation:

$$M\colon \Pi H_R^1 \to MH_R^1.$$

Then we develop some tools that will allow the computation of ΠH_R^1 in certain cases.

In Chapter III finally, we apply the results of the first two chapters to some interesting examples, one of which is the group C^*-algebra of a group *p4gm*, which was computed by Raeburn [Ra; 82]. This algebra has a highly non-Hausdorff spectrum. We are able to give a shorter description of this algebra in terms of MH_R^1 and, moreover, this description characterizes the algebra completely (III.2.2).

In all the examples given in Chapter III, we express the isomorphism classes of C^*-bundles in terms of MH_R^1 and quotients of homotopy groups of imbeddings of finite dimensional C^*-algebras.

At the end of Chapter III, we briefly discuss the open problems and the limits of the methods developed in the first two chapters.

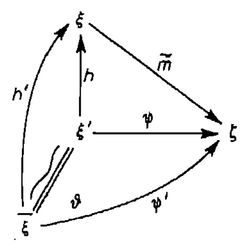
We want to remark that one might ask why we use C^*-algebras instead of more general Banach-$*$-algebras, since we are interested in group algebras and our bundles have matrix algebras as fibres, which could be given a norm different from the C^*-norm. The answer is that in Chapter I the proofs of 2.4., 3.16. and 5.1. rely heavily on results on C^*-algebras and the classification in Chapters II and III is so far only available for C^*-bundles.

I. C^*-semigroup bundles and C^*-algebras whose irreducible representations are all finite dimensional

1. C^*-semigroup bundles and their morphisms

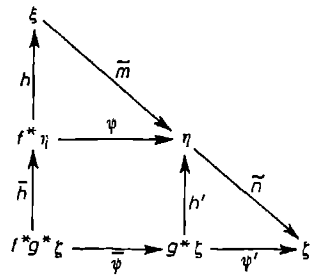
The appropriate setting for the concept of C^*-semigroup bundles is the category of C^*-bundles and C^*-bundle maps in the sense of Fell. For the definition and the most important facts about these bundles, we refer to [DG; 83] whose notation will be used here. We assume also that all C^*-bundles are full, i.e. each point in the total space of the bundle is in the image of some continuous section of the bundle.

For notational reasons it is convenient to also introduce the category $\mathbf{C^*\tilde{B}}$, which has C^*-bundles as its objects, but a morphism $\tilde{m}$: $\xi \to \zeta$ covering a map f: $X \to Y$ is an equivalence class of pairs (h, ψ), where ψ: $\xi' \to \zeta$ is a strong C^*-bundle map covering f for some C^*-bundle ξ' over X and h: $\xi' \to \xi$ covers the identity map of X. If $\bar{\xi}$ is another C^*-bundle over X, ψ': $\bar{\xi} \to \zeta$ strong and h': $\bar{\xi} \to \zeta$ covers id_X, then (h', ψ') is considered to define the same morphism $\tilde{m}$, iff there is an isomorphism θ: $\bar{\xi} \cong \xi'$ such that $\psi\theta = \psi'$ and $h\theta = h'$, i.e. the following diagram commutes:



We write $\tilde{m} = [h, \psi]$. Note that $\xi' \cong f^*\zeta$.

If $\tilde{m} = [h, \psi]$: $\xi \to \eta$, $\tilde{n} = [h', \psi']$: $\eta \to \zeta$, $\tilde{m}$ over f: $X \to Y$ and $\tilde{n}$ over g: $Y \to Z$, then we can assume that ψ: $f^*\eta \to \eta$ and ψ': $g^*\zeta \to \zeta$ and we let $\bar{\psi}$ be the strong bundle map $f^*g^*\zeta \to g^*\zeta$ and $\bar{h}$ the unique factorization of $\bar{\psi}h'$ through ψ. We define $\tilde{n}\circ\tilde{m}$ to be the morphism $[h\circ\bar{h}, \psi'\circ\bar{\psi}]$, which is independent of the choice of representants for $\tilde{m}$ and $\tilde{n}$. This is illustrated in the following diagram:



For any $x \in X$, $\tilde{m} = [h, \psi]$: $\xi \to \zeta$ over f: $X \to Y$ defines a C^*-homomorphism on fibres $\tilde{m}(x)$: $\zeta(f(x)) \to \xi(x)$, namely $\tilde{m}(x)$: $h_x \circ (\psi_x)^{-1}$, which is easily seen to be independent of the representant (h, ψ). Moreover if $\tilde{m}' = [h', \psi']$: $\xi \to \zeta$ satisfies for all $x \in X$ the condition $\tilde{m}(x) = \tilde{m}'(x)$, then $\tilde{m} = \tilde{m}'$, because without loss of generality we can assume $\psi = \psi'$: $f^*\zeta \to \zeta$ and thus

$$h'_x = \tilde{m}'(x) \circ \psi_x = \tilde{m}(x) \circ \psi_x = h_x$$

and $h' = h$.

Also, if we define for a selection $s \in \Pi\zeta$ the selection $\tilde{m}^* s = h_* f^* s$, then we have for all $x \in X$:

$$(\tilde{m}^* s)(x) = \tilde{m}(x)\big(s(f(x))\big)$$

and hence $\tilde{m} = \tilde{m}'$ iff for all $s \in \Pi\zeta$

$$\tilde{m}^* s = \tilde{m}'^* s.$$

Note, that $\tilde{m}^*$: $\Pi_b\zeta \to \Pi_b\xi$, $s \to \tilde{m}^* s$ is normdecreasing!

1.1. Definition. Let (X, m) be a topological semigroup and ξ_0 a C^*-bundle over X. A *C^*-semigroup bundle* is a pair $\xi = (\xi_0, \tilde{m})$, where $\tilde{m} = [h, \psi]$: $\xi_0 \times \xi_0 \to \xi_0$ is a morphism covering m: $X \times X \to X$ and h is injective. Here $\xi_0 \times \xi_0$ is the cartesian product in the sense of [Hu; 75: Def. 4.1, pg. 15]. ξ_0 is called the *underlying C^*-bundle* of ξ and h is called the *structure map*. (It is unique up to isomorphism!) ■

For any C^*semigroup bundle we will always denote the underlying C^*-bundle by a subscript "0".

For two C^*-semigroup bundles $\xi = (\xi_0, \tilde{m})$ and $\eta = (\eta_0, \tilde{n})$ over (X, m) and (Y, n) respectively, we define a *C^*-semigroup bundle map* to be a morphism $\tilde{\varphi}$: $\xi_0 \to \eta_0$ covering a (continuous) semigroup homomorphism φ: $X \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} \xi_0 \times \xi_0 & \xrightarrow{\tilde{m}} & \xi_0 \\ {\scriptstyle \tilde{\varphi} \times \tilde{\varphi}} \downarrow & & \downarrow {\scriptstyle \tilde{\varphi}} \\ \eta_0 \times \eta_0 & \xrightarrow[\tilde{n}]{} & \eta_0 \end{array}$$

Here the cartesian product of two morphisms $[h_1, \psi_1]$ and $[h_2, \psi_2]$ is defined to be $[h_1, \psi_1]\times[h_2, \psi_2] = [h_1 \times h_2, \psi_1 \times \psi_2]$.

A C^*-semigroup bundle map covering the identity on X corresponds to a morphism $[\alpha, \mathrm{id}_{\eta_0}]$: $\xi_0 \to \eta_0$, where α is an ordinary C^*-bundle map satisfying $(\alpha\times\alpha)h_\eta = h_\xi\bar{\alpha}$ if $\tilde{m} = [h_\xi, \psi_\xi]$, $\tilde{n} = [h_\eta, \psi_\eta]$ and $\bar{\alpha}$ is the unique factorization of $\alpha\psi_\eta$ through ψ_ξ.

A selection $s \in \Pi\xi_0$ is said to be *compatible* if $\tilde{m}^* s = s \times s$ and the set of compatible selections is denoted $\Pi\xi$. Similarly, if we write as in [DG; 83] $\Pi_b\xi_0$ for the set of bounded selections, then $\Pi_b\xi = \Pi\xi \cap \Pi_b\xi_0$ denotes the set of bounded compatible selections of ξ.

For a subset $C \subseteq X$, a selection $s \in \Pi\xi_0|C$ is said to be compatible over C if it extends to a compatible selection and $\Pi\xi|C$ and $\Pi_b\xi|C$ denote – by abuse of notation – the sets of compatible and bounded compatible selections over C, respectively. $\Gamma(\xi)$, $\Gamma_b(\xi)$, $\Gamma(\xi|C)$ and $\Gamma_b(\xi|C)$ are the symbols for the algebras of compatible and bounded compatible sections and the algebras of compatible and bounded compatible selections over C, that are continuous, respectively.

If C generates X, then $\Gamma_b(\xi|C)$ is a C^*-subalgebra of the C^*-algebra $\Gamma_b(\xi_0|C)$ of bounded sections of $\xi_0|C$.

Proof. First observe that the injectivity of h implies for any $s \in \Pi\xi$ and $x, y \in X$ that

$$\max\{\|s(x)\|, \|s(y)\|\} = \|(s\times s)(x, y)\| = \|h((m^* s)(x, y))\|$$
$$= \|(m^* s)(x, y)\| = \|s(m(x, y))\|$$

or

$$\sup\{\|s(x)\|: x \in C\} = \sup\{\|s(x)\|: x \in X\}$$

or restriction $\Pi_b\xi \to \Pi_b\xi|C$ is an isometry. As $\Gamma_b(\xi|C) = (\Pi_b\xi|C) \cap \Gamma_b(\xi_0|C)$, the proof is finished once we have shown that $\Pi_b\xi \subseteq \Pi_b\xi_0$ is closed. But this follows from the continuity of the map $s \mapsto \tilde{m}^* s$. Therefore $\Gamma_b(\xi|C)$ is a closed involutive subalgebra of the C^*-algebra $\Gamma_b(\xi_0|C)$ and hence a C^*-algebra itself. ∎

Now consider topological semigroups (X, m) and (Y, n), a C^*-semigroup bundle $\xi = (\xi_0, \tilde{m})$ over X and a continuous semigroup homomorphism φ: $Y \to X$. The C^*-bundle $\varphi^*\xi_0$ inherits a C^*-semigroup structure $\tilde{n} = [h', \psi']$ whose construction we are going to describe now. So let ψ': $n^*\varphi^*\xi_0 \to \varphi^*\xi_0$, ψ_φ: $\varphi^*\xi_0 \to \xi_0$ and ψ: $m^*\xi_0 \to \xi_0$ be the strong bundle maps covering the respective maps on the base spaces. As

$$n^*\varphi^*\xi_0 \cong (\varphi \circ n)^*\xi_0 = (m \circ (\varphi\times\varphi))^*\xi_0 \cong (\varphi\times\varphi)^* m^*\xi_0,$$

there is a strong map $\bar{\psi}$: $n^*\varphi^*\xi_0 \to m^*\xi_0$, so that $\psi \circ \bar{\psi} = \psi_\varphi \circ \psi'$. Furthermore, we get a unique factorization h' of $h\bar{\psi}$ through $\psi_\varphi \times \psi_\varphi$, where h is the structure map of ξ. The following diagram illustrates this:

$$\begin{array}{ccc}
\varphi^*\xi_0 \times \varphi^*\xi_0 & \xrightarrow{\psi_\varphi \times \psi_\varphi} & \xi_0 \times \xi_0 \\
{\scriptstyle h'}\uparrow & & \uparrow{\scriptstyle h} \\
n^*\varphi^*\xi_0 & \xrightarrow{\bar{\psi}} & m^*\xi_0 \\
{\scriptstyle \psi'}\downarrow & & \downarrow{\scriptstyle \psi} \\
\varphi^*\xi_0 & \xrightarrow[\psi_\varphi]{} & \xi_0 .
\end{array}$$

We denote the resulting C^*-semigroup bundle $(\varphi^*\xi_0, \tilde{n})$ by $\varphi^*\xi$. There is then a morphism $\tilde{\varphi}$: $\varphi^*\xi_0 \to \xi_0$ defined by $\tilde{\varphi} = [\mathrm{id}_{\varphi^*\xi_0}, \psi_\varphi]$, which is actually a C^*-semigroup bundle map, since

$$\tilde{\varphi} \circ \tilde{n} = [h', \psi_\varphi \psi'] = [h', \bar{\psi}\psi] = \tilde{m} \circ (\tilde{\varphi} \times \tilde{\varphi}).$$

By the functorial properties of pullbacks or by direct calculations, we obtain furthermore that $\varphi^*\Gamma(\xi) \subseteq \Gamma(\varphi^*\xi)$.

2. The universal C^*-semigroup bundle of a C^*-algebra

Let A be any C^*-algebra and $\mathrm{Id}(A)$ the set of (closed twosided) ideals. By identifying each ideal $I \in \mathrm{Id}(A)$ with a normfunction n_I: $A \to \mathbf{R}$, $a \mapsto n_I(a) = \|a+I\| \in [0, \|a\|]$, $\mathrm{Id}(A)$ inherits the subspace topology of $\prod_{a \in A} [0, \|a\|]$, which makes it a compact Hausdorff space, since it is a closed subspace. (For details see [DG; 83].) Moreover, the map $\cap$: $\mathrm{Id}(A) \times \mathrm{Id}(A) \to \mathrm{Id}(A)$, $(I, J) \mapsto I \cap J$ is continuous — because for all $a \in A$ and $I, J \in \mathrm{Id}(A)$, $\|a+I \cap J\| = \max\{\|a+I\|, \|a+J\|\}$ — and defines a semigroup structure on $\mathrm{Id}(A)$.

We construct a C^*-family over $\mathrm{Id}(A)$ having A/I as fibre over $I \in \mathrm{Id}(A)$. Let $\tilde{A} = \{\tilde{a}\colon a \in A\} \subseteq \prod_{I \in \mathrm{Id}(A)} A/I$, where $\tilde{a}$ sends I to $a+I$. $\tilde{A}$ is full and for each $a \in A$, the map $I \mapsto \|\tilde{a}(I)\| = \|a+I\|$ is continuous by the definition of the topology of $\mathrm{Id}(A)$. By [DG; 83], there is a unique topology on $\sum_{I \in \mathrm{Id}(A)} A/I$ making it a C^*-bundle $\eta(A)_0$ and having $\tilde{A} \subseteq \Gamma(\eta(A)_0)$.

This fibre of $\cap^*\eta(A)_0$ over (I, J) is $\{(I, J)\} \times (A/(I \cap J))$ and we define h: $\cap^*\eta(A)_0 \to \eta(A)_0 \times \eta(A)_0$ by $(I, J, a+I \cap J) \mapsto (a+I, a+J)$. The map h is clearly injective, thus has fibrewise norm one and as $h_* \cap^* \tilde{a} = \tilde{a} \times \tilde{a}$ is in $\Gamma_b(\eta(A)_0 \times \eta(A)_0)$, h must be continuous ([DG; 83: Prop. 1.4]). If we let ψ: $\cap^*\eta(A)_0 \to \eta(A)_0$ denote the strong bundle map covering $\cap$: $\mathrm{Id}(A) \times \mathrm{Id}(A) \to \mathrm{Id}(A)$ and $\tilde{\cap} = [h, \psi]$, then we get a C^*-semigroup bundle $\eta(A) = (\eta(A)_0, \tilde{\cap})$.

2.1. Definition. We call $\eta(A)$ the *universal C^*-semigroup bundle of A.*

$\Gamma(\eta(A))$ is a C^*-subalgebra of $\Gamma(\eta(A)_0)$, full for $\eta(A)_0$ and $A \cong \tilde{A} \subseteq \Gamma(\eta(A))$

by definition. The following proposition allows us to conclude that actually $\tilde{A} = \Gamma(\eta(a))$, thus characterizing those elements of $\Gamma(\eta(A)_0)$, that arise as $\tilde{a}$ for some $a \in A$.

2.2. PROPOSITION. *Let $\xi = (\xi_0, \tilde{m})$ be a C^*-semigroup bundle over the compact Hausdorff semigroup (X, m) and let $A \subseteq \Gamma(\xi)$ be a C^*-subalgebra, which is full for ξ_0. Then $A = \Gamma(\xi)$.*

Proof. Let $x, y \in X$ and $v \in \xi_0(m(x, y))$. Then there exist $a \in A$ and $s \in \Gamma(\xi)$ such that $a(m(x, y)) = s(m(x, y)) = v$. Thus

$$\begin{aligned}(a(x), a(y)) &= (\tilde{m}^* a)(x, y) = \tilde{m}(x, y)\big(a(m(x, y))\big)\\ &= \tilde{m}(x, y)\big(s(m(x, y))\big) = (\tilde{m}^* s)(x, y) = \big(s(x), s(y)\big).\end{aligned}$$

Hence $A|\{x, y\} = \Gamma(\xi)|\{x, y\}$ for all $x, y \in X$ and as X is compact Hausdorff, the Stone–Weierstrass–Glimm Theorem for bundles [DG; 83: Prop. 2.18] applies to yield $A = \Gamma(\xi)$. ∎

2.3. COROLLARY. *Let A be a C^*-algebra and $\eta(A)$ its universal C^*-semigroup bundle, then $A \cong \Gamma(\eta(A))$.* ∎

Notice that the corollary could also be proved directly by the following argument pointed out to me by Klaus Keimel:

Let $s \in \Gamma(\eta(A))$, $I \in \mathrm{Id}(A)$ and $a = s(0) \in A$. Then $(s(I), s(0)) = (\tilde{m}^* s)(I, 0) = h((I, 0, s(I \cap 0)) = (a + I, a)$ or $s = \tilde{a}$. The next question is: "How is an arbitrary C^*-semigroup bundle related to the universal C^*-semigroup bundle of the algebra of its compatible section?", in other words, why is $\eta(A)$ called "universal"?

2.4. PROPOSITION. *Let $\xi = (\xi_0, \tilde{m})$ be a C^*-semigroup bundle over (X, m) such that $A = \Gamma_b(\xi)$ is full for ξ_0 and let $\eta(A)$ be the universal C^*-semigroup bundle of A. Then there is a continuous semigroup homomorphism $\varphi\colon X \to \mathrm{Id}(A)$ such that $\xi \cong \varphi^* \eta(A)$ and – identifying these two C^*-semigroup bundles – $A = \varphi^* \Gamma(\eta(A))$.*

Proof. For $x \in X$ define $\varphi(x) = \{a \in A\colon a(x) = 0\}$. Then φ is a semigroup homomorphism by injectivity of the structure map. As point evaluation at $x \in X$ defines a C^*-homomorphism $A \to \xi_0(x)$ with kernel $\varphi(x)$ and $\|a(x)\| = \|a + \varphi(x)\|$ for $a \in A$, continuity of the maps $x \mapsto \|a(x)\|$, $a \in A$, implies the continuity of φ.

For every $x \in X$ we can define the map

$$\psi_x\colon \xi_0(x) \to A/\varphi(x), \qquad a(x) \mapsto a + \varphi(x), \qquad a \in A,$$

using the fact that A is full for ξ_0, and it is easily checked that the ψ_x are isomorphisms of C^*-algebras. Furthermore, the Banach family $\psi = (\psi_x)_{x \in X}$ satisfies the hypotheses of [DG; 83: Prop. 1.4] and is therefore continuous. Hence $\xi_0 \cong \varphi^* \eta(A)_0$ and we identify these two bundles without loss of generality.

As before, functoriality of pullbacks or direct calculations guarantee that $\xi \cong \varphi^*\eta(A)$ and thus $\varphi^*\Gamma(\eta(A)) \subseteq A$. But as $A \cong \tilde{A} = \Gamma(\eta(A))$ by 2.3, $a = \varphi^*\tilde{a}$ and thus $A = \varphi^*\Gamma(\eta(A))$. ■

2.5. EXAMPLE. For each positive integer n let $M_n(C)$ be the C^*-algebra of complex $n \times n$ matrices and $U(n) \subseteq M_n(C)$ its unitary group. Consider an arbitrary C^*-algebra A. For each positive integer n let $\mathrm{Rep}_n(A)$ be the set of all representations of A on C^n, topologized by pointwise normconvergence. As a closed subset of the compact set

$$\prod_{a\in A} \{m \in M_n(C)\colon \ \|m\| \leqslant \|a\|\},$$

$\mathrm{Rep}_n(A)$ is a compact Hausdorff space. Unitary equivalence is a closed relation and the quotient map $p_n\colon \mathrm{Rep}_n(A) \to \mathrm{Rep}_n(A)/U(n)$ is open, so that the quotient $\mathrm{Rep}_n(A)/U(n)$ is still a compact Hausdorff space. We let S_A be the topological sum of the $\mathrm{Rep}_n(A)/U(n)$, where $n = 1, 2, \ldots$, and note that S_A is a locally compact Hausdorff space, countable at infinity.

Direct sum operation defines an addition on S_A, whose continuity follows from the continuity of the direct sum operation on $\mathrm{Rep}_n(A) \times \mathrm{Rep}_m(A) \to \mathrm{Rep}_{n+m}(A)$ for positive integers n and m, the continuity of the quotient map p_{n+m}, the fact that $p_n \times p_m$ is a quotient map since p_n and p_m are open, and commutativity for the diagram

$$\begin{array}{ccc} \mathrm{Rep}_n(A) \times \mathrm{Rep}_m(A) & \xrightarrow{\ \oplus\ } & \mathrm{Rep}_{n+m}(A) \\ {\scriptstyle p_n \times p_m}\downarrow & & \downarrow{\scriptstyle p_{n+m}} \\ \mathrm{Rep}_n(A)/U(n) \times \mathrm{Rep}_m(A)/U(m) & \xrightarrow[\oplus]{} & \mathrm{Rep}_{n+m}(A)/U(n+m). \end{array}$$

In this way S_A carries the structure of an abelian locally compact Hausdorff semigroup.

As every finite dimensional representation can be uniquely expressed as a finite direct sum of irreducible and trivial representations up to unitary equivalence, S_A is a cancellation semigroup generated by

$$\{\pi \in \hat{A}\colon \ \dim \pi < \infty\} \cup \{\pi\colon \ A \to C\colon \ \pi a = 0 \text{ for all } a \in A\}.$$

The map dim: $S_A \to N$, $\pi \mapsto \dim \pi$ is a continuous semigroup homomorphism and for all $n \in N$, $\dim^{-1}(n) = \mathrm{Rep}_n/U(n)$ is compact Hausdorff. We call S_A the (finite dimensional) representation semigroup of A.

A C^*-algebra homomorphism $\varphi\colon A \to B$ yields a continuous semigroup homomorphism $S_\varphi\colon S_B \to S_A$ defined by $S_\varphi(\pi) = \pi \circ \varphi$ and one can easily check that S defines a contravariant functor from the category of C^*-algebras to the category of topological semigroups. Also note that S_φ preserves the dimensions of representations.

Now observe that the map Ker: $S_A \to \mathrm{Id}(A)$, $\pi \mapsto \mathrm{Ker}\, \pi$ is a continuous semigroup homomorphism, since for any $\pi \in S_A$ and $a \in A$, $\|a + \mathrm{Ker}\, \pi\|$

$= \|\pi(a)\|$ and $\operatorname{Ker}(\pi \oplus \varrho) = (\operatorname{Ker}\pi) \cap (\operatorname{Ker}\varrho)$. Hence we can form $\xi(A) = \operatorname{Ker}^*\eta(A)$ and by our previous remarks $\operatorname{Ker}^*\Gamma(\eta(A)) \subseteq \Gamma_b(\xi(A))$. If all irreducible representations of A are finite dimensional, then $\operatorname{Ker}^*\tilde{a} = 0$ iff $a \in \operatorname{Ker}\pi$ for all $\pi \in \hat{A}$, which is in turn equivalent to $a = 0$ and we obtain $A \cong \operatorname{Ker}^*\Gamma(\eta(A))$ for these C^*-algebras.

In Section 5 we will establish a duality theory for C^*-algebras having a finite upper bound on the dimensions of its irreducible representations — we denote the class of those C^*-algebras **BD** — on one hand and certain C^*-semigroup bundles on the other hand.

3. Abelian and associative C^*-semigroup bundles and the extension of compatible sections

The main part of this section is devoted to the extension of compatible sections and the existence of enough compatible sections under suitable conditions. A major difficulty in the extension of compatible sections is that the algebra of compatible sections is not a module over the ring of bounded continuous functions on the base space in general.

In fact, let $\xi = (\xi_0, \tilde{m})$ be a C^*-semigroup bundle over the completely regular semigroup (X, m), $A = \Gamma_b(\xi)$, and assume that for all x, y in the support, $\operatorname{supp}\xi_0$, of ξ_0 there is an $a \in A$ with $a(x) \neq 0$ and $a(y) \neq 0$ — it will turn out later that this assumption is reasonable — and let $f \in C_b(X)$. If $f \cdot a$ is also in A, then for all $x, y \in X$,

$$\begin{aligned}(f(x)a(x), f(y)a(y)) &= (\tilde{m}^*(f \cdot a))(x, y)\\ &= f(m(x, y)) \cdot (\tilde{m}^* a)(x, y)\\ &= f(m(x, y))(a(x), a(y)).\end{aligned}$$

Thus if A is a $C_b(X)$ module, this implies $f(m(x, y)) = f(x) = f(y)$ for all $x, y \in \operatorname{supp}\xi_0$ and $f \in C_b(X)$. Since X is completely regular, this implies that $\operatorname{supp}\xi_0$ consists of at most one point.

We will use, however, the fact that $\Gamma_b(\xi_0)$ is a $C_b(X)$-module, but first we need to prove some preparatory results and introduce some further concepts.

3.1. Lemma. *Let $\xi = (\xi_0, \tilde{m})$ be a C^*-semigroup bundle over (X, m). For any $s, t \in \Pi\xi$, the set $S_{s,t} = \{x \in X\colon s(x) = t(x)\}$ is a subsemigroup of X.*

Proof. If $s, t \in \Pi\xi$ and $x, y \in S_{s,t}$, then $s(x) = t(x)$ and $s(y) = t(y)$. Thus

$$(\tilde{m}^* s)(x, y) = (s(x), s(y)) = (t(x), t(y)) = (\tilde{m}^* t)(x, y)$$

and as h is injective

$$s(m(x, y)) = t(m(x, y))$$

or $m(x, y) \in S_{s,t}$. ■

3.2. PROPOSITION. *Let* $\xi = (\xi_0, \tilde{m})$ *be a* C^**-semigroup bundle over* (X, m), $C \subseteq X$ *and denote by* $X(C)$ *the subsemigroup generated by* C *algebraically, and by* $i: X(C) \to X$ *and* $j: C \to X(C)$ *the inclusion maps. Then* j^*: $\Gamma(\xi|X(C)) = \Gamma(i^*\xi) \to \Gamma(\xi|C)$ *is injective.*

Proof. Suppose $s \in \Gamma(\xi|X(C))$ with $j^*s = s|C = 0$. By 3.1 the set $S = \{x \in X(C): s(x) = 0\}$ is a subsemigroup of $X(C)$ containing C. As C generates $X(C)$, $S = X(C)$ and thus $s \equiv 0$. ■

For a semigroup X, let us denote the set of irreducible elements of X by $\mathrm{Irr}(X)$. From now on, we assume that our semigroups – except $\mathrm{Id}(A)$ – are abelian locally compact Hausdorff cancellation semigroups generated by their irreducible elements and we write for $x, y \in X$: $m(x, y) = x + y$. Thus, every $x \in X$ can be expressed uniquely – up to rearrangement – as $x = x_1 + x_2 + \ldots + x_n$ for some $x_1, x_2, \ldots, x_n \in \mathrm{Irr}(X)$.

Our next example shows that a C^*-semigroup bundle does not necessarily have "many" compatible sections:

3.3. EXAMPLE. Let $\xi_0 = \varepsilon(N, C)$ the trivial line bundle over the positive integers. Define ψ: $+^*\xi_0 \to \xi_0$ by $(n, m, z) \to (n+m, z)$ and h: $+^*\xi_0 \to \xi_0 \times \xi_0$ by

$$h(n, m, z) = ((n, z), (m, 0)).$$

Let $\tilde{m} = [h, \psi]$ and $\xi = (\xi_0, \tilde{m})$. Take any $s \in \Gamma(\xi)$ and let $s(1) = (1, z)$ and $s(2) = (2, u)$. As s is compatible

$$\begin{aligned}((1, z), (1, z)) &= (s \times s)(1, 1) = (\tilde{m}^* s)(1, 1)\\ &= h(1, 1, u) = ((1, u), (1, 0)).\end{aligned}$$

Therefore, $(1, z) = (1, 0) = (1, u)$ or $z = u = 0$. As 1 generates N, 3.2 yields $s \equiv 0$ or $\Gamma(\xi) = 0$. ■

Now let $\xi = (\xi_0, \tilde{m})$ be a C^*-semigroup bundle over (X, m), $\tilde{m} = [h, \psi]$ and consider the map flip: $X \times X \to X \times X$, $(x, y) \mapsto (y, x)$. Suppose $v \in \xi_0(x+y)$, $u \in (m^*\xi_0)(x, y)$, $w \in (m^*\xi_0)(y, x)$ and $(r, s) \in \xi_0(x) \times \xi_0(y)$, where $\psi u = \psi w = v$. Furthermore, denote by ψ_{flip}: $m^*\xi_0 \to m^*\xi_0$ and $\bar{\psi}_{\mathrm{flip}}$: $\xi_0 \times \xi_0 \to \xi_0 \times \xi_0$ the strong bundle maps covering flip, defined by

$$\psi_{\mathrm{flip}} u = w \quad \text{and} \quad \bar{\psi}_{\mathrm{flip}}(r, s) = (s, r).$$

Note that $\mathrm{flip}^* m^* \xi_0 \cong m^* \xi_0$, $\mathrm{flip}^*(\xi_0 \times \xi_0) \cong \xi_0 \times \xi_0$ and $\psi \circ \psi_{\mathrm{flip}} u = \psi w = v = \psi u$, so that $\psi \circ \psi_{\mathrm{flip}} = \psi$. Let us denote the morphism $[\mathrm{id}_{\xi_0 \times \xi_0}, \psi_{\mathrm{flip}}]$ by $\widetilde{\mathrm{flip}}$.

3.4. DEFINITION. We say that the C^*-semigroup bundle $\xi = (\xi_0, \tilde{m})$ is *abelian*, if $\tilde{m} \circ \widetilde{\mathrm{flip}} = \tilde{m}$ and that ξ is associative, if $\tilde{m} \circ (\mathrm{id}_\xi \times \tilde{m}) = \tilde{m} \circ (\tilde{m} \times \mathrm{id}_\xi)$. ■

Now, with the notation preceding 3.4, suppose that $hu = (u_x, u_y)$ and $hw = (w_y, w_x)$. In order to construct compatible selections, we need at least that

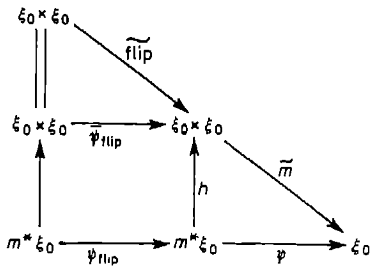
$$h\circ\psi_{\mathrm{flip}}u = hw = (w_y, w_x) = (u_y, u_x) = \bar{\psi}_{\mathrm{flip}}(u_x, y_y) = \bar{\psi}_{\mathrm{flip}}\circ hu$$

or

$$h\circ\psi_{\mathrm{flip}} = \bar{\psi}_{\mathrm{flip}}\circ h.$$

In other words, if ϑ is the unique factorization of $h\circ\psi_{\mathrm{flip}}$ through $\bar{\psi}_{\mathrm{flip}}$, then we need $h = \vartheta$.

Consider the following diagram and recall that $\mathrm{flip}^* m^* \xi_0 \cong m^* \xi_0$



From this it is clear that $h = \vartheta$ is equivalent to

$$\tilde{m}\circ\widetilde{\mathrm{flip}} = [\vartheta, \psi\psi_{\mathrm{flip}}] = [\vartheta, \psi] = [h, \psi] = \tilde{m}.$$

Or ξ is abelian iff $h = \vartheta$.

Note that $\eta(A)$ is both abelian and associative for any C^*-algebra A.

3.5. Proposition. *Let $\xi = (\xi_0, \tilde{m})$ be a C^*-semigroup bundle over (X, m), such that $\Pi\xi$ is full for ξ_0. Then ξ is abelian and associative.*

Proof. To show that ξ is abelian, note that for any $s, t \in \Pi\xi_0$, $\widetilde{\mathrm{flip}}^*(s \times t) = t \times s$. Hence, for $s \in \Pi\xi$,

$$\widetilde{\mathrm{flip}}^* \tilde{m}^* s = \widetilde{\mathrm{flip}}^*(s \times s) = s \times s = \tilde{m}^* s$$

and by the remarks preceding 1.1,

$$\tilde{m}\circ\widetilde{\mathrm{flip}} = \tilde{m}$$

and ξ is abelian. Similarly, $s \in \Pi\xi$ implies

$$[\tilde{m}\circ(\tilde{m} \times \mathrm{id}_\xi)]^* s = (\tilde{m} \times \mathrm{id}_\xi)^* \tilde{m}^* s = (\tilde{m} \times \mathrm{id}_\xi)^*(s \times s)$$

$$= (\tilde{m}^* s) \times s = s \times s \times s = s \times \tilde{m}^* s = [\tilde{m}\circ(\mathrm{id}_\xi \times \tilde{m})]^* s$$

and again $\tilde{m}\circ(\tilde{m} \times \mathrm{id}_\xi) = \tilde{m}\circ(\mathrm{id}_\xi \times \tilde{m})$ or ξ is associative. ∎

Now we define inductively morphisms $\tilde{m}_n$: $\xi^n \to \xi$, $\tilde{m}_n = [h_n, \psi_n]$, by setting $\tilde{m}_1 = \mathrm{id}_\xi$ and $\tilde{m}_{n+1} = \tilde{m}\circ(\tilde{m}_n \times \mathrm{id}_\xi)$ for $n \geqslant 1$. Then $\tilde{m}_n$ covers the map $X^n \to X$, $(x_1, \ldots, x_n) \mapsto x_1 + \ldots + x_n$ and h_n is injective for each n. For $x_1, \ldots, x_n \in X$ we get a map

$$\tilde{m}(x_1, \ldots, x_n)\colon \xi_0(x_1+\ldots+x_n)\to\xi_0(x_1)\times\ldots\times\xi_0(x_n)$$

such that $s\in\Pi\xi$ iff for all $x_1, \ldots, x_n\in X$

$$(\tilde{m}_n^* s)(x_1, \ldots, x_n) = \big(s(x_1), \ldots, s(x_n)\big) = \tilde{m}_n(x_1, \ldots, x_n)\big(s(x_1+\ldots+x_n)\big).$$

3.6. Lemma. *Let $\xi = (\xi_0, \tilde{m})$ be an associative C^*-semigroup bundle. For every $n = 2, 3, \ldots$ and $i = 1, 2, \ldots, n-1$, we have*

(i) $\tilde{m}_n = \tilde{m}\circ(\tilde{m}_{n-1}\times \mathrm{id}_\xi) = \tilde{m}\circ(\mathrm{id}_\xi\times\tilde{m}_{n-1})$ *and*
(ii) $\tilde{m}_n = \tilde{m}_{n-1}\circ(\mathrm{id}_{\xi^{i-1}}\times\tilde{m}\times\mathrm{id}_{\xi^{n-i-1}})$

where $\tilde{m}\times\mathrm{id}_{\xi^0} = \tilde{m} = \mathrm{id}_{\xi^0}\times\tilde{m}$.

Proof. By induction. Clearly, (i) and (ii) hold for $n = 2$. Now assume that (i) and (ii) are true for $n\geqslant 2$. We use the abbreviation $\mathrm{id}_{\xi^k} = \mathrm{id}_k$. Then

$$\begin{aligned}
\tilde{m}_{n+1} &= \tilde{m}\circ(\tilde{m}_n\times\mathrm{id}_1) = \tilde{m}\circ\big[\big(\tilde{m}\circ(\tilde{m}_{n-1}\times\mathrm{id}_1)\big)\times\mathrm{id}_1\big]\\
&= \tilde{m}\circ\big[\big(\tilde{m}\circ(\mathrm{id}_1\times\tilde{m}_{n-1})\big)\times\mathrm{id}_1\big] && \text{by hypothesis}\\
&= \tilde{m}\circ(\tilde{m}\times\mathrm{id}_1)\circ(\mathrm{id}_1\times\tilde{m}_{n-1}\times\mathrm{id}_1)\\
&= \tilde{m}\circ(\mathrm{id}_1\times\tilde{m})\circ(\mathrm{id}_1\times\tilde{m}_{n-1}\times\mathrm{id}_1) && \text{by associativity}\\
&= \tilde{m}\circ\big[\mathrm{id}_1\times\big(\tilde{m}\circ(\tilde{m}_{n-1}\times\mathrm{id}_1)\big)\big]\\
&= \tilde{m}\circ(\mathrm{id}_1\times\tilde{m}_n)
\end{aligned}$$

and for $i = 1, \ldots, n-1$:

$$\begin{aligned}
\tilde{m}_{n+1} &= \tilde{m}\circ(\tilde{m}_n\times\mathrm{id}_1)\\
&= \tilde{m}\circ[(\tilde{m}_{n-1}\times\mathrm{id}_1)\circ(\mathrm{id}_{i-1}\times\tilde{m}\times\mathrm{id}_{n-i})] && \text{by hypothesis}\\
&= \tilde{m}_n\circ(\mathrm{id}_{i-1}\times\tilde{m}\times\mathrm{id}_{n-i})
\end{aligned}$$

and for $i = n$

$$\begin{aligned}
\tilde{m}_{n+1} &= \tilde{m}\circ[\mathrm{id}_1\times\tilde{m}_n] && \text{by (i)}\\
&= \tilde{m}\circ\big[\mathrm{id}_1\times\big(\tilde{m}_{n-1}\circ(\mathrm{id}_{n-2}\times\tilde{m})\big)\big] && \text{by hypothesis}\\
&= \tilde{m}\circ(\mathrm{id}_1\times\tilde{m}_{n-1})\circ(\mathrm{id}_{n-1}\times\tilde{m})\\
&= \tilde{m}_n(\mathrm{id}_{n-1}\times\tilde{m}) && \text{by hypothesis.}
\end{aligned}$$ ∎

For the next proposition let us introduce the maps $\mathrm{flip}^n_{i,k}\colon X^n\to X^n$,

$$(x_1, \ldots, x_i, \ldots, x_k, \ldots, x_n)\mapsto(x_1, \ldots, x_k, \ldots, x_i, \ldots, x_n),$$

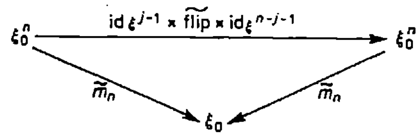
for all n and $1\leqslant i, k\leqslant n$ and for $j = 1, \ldots, n$, let $p_j\colon X^n\to X$ be the jth-factor projection.

3.7. Proposition. *Let* $\xi = (\xi_0, \tilde{m})$ *be an abelian associative* C^**-semigroup bundle,* $n \in N$ *and* $x \in X^n$. *For* $i \neq k$, $1 \leqslant i, k \leqslant n$, *let* $\varphi_x = \tilde{m}_n(x)$, $\varphi'_x = \tilde{m}_n(\mathrm{flip}^n_{i,k} x)$ *and* π_j: $(\xi_0)^n(x) \to \xi_0(p_j x)$ *and* π'_j: $(\xi_0)^n(\mathrm{flip}^n_{i,k} x) \to \xi_0(p_j \circ \mathrm{flip}^n_{i,k}(x))$ *be the j-th-factor projections. Then*

$$\pi'_k \circ \varphi'_x = \pi_i \circ \varphi_x.$$

In particular, if $p_i x = p_k x$, *then* $\pi_k \circ \varphi_x = \pi_i \circ \varphi_x$.

Proof. For $j = 1, \ldots, n-1$ consider the following diagrams:



Since ξ is abelian and associative, 3.6 implies that

$$\tilde{m}_n \circ (\mathrm{id}_{\xi^{j-1}} \times \widetilde{\mathrm{flip}} \times \mathrm{id}_{\xi^{n-j-1}}) = \tilde{m}_{n-1} \circ (\mathrm{id}_{\xi^{j-1}} \times (\tilde{m} \circ \widetilde{\mathrm{flip}}) \times \mathrm{id}_{\xi^{n-j-1}})$$
$$= \tilde{m}_{n-1} \circ (\mathrm{id}_{\xi^{j-1}} \times \tilde{m} \times \mathrm{id}_{\xi^{n-j-1}}) = \tilde{m}_n,$$

so the diagrams commute. Composing several of those $\mathrm{id}_{\xi^{j-1}} \times \widetilde{\mathrm{flip}} \times \mathrm{id}_{\xi^{n-j-1}}$, we can cover $\mathrm{flip}^n_{i,k}$ by a morphism $\widetilde{\mathrm{flip}}^n_{i,k} = [\mathrm{id}_{\xi_0^n}, \psi^n_{i,k}]$: $\xi_0^n \to \xi_0^n$, such that $\tilde{m}_n \circ \widetilde{\mathrm{flip}}^n_{i,k} = \tilde{m}_n$.

Using arguments almost identical to those of the discussion following 3.4 for $\tilde{m}_n \circ \widetilde{\mathrm{flip}}^n_{i,k} = \tilde{m}_n$ instead of $\tilde{m} \circ \widetilde{\mathrm{flip}} = \tilde{m}$ yields $\pi_i \circ \varphi_x = \pi'_k \circ \varphi'_x$ and clearly $\pi_k \circ \varphi_x = \pi_i \circ \varphi_x$ if $p_i x = p_k x$. ■

In the next proposition and lemmas we investigate the connection between the extendability and fullness of compatible sections and selections. For technical reasons, we will have to restrict the base semigroups to a class closely resembling the representation semigroup S_A of a C^*-algebra A introduced in 2.5.

3.8. Definition. An abelian locally compact Hausdorff cancellation semigroup (X, m), which is generated by the set $\mathrm{Irr}(X)$ of its irreducible elements is called a *representation semigroup* if there is a continuous semigroup homomorphism d: $X \to N$ such that for each $n \in N$, $d^{-1}(n)$ is compact in X. We call d the *dimension function* of X and say that *d is bounded* if $d|\mathrm{Irr}(X)$ is bounded.

A continuous semigroup homomorphism φ between two representation semigroups is called a *representation semigroup map* if it preserves the dimension functions. ■

3.9. Example. If A is a C^*-algebra, then S_A is a representation semigroup in the sense of 3.8 with dimension function dim, and for another C^*-algebra B and a C^*-homomorphism φ: $A \to B$, S_φ is a representation semigroup map. If $A \in \mathbf{BD}$, then the dimension function of S_A is bounded. ■

3.10. Proposition. *Let $\xi = (\xi_0, \tilde{m})$ be a C^*-semigroup bundle over the representation semigroup (X, m) with dimension function d.*

Any compatible selection s which is continuous over $\overline{\operatorname{Irr}(X)}$ is continuous over all of X.

Proof. Denote by $Y_k = \operatorname{Irr}(X) \cap d^{-1}\{1, \ldots, k\}$ and by $Y_{k,n}$ the image of $(Y_k)^n$ under addition, so $Y_{k,n} = \underbrace{Y_k + \;\ldots\; + Y_k}_{n\text{-terms}} \subseteq X$. As a closed subset of a finite union of compact Hausdorff spaces, $\bar{Y}_k$ is compact Hausdorff, so that $m_n\colon (\bar{Y}_k)^n \to X$, $(x_1, \ldots, x_n) \mapsto x_1 + \ldots + x_n$, is closed and the image of $(\bar{Y}_k)^n$ is $\bar{Y}_{k,n}$ since $Y_{k,n} \subseteq m_n(\bar{Y}_k^n) \subseteq \bar{Y}_{k,n}$.

Remarks. (1) Suppose $x \in \bar{Y}_{k,n}$, $x = x_1 + \ldots + x_n$, where $x_i \in Y_k$, then $dx = dx_1 + \ldots + dx_n \leqslant n \cdot k$, so that

$$\bar{Y}_{k,n} \subseteq \bigcup_{r \leqslant n \cdot k} d^{-1}(r).$$

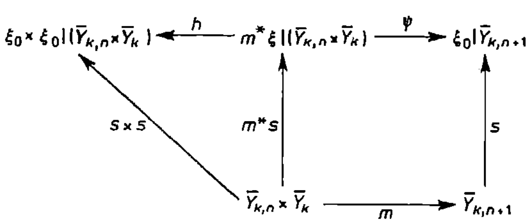
(2) Suppose $dx = r$. Then $x = x_1 + \ldots + x_n$, where $x_i \in \operatorname{Irr}(X)$, and with $k = \max\{dx_i\colon 1 \leqslant i \leqslant n\}$, we have for each i: $x_i \in Y_k$, so that $x \in Y_{k,n}$. As $r = dx_1 + \ldots + dx_n \leqslant k \cdot n \leqslant r^2$:

$$d^{-1}(r) \subseteq \bigcup_{\substack{(k,n) \\ r \leqslant k \cdot n \leqslant r^2}} \bar{Y}_{k,n}. \tag{3.10.1}$$

Note that $X = \bigcup_{(k,n) \in N^2} \bar{Y}_{k,n}$ and $\overline{\operatorname{Irr}(X)} = \bigcup_{k \in N} \bar{Y}_{k,1} = \bigcup_{k \in N} \bar{Y}_k$. Since the $d^{-1}(r)$ are open in X and $X = \bigcup_{r \in N} d^{-1}(r)$, we only need to show that a compatible selection s, which is continuous over $\operatorname{Irr}(X)$ is continuous over $d^{-1}(r)$ for each $r \in N$.

Using (3.10.1), we will be done if we can show that s is continuous over each of the $\bar{Y}_{k,n}$, because a function is continuous on a finite union of closed sets if it is continuous on each of those closed sets.

Now, if $n = 1$, then $\bar{Y}_{k,1} = \bar{Y}_k \subseteq \overline{\operatorname{Irr}(X)}$ and s is continuous by assumption. Suppose that s is continuous on $\bar{Y}_{k,n}$. Consider the following diagram:



with the obvious restrictions of maps and $[h, \psi] = \tilde{m}$. $s \times s$ is continuous on $\bar{Y}_{k,n} \times \bar{Y}_k$ by induction hypothesis. As $s \times s = h \circ (m^* s)$ by compatibility, $(s \times s)(\bar{Y}_{k,n} \times \bar{Y}_k) \subseteq \operatorname{Im}(h)$, and h being open onto its image implies that $m^* s|(\bar{Y}_{k,n} \times \bar{Y}_k)$ is continuous and hence $(\psi \circ m^* s)|(\bar{Y}_{k,n} \times \bar{Y}_k)$. But m is a closed map and $m(\overline{Y}_{k,n}\times \overline{Y}_k)=\overline{Y}_{k,n+1}$ – that is, the restriction of m to $\overline{Y}_{k,n}\times \overline{Y}_k$ and co-restriction to $\overline{Y_{k,n+1}}$ yields a surjection – so that $s|\overline{Y}_{k,n+1}$ is continuous. ■

3.11. COROLLARY. *Let (X, m) be a representation semigroup, ξ a C^*-semigroup bundle over X and i: $\overline{\mathrm{Irr}(X)}\to X$ the inclusion.*

Then the restriction map i^*: $\Gamma(\xi)\to\Gamma(\xi|\overline{\mathrm{Irr}(X)})$ is an isomorphism of involutive algebras and its restriction i^*: $\Gamma_b(\xi)\to\Gamma_b(\xi|\overline{\mathrm{Irr}(X)})$ is an isomorphism of C^*-algebras.

Proof. By definition of compatibility over $\overline{\mathrm{Irr}(X)}$, every $s\in\Gamma(\xi|\overline{\mathrm{Irr}(X)})$ extends to a compatible selection, which is continuous by 3.10 and bounded if s is. Thus the restriction maps are surjective. Injectivity follows from 3.2. ■

The next lemmas and propositions investigate the existence of "enough" compatible sections. We assume until Definition 3.15 that ξ is a C^*-semigroup bundle over the representation semigroup (X, m) with dimension function d, such that

(a) every selection over $Y=\mathrm{Irr}(X)$ extends to a (necessarily unique) compatible selection and

(b) compatible selections are full.

We use the notation of 3.10 and denote in addition by Y_0 the empty set and for $n\in N$ by X_n the semigroup generated by Y_n. Furthermore, let n_0 be the smallest integer for which $Y_n\neq\emptyset$, i.e. $n_0=\min\{n\in N\colon Y_n\neq\emptyset\}$.

3.12. LEMMA. *Let $C\subseteq Y$ be a closed subset, $k\in N_0=N\cup\{0\}$ and $\tilde{s}\in\Gamma_b(\xi|C\cup\overline{Y}_k)$ a compatible section. Then there exists for each $n\geqslant k$ a compatible section $s_n\in\Gamma_b(\xi|C\cup\overline{Y}_n)$ such that $s_k=\tilde{s}$ and*

(i) $s_n|C\cup\overline{Y}_i\equiv s_i$ *for all* $i=k, k+1,\dots, n-1$;

(ii) $\|s_n\|=\|\tilde{s}\|$.

Proof. First observe that Y is open in X by (3.10.1) so that $C\subseteq\mathrm{int}(\overline{Y})$.

Now define for $n\in N$ the set $C_n=\{x\in C\colon dx=n\}$. Clearly, C_n is compact Hausdorff and contained in $\mathrm{int}(\overline{Y}_n\cap d^{-1}n)$. There are two cases:

(1) $k<n_0$,

(2) $k\geqslant n_0$.

In the first case, $Y_k=\emptyset=C_k$ and for $i=k, k+1,\dots, n_0-1$, $C\cup\overline{Y}_i=C$. We define for $k\leqslant i\leqslant n_0-1$, $s_i=\tilde{s}$. Thus we have to find s_n now for $n\geqslant n_1=\max\{k, n_0\}$ in both cases and proceed by induction. Let $n=n_1$. If $n_1=k$, define $s_{n_1}=\tilde{s}$. If $n_1=n_0>k$, let $r=\tilde{s}|C_{n_0}$ if $C_{n_0}\neq\emptyset$ and $r\equiv 0\in\Gamma(\xi_0|\overline{Y}_{n_0})$ if $C_{n_0}=\emptyset$. By [DG; 83: pg. 11], r extends to a section r' over all of $\overline{Y}_{n_0}=Y_{n_0}$ in such a way that $\|r'\|=\|r\|$. Now set

$$s_{n_0}(x)=\begin{cases} r'(x) & \text{if } x\in\overline{Y}_{n_0},\\ \tilde{s}(x) & \text{if } x\in C\setminus\overline{Y}_{n_0}.\end{cases}$$

Then $s_{n_0}|C\cup \bar{Y}_k = s_{n_0}|C = \tilde{s} = s_k$ as $\bar{Y}_k = \emptyset$ in this case and $\|s_{n_0}\| = \|\tilde{s}\|$. The selection t defined by

$$t(x) = \begin{cases} s_{n_0}(x) & \text{if } x \in C \cup Y_{n_0} = C \cup \bar{Y}_{n_0}, \\ 0 & \text{if } x \in Y \backslash (C \cup Y_{n_0}). \end{cases}$$

extends to a compatible selection by assumption since $C \cup \bar{Y}_{n_0} \subseteq Y = \operatorname{Irr}(X)$ and s_{n_0} is thus compatible.

Now assume that s_n is defined for $n \geqslant n_1$. If $Y_{n+1} \cap d^{-1}(n+1) = \emptyset$, define $s_{n+1} \equiv s_n$. Otherwise let $t \in \Gamma_b(\xi|X_n)$ such that $t|\bar{Y}_n = s_n|\bar{Y}_n$. This t exists by 3.11 applied to $\xi|X_n$. Let

$$D = X_n \cap (\bar{Y}_{n+1} \cap d^{-1}(n+1)) = \{y \in \bar{Y}_{n+1} \cap d^{-1}(n+1)\colon\ y \notin Y = \operatorname{Irr}(X)\}.$$

Because of (3.10.1), Y is open in X and thus D is closed. $t|D$ is continuous and can thus be extended to a section t' over $\bar{Y}_{n+1} \cap d^{-1}(n+1)$ such that $\|t'\| = \|t|D\| \leqslant \|s_n\|$, using the fact that $\bar{Y}_{n+1} \cap d^{-1}(n+1)$ is paracompact and Hausdorff.

For the same reason we can extend $r = \tilde{s}|C_{n+1}$ to a section r' over $\bar{Y}_{n+1} \cap d^{-1}(n+1)$, such that $\|r'\| = \|r\| \leqslant \|\tilde{s}\|$ if $C_{n+1} \neq \emptyset$ (else we let $r' \equiv 0$).

Using normality of $\bar{Y}_{n+1} \cap d^{-1}(n+1)$ we can find disjoint neigborhoods U of D and V of C and functions $f, g\colon \bar{Y}_{n+1} \cap d^{-1}(n+1) \to [0, 1]$, such that $f|D \equiv 1$, $g|C \equiv 1$ and for $x \notin U$, $f(x) = 0$ and for $x \notin V$, $g(x) = 0$. Define

$$s_{n+1}(x) = \begin{cases} f(x)t'(x) + g(x)r'(x) & \text{if } x \in \bar{Y}_{n+1} \cap d^{-1}(n+1), \\ s_n(x) & \text{if } x \in (C \cup \bar{Y}_n) \backslash (\bar{Y}_{n+1} \cap d^{-1}(n+1)). \end{cases}$$

s_{n+1} is clearly continuous, since $\bar{Y}_{n+1} \cap d^{-1}(n+1)$ and $(C \cup \bar{Y}_n) \backslash (\bar{Y}_{n+1} \cap d^{-1}(n+1))$ are both closed and disjoint, $\|s_{n+1}\| = \|\tilde{s}\|$ and $s_{n+1}|C \cup \bar{Y}_i = s_n|C \cup \bar{Y}_i = s_i$ for $i = k_1, \ldots, n$.

As in the case $n = n_1 = n_0 > k$, we see that s_{n+1} is compatible. ∎

3.13. Proposition. *Let $C \subset Y = \operatorname{Irr}(X)$ be closed and $k \in N_0$ and $\tilde{s} \in \Gamma_b(\xi|C \cup \bar{Y}_k)$. There is a section $s \in \Gamma_b(\xi)$, such that $s|C \cup \bar{Y}_k = \tilde{s}$ and $\|s\| = \|\tilde{s}\|$.*

Proof. For $x \in \bar{Y}_n$ define $s(x) = s_n(x)$, where s_n is an extension of $\tilde{s}$ as in 3.12. s is well defined by 3.12 (i) and continuous on $\bar{Y}$ since $s|\bar{Y}$ is continuous on each open and closed set $\bar{Y}_n \backslash \bar{Y}_{n-1} = \bar{Y} \cap d^{-1}(n)$. s is also compatible, since each $s_n|\bar{Y}_n$ extends to a compatible selection t_n of $\xi|X_n$, so that $t_n|X_{n-1} = t_{n-1}$, and the compatible selection t defined by $t(x) = t_n(x)$ for $x \in X_n$ extends s. Also, $\|s\| = \sup\{\|s_n\|\colon n \in N\} = \|\tilde{s}\|$ by 3.12 (ii).

Now extend s to all of X *via* 3.11 to complete the proof. ∎

3.14. Corollary. *Suppose that every selection over $\operatorname{Irr}(X)$ extends to a compatible selection and that compatible selections are full for ξ_0. Then $\Gamma_b(\xi)$ is full for ξ_0 and separates any two irreducible points of X over at least one of which the fibre is nonzero.*

Proof. Let $x \in X$ and $x = x_1 + \ldots + x_n$ be its unique decomposition into irreducibles and let t be any compatible selection. Let $C = \{x_1, \ldots, x_n\}$, $k = 0$ and $\tilde{s} = t|C \cup \bar{Y}_k$. By 3.13, there is an $s \in \Gamma_b(\xi)$ such that $s(x_i) = t(x_i)$ for $i = 1, \ldots, n$. By compatibility, $s(x) = t(x)$. As compatible selections are full, $\Gamma_b(\xi)$ is full for ξ_0.

If $x \neq y \in \mathrm{Irr}(X)$, $\xi_0(x) \neq 0$, pick any compatible selection t such that $t(x) \neq 0$ and $t(y) = 0$. Using 3.13 as before, we can find $s \in \Gamma_b(\xi)$ with $s(x) = t(x) \neq 0$ and $s(y) = 0$. ■

3.15. Definition. A C^*-semigroup bundle ξ such that $\Gamma_b(\xi)$ is full and separates any two irreducible points over at least one of which the fibre is nonzero is called a *full C^*-semigroup bundle.*

A C^*-semigroup bundle which has simple dual C^*-algebras as its fibres over $\mathrm{Irr}(X)$ is called a *simple dual C^*-semigroup bundle* or briefly *SC^*--semigroup bundle.* ■

3.16. Theorem. *Let $\xi = (\xi_0, \tilde{m})$ be a SC^*-semigroup bundle over (X, m). The following are equivalent:*

(i) *every selection over* $\mathrm{Irr}(X)$ *extends to a compatible selection and $\Pi\xi$ is full for ξ_0.*

(ii) *ξ is abelian and associative and for $x_1, x_2 \in X$, the maps*

$$\pi_i \circ \tilde{m}(x_1, x_2)\colon\ \xi_0(x_1 + x_2) \to \xi_0(x_i)$$

are surjective, where π_i is the i-th factor projection and $i = 1, 2$. Furthermore, for $x_1, x_2 \in \mathrm{Irr}(X)$

$$\mathrm{Ker}\, \pi_1 \circ \tilde{m}(x_1, x_2) = \mathrm{Ker}\, \pi_2 \circ \tilde{m}(x_1, x_2)$$

implies $x_1 = x_2$ or $\xi(x_1) = \xi(x_2) = 0$.

If, in addition, X is a representation semigroup, then (i) *and* (ii) *are equivalent to*

(iii) *ξ is full.*

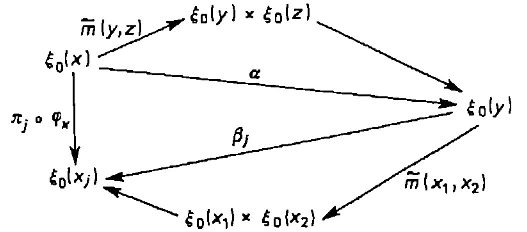
Proof. First we prove the implication "(i)$\Rightarrow$(ii)". By 3.5, ξ is abelian and associative. So let $x, y \in X$ and $v \in \xi(x)$. By (i), there is an $s \in \Pi\xi$ such that $s(x) = v$. Thus $\pi_1 \circ \tilde{m}(x, y)(s(x+y)) = \pi_1(s(x), s(y)) = s(x) = v$, or $\pi_1 \circ \tilde{m}(x, y)$ is surjective and so is $\pi_2 \circ \tilde{m}(x, y)$ by a similar argument.

Now assume $x, y \in \mathrm{Irr}(X)$, $x \neq y$ and without loss of generality, $\xi_0(x) \neq 0$. There is $v \in \xi_0(x)$, $v \neq 0$, and the selection over $\mathrm{Irr}(X)$, which sends x to v and everything else to 0, extends to some $s \in \Pi\xi$. Then $\pi_1 \circ \tilde{m}(x, y)(s(x+y)) = v \neq 0$ and $\pi_2 \circ \tilde{m}(x, y)(s(x+y)) = s(y) = 0$ and the maps have different kernels.

To prove the implication (ii)$\Rightarrow$(i), let $s \in \Pi\xi_0|\mathrm{Irr}(X)$, $x \in X$ and $x = x_1 + \ldots + x_n$ its decomposition into irreducibles. For $k = 1, \ldots, n$, the fibres $\xi_0(x_k)$ are simple dual by assumption and we can consider the maps $\pi_k \circ \varphi_x$: $\xi_0(x) \to \xi_0(x_k)$ as representations of $\xi_0(x)$. (Here π_k: $\xi_0(x_1) \times \ldots \times \xi_0(x_n) \to \xi_0(x_k)$ is again the kth-factor projection and $\varphi_x = \tilde{m}_n(x_1, \ldots, x_n)$.)

By assumption and associativity of ξ, $\pi_1 \circ \varphi_x$ is surjective and thus, applying 3.7, $\pi_k \circ \varphi_x$ is surjective for $k = 1, \ldots, n$ and can thus be considered an irreducible representation.

Now assume $\pi_i \circ \varphi_x$ is equivalent to $\pi_k \circ \varphi_x$. Then $I = \operatorname{Ker} \pi_i \circ \varphi_x = \operatorname{Ker} \pi_k \circ \varphi_x$. Again by 3.7, we assume without loss of generality that $i = 1$, $k = 2$, and $n \geqslant 3$. Set $y = x_1 + x_2$, $z = x_3 + \ldots + x_n$ and consider the commutative diagrams



were the unlabelled arrows represent the corresponding projections. Clearly $\alpha(I) \subseteq \operatorname{Ker} \beta_j$ for $j = 1, 2$. Now let $v \in \operatorname{Ker} \beta_j$. As α is surjective, there is $w \in \xi_0(x)$ such that $\alpha(w) = v$. But then

$$\pi_j \circ \varphi_x(w) = \beta_j(\alpha(w)) = \beta_j(v) = 0.$$

Thus $w \in I$ or $\operatorname{Ker} \beta_j \subseteq \alpha(I)$. Hence $\operatorname{Ker} \beta_1 = \alpha(I) = \operatorname{Ker} \beta_2$. As x_1 and x_2 are irreducible, $x_1 = x_2$ and hence $\pi_1 \circ \varphi_x = \pi_2 \circ \varphi_x$. Therefore, any two of the $\pi_j \circ \varphi_x$, $j = 1, \ldots, n$, are as representations either inequivalent or equal.

Given a selection s over $\operatorname{Irr}(X)$, we can apply [Di; 69: 4.2.5] to see the existence of an $s(x) \in \xi_0(x)$, such that for $j = 1, \ldots, n$, $\pi_j \circ \varphi_x(s(x)) = s(x_j)$. Furthermore, $s(x)$ is unique by the injectivity of φ_x and since ξ is abelian and associative, we get a well defined extension of s, which is compatible by construction.

Now take any $v \in \xi_0(x)$. Define a selection s over $\operatorname{Irr}(X)$ by setting $s(x_j) = \pi_j \circ \varphi_x(v)$ for $j = 1, \ldots, n$ and $s(y) = 0$ for $y \notin \{x_1, \ldots, x_n\}$. Extend s to a compatible selection. Obviously

$$\varphi_x(s(x)) = (s(x_1), \ldots, s(x_n)) = \varphi_x(v)$$

and as φ_x is injective, $v = s(x)$. Thus compatible selections are also full.

Finally, assume that X is in addition a representation semigroup. "(i)$\Rightarrow$(iii)" follows from 3.13 and 3.14, so we prove "(iii)$\Rightarrow$(ii)":

As compatible sections are full, 3.5 implies that ξ is abelian and associative. The surjectivity of the maps $\pi_i \circ \tilde{m}(x, y)$ follows as in the proof of "(i)$\Rightarrow$(ii)".

Let us assume again that $x, y \in \operatorname{Irr}(X)$, $x \neq y$ and $\xi_0(x) \neq 0$. Substituting "selection" by "section" in "(i)$\Rightarrow$(ii)", we find that $\pi_1 \circ \tilde{m}(x, y)$ and $\pi_2 \circ \tilde{m}(x, y)$ have different kernels. ■

The following example illustrates condition (ii) of 3.16.

3.17. Example. Let (X, m) be the semigroup $N \times N$ with coordinatewise addition. The dimension function $N \times N \to N$, $(n, m) \mapsto n+m$, gives $N \times N$ the structure of a representation semigroup. Let $F = M_2(C)$ and denote by ξ_0 the (trivial) bundle over $N \times N$ with fibre F. Define $h\colon m^*\xi_0 \to \xi_0 \times \xi_0$ in the following way: For (n, m), $(n', m') \in N \times N$ and $v \in \xi_0(n+n', m+m')$, let $h(n, m, n', m', v) = ((n, m, v), (n', m', v))$. Then h is injective and defines the structure of an SC^*-semigroup bundle on ξ_0, which is obviously associative and abelian. $N \times N$ is generated by the irreducible elements $(1, 0)$ and $(0, 1)$. The section s over $\mathrm{Irr}(N \times N)$ such that $s(1, 0) = (1, 0, v)$ and $s(0, 1) = (0, 1, 0)$ for some $v \neq 0$ clearly does not extend to a compatible section, because

$$\begin{aligned}((1, 0, w), (0, 1, w)) &= \tilde{m}((1, 0), (0, 1))(s(1, 1)) \\ &= (s(1, 0), s(0, 1)) = ((1, 0, v), (0, 1, 0))\end{aligned}$$

from which we conclude that $v = w = 0$.

Notice that the surjectivity in (ii) of 3.16 is satisfied and that we could replace $M_2(C)$ by the compact operators $K(H)$ for any Hilbert space $H \neq 0$. ■

This finishes our investigation on the extendability of compatible sections.

4. Existence and "uniqueness" of representation semigroups and C^*-semigroup bundles

4.1. *Notation.* Let G be a set and X the free abelian cancellation semigroup generated by G.

Furthermore, let $Y_0 \subseteq X$ be a topological space containing G as a dense subset and let Y denote the set underlying Y_0.

Define a semigroup $Z = \sum_{k=1}^{\infty} Y_0^k$ with juxtaposition as continuous multiplication. Let $p\colon Z \to X$, $(x_1, \ldots, x_n) \mapsto x_1 + \ldots + x_n$ and endow X with the quotient topology from p. Let Y_1 be the set Y endowed with the topology induced by that quotient topology on X.

In what follows, we will omit all obvious restrictions and corestrictions of maps. To avoid confusion, let $q = p|p^{-1}(Y)\colon p^{-1}(Y) \to Y$ with respect to the induced topology of $p^{-1}(Y)$ in Z and Y with the topology of Y_0, and consider $p|p^{-1}(Y)$ as a map into Y with the topology of Y_1.

4.2. Proposition. *If $p^{-1}(Y)$ is closed in Z and q is continuous, then* $Y_1 = Y_0 = \bar{G}$.

Proof. As $p^{-1}(Y_0) = p^{-1}(Y_1)$ is closed in Z, Y_1 is closed in X and $p\colon p^{-1}(Y_1) \to Y_1$ is a quotient map. Since q is continuous, the topology of Y_1 is finer than the topology of Y_0 (i.e. has more open sets).

Now let $V \subseteq Y_1$ be open. Then $p^{-1}(V) = q^{-1}(V)$ is open in $p^{-1}(Y)$ and thus $q^{-1}(V) \cap Y_0^1$ is open in Z. But Y_0 and Y_0^1 are homeomorphic and $V \subseteq Y_0$

corresponds to $q^{-1}(V)\cap Y_0^1$ and is open. The topology of Y_0 is hence finer than the topology of Y_1 and the topologies agree.

As $G \subseteq Y_1 = Y_0$ is dense and Y_1 is closed in X, $G = Y_1$. ∎

In what follows, we assume that $p^{-1}(Y)$ is closed and q is continuous. We can thus, by 4.2, drop the subscripts and write $Y = Y_0 = Y_1$.

Now assume, in addition, that there is a semigroup homomorphism $d\colon X \to N$ such that $d|Y$ is continuous and for all $n \in N$, $d^{-1}(n)\cap Y$ is compact Hausdorff. It is easy to show that

$$Y^n \cap p^{-1}d^{-1}(k) = \bigcup_{l_1+\ldots+l_n=k}\prod_{i=1}^{n}(Y\cap d^{-1}(l_i)),$$

which is compact Hausdorff and open as a finite union of disjoint open compact Hausdorff subspaces of Z. Furthermore, $Y^n \cap p^{-1}d^{-1}(k) = \emptyset$ for all $n > k$, so that

$$p^{-1}d^{-1}(k) = \sum_{n=1}^{k} Y^n \cap p^{-1}d^{-1}(k)$$

is open and compact Hausdorff. Hence, $d^{-1}(k)$ is open, closed and compact in X, so that $p\colon p^{-1}d^{-1}(k) \to d^{-1}(k)$ is a quotient map.

Call a topological space T_4, if its points are closed and closed sets possess disjoint neighborhoods. By [Qu; 76: pg. 73], the quotient space of a T_4-space is T_4, if the quotient map is closed. X is therefore Hausdorff if, for each $k \in N$, $p|p^{-1}d^{-1}(k)$ is a closed map, because T_4 clearly implies Hausdorff and compact Hausdorff spaces are T_4. So we show that $p(B)$ is closed for all closed $B \subseteq p^{-1}d^{-1}(k)$.

We assume without loss of generality

$$B \subseteq \prod_{i=1}^{n}(Y\cap d^{-1}(l_i)) \subseteq Y^n \subseteq Z, \qquad \text{where } l_1+\ldots+l_n = k.$$

Let $f\colon p^{-1}(Y) \to Y^1 \subseteq Z$ be the map that assigns to each $x \in p^{-1}(Y)$ the unique $y \in Y^1$ such that $px = py$. f is clearly continuous as $p|Y^1\colon Y^1 \to Y$ is a homeomorphism. Then $(f^n)^{-1}(B) = (\underbrace{f\times \ldots \times f}_{n\text{-times}})^{-1}(B) \subseteq \sum_{r=n}^{k} Y^r$ is closed and $f^{-1}(Y^1) = p^{-1}(Y)$.

There is a natural action of $\mathbf{S} = S_n \times \ldots \times S_k$ on $\sum_{r=n}^{k} Y^r$, where S_i is the permutation group of i elements and $\sigma \in \mathbf{S}$ is a homeomorphism permuting the components of the elements of $\sum_{r=n}^{k} Y^r$. We claim:

4.3.

$$p^{-1}(p(B)) = \bigcup_{\sigma\in\mathbf{S}}\bigcup_{s=1}^{k} f^s\big[\sigma\big(((f^n)^{-1}(B))\cap(f^{-1}(Y^1))^s\big)\big].$$

Proof. First observe that for all $s \in N, \sigma \in S$

(4.3.1) $$p \circ f^s \circ \sigma = p.$$

Now suppose $(x_1, \ldots, x_l) \in p^{-1}(p(B))$. There is a $y \in B$ and irreducibles $z_1, \ldots, z_m$, such that

$$py = x_1 + \ldots + x_l = z_1 + \ldots + z_m.$$

Consequently, there is a $\sigma \in S$, so that $f^l \circ \sigma(z_1, \ldots, z_m) = (x_1, \ldots, x_l)$ and $f^n(z_1, \ldots, z_m) = y$.

Conversely, assume $(y_1, \ldots, y_s) \in (f^{-1}(Y^1))^s$, such that $f^n \circ \sigma(y_1, \ldots, y_s) \in B$. Then, by 4.3.1,

$$\begin{aligned} p \circ f^s(y_1, \ldots, y_s) &= p(y_1, \ldots, y_s) \\ &= p \circ f^n \circ \sigma(y_1, \ldots, y_s). \ \blacksquare \end{aligned}$$

As $\sigma\left[(f^n)^{-1}\left(\prod_{i=1}^{n} (Y \cap d^{-1}(l_i))\right)\right]$ is compact and Y^1 is Hausdorff, f^s is a closed map for each s. $p^{-1}(p(B))$ is therefore closed as a finite union of closed sets and $p(B)$ is closed in X. Thus, $d^{-1}(k)$ is compact Hausdorff for all $k \in N$ and $p\colon Z \to X$ is closed.

Now a closed continuous surjective map is a quotient map, $p \times p$ is closed – since its restriction to $(p \times p)^{-1}(d^{-1}(k) \times d^{-1}(l))$ is closed for each k and l – and commutativity of

$$\begin{array}{ccc} Z \times Z & \xrightarrow{\text{juxtaposition}} & Z \\ {\scriptstyle p \times p}\downarrow & & \downarrow{\scriptstyle p} \\ X \times X & \xrightarrow[+]{} & X \end{array}$$

yields the continuity of addition in X. We conclude that X is a representation semigroup.

If we endow X with another topology which induces the original topology on Y, in which $\bar{G} = Y$ and which makes it a representation semigroup – let us denote it by X_1 and by $p_1\colon Z \to X_1$ the map $(x_1, \ldots, x_k) \mapsto x_1 + \ldots + x_k$ – then p_1 is clearly continuous by continuity of addition, so that the topology of X_1 is coarser than the quotient topology on X. But then id: $d^{-1}(n) \subseteq X \to d^{-1}(n) \subseteq X_1$ is a continuous bijection between compact Hausdorff spaces and the two topologies are equal.

We summarize the results in the following:

4.4. Proposition. *Let G be a set, X the free abelian cancellation semigroup generated by G and $Y \subseteq X$ a topological space containing G as a dense subset. Assume also the existence of a semigroup homomorphism $d\colon X \to N$ such that $d|Y$*

is continuous and, for $n \in N$, $d^{-1}(n) \cap Y$ *is compact Hausdorff. Furthermore, with the notation of* 4.1, *assume that* $p^{-1}(Y)$ *is closed and* q *is continuous.*

Then there exists a unique topology on X *that makes* X *a representation semigroup, induces the original topology on* Y *and in which* $Y = \bar{G}$. ■

If X' is any representation semigroup and $G = \mathrm{Irr}(X')$, then we can identify X' and X as sets. Then let $Y = \bar{G}$. The conditions of 4.4 are clearly satisfied because of the continuity of addition and hence:

4.5. COROLLARY. *A representation semigroup is completely determined by the closure of the irreducibles.* ■

4.6. COROLLARY. *Let* X *be a representation semigroup,* $Y = \overline{\mathrm{Irr}(X)}$, S *an abelian topological semigroup and* $f\colon X \to S$ *a semigroup homomorphism such that* $f|Y$ *is continuous.*

Then f *is continuous.*

Proof. Let $G = \mathrm{Irr}(X)$, Z and p as in 4.1 and define $\bar{f}\colon Z \to S$ by $\bar{f}(y_1, \ldots, y_k) = f(y_1) + \ldots + f(y_k)$. As a composition of continuous maps, namely $f|Y$ to some power and addition in S, $\bar{f} = f \circ p$ is continuous. Since p is a quotient map, f is continuous too. ■

4.7. *Remarks.*

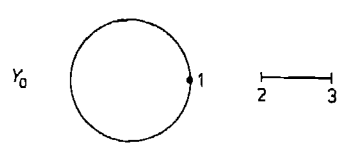
4.7.1. If the sets Y_0 and G in 4.1 agree, then the conditions of 4.2 are automatically satisfied and $X \cong \sum_{n=1}^{\infty} Y_0^n/S_n$, where S_n is the group of permutations of n elements, acting by permuting the coordinates.

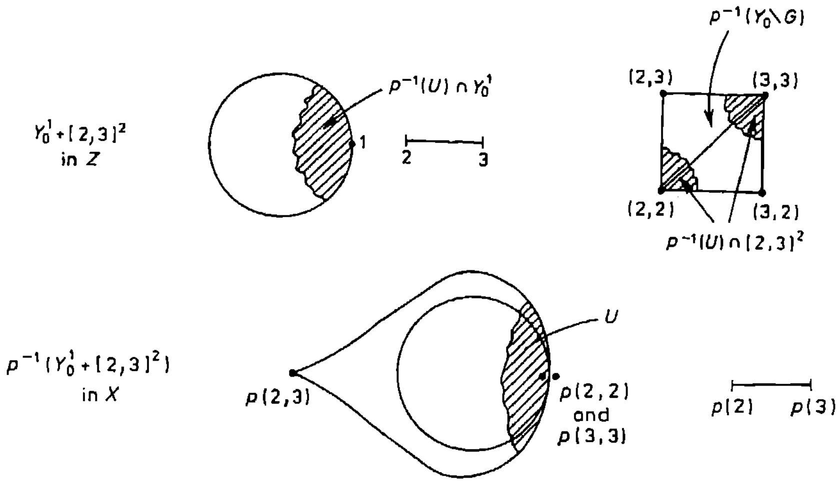
4.7.2. The condition that $p^{-1}(Y_0)$ be closed is clearly needed to make $Y_0 = \bar{G}$ and continuity of q says that "addition is continuous on Y_0 where defined".

The following examples illustrate what can happen if the conditions on p and q are not satisfied:

4.8. EXAMPLES (with the notation of 4.1).

4.8.1. In this example $p^{-1}(Y)$ will not be closed and $Y_0 \neq \bar{G}$, although q is continuous. So let $B_1 = \{z \in C\colon |z| < 1\}$, $G = B_1 \cup [2, 3]$ and $(X, \oplus)$ the free abelian cancellation semigroup generated by G. For $t \in [2, 3)$, we identify $t \oplus t \in X$ with $e^{2\pi i(t-2)}$ and let $Y_0 = \bar{B}_1 \cup [2, 3]$ with the topology induced by C. The map $d\colon Y_0 \to N$, $dz = 1$, if $z \in [2, 3]$ and $dz = 2$ if $z \in \bar{B}_1$ is continuous, $d^{-1}(m)$ is compact Hausdorff for $m = 1, 2$ and d extends to a semigroup homomorphism $d\colon X \to N$. The following pictures illustrate the situation:



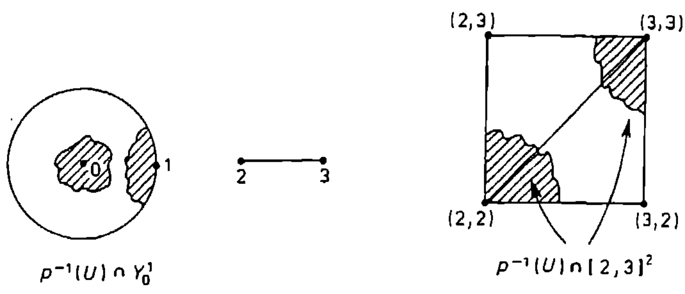


Here U is a typical neighborhood of $p(2, 2)$ and $p(3, 3)$. $p^{-1}(Y)$ is not closed, because it does not contain $(3, 3)$ and Y_0 is not the closure of G in X since $p(3, 3)\in\bar{G}$ in X. Moreover, X is not Hausdorff.

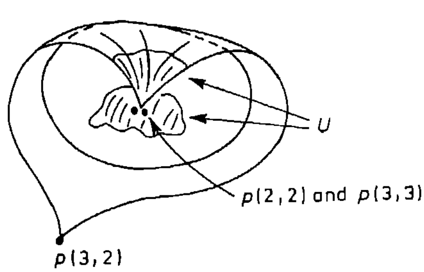
4.8.2. Now we construct another example, in which $p^{-1}(Y)$ is closed, but Y_0 and Y_1 are not homeomorphic, because q is not continuous.

So let $G = (B_1\setminus\{0\})\cup[2, 3]$ and let $(X, \oplus)$ be the free abelian cancellation semigroup generated by G, identify $t\oplus t\in X$ again with $e^{2\pi i(t-2)}$ for $t\in[2, 3)$ and identify $3\oplus 3$ with 0. Let $Y_0 = \bar{B}_1\cup[2, 3]$ with the topology inherited from C and define d: $X\to N$ as in 4.8.1. Again, a picture illustrates the situation:

$Y_0^1+[2, 3]^2\subseteq Z$



$p(Y_0^1+[2, 3]^2)\subseteq X$



Again, U is a typical neighborhood of $p(2, 2)$ and $p(3, 3)$. Although $p^{-1}(Y)$ is closed here, Y_0 and Y_1 are not homeomorphic, because $\lim_{n\to\infty} q(3-1/n, 3-1/n) = 1 \neq 0 = q(3, 3)$, i.e., because q is not continuous. ∎

Let us now consider the question of constructing a C^*-semigroup bundle over a representation semigroup X from a given C^*-bundle over $\overline{\mathrm{Irr}(X)}$.

So let η be a C^*-bundle over $Y = \overline{\mathrm{Irr}(X)}$, where X is a representation semigroup. Let Z and $p\colon Z \to X$ be as in 4.1 and ζ the C^*-bundle over Z such that for $k \in N$, $\zeta | Y^k = \eta^k$.

If we want to extend η to a C^*-semigroup bundle over X, then we certainly need a condition that reflects the "C^*-semigroup bundle structure over Y given by η". So we assume the existence of an injective bundle map $\alpha\colon p^*\eta \to \zeta | p^{-1}(Y)$. In addition, we assume that

$$A = \{a \in \Gamma_b(\eta)\colon \alpha_* p^* a | Y^k = \underbrace{a \times \ldots \times a}_{k\text{-times}} \text{ for all } k \in N\}$$

is full for η.

Define a semigroup homomorphism $f\colon X \to \mathrm{Id}(A)$, the ideal space of A, by setting

$$f(x) = \begin{cases} \{a \in A\colon a(x) = 0\} & \text{if } x \in Y, \\ f(x_1) \cap \ldots \cap f(x_n) & \text{if } x = x_1 + \ldots + x_n,\ x_i \in \mathrm{Irr}(X). \end{cases}$$

If $x = x_1 + \ldots + x_n \in Y$ and $a(x) = 0$, then $(a(x_1), \ldots, a(x_n)) = (\alpha_* p^* a)(x_1, \ldots, x_n) = 0$, or $a \in f(x_1) \cap \ldots \cap f(x_n)$. Conversely, if $a(x_1) = 0, \ldots, a(x_n) = 0$, then $(p^*a)(x_1, \ldots, x_n) = 0$, since α is injective and thus $a(x) = 0$ and $a \in f(x)$. We conclude that f is well defined.

Again, since $\|a + f(x)\| = \|a(x)\|$ for $x \in X$, $f | Y$ is continuous and by 4.6, f is continuous.

Let $\xi = f^*\eta(A)$, where $\eta(A)$ is the universal C^*-semigroup bundle of A. As A is full, $\eta(x) \cong A/f(x)$ for all $x \in Y$ and $f^*\tilde{a}$ is a continuous section of ξ_0. As in 2.4, $\xi_0 | Y \cong \eta$ and $A \subseteq \Gamma_b(\xi | Y)$, since $(f^*\tilde{a})(x) = (x, a + f(x))$ for $x \in X$, so that $f^*\tilde{a} | Y = a$ under the identification $\xi_0 | Y = \eta$.

On the other hand, if $\eta = \bar{\xi}_0 | Y$ for some full SC^*-semigroup bundle $\bar{\xi}$ over X with structure map $h - \bar{\xi}$ is abelian and associative by 3.5 — and if $\alpha\colon p^*\eta \to \zeta$ is defined by the maps $h_n\colon m_n^* \bar{\xi}_0 \to \bar{\xi}_0^n$, then clearly $\Gamma_b(\bar{\xi} | Y) \subseteq A$. By 3.16, $a | \mathrm{Irr}(X)$ extends to a compatible selection which agrees on Y with a for all $a \in A$, and with 3.10 we conclude that $a \in \Gamma_b(\bar{\xi} | Y)$. Thus, $A = \Gamma_b(\bar{\xi} | Y) \cong \Gamma_b(\bar{\xi})$ and by 2.4, $\bar{\xi} \cong f^*\eta(A)$. We summarize this in

4.9. Proposition. *Let X be a representation semigroup, $Y = \overline{\mathrm{Irr}(X)}$, p, Z as in* 4.1 *and η and ζ as before. Assume the existence of an injective map $\alpha\colon p^*\eta \to \zeta$ covering the inclusion $p^{-1}(Y) \subsetneq Z$, such that*

$$A = \{a \in \Gamma_b(\eta)\colon \textit{for all } k \in N\colon \alpha_* p^* a | Y^k = \underbrace{a \times \ldots \times a}_{k\text{-times}}\}$$

is full for η.

Then there is a full C^*-semigroup bundle ξ over X with injective structure map, so that $\xi_0 | Y \cong \eta$ and under this identification $A \subseteq \Gamma_b(\xi | Y)$.

Conversely, if $\eta = \bar{\xi}_0 | Y$ for some full SC^*-semigroup bundle $\bar{\xi}$ and α is defined by the structure map, then $A \cong \Gamma_b(\bar{\xi})$ and $\bar{\xi} \cong \xi$. ∎

Proposition 4.9 allows us now to construct more examples and counterexamples of C^*-semigroup bundles by just specifying the restriction of the C^*-semigroup bundle to the closure of the irreducibles.

5. Duality between certain C^*-semigroup bundles and certain C^*-algebras

Before we begin our investigations of the duality between certain C^*-semigroup bundles and certain C^*-algebras, let us begin with generalizing 2.2.

5.1. PROPOSITION. *Let (X, m) be a representation semigroup and ξ a C^*-semigroup bundle over X and $i\colon \overline{\mathrm{Irr}(X)} \hookrightarrow X$ the inclusion. Suppose that the dimension function d is bounded — i.e. $\overline{\mathrm{Irr}(X)}$ is compact — and that $A \subseteq \Gamma_b(\xi)$ is a C^*-subalgebra which is full for ξ_0. Then*

$$A = \Gamma_b(\xi).$$

Proof. Let $(x, y) \in X \times X$ and pick $s \in \Gamma_b(\xi)$. By assumption, there is $a \in A$ with $a(x+y) = s(x+y)$. As in the proof of 2.2, it follows that $a(x) = s(x)$ and $a(y) = s(y)$, or

$$A | \{x, y\} = \Gamma_b(\xi) | \{x, y\}$$

for all $x, y \in X$. As $Y = \overline{\mathrm{Irr}(X)}$ is compact Hausdorff and by 3.11 $\Gamma_b(\xi) \cong \Gamma_b(\xi | Y)$ — hence also $A \cong A | Y$ — an application of the Stone–Weierstrass–Glimm Theorem for bundles [DG; 83: 2.18] finishes the proof. ∎

In Example 2.5 we constructed for an arbitrary C^*-algebra A its representation semigroup S_A and the pullback $\xi(A)$ of the universal C^*-semigroup bundle $\eta(A)$ via the kernel map. With this notation, we obtain as a corollary of 5.1:

5.2. THEOREM. *If $A \in$ **BD**, then $A \cong \Gamma_b(\xi(A))$.*

Proof. As in 2.5, $A \cong \mathrm{Ker}^* \Gamma(\eta(A)) \subseteq \Gamma_b(\xi(A))$ and we identify A with the C^*-subalgebra $\mathrm{Ker}^* \Gamma(\eta(A))$ which is full for $\xi(A)_0$. The dimension function of S_A is bounded, since $A \in$ **BD**. An application of 5.1 yields $A \cong \Gamma_b(\xi(A))$. ∎

5.2 gives us the first part of the duality theory, namely that for algebras in **BD**, the process

BD-algebras $\to$ full SC^*-semigroup bundle
$\to$ algebra of bounded compatible sections

gets us back to the original algebra. The opposite part of the duality, namely the problem, if the process

$$\text{full } SC^*\text{-semigroup bundle} \to \text{algebra of bounded compatible sections} \to \text{full } SC^*\text{-semigroup bundle}$$

yields equivalent SC^*-semigroup bundles, is more difficult to solve. In fact, it is quite easy to construct counterexamples for this level of generality. Obviously, we need restrictions on the fibers, but it turns out that there are also topological obstructions for the bundle. So let us restrict our attention to full SC^*-semigroup bundles over representation semigroups satisfying the following conditions:

5.3.1. There is exactly one irreducible element x_0 in the base semigroup over which the fibre is equal to 0 and the value of the dimension function at x_0 is 1.

5.3.2. The fibre over any irreducible element $x \neq x_0$ of the base semigroup is $M_n(C)$, where n is the value of the dimension function at x.

Now let ξ be a C^*-semigroup bundle satisfying 5.3.1 and 5.3.2 over the representation semigroup X, h the structure map, d the dimension function and $x_0 \in \mathrm{Irr}(X)$, such that $\xi_0(x_0) = 0$ and $dx_0 = 1$. For each $x = x_1 + \ldots + x_n$, $x_i \in \mathrm{Irr}(X)$ and possibly equal to x_0, the map

$$\xi_0(x) \xrightarrow{\tilde{m}_n(x_1,\ldots,x_n)} \xi_0(x_1) \times \ldots \times \xi_0(x_n) \to M_{dx}(C),$$

$$(a_1, \ldots, a_n) \mapsto \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix}$$

defines a representation π_x of $\xi_0(x)$ of dimension dx. We define a map $f\colon X \to S_A$, where $A = \Gamma_b(\xi)$, by

5.3.3. $f(x) = [\pi_x \circ \mathrm{ev}_x]$.

5.4. Definition. A full C^*-semigroup bundle ξ over X is called *proper* if it satisfies 5.3.1 and 5.3.2, and if $f\colon X \to S_A$ – as defined above – is continuous.

5.5. *Remarks*

5.5.1. Since we assumed that ξ is full, ξ is abelian and associative and hence f is well defined and a semigroup homomorphism.

5.5.2. The element x_0 in 5.3.1 corresponds to the one dimensional zero representation of $\Gamma_b(\xi)$ and is needed to allow the application of the duality theory to C^*-algebras without identity.

5.5.3. As ξ is full and x_0 is the only irreducible point x with $\xi_0(x) = 0$, $\Gamma_b(\xi)$ separates irreducible points so that $f|\mathrm{Irr}(X)$ is injective. But f carries irreducible points to irreducible representations and as both X and S_A are cancellation semigroups, f has to be injective. Note that the dimension of $f(x)$ is exactly dx.

5.6. LEMMA. *$f|\mathrm{Irr}(X)$ is continuous.*

Proof. Let $\{y_\alpha\}_{\alpha\in\mathfrak{A}}$ be a net in $\mathrm{Irr}(X)$ and $y = \lim y_\alpha$. Without loss of generality assume that for all $\alpha\in\mathfrak{A}$, $dy_\alpha = dy = n$. We have to show that $f(y_\alpha)\to f(y)$.

Case 1. $y \neq x_0$. As $\xi_0|\mathrm{Irr}(X)\cap(d^{-1}(n)\backslash\{x_0\})$ is homogeneous, there is a neighborhood U of y so that $\xi_0|U \cong U\times M_n(\boldsymbol{C})$ and we can consider $a|U$ as a function $U\to M_n(\boldsymbol{C})$. The assignment $x\mapsto a(x) = \mathrm{ev}_x(a)$ defines a continuous function $F\colon U\to \mathrm{Rep}_n(A)$ and denoting by $p_n\colon \mathrm{Rep}_n(A)\to\mathrm{Rep}_n(A)/U(n)$ the canonical projection $f|U = p_n\circ F$ is continuous.

Case 2. $y = x_0$. Then $\pi_y\mathrm{o}\,\mathrm{ev}_y(a) = 0$ for all $a\in A$, so $f(y)$ is the one dimensional zero representation. For $a\in A$ we can define $\|f(y_\alpha)(a)\|$ independently of the representant of $f(y_\alpha)$ and $\|f(y_\alpha)(a)\| = \|a(y_\alpha)\|\to 0$ by continuity of a, so that $\lim f(y_\alpha) = f(y)$ in S_A. ∎

The following example shows that f does not have to be continuous even if conditions 5.3.1 and 5.3.2 are satisfied.

5.6. EXAMPLE. Let $G = \{-1, -2\}\cup(0, 1]$ and let $(X, \oplus)$ be the free abelian semigroup generated by G and identify $(-1)\oplus(-1)$ with $0\in\boldsymbol{R}$. Now endow X with the structure of a representation semigroup via 4.4 given by $Y = \bar{G} = \{-1, -2\}\cup[0, 1]$ and the dimension function d defined by

$$d(t) = \begin{cases} 1 & \text{if } t\in\{-1, -2\}, \\ 2 & \text{if } t\in[0, 1]. \end{cases}$$

Now let

$$A = \{g = \begin{bmatrix} g_{11} g_{12} \\ g_{21} g_{22}\end{bmatrix} \in C([0, 1], M_2(\boldsymbol{C}))\colon\ g_{ij}(0) \neq 0 \Rightarrow i = j = 1\}$$

and η the bundle over $\bar{G}$ with fibre 0 over -2, $\boldsymbol{C}$ over -1 and $\eta|[0, 1]$ is the subbundle of the trivial bundle over $[0, 1]$ with fibre $M_2(\boldsymbol{C})$, defined by A. With $p\colon Z\to X$ as in 4.1, and ζ as in the discussion preceding 4.9, we get that $p^{-1}(\bar{G}) = \bar{G}^1 + \{(-1, -1)\}$. We define $\alpha|\bar{G}^1$ to be the identity and

$$\alpha(-1, -1)(\begin{bmatrix} \lambda & 0 \\ 0 & 0\end{bmatrix}) = ((-1, -1), (\lambda, \lambda)).$$

α is clearly injective and $A \cong \{s\in\Gamma_b(\eta)\colon\ \alpha_* p^* s|\bar{G}^1 = s$ and $(\alpha_* p^* s)\times(-1, -1) = (s(-1), s(-1))\}$ is full for η. We thus get a C^*-semigroup bundle ξ over X such that $\xi_0|\bar{G}\cong\eta$ and $A\subseteq\Gamma_b(\xi|\bar{G})$ by 4.9 and by 5.1, $A = \Gamma_b(\xi|\bar{G})$.

With the map $f\colon X \to S_A$ defined as before, we have $f(0) = f(-1) \oplus f(-1)$. Now let $t \in (0, 1]$. Then

$$\lim_{t\to 0} f(t) = f(-2) \oplus f(-1) \neq f(-1) \oplus f(-1),$$

since $f(-2)$ is the one dimensional zero representation and f is not continuous at $(-1) \oplus (-1) = 0$. ∎

This example indicates, however, that this "pathology" is due to "discrepancy" between the topology of the bundle and the topology of the base semigroup.

5.8. Proposition. *Let ξ be a C^*-bundle over X, $B = \Gamma_b(\xi)$ and $A \subseteq B$ a C^*-subalgebra which is full for ξ. Assume that there is a natural number n and for each $x \in X$ an n-dimensional representation π_x of $\xi(x)$. Define $f\colon X \to S_A$ and $g\colon X \to S_B$ by $g = [\pi_x \circ \mathrm{ev}_x]$ and $f = [\pi_x \circ \mathrm{ev}_x | A]$. Then*

f is continuous if and only if g is.

Proof. Let $i\colon A \hookrightarrow B$ be the inclusion and $r = S_i\colon S_B \to S_A$ the restriction map. r is continuous and $f = r \circ g$. Continuity of g implies thus continuity of f.

For the converse, we use the following fact:

5.8.1. If $p\colon Y \to Z$ is a quotient map, $\{x_\alpha\}_{\alpha \in \mathfrak{A}} \subseteq Z$ a net and $z \in Z$, then $z = \lim z_\alpha$ iff $\forall y \in p^{-1}(z)$ $\forall$ open neighborhoods U of y $\exists \alpha_0$ $\forall \alpha \geqslant \alpha_0$:

$$p^{-1}(z_\alpha) \cap U \neq \emptyset.$$

So let $\{x_\alpha\}_{\alpha \in \mathfrak{A}} \subseteq X$ be a net and $x = \lim x_\alpha$. Choose a representant $\pi_x \circ \mathrm{ev}_x$ of $g(x)$ and $\varepsilon_1, \ldots, \varepsilon_n > 0$ and $b_1, \ldots, b_n \in B$. Let $U = \{\varrho \in \mathrm{Rep}_n(A)\colon \|\varrho(b_i) - \pi_x(b_i(x))\| < \varepsilon_i, i = 1, \ldots, n\}$. Since A is full for ξ, there are $a_1, \ldots, a_n \in A$ with $a_i(x) = b_i(x)$, $i = 1, \ldots, n$. By 5.8.1, and since f is continuous, there exists α_0 and for all $\alpha \geqslant \alpha_0$ representants $\pi_{x_\alpha} \circ (\mathrm{ev}_{x_\alpha} | A)$ so that for all $\alpha \geqslant \alpha_0$, $i = 1, \ldots, n$

$$\|\pi_{x_\alpha}(a_i(x_\alpha)) - \pi_x(a_i(x))\| < \frac{\varepsilon_i}{2} \quad \text{and} \quad \|a_i(x_\alpha) - b_i(x_\alpha)\| < \frac{\varepsilon_i}{2}$$

(since $x \mapsto \|(a_i - b_i)(x)\|$ is continuous!).

Thus, for $\alpha \geqslant \alpha_0$ and $i = 1, \ldots, n$

$$\|\pi_{x_\alpha}(b_i(x_\alpha)) - \pi_x(b_i(x))\|$$
$$\leqslant \|\pi_{x_\alpha}(b_i(x_\alpha)) - \pi_{x_\alpha}(a_i(x_\alpha))\| + \|\pi_{x_\alpha}(a_i(x_\alpha)) - \pi_x(a_i(x))\| < \varepsilon_i$$

and by 5.8.1 again $g(x) = \lim g(x_\alpha)$ and g is continuous. ∎

5.9. Corollary. *Let ξ be a full SC^*-semigroup bundle over the representation semigroup X, satisfying* 5.3.1 *and* 5.3.2, *$A = \Gamma_b(\xi)$ and $B = \Gamma_b(\xi_0)$. Let $f\colon X \to S_A$ and $g\colon X \to S_B$ the maps defined through point evaluation. Then f is continuous if and only if g is.* ∎

5.9 establishes that the continuity of f depends (for fixed structure map and base semigroup) on the topology of ξ_0. In order to show the continuity of f, it suffices to show the continuity of $f|\overline{\mathrm{Irr}(X)}$ by 4.6 and hence only on the continuity of $g|\overline{\mathrm{Irr}(X)}$. Now $f|\mathrm{Irr}(X)$ and hence $g|\mathrm{Irr}(X)$ are already continuous. The next proposition will provide some condition that will enable us to deduce the continuity on $\overline{\mathrm{Irr}(X)}$ from that on $\mathrm{Irr}(X)$ under certain hypotheses:

5.10. PROPOSITION. *Let X, Y be Hausdorff spaces, Y regular and $A \subseteq X$ dense. Let $f\colon X \to Y$, so that $f|A$ is continuous and assume that for any $x \in X \backslash A$ and net $\{x_\alpha\}_{\alpha \in \mathfrak{A}} \subseteq A$, $x = \lim x_\alpha$ implies $f(x) = \lim f(x_\alpha)$. Then f is continuous.*

Proof. Let $x \in X \backslash A$ and W a neighborhood of $f(x)$. There is then, since Y is regular, a closed neighborhood V of $f(x)$ with $V \subseteq W$.

5.10.1. CLAIM. *There is an open neighborhood U of x so that $f(U \cap A) \subseteq V$.*

If not, then there were for any open neighborhood U of x an $x_u \in U \cap A$ with $f(x_u) \notin V$. The net $\{x_u\}$ would converge to x and thus by assumption $f(x_u)$ to $f(x)$. As $f(x) \in \mathrm{int}(V)$, this is impossible.

So let U be an open neighborhood of x with $f(U \cap A) \subseteq V$. For all $u \in U$, there is a net $\{u_\beta\}_{\beta \in \mathfrak{B}} \subseteq U \cap A$ with $u = \lim u_\beta$, so that $f(u) = \lim f(u_\beta) \in V = \bar{V}$. Thus $f(U) \subseteq V \subseteq W$ and f is continuous. ■

The proposition tells us that f is continuous if for each $x \in \overline{\mathrm{Irr}(X)} \backslash \mathrm{Irr}(X)$, the fibre $\xi_0(x)$ is "imbedded into $\xi_0|\overline{\mathrm{Irr}(X)}$ with the multiplicities of the irreducible representations in $f(x)$". To see this, let $\{x_\alpha\}_{\alpha \in \mathfrak{A}} \subseteq \mathrm{Irr}(X)$ be a net with $x = \lim x_\alpha$. Then $\{f(x_\alpha)\}_{\alpha \in \mathfrak{A}}$ has a convergent subnet whether f is continuous or not. So let us, without loss of generality, assume that $\pi = \lim f(x_\alpha)$. Then π and $f(x)$ must contain the same irreducible representations since they have the same kernel, or $\pi = f(x)$, iff the irreducible representations occur with the same multiplicities in π as in $f(x)$. But the multiplicities of π define the multiplicities of the embedding of $\xi_0(x)$ into $\xi_0|\overline{\mathrm{Irr}(X)}$ as defined by the cocycle of $\xi_0|\overline{\mathrm{Irr}(X)}$ (see [Du; 1], [Du; 2], and Chapter II.5.2).

5.11. THEOREM. *Let ξ be a proper C^*-semigroup bundle over X and $A = \Gamma_b(\xi)$. Then there is a strong invertible C^*-semigroup bundle map*

$$\tilde{\theta}\colon \xi \to \xi(A)|f(X)$$

covering f and $f\colon X \to f(X)$ is a representation semigroup isomorphism.

Proof. Let d be the dimension function of X. As f is continuous, as S_A is Hausdorff, and as for each $n \in N$, $d^{-1}(n)$ is compact, $f|d^{-1}(n)$ is a closed map. Because f is injective, $f|d^{-1}(n)$ is a homeomorphism onto its image, and since f preserves dimension functions, we conclude that f is a homeomorphism onto its image.

Now let $\varphi\colon X \to \mathrm{Id}(A)$ be the semigroup homomorphism such that $\xi = \varphi^*\eta(A)$ (2.4) and again let $\xi(A) = \mathrm{Ker}^*\,\eta(A)$, where Ker: $S_A \to \mathrm{Id}(A)$. Then

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & S_A \\ & \searrow^{\varphi} \quad \swarrow_{\mathrm{Ker}} & \\ & \mathrm{Id}(A) & \end{array}$$

commutes and

$$\xi \cong \varphi^*\eta(A) = (\mathrm{Ker} \circ f)^*\eta(A) \cong f^*\mathrm{Ker}^*\eta(A) = f^*\xi(A) = f^*\big(\xi(A) | f(X)\big).$$

So there exists a strong C^*-semigroup bundle map $\tilde{\theta}\colon \xi \to \xi(A) | f(X)$, which is invertible since f is a homeomorphism onto its image. (See Section 1.) ∎

5.12. COROLLARY. *If the dimension function d is bounded, then there is an invertible C*semigroup bundle map* $\tilde{\theta}\colon \xi \to \xi(A)$.

Proof. If d is bounded, then $\overline{\mathrm{Irr}(X)}$ is compact. Hence, every irreducible representation of $\Gamma_b\big(\xi_0 | \overline{\mathrm{Irr}(X)}\big)$ factors through a unique point evaluation, and so does every irreducible representation of $A \cong \Gamma_b\big(\xi | \overline{\mathrm{Irr}(X)}\big) \subseteq \Gamma_b\big(\xi_0 | \overline{\mathrm{Irr}(X)}\big)$. $f | \mathrm{Irr}(X)\colon \mathrm{Irr}(X) \to \mathrm{Irr}(S_A)$ – and hence f – is therefore surjective. An application of 5.11 finishes the proof. ∎

Now let ξ_1 and ξ_2 be C^*-semigroup bundles over X and $\tilde{\varphi}\colon \xi_1 \to \xi_2$ an isomorphism of C^*-semigroup bundles. Then $\tilde{\varphi}^*\colon \Gamma_b(\xi_2) \to \Gamma_b(\xi_1)$ is an isomorphism of C^*-algebras. 5.12 allows a converse conclusion for proper C^*-semigroup bundles. First, however, we want to make the following observation:

If $\varphi\colon A \to B$ is a C^*-algebra homomorphism, we can define $\mathrm{Id}(\varphi)\colon \mathrm{Id}(B) \to \mathrm{Id}(A)$, $I \to \varphi^{-1}(I)$, which is a continuous semigroup homomorphism and it can be checked easily that we can consider Id a contravariant functor from the category of C^*-algebras to the category of compact Hausdorff semigroups and their respective morphisms.

5.13. COROLLARY. *Let ξ_1 and ξ_2 be proper C^*-semigroups bundles over X, for $i = 1, 2$, $A_i = \Gamma_b(\xi_i)$ and $f_i\colon X \to S_{A_i}$ the maps as defined before. Suppose that $\varphi\colon A_1 \to A_2$ is an isomorphism of C^*-algebras and assume that the dimension function is bounded. Then there is a bicontinuous automorphism g of X such that*

$$\xi_2 \cong g^*\xi_1.$$

Proof. By functorial properties $\mathrm{Id}(\varphi)$ and S_φ, define isomorphisms of semigroups. It is also routine to check the commutativity of

$$\begin{array}{ccc} S_{A_2} & \xrightarrow{\ S_\varphi\ } & S_{A_1} \\ {\scriptstyle \mathrm{Ker}}\downarrow & & \downarrow{\scriptstyle \mathrm{Ker}} \\ \mathrm{Id}(A_2) & \xrightarrow[\mathrm{Id}(\varphi)]{} & \mathrm{Id}(A_1) \end{array}$$

and we define $g\colon X \to X$ by $g = f_1^{-1} \circ S_\varphi \circ f_2$, using that the f_i are isomorphisms.

By universality of the $\eta(A_i)$, 5.12 and definition of g,

$$\xi_2 \cong (\mathrm{Id}(\varphi) \circ \mathrm{Ker} \circ f_2)^* \eta(A_1)$$
$$\cong (\mathrm{Ker} \circ S_\varphi \circ f_2)^* \eta(A_1) = (\mathrm{Ker} \circ f_1 \circ g)^* \eta(A_1) \cong g^* \xi_1. \ \blacksquare$$

If $A, B \in \mathbf{BD}$ and $\varphi\colon A \to B$ is a C^*-map, we can define a C^*-semigroup bundle morphism $\xi(\varphi)\colon \xi(B) \to \xi(A)$, $\xi(\varphi) = [\theta, \psi]$, where ψ is the strong bundle map covering S_φ and $\theta\colon S_\varphi^* \xi(A) \to \xi(B)$ is defined by $\theta(\pi \circ \varphi, a + \mathrm{Ker}(\pi \circ \varphi)) = (\pi, \varphi(a) + \mathrm{Ker}\,\pi)$. Similarly, if ξ, ζ are C^*-semigroup bundles and $\tilde{\varphi}\colon \xi \to \zeta$ a C^*-semigroup morphism, we define a *C^*-map* $\Gamma_b(\tilde{\varphi})\colon \Gamma_b(\zeta) \to \Gamma_b(\xi)$ by $a \to \tilde{\varphi}^* a$. It is easy to check that with these definitions Γ_b and $\xi(\cdot)$ become contravariant functors and we have:

5.14. Duality Theorem. *There is a complete duality between the categories* **BD** *of* **BD**-*C^*-algebras and* **BPC*S** *of proper C^*-semigroup bundles over representation groups with bounded dimension functions, i.e. there are natural equivalences*

$$\mathrm{id}_{\mathbf{BD}} \to \Gamma_b \circ \xi(\cdot) \quad \textit{and}$$
$$\mathrm{id}_{\mathbf{BPC^*S}} \to \xi(\cdot) \circ \Gamma_b. \ \blacksquare$$

If we want to extend duality to proper C^*-semigroup bundles over a representation semigroup X with unbounded dimension function, then we have to find an appropriate C^*-subalgebra of the algebra of compatible sections. In Section 6 we will give examples of representation semigroups (6.2.4 and 6.2.5) which suggest that the concept of "compatible sections vanishing at ∞" – or better "compatible sections whose restriction to $\overline{\mathrm{Irr}(X)}$ vanishes at ∞" – may be inappropriate because only the zero section may satisfy this condition.

6. The core of a representation semigroup

In Sections 3 and 4 we discussed, to a certain extent, how far a C^*-semigroup bundle is determined by its restriction to the closure of the irreducible elements of the base semigroup. In this section, we discuss the restriction to an even smaller subset of the base semigroup.

For a representation semigroup X, with dimension function d, we define the set

$$\mathrm{Core}_0(X) = \{y \in \overline{\mathrm{Irr}(X)}\colon \forall x \in X\colon x + y \notin \overline{\mathrm{Irr}(X)}\}$$

and a relation $\leqslant$ on X by letting

$$a \leqslant b \quad \text{if either } \begin{cases} a = b & \text{or} \\ b = a + c & \text{for some } c \in X. \end{cases}$$

This relation is a partial order since X is a cancellation semigroup. For any $y \in \overline{\mathrm{Irr}(X)} \backslash \mathrm{Core}_0(X)$, let

$$M_y = \{x \in X\colon\ x+y \in \overline{\mathrm{Irr}(X)}\}.$$

The following lemma lists some properties of $\mathrm{Core}_0(X)$.

6.1. LEMMA. (i) $\mathrm{Core}_0(X) = \overline{\mathrm{Irr}(X)} \Leftrightarrow \overline{\mathrm{Irr}(X)} = \mathrm{Irr}(X)$;

(ii) $\mathrm{Core}_0(X) = \emptyset \Rightarrow d$ *is unbounded*;

(iii) *If* d *is bounded, then* M_y *has a maximal element* a *for every* $y \in \overline{\mathrm{Irr}(X)} \backslash \mathrm{Core}_0(X)$ *and* $y+a \in \mathrm{Core}_0(X)$.

(iv) *If every chain* $\mathbf{C}$ *in* $(M_y, \leqslant |M_y)$ *has a maximal element for* $y \in \overline{\mathrm{Irr}(X)} \backslash \mathrm{Core}_0(X)$, *then there is an* $a \in M_y$ *with* $y+a \in \mathrm{Core}_0(X)$.

(v) *If* d *is bounded, then* $\mathrm{Core}_0(X)$ *is open in* $\overline{\mathrm{Irr}(X)}$.

Proof. Let $Y = \mathrm{Irr}(X)$ and $Z = \mathrm{Core}_0(X)$.

(i) If $Z = \bar{Y}$ and $x_1 + \ldots + x_n \in \bar{Y}$, $x_i \in Y$, then $x_i \in \bar{Y} \backslash Z = \emptyset$, a contradiction. Hence, every $y \in \bar{Y}$ is irreducible.

If every $y \in \bar{Y}$ is irreducible, then, for every $x \in X$ and $y \in \bar{Y}$, $y+x \notin \bar{Y}$. Thus $y \in Z$ or $Z = \bar{Y}$.

(ii) Let $y \in Y$. Since $Z = \emptyset$, there is $x_1 \in X$ so that $z_1 = y + x_1 \in Y$. Again, there is $x_2 \in X$ so that $z_2 = z_1 + x_2 = y + x_1 + x_2 \in Y$. In this way we can define sequences $\{x_n\}_{n \in N}$ and $\{z_n\}_{n \in N} \subseteq Y$ so that $z_{n+1} = z_n + x_{n+1} \in Y$.

Then $dz_n = dy + dx_1 + \ldots + dx_n \geqslant n+1$ for all $n \in N$ and d is unbounded.

(iii) Let $y \in \bar{Y} \backslash Z$ and $\mathbf{C} \subseteq M_y$ a chain. As $y \in \bar{Y} \backslash Z$, $M_y \neq \emptyset$. Now for all $a \in \mathbf{C}$, $y + a \in \bar{Y}$. If $\mathbf{C}$ had no maximal element, then, for every $a \in \mathbf{C}$, there were a $c_a \in X$ with $y + a + c_a \in \bar{Y}$, or there were a sequence $x_1 < x_2 < \ldots$ in $\mathbf{C}$ with $y + x_n \in \bar{Y}$ and $d(y + x_n) = dy + dx_n \geqslant n+1$, so that d were unbounded. This is a contradiction and $\mathbf{C}$ has a maximal element. By Zorn's lemma, there is a maximal element $a \in M_y$.

Suppose $y + a \notin Z$. Then there is $x \in X$ with $y + a + x \in \bar{Y}$, so that $a + x \in M_y$, and $a + x > a$, which contradicts the maximality of a. Therefore $y + a \in Z$.

(iv) Repeat the last arguments of (iii).

(v) Suppose d is bounded. This is equivalent to $\bar{Y}$ being compact. Denote by m and π the maps

$$m\colon X \times X \to X, \quad (x, y) \mapsto x + y \quad \text{and}$$
$$\pi\colon X \times X \to X, \quad (x, y) \mapsto x.$$

Then we can write

$$\begin{aligned} Z &= \{y \in \bar{Y}\colon\ \forall x \in X\colon\ x + y \notin \bar{Y}\} \\ &= \bar{Y} \backslash \{y \in \bar{Y}\colon\ \exists x \in X\colon\ x + y \in \bar{Y}\} \\ &= \bar{Y} \backslash \pi(m^{-1}(\bar{Y})). \end{aligned}$$

Now, for any $(x, y) \in m^{-1}(\bar{Y})$, dx and $dy < \max\{dz\colon\ z \in \bar{Y}\}$, or $m^{-1}(\bar{Y})$ is compact. $\pi(m^{-1}(\bar{Y})) \cap \bar{Y}$ is thus closed and Z open in $\bar{Y}$. ∎

6.2. EXAMPLES.

6.2.1. Let $A = \{f \in C([0, 1], M_2(C))\colon f(0)$ and $f(1)$ are diagonal$\}$. A is isomorphic to the group C^*-algebra of the infinite dihedral group. Let $X = S_A$. $\mathrm{Core}_0(X)$ is homeomorphic to $[0, 1] + \{\text{point}\}$, where the isolated point corresponds to the one dimensional zero-representation.

6.2.2. Let $A = C^*(p4gm)$ (compare [Ra; 82], $\Omega = \{(\alpha, \beta) \in [0, \frac{1}{2}] \times [0, \frac{1}{2}]\colon \beta \leqslant \alpha\}$ the Brillouin zone of the group $p4gm$ and $X = S_A$. $\mathrm{Core}_0(X)$ is homeomorphic to $\Omega + \{\text{point}\}$ and the isolated point has the same meaning as in 6.2.1.

6.2.3. Let $G = N + \sum_{n=2}^{\infty} ([0, 1) \times \{n\})$, $(X, \oplus)$ the free abelian cancellation semigroup generated by G and identify $n \oplus 1$ with $(1, n) \in [0, 1] \times \{n\}$ for $n \geqslant 2$. Let $Y = N + \sum_{n=2}^{\infty} ([0, 1] \times \{n\})$ with the topology obtained from the usual topology on $[0, 1]$ and the discrete topology on N. Define a dimension function d by setting $d(1) = d(2) = 1$, $d(n) = n-1$ for $n \geqslant 2$ and $d|[0, 1] \times \{n\} \equiv n$. We use 4.4 to give X the topology of a representation semigroup so that $G = \mathrm{Irr}(X)$ and $Y = \bar{G}$.

Then $\mathrm{Core}_0(X) = \sum_{n=2}^{\infty} ([0, 1] \times \{n\})$, $\overline{\mathrm{Irr}(X)} = Y$, $Y \backslash \mathrm{Core}_0(X) = N$, and the sum of 1 and n in X is in $\mathrm{Core}_0(X)$ for every $n \geqslant 2$.

$M_1 = \{x \in X\colon x \oplus 1 \in \overline{\mathrm{Irr}(X)}\}$ and every chain in M_1 is of the form $\{n\}$ for some $n \geqslant 2$ and for $k \geqslant 2$, $M_k = \{1\}$. Thus, the hypothesis of 6.1 (iv) holds and the conclusion is true, despite the fact that d is not bounded.

6.2.4. Let G and X as in 6.2.3 and identify $1 \oplus 2 \oplus \ldots \oplus n$ with $(1, n) \in [0, 1] \times \{n\}$. Let $Y = N \cup \sum_{n=2}^{\infty} ([0, 1] \times \{n\})$ with the natural topology and define the dimension function by $d|N \equiv 1$ and $d|[0, 1] \times \{n\} \equiv n$. Again, use 4.4 to make X a representation semigroup with $Y = \bar{G} = \overline{\mathrm{Irr}(X)}$.

Then $\mathrm{Core}_0(X) = \sum_{n=2}^{\infty} ([0, 1) \times \{n\})$. For any $y = (1, n) \in [0, 1] \times \{n\}$, $y \notin \mathrm{Core}_0(X)$ and there is no $a \in M_y$ with $y + a \in \mathrm{Core}_0(X)$, because $M_y = \{(n+1) \oplus (n+2) \oplus \ldots \oplus (n+r)\colon r \in N\}$. For all $y \in \overline{\mathrm{Irr}(X)} \backslash \overline{\mathrm{Core}_0(X)}$ however, there is an $a \in X$ such that $y + a \in \overline{\mathrm{Core}_0(X)}$, as $\overline{\mathrm{Irr}(X)} \backslash \overline{\mathrm{Core}_0(X)} = N$.

6.2.5. Define G in the following way: for all $n \in N_0$, let $\Delta_n = \{(x_0, \ldots, x_n) \in R^{n+1}\colon \sum_{i=0}^{n} x_i = 1\}$ be the standard n-simplex in R^{n+1} and consider Δ_n a subset of Δ_{n+1} via the imbedding

$$(x_0, \ldots, x_n) \mapsto (x_0, \ldots, x_n, 0).$$

Let $G = \Delta_0 + \Delta_1 + \sum_{n=2}^{\infty} (\Delta_n \backslash \Delta_{n-1})$. Again let $(X, \oplus)$ be the free abelian cancellation semigroup generated by G. For $n \geqslant 2$, $p \in \Delta_0$ and $v = (v_0, \ldots, v_{n-1}) \in \Delta_{n-1}$, identify $p \oplus v$ with $(v_0, \ldots, v_{n-1}, 0) \in \Delta_n$. Let $Y = \sum_{n=0}^{\infty} \Delta_n$ with the natural topology and define the dimension function by $d|\Delta_n \equiv n$ for $n \in N$ and $d|\Delta_0 \equiv 1$. Again, we use 4.4 to endow X with the structure of a representation semigroup in which $G = \mathrm{Irr}(X)$ and $Y = \bar{G}$. Obviously, $\mathrm{Core}_0(X) = \varnothing$. ∎

6.3. Definition. Let X be a representation semigroup with bounded dimension function. We call $\mathrm{Core}(X) = \overline{\mathrm{Core}_0(X)}$ the *core of* X. ∎

The following two propositions will be useful in the classification of some **BD**-algebras.

6.4. Proposition. *Let ξ be a C^*-semigroup bundle over the representation semigroup X with bounded dimension function d. Then restriction defines an isomorphism*

$$\Gamma_b(\xi|\mathrm{Core}(X)) \cong \Gamma_b(\xi).$$

Proof. Let $Z = \mathrm{Core}_0(X)$ and $Y = \overline{\mathrm{Irr}(X)}$. By 3.11, we only need to show that $\Gamma_b(\xi|\bar{Z}) \cong \Gamma_b(\xi|Y)$. So let $s \in \Gamma_b(\xi|Y)$, such that $s|\bar{Z} \equiv 0$. Now let $y \in Y \backslash \bar{Z}$. By 6.1 (iii), there is an $x \in X$ so that $y + x \in Z$. Then, with $t \in \Pi\xi$ satisfying $t|Y = s$, we obtain

$$\begin{aligned}(t(y), t(x)) &= (\tilde{m}^* t)(y, x) = \tilde{m}(y, x)(t(x+y)) \\ &= \tilde{m}(y, x)(s(x+y)) = (0, 0).\end{aligned}$$

Thus, $s(y) = t(y) = 0$ and restriction is injective.

Now let $s \in \Gamma_b(\xi|\bar{Z})$ and $t \in \Pi\xi$, such that $t|\bar{Z} = s$. Consider the diagram (with the obvious restrictions)

$$\begin{array}{ccccccc}
\xi_0|\pi(m^{-1}(\bar{Z})) & \xleftarrow{\varrho} \xi_0 \times \xi_0|m^{-1}(\bar{Z}) & \xleftarrow{h} & m^*\xi_0|m^{-1}(\bar{Z}) & \xrightarrow{\psi} & \xi_0|\bar{Z} \\
{\scriptstyle t}\uparrow & \uparrow{\scriptstyle \tilde{m}^* t} & & \uparrow{\scriptstyle m^* t} & & \uparrow{\scriptstyle t} \\
Y\backslash Z = Y \cap \pi(m^{-1}(Z)) \subseteq \pi(m^{-1}(\bar{Z})) & \xleftarrow[\pi]{} m^{-1}(\bar{Z}) & = & m^{-1}(\bar{Z}) & \xrightarrow[m]{} & \bar{Z}
\end{array}$$

where h is the structure map of ξ, m multiplication in X and π and ϱ are the corresponding first factor projections. By 6.1 (v), Z is open in Y (hence $Y \backslash Z$ compact Hausdorff!). As h and ϱ are continuous, so is

$$t|\pi(m^{-1}(\bar{Z})) \circ \pi = \varrho_* \circ h_*(m^* s) = \varrho_*(\tilde{m}^* s).$$

But again by 6.1 (v), π: $m^{-1}(\bar{Z}) \cap \pi^{-1}(Y \backslash Z) \to Y \backslash Z$ is a surjection and a quotient map, since both domain and range are compact Hausdorff. Thus $t|Y\backslash Z$ is continuous. But

$$t|[(Y\backslash Z) \cap \bar{Z}] = s|[(Y \backslash Z) \cap \bar{Z}]$$

and $t|\bar{Z} = s$, so that $t|Y$ is continuous. This completes the proof. ∎

6.5. PROPOSITION. *Let ξ be a full SC*-semigroup bundle over the representation semigroup X with bounded dimension function. Let $Y = \overline{\mathrm{Irr}(X)}$ and $Z = \mathrm{Core}_0(X)$.*

If for all $y \in Y$ and $x_1, x_2 \in X$, $y + x_1 \in \bar{Z}$ and $y + x_2 \in \bar{Z}$ implies $x_1 = x_2$ and $y \in Z$ implies $x_1 + y \notin Z$, then $\Gamma_b(\xi) \cong \Gamma_b(\xi_0 | \bar{Z})$ and the isomorphism is given by restriction $s \mapsto s | \bar{Z}$.

Proof. By 6.4, all we need to show is that $\Gamma_b(\xi_0 | \bar{Z}) \subseteq \Gamma_b(\xi | \bar{Z})$.
So let $s \in \Gamma_b(\xi_0 | \bar{Z})$. For $y \in \mathrm{Irr}(X) \backslash \bar{Z}$ define

$$t(y) = \pi_1((\tilde{m}^* s)(x, y)), \tag{6.5.1}$$

where x is the unique element in X, so that $x + y \in \bar{Z}_n$ (assumption and 6.1 (iii)) and $\pi_1\colon \xi_0(y) \times \xi_0(x) \to \xi_0(y)$ is first factor projection. For $y \in \mathrm{Irr}(X) \cap \bar{Z}$, we let $t(y) = s(y)$. By 3.16, t extends to a compatible selection which will also be denoted by t.

Now let $y = x_1 + \ldots + x_n \in \bar{Z}$, $x_i \in \mathrm{Irr}(X)$ and $x = x_2 + \ldots + x_n$. By assumption, $x_i \notin \bar{Z}$. Thus, by 3.5,

$$\begin{aligned} \tilde{m}(x_1, x)(s(y)) &= (t(x_1), t(x)) \quad \text{by (6.5.1)}, \\ &= \tilde{m}(x_1, x)(t(y)) \end{aligned}$$

and by injectivity of the structure map, $s = t | \bar{Z}$, or s is compatible. ∎

6.5 means that if the base semigroup satisfies the condition in the hypothesis of 6.5, then two proper C^*-semigroup bundles whose underlying C^*-bundles restrict to isomorphic C^*-bundles over the core of the base semigroup are, in fact, isomorphic as C^*-semigroup bundles. (5.12).

II. The calculation of ΠH^1_R for certain C^*-bundles

1. The functor ΠH^1_R

In this section we want to briefly recall the definitions of ΠH^1_R from [Du; 1], to which we also refer for further details.

So let D be a set and R a relation on D, which is reflexive and transitive. Let TOP_R be the category, whose objects are triples $\mathbf{X} = (X, (A_i)_D, (U_i)_D)$, such that X is a topological space, $(U_i)_D$ is an open cover of X and $(A_i)_D$ is a family of subsets of X satisfying

1.1.1. if $i \neq j$, then $A_i \cap A_j = \emptyset$ and $A_i \subseteq U_i$;
1.1.2. for all $i \in D$, $\bigcup \{A_j\colon (j, i) \in R, j \neq i\}$ is closed in X and disjoint from U_i;
1.1.3. if $i, j \in D$ and $U_i \cap U_j \neq \emptyset$, then

$$(i, j) \in R \cup R^{-1}.$$

A morphism $f\colon (X,(A_i)_D,(U_i)_D)\to(Y,(B_i)_D,(V_i)_D)$ is a continuous map $f\colon X\to Y$ such that for all $i\in D$, $f(A_i)\subseteq B_i$ and $f(U_i)\subseteq V_i$.

Also let $S(C_R^*)$ be the category whose objects are pairs $\xi=(\mathbf{X},(\xi_i)_D)$, where $\mathbf{X}$ is an object of TOP_R and ξ_i is a C^*-bundle over U_i. A morphism $\psi\colon(\mathbf{X},(\xi_i)_D)\to(\mathbf{Y},(\eta_i)_D)$ is a family $(\psi_i)_D$ of strong C^*-bundle maps $\psi_i\colon\ \xi_i\to\eta$ covering $f|U_i$ for some $f\colon (X,(A_i)_D,(U_i)_D)\to(Y,(B_i)_D,(V_i)_D)$.

For $\mathbf{X}=(X,(A_i)_D,(U_i)_D)$, let $N(\mathbf{X})$ be the set of covers $(V_i)_D$ of X indexed by D such that for each $i\in D$, $A_i\subseteq V_i\subseteq U_i$ and V_i is a neighborhood of A_i.

If $\xi=(\mathbf{X},(\xi_i)_D)$ is an object of $S(C_R^*)$ and $\mathbf{V}=(V_i)_D\in N(\mathbf{X})$, set

$$\xi|\mathbf{V}=\xi|(X,(A_i)_D,(V_i)_D)=(\mathbf{X},(\xi_i|V_i)_D).$$

We denote by $C_R^0((V_i)_D,(\xi_i)_D)$ the set $\prod_{i\in D}\mathrm{Aut}(\xi_i|V_i)$ and refer to its elements as "zero chains".

Let $Z_R^1((V_i)_D,(\xi_i)_D)$ be the set of families $(\varphi_{ij})_{(j,i)\in R}$, of maps φ_{ij}: $\xi_j|V_{ij}<\xi_i|V_{ij}$, where $V_{ij}=V_i\cap V_j$, such that

1.2.1. For all $i\in D$, φ_{ii} is the identity on $\xi_i|V_i$;

1.2.2. If (k,j), $(j,i)\in R$, then

$$(\varphi_{ij}|V_{ijk})\circ(\varphi_{jk}|V_{ijk})=(\varphi_{ij}|V_{ijk})$$

where $V_{ijk}=V_i\cap V_j\cap V_k$. We refer to the elements of $Z_R^1((V_i)_D,\xi_i)_D)$ as cocycles because of condition 1.2.2. Then $C_R^0((V_i)_D,(\xi_i)_D)$ acts on $Z_R^1((V_i)_D,(\xi_i)_D)$ in an obvious way and the resulting orbit set is denoted $H_R^1((V_i)_D,(\xi_i)_D)$. Refinement makes $N(X)$ a directed set and we can form

$$H_R^1(\xi)=\varinjlim H_R^1((V_i)_D,(\xi_i)_D)$$

where the cover $(V_i)_D$ converges to $(A_i)_D$. In this way, H_R^1 defines a contravariant functor from $S(C_R^*)$ to the category of sets.

Remark. Dupré required that $N(\mathbf{X})$ consists of those $(V_i)_D$ such that V_i is open for each $i\in D$. Since these $(V_i)_D$ are cofinal in the collection of all $(V_i)_D$ such that $A_i\subseteq V_i\subseteq U_i$ and V_i is a neighborhood of A_i, we obtain the same functor H_R^1 as Dupré in [Du; 1].

Now let $I=[0,1]$ be the unit interval. Set $\mathbf{X}\times I=(X\times I,(A_i\times I)_D,(U_i\times I)_D)$ and let $\pi\colon\mathbf{X}\times I\to\mathbf{X}$ be first factor projection. Furthermore, define $\mu_t\colon\mathbf{X}\to\mathbf{X}\times I$ for all $t\in I$ by $\mu_t(x)=(x,t)$. Let $\xi\times I=\pi^*\xi$. Then $\mu_t^*(\xi\times I)=\xi$ for every $t\in I$. Let $\psi_t=(\psi_{t,i})\colon\ \mu_t^*(\xi\times I)\to\xi\times I$ be defined by letting $\psi_{t,i}\colon\ \mu_t^*(\xi_i\times I)\to\xi_i\times I$ be the strong map covering μ_t.

Finally, we define $\Pi(\xi)\colon H_R^1(\xi)\to\Pi H_R^1(\xi)$ to be the coequalizer in the category of sets of the two maps

$$H_R^1(\psi_0),\ H_R^1(\psi_1)\colon\ H_R^1(\xi\times I)\to H_R^1(\xi).$$

In this way, ΠH_R^1 is a contravariant functor from $S(C_R^*)$ to the category of sets. Another contravariant functor B_R from $S(C_R^*)$ to the category of sets can be defined by assigning to $\xi = (\mathbf{X}, (\xi_i)_D)$ the isomorphism class of C^*-bundles η over X, such that $\eta | A_i = \xi_i | A_i$ for all $i \in D$. Dupré announced in [Du; 2] the following theorem:

1.1. THEOREM. *The natural map*

$$\Pi H^1_R \to B_R$$

defined in [Du; 1], *is well defined and a natural isomorphism for the classification of finite order C^*-bundles all of whose fibres are finite dimensional over hereditarily paracompact Hausdorff spaces.* ■

For the convenience of the reader, we have added a proof of a restricted version of this theorem for our examples in II.2 as an appendix.

The task of this chapter is to compute ΠH^1_R for certain C^*-bundles. In order to accomplish this, we are following the strategy suggested in [Du; 1: Prop. 2.1]. The difficulty, however, is that we have to compute homotopy classes of cocycles via homotopies that also have to satisfy the cocycle condition. We say that two cocycles in the same class are cocycle homotopic. Before we state the details of the strategy, let us first introduce some new notation and prove a lemma.

For $\xi = (\mathbf{X}, (\xi_i)_D) \in S(C^*_R)$, $\mathbf{X} = (X, (A_i)_D, (U_i)_D)$ and $\mathbf{V} = (V_i)_D \in N(\mathbf{X})$, we define

$$\Pi(\xi | \mathbf{V})\colon\ H^1_R((V_i)_D, (\xi_i)_D) \to \Pi H^1_R((V_i)_D, (\xi_i)_D)$$

to be the coequalizer of the maps (in the category of sets)

$$H^1_R((V_i \times I)_D, (\xi_i \times I)_D) \to H^1_R((V_i)_D, (\xi_i)_D)$$

induced by ψ_0 and ψ_1.

1.2. LEMMA. *For any $\xi = (\mathbf{X}, (\xi_i)_D) \in S(C^*_R)$, $\mathbf{X} = (X, (A_i)_D, (U_i)_D)$, we have*

$$\Pi H^1_R(\xi) = \varinjlim \Pi H^1_R((V_i)_D, (\xi_i)_D)$$

where $(V_i)_D$ runs through all elements of $N(\mathbf{X})$.

Proof. We abbreviate in the following way:

$$V = (V_i)_D \in N(\mathbf{X}), \qquad U = (U_i)_D \in N(\mathbf{X}), \qquad \text{etc.}$$

$$H(V) = H^1_R(V, (\xi_i)_D), \qquad H(V \times I) = H^1_R((V_i \times I)_D, (\xi_i \times I)_D),$$

$$H(\xi) = H^1_R(\xi), \qquad H(\xi \times I) = H^1_R(\xi \times I) \qquad \text{and}$$

$$\overline{\Pi H(\xi)} = \varinjlim \Pi H(V), \qquad \text{where } V \text{ runs through } N(\mathbf{X}).$$

Note that elements of the form $V \times I$ are cofinal in $N(\mathbf{X} \times I)$ since I is compact.

Consider the following commutative diagrams for $U, V \in N(\mathbf{X})$:

$$\begin{array}{ccccccc}
H(V\times I) & \xrightarrow{\bar{\varphi}_{UV}} & H(U\times I) & \xrightarrow{\bar{\varphi}_{U}} & H(\xi\times I) & & \\
\downarrow{\scriptstyle \mu^*_{t,V}} & & \downarrow{\scriptstyle \mu^*_{t,U}} & & \downarrow{\scriptstyle \mu^*_{t}} & & \\
H(V) & \xrightarrow{\varphi_{UV}} & H(U) & \xrightarrow{\varphi_{U}} & H(\xi) & \xrightarrow{\Pi} & \Pi H(\xi) \\
\downarrow{\scriptstyle \Pi_V} & & \downarrow{\scriptstyle \Pi_U} & & \downarrow{\scriptstyle \bar{\Pi}} & & \\
\Pi H(V) & \xrightarrow[\psi_{UV}]{} & \Pi H(U) & \xrightarrow[\psi_{U}]{} & \overline{\Pi H(\xi)} & &
\end{array}$$

where the maps are the obvious restrictions, pullbacks, quotient maps, etc.

For $U \in N(\mathbf{X})$, define θ_U: $\Pi H(U) \to \Pi H(\xi)$ by

$$\theta_U \circ \Pi_U(z) = \Pi \circ \varphi_U(z), \qquad z \in H(U).$$

θ_U is well defined, since in the case that $\Pi_U(z^0) = \Pi_U(z^1)$, there is a $z \in H(U \times I)$, such that $\mu^*_{t,U}(z) = z^t$, $t = 0, 1$. Then

$$\theta_U \circ \Pi_U(z^t) = \Pi \circ \varphi_U \circ \mu^*_{t,U}(z) = \Pi \circ \mu^*_t \circ \bar{\varphi}_U(z)$$

for $t = 0, 1$ and θ_U is well defined by definition of Π. Also $\theta_U \circ \psi_{UV} = \theta_V$, because for $z \in H(V)$,

$$\begin{aligned}
\theta_U \circ \psi_{UV} \circ \Pi_V(z) &= \theta_U \circ \Pi_U \circ \varphi_{UV}(z) \\
&= \Pi \circ \varphi_U \circ \varphi_{UV}(z) = \Pi \circ \varphi_V(z) = \theta_V \circ \Pi_V(z).
\end{aligned}$$

Thus we get – by the universal property of direct limits – a unique θ: $\overline{\Pi H(\xi)} \to \Pi H(\xi)$ such that

$$\theta \circ \psi_U \circ \Pi_U = \Pi \circ \varphi_U$$

and θ is obviously onto.

Now suppose that $\theta \circ \psi_{U(0)} \circ \Pi_{U(0)}(z^0) = \theta \circ \psi_{U(1)} \circ \Pi_{U(1)}(z^1)$. Then there are $U(2) \in N(\mathbf{X})$ and $z \in H(U(2) \times I)$, such that

$$\mu^*_t \circ \bar{\varphi}_{U(2)}(z) = \varphi_{U(t)}(z^t), \qquad t = 0, 1$$

by definition of Π and direct limits. Now

$$\begin{aligned}
\varphi_{U(t)}(z^t) &= \mu^*_t \circ \bar{\varphi}_{U(2)}(z) \\
&= \varphi_{U(2)} \circ \mu^*_{t,U(2)}(z), \qquad t = 0, 1.
\end{aligned}$$

By definition of direct limits, there are $V(0)$, $V(1) \in N(\mathbf{X})$, so that

$$\varphi_{V(t)U(2)} \circ \mu^*_{t,U(2)}(z) = \varphi_{V(t)U(t)}(z^t), \qquad t = 0, 1$$

and with $V = V(0) \cap V(1)$

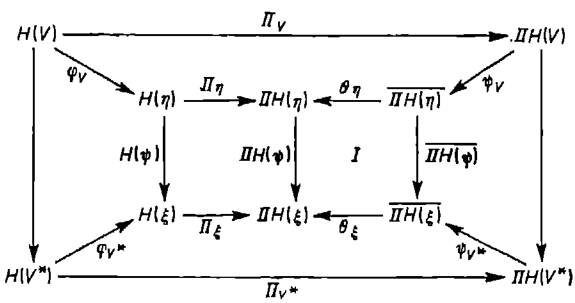
$$\begin{aligned}
\varphi_{VU(t)}(z^t) &= \varphi_{VV(t)} \circ \varphi_{V(t)U(t)}(z^t) \\
&= \varphi_{VV(t)} \circ \varphi_{V(t)U(2)} \circ \mu^*_{t,U(2)}(z) \\
&= \varphi_{VU(2)} \circ \mu^*_{t,U(2)}(z) = \mu^*_{t,V} \circ \bar{\varphi}_{VU(2)}(z), \qquad t = 0, 1.
\end{aligned}$$

Hence $\Pi_V \circ \varphi_{VU(0)}(z^0) = \Pi_V \circ \varphi_{VU(1)}(z^1)$ or

$$\psi_{U(0)} \circ \Pi_{U(0)}(z^0) = \psi_V \circ \psi_{VU(0)} \circ \Pi_{U(0)}(z^0) = \psi_V \circ \Pi_V \circ \varphi_{VU(0)}(z^0)$$
$$= \psi_V \circ \Pi_V \circ \varphi_{VU(1)}(z^1) = \psi_{U(1)} \circ \Pi_{U(1)}(z^1)$$

and θ is injective. Thus, θ is a bijection. ■

1.3. *Remark.* The θ in the proof of 1.2 defines actually a natural equivalence of functors. This can be seen in the following way: Let ψ: $\xi = (\mathbf{X}, (\xi_i)_D) \to \eta = (\mathbf{Y}, (\eta_i)_D)$ be a morphism in $S(C^*_{\mathbb{R}})$ covering f: $\mathbf{X} = (X, (A_i)_D, (U_i)_D) \to \mathbf{Y}$ and $V = (V_i)_D \in N(\mathbf{Y})$. With $V^* = ((f^{-1}(V_i) \cap U_i)_D) \in N(\mathbf{X})$, ψ induces obvious maps $H(V) \to H(V^*)$ and $\Pi H(V) \to \Pi H(V^*)$. Consider the following diagram



Since all square commute — except possibly the square labelled I — a diagram chasing yields

$$\theta_\xi \circ \overline{\Pi H(\psi)} \circ \psi_V \circ \Pi_V = \Pi H(\psi) \circ \theta_\eta \circ \psi_V \circ \Pi_V$$

and by the definition of direct limits and by surjectivity of Π_V

$$\theta_\xi \circ \overline{\Pi H(\psi)} = \Pi H(\psi) \circ \theta_\eta$$

so that square I commutes.

It is in the sense of this remark that we have to interpret equality in the statement of 1.2 and we will use this identification throughout the rest of this paper. ■

Our strategy to compute $\Pi H^1_R(\xi)$ for certain ξ then is to compute $\overline{\Pi H^1_R(\xi)}$ instead, by first finding a cofinal family in $N(\mathbf{X})$, on which the $\Pi H^1_R((V_i)_D, (\xi_i)_D)$ are constant. We then try to represent every element in $\Pi H^1_R((V_i)_D, (\xi_i)_D)$ by a cocycle of a particularly simple type and then study the relations between them.

First, however, we will investigate the spaces of embeddings of one finite dimensional C^*-algebra into another one — these spaces are the analog of the group of a bundle in classical fibre bundle theory.

2. C^*-bundle embeddings, multiplicity bundles and MH^1_R

2.1. *Notation.* For $p \in \boldsymbol{N}$, let $M(p)$ be the algebra of complex $p \times p$-matrices. For $p = (p_1, \ldots, p_n) \in \boldsymbol{N}^n$, let $M(p) = M(p_1) \times \ldots \times M(p_n)$ and $U(p) = U(M(p)) = U(p_1) \times \ldots \times U(p_n)$, where $U(p_i)$ is the group of unitaries in $M(p_i)$. For a matrix $a = (a_{ij})$ with nonnegative integer coefficients, we define

$$U(a) = \prod_{(i,j), a_{ij} \neq 0} U(a_{ij}).$$

Clearly, each finite dimensional C^*-algebra is isomorphic to $M(p)$ for some $n \in \boldsymbol{N}$, $p \in \boldsymbol{N}^n$.

For $p \in \boldsymbol{N}^n$, $q \in \boldsymbol{N}^m$ and a C^*-map $\varphi\colon M(p) \to M(q)$, we let $m_{q,p}(\varphi)$ be the multiplicity matrix of φ, $m_{q,p}(\varphi) \in M(m \times n, \boldsymbol{N}_0)$, where $(m_{q,p}(\varphi))_{ij}$ is the multiplicity of the map $M(p_j) \to M(q_i)$ induced by φ, considered as a representation of $M(p_j)$ on $\boldsymbol{C}^{q_i}$.

The image of $m_{q,p}\colon \operatorname{Hom}^*(M(p), M(q)) \to M(m \times n, \boldsymbol{N}_0)$ is denoted by $\mathbf{M}(p, q)$. Then $\mathbf{M}(p, q) = \{a \in M(m \times n, \boldsymbol{N}_0)\colon ap \leqslant q\}$, were "$\leqslant$" is applied componentwise, because clearly $m_{q,p}(\varphi)p \leqslant q$ for all $\varphi \in \operatorname{Hom}^*(M(p), M(q))$ and conversely, for $a = (a_{ij}) \in M(m \times n, \boldsymbol{N}_0)$ such that $ap \leqslant q$ and $(A_1, \ldots, A_n) \in M(p)$, we can define

(2.1.1) $\alpha_a(A_1, \ldots, A_n) =$

$$\left(\begin{bmatrix} A_1 {}_{\ddots}^{a_{11}} & & & & & \bigcirc \\ & A_1 & & & & \\ & & \ddots & & & \\ & & & A_n {}_{\ddots}^{a_{1n}} & & \\ & & & & A_n & \\ \bigcirc & & & & & \bigcirc_{(\delta a)_1} \end{bmatrix}, \ldots, \begin{bmatrix} A_1 {}_{\ddots}^{a_{m1}} & & & & & \bigcirc \\ & A_1 & & & & \\ & & \ddots & & & \\ & & & A_n {}_{\ddots}^{a_{mn}} & & \\ & & & & A_n & \\ \bigcirc & & & & & \bigcirc_{(\delta a)_m} \end{bmatrix} \right)$$

where $(\delta a)_i$ is the ith component of the defect $\delta a = q - ap$ of a. Thus the map $\alpha\colon \mathbf{M}(p, q) \to \operatorname{Hom}^*(M(p), M(q))$ is a "cross-section" of $m_{q,p}$. Also note that for $a \in \mathbf{M}(p, q)$, $b \in \mathbf{M}(r, p)$, the defect of $a \cdot b$ satisfies

$$\begin{aligned} \delta(a \cdot b) &= q - a \cdot br \\ &= q - ap + a(p - br) \\ &= \delta a + a \delta b. \end{aligned} \tag{2.1.2}$$

Note that for $r \in \boldsymbol{N}^l$, $\varphi \in \operatorname{Hom}^*(M(p), M(q))$ and $\psi \in \operatorname{Hom}^*(M(q), M(r))$,

$$m_{r,p}(\psi \circ \varphi) = m_{r,q}(\psi) \cdot m_{q,p}(\varphi) \tag{2.1.3}$$

under the usual matrix multiplication.

2.2. LEMMA. *Let* $m, n, l \in N$, $p \in N^m$, $q \in N^n$, $r \in N^l$, $a \in \mathbf{M}(p, q)$ and $b \in \mathbf{M}(q, r)$. *Then there is an* l*-tuple* $u = u(b, a)$ *of permutation matrices* $u = (u_1, \ldots, u_l) \in U(r)$, *so that*

$$\text{(2.2.1)} \qquad \mathrm{Ad}(u) \circ \alpha_{ba} = \alpha_b \circ \alpha_a .$$

Furthermore, if $l = n$, $q = r$ *and* b *is invertible, then*

$$\text{(2.2.2)} \qquad \alpha_{ba} = \alpha_b \circ \alpha_a .$$

Proof. Since both α_{ba} and $\alpha_b \circ \alpha_a$ have the same multiplicity matrix, namely ba, an element $A = (A_1, \ldots, A_m)$ of $M(p)$ gets mapped by both α_{ba} and $\alpha_b \circ \alpha_a$ into $M(r)$, so that A_j appears with multiplicity $\sum_{k=1}^{n} b_{ik} a_{kj}$ in $M(r_i)$. Since in the definition of the α_{ba}, α_b, α_a no nontrivial unitary action is involved, it is clear that the image of A under both α_{ba} and $\alpha_b \circ \alpha_a$ can only differ by rearrangements of the blocks of A in each component of $M(r)$. This proves the first claim.

If $q = r$ and b is invertible, then b is a permutation matrix, that is, $b_{ij} = \delta_{i\sigma^{-1}(j)}$ for some appropriate permutation σ. Then $(ba)_{ij} = \sum_{k=1}^{n} \delta_{i\sigma^{-1}(k)} \cdot a_{kj} = a_{\sigma(i)j}$ and for $A = (A_1, \ldots, A_m) \in M(p)$,

$$\alpha_{ba}(A) = \left(\begin{bmatrix} A_1 \;_{a_{\sigma(1)1}} & & & & \bigcirc \\ & \ddots & & & \\ & & A_1 & & \\ & & & \ddots & \\ & & & & A_m \;_{a_{\sigma(1)m}} \\ & & & & \quad \ddots \\ \bigcirc & & & & \qquad A_m \end{bmatrix}, \ldots, \begin{bmatrix} A_1 \;_{a_{\sigma(n)1}} & & & & \bigcirc \\ & \ddots & & & \\ & & A_1 & & \\ & & & \ddots & \\ & & & & A_n \;_{a_{\sigma(n)m}} \\ & & & & \quad \ddots \\ \bigcirc & & & & \qquad A_n \end{bmatrix} \right)$$

$$= \alpha_b \circ \alpha_a(A). \quad ■$$

Note that the assignment $(b, a) \mapsto u(b, a)$ has the following properties:

$$\text{(2.2.3)} \qquad u(b, a) = \text{identity matrix if } b \text{ is invertible},$$

$$\begin{aligned} \text{(2.2.4)} \qquad \mathrm{Ad}\big(u(c, b) \cdot u(cb, a)\big) \circ \alpha_{cba} &= \mathrm{Ad}\big(u(c, b)\big) \circ \alpha_{cb} \circ \alpha_a = \alpha_c \circ \alpha_b \circ \alpha_a \\ &= \alpha_c \circ \mathrm{Ad}\big(u(b, a)\big) \circ \alpha_{ba} \\ &= \mathrm{Ad}\big[\alpha_c\big(u(b, a)\big)\big] \circ \alpha_c \circ \alpha_{ba} \\ &= \mathrm{Ad}\big[\alpha_c\big(u(b, a)\big) \cdot u(c, ba)\big] \circ \alpha_{cba} \end{aligned}$$

for c such that $\delta c = 0$.

Now $\varphi \in \mathrm{Hom}^*(M(p), M(q))$ is injective iff, for all $1 \leqslant i \leqslant n$, there is $1 \leqslant j \leqslant m$ such that $(m_{q,p}(\varphi))_{ij} \neq 0$. Denote the image of $C^*U(M(p), M(q))$ – the set of injective C^*-maps $M(p) \to M(q)$ – under $m_{q,p}$ by $\mathbf{M}_0(p, q)$. By abuse of notation, we will denote the restriction and corestriction of $m_{q,p}$ to $C^*U(M(p), M(q))$ and $\mathbf{M}_0(p, q)$ by $m_{q,p}$ also.

The adjoint map defines an action of $U(q)$ on $C^*U(M(p), M(q))$ and by unique decomposition of finite-dimensional representations into irreducible representations modulo unitary equivalence, we obtain for $u \in U(q)$ and $\varphi \in C^*U(M(p), M(q))$,

$$m_{q,p}(\mathrm{Ad}(u) \circ \varphi) = m_{q,p}(\varphi).$$

Thus we get a well-defined surjective map

$$C^*U(M(p), M(q))/U(q) \xrightarrow{\bar{m}_{q,p}} \mathbf{M}_0(p, q).$$

Again, by uniqueness of the decomposition of finite dimensional representations, there is a unitary $u \in U(q)$ for every $\varphi \in C^*U(M(p), M(q))$, such that

$$\mathrm{Ad}(u) \circ \varphi = \alpha_a$$

where $a = m_{q,p}(\varphi)$. Hence $\bar{m}_{q,p}$ is injective, or

$$\bar{m}_{q,p}\colon\ C^*U(M(p), M(q))/U(q) \cong \mathbf{M}_0(p, q).$$

Now suppose that $u = (u_1, \ldots, u_m) \in U(q)$ and $a \in \mathbf{M}_0(p, q)$ satisfy

$$\mathrm{Ad}(u) \circ \alpha_a = \alpha_a.$$

Then clearly $\mathrm{Ad}(u_i) \circ \pi_i \circ \alpha_a = \pi_i \circ \alpha_a$, where $\pi_i\colon M(q) \to M(q_i)$ is the ith-factor projection. In particular, for all j, u_i has to commute with $\pi_i \circ \alpha_a(A_1, \ldots, A_n)$, where $A_k = 0$ if $k \neq j$ and $A_k = \mathrm{Id}_{p_k}$ if $k = j$. Thus, for all $a_{ij} \neq 0$, there are unitaries u_{ij} – and if $\sum_{k=1}^{n} a_{ik} p_k \neq q_i$ also a unitary u_{in+1} – such that

$$U_i = \begin{bmatrix} u_{i1} & & & & \bigcirc \\ & u_{i2} & & & \\ & & \ddots & & \\ & & & u_{in} & \\ \bigcirc & & & & u_{in+1} \end{bmatrix}$$

(ignoring the u_{ij} for which $a_{ij} = 0$ or $\sum_{k=1}^{n} a_{ik} p_k = q_i$). Then, for $1 \leqslant j \leqslant n$, u_{ij} commutes with

$$\begin{bmatrix} A & & 0 \\ & \ddots & \\ 0 & & A \end{bmatrix} \in M(p_j) \otimes \mathbb{C}\mathrm{Id}_{a_{ij}}$$

for all $A \in M(p_j)$ and hence $u_{ij} \in (M(p_j) \otimes \mathbb{C}\mathrm{Id}_{a_{ij}})' = \mathbb{C}\mathrm{Id}_{p_j} \otimes M(a_{ij})$.

Conversely, if u satisfies these conditions, then

$$\mathrm{Ad}(u)\circ\alpha_a = \alpha_a.$$

Thus, the stabilizer $S_a = \{u \in U(q)\colon \mathrm{Ad}(u)\circ\alpha_a = \alpha_a\}$ of α_a under the action of $U(q)$ on $C^*U(M(p), M(q))$ is isomorphic to

$$\prod_{i=1}^{m}\Big[\prod_{j=1}^{n} U(a_{ij}) \times U\big(q_i - \sum_{k=1}^{n} a_{ik}p_k\big)\Big],$$

where we ignore the factors for which either $a_{ij} = 0$ or $q_i = \sum_{k=1}^{n} a_{ik}p_k$.

If we denote for $a \in \mathbf{M}_0(p, q)$ the extended matrix $(a|\delta a)$ – where $\delta a = q - ap$ is the defect of a – by $\bar{a}$, then

2.3. PROPOSITION. $C^*U(M(p), M(q)) \cong \sum_{a\in\mathbf{M}_0(p,q)} U(q)/U(\bar{a})$. ∎

This description of $C^*U(M(p), M(q))$ enables us now to calculate homotopy groups of $C^*U(M(p), M(q))$. For example:

2.4. COROLLARY. $\pi_0\big(C^*U(M(p), M(q))\big) \cong \mathbf{M}_0(p, q)$. ∎

2.5. COROLLARY. *For every* $a \in \mathbf{M}_0(p, q)$

$$\pi_1\big(C^*U(M(p), M(q)), \alpha_a\big) \cong \mathbf{Z}^m/d\mathbf{Z}^m$$

for some $d = (d_1, \ldots, d_m) \in \mathbf{Z}^m$ *and multiplication in* $\mathbf{Z}^m$ *is meant to be coordinatewise.*

Proof. Since application of π_1 commutes with taking cartesian products, we assume without loss of generality that $m = 1$. We now use the long exact homotopy sequence applied to the fibre bundle $\varrho\colon U(q) \to U(q)/U(\bar{a})$. Let $i\colon U(\bar{a}) \hookrightarrow U(q)$ be the injection according to the discussion preceding 2.3. The sequence then is

$$\pi_1(U(\bar{a})) \xrightarrow{i_*} \pi_1(U(q)) \xrightarrow{\varrho_*} \pi_1(U(q)/u(\bar{a})) \to \pi_0(U(\bar{a})) = 0$$
$$\pi_1(U(q)) \cong \mathbf{Z}$$

We identify $\pi_1(U(q))$ with $\mathbf{Z}$ and $\pi_1(U(\bar{a}))$ with the set

$$\{(r_1, \ldots, r_n, r_{n+1}) \in \mathbf{Z}^n \times \mathbf{Z}\colon r_k = 0 \text{ if } \bar{a}_{ik} = 0\}.$$

Then

(2.5.1) $$i_*(r_1, \ldots, r_{n+1}) = r_1p_1 + \ldots + r_np_n + r_{n+1}.$$

Now let

(2.5.2) $$d = \min\{i_*(r)\colon r \in \pi_1(U(\bar{a})) \text{ and } i_*(r) > 0\}.$$

Then, since ϱ_* is onto, we have $\pi_1(U(q)/U(\bar{a})) \cong \mathbf{Z}/d\mathbf{Z}$. Noticing that $\pi_1\big(C^*U(M(p), M(q)), \alpha_a\big) \cong \pi_1(U(q)/U(\bar{a}))$ finishes the proof. ∎

The following lemma provides a few facts used in the proofs of the next section. For a pointed space (X, x_0), we consider a loop with base point x_0 to be a map $(I, \dot{I}, \{1\}) \to (X, \{x_0\}, \{x_0\})$ where $\dot{I} = \{0, 1\}$.

2.6. LEMMA. *Let $m, n \in N$, $p \in N^n$, $q \in N^m$, $a \in \mathbf{M}_0(p, q)$, ω, ν loops in $U(q)$ with basepoint the identity matrix and γ a path in $U(q)$. Let ϱ_*: $\pi_1(U(q)) \to \pi_1(C^* U(M(p), M(q)), \alpha_a)$ as defined in the proof of 2.5. Then we have*

$$[\omega \cdot \nu] = [\omega] + [\nu] \quad \text{in } \pi_1(U(q)) \tag{2.6.1}$$

$$[\gamma \cdot \omega \cdot \gamma^*] = [\omega] \quad \text{in } \pi_1(U(q)) \tag{2.6.2}$$

(2.6.3) $[\mathrm{Ad}(\gamma \cdot \omega) \circ \alpha_a] = \varrho_*([\omega]) + [\mathrm{Ad}(\gamma) \circ \alpha_a]$

in $\pi_1(C^* U(M(p), M(q)), \alpha_a)$ in the case that $\gamma(\dot{I}) \subseteq U(\bar{a}) \subseteq U(q)$.

Proof. (2.6.1) follows from [St; 51: Lemma 16.7], or by looking directly at the canonical generators of $\pi_1(U(q))$.

(2.6.2) For $s = 0, 1$, connect $\gamma(s)$ to the identity matrix via a path τ_s. The map H_0: $(I \times \{0\}) \cup (\dot{I} \times I) \to U(q)$ defined by

$$H_0(s, t) = \begin{cases} \gamma(s) & \text{if } t = 0, \\ \tau_s(t) & \text{if } s = 0, 1 \end{cases}$$

extends to a map H: $I \times I \to U(q)$ such that $\mu_1^* H$ is a loop γ_0 and $\mu_0^* H = \gamma$, because $(I \times \{0\}) \cup (\dot{I} \times I)$ is a retract of $I \times I$. Thus H defines a homotopy $\gamma \omega \gamma^* \simeq \gamma_0 \omega \gamma_0^*$, or by (2.6.1)

$$[\gamma \omega \gamma^*] = [\gamma_0 \omega \gamma_0^*] = [\gamma_0] + [\omega] - [\gamma_0] = [\omega].$$

(2.6.3) Suppose $\gamma(\dot{I}) \subseteq U(\bar{a})$, i.e. $\mathrm{Ad}(\gamma) \circ \alpha_a$ defines a loop in $C^* U(M(p), M(q))$ with basepoint α_a. Again, we connect for $s = 0, 1$, $\gamma(s)$ to the identity matrix via a path τ_s, this time in $U(\bar{a})$ though. This way we get, just as in the proof of 2.6.2, a homotopy H: $\gamma \simeq \gamma_0 = \mu_1^* H$. Then $\mathrm{Ad}(H \cdot \omega) \circ \alpha_a$ provides a homotopy $\mathrm{Ad}(\gamma \cdot \omega) \circ \alpha_a \simeq \mathrm{Ad}(\gamma_0 \circ \omega) \circ \alpha_a$ and $\mathrm{Ad}(H) \circ \alpha_a$ a homotopy $\mathrm{Ad}(\gamma) \circ \alpha_a \simeq \mathrm{Ad}(\gamma_0) \circ \alpha_a$. Thus

$$\begin{aligned} [\mathrm{Ad}(\gamma \cdot \omega) \circ \alpha_a] &= [\mathrm{Ad}(\gamma_0 \cdot \omega) \circ \alpha_a] = \varrho_*([\gamma_0 \cdot \omega]) \\ &= \varrho_*([\omega]) + \varrho_*([\gamma_0]) = \varrho_*([\omega]) + [\mathrm{Ad}(\gamma_0) \circ \alpha_a] \\ &= \varrho_*([\omega]) + [\mathrm{Ad}(\gamma) \circ \alpha_a]. \ \blacksquare \end{aligned}$$

We are now ready to introduce the concept of mulitplicity bundles and their application to the computation of ΠH_R^1. So let ξ and η be (locally trivial) C^*-bundles over a space X, having finite dimensional fibres F and G, respectively, say, $F = M(p)$ and $G = M(q)$ for some $p \in N^n$, $q \in N^m$. ξ and η are bundles with group $\mathrm{Aut}(F)$ and $\mathrm{Aut}(G)$, respectively, and we can find an open cover $\{U_i: i \in J\}$, of X, so that $\xi | U_i$ and $\eta | U_i$ are trivial and we can find coordinate transformations f_{ij}: $U_{ij} \to \mathrm{Aut}(F)$ and g_{ij}: $U_{ij} \to \mathrm{Aut}(G)$, so that ξ and η are isomorphic to C^*-bundles defined by these coordinate transformations.

The group $\mathrm{Aut}(G) \times \mathrm{Aut}(F)$ acts in an obvious way on $C^*U(F, G)$. If S is the isotropy subgroup of $\mathrm{Aut}(G) \times \mathrm{Aut}(F)$ for this action, then the bundle $C^*U(\xi, \eta)$ of C^*-bundle imbeddings $\xi < \eta$ is a fibre bundle with fibre $C^*U(F, G)$ and group $[\mathrm{Aut}(G) \times \mathrm{Aut}(F)]/S$ and $C^*U(\xi, \eta)|U_i$ is trivial. Moreover, the coordinate transformations of $C^*U(\xi, \eta)$ are given by the canonical images of the (g_{ij}, f_{ij}) in $[\mathrm{Aut}(G) \times \mathrm{Aut}(F)]/S$ and by [St; 51: Lemma 2.8], different trivializations of ξ and η define the same bundle $C^*U(\xi, \eta)$.

Analogously, the group $\mathbf{M}_0(q, q) \times \mathbf{M}_0(p, p)$ acts on $\mathbf{M}_0(p, q)$ — an element (b, c) sends $a \in \mathbf{M}_0(p, q)$ to bac^{-1} — and we define Λ to be the isotropy subgroup of this action and $\Gamma = [\mathbf{M}_0(q, q) \times \mathbf{M}_0(p, p)]/\Lambda$. The map

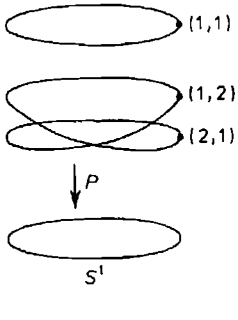
$$m_{q,q} \times m_{p,p}\colon \mathrm{Aut}(G) \times \mathrm{Aut}(F) \to \mathbf{M}_0(q, q) \times \mathbf{M}_0(p, p)$$

maps S into Λ and thus defines coordinate functions γ_{ij}: $U_{ij} \to \Gamma$. We endow Γ as well as $\mathbf{M}_0(p, q)$ with the discrete topologies. Then γ_{ij} are continuous and define a bundle $\mathbf{M}_0(\xi, \eta)$ with fibre $\mathbf{M}_0(p, q)$ and group Γ over X, which is actually a covering space. Again, by [St; 51: Lemma 2.8], the construction is independent of the trivialization of ξ and η.

2.7. DEFINITION. The bundle $\mathbf{M}_0(\xi, \eta)$ is called the *multiplicity bundle* of ξ and η and we write $\mathbf{M}_0(\xi) = \mathbf{M}_0(\xi, \xi)$. ■

Now we can define a map $M^0(\xi, \eta)$: $C^*U(\xi, \eta) \to \mathbf{M}_0(\xi, \eta)$ in the natural way assigning multiplicities over each trivial part $C^*U(\xi, \eta)|U_i$ and $\mathbf{M}_0(\xi, \eta)|U_i$ and observing that by the choice of the γ_{ij} this map is well defined. Thus we obtain a map $M^0(\xi, \eta)_*$: $\Gamma(C^*U(\xi, \eta)) \to \Gamma(\mathbf{M}_0(\xi, \eta))$, which is constant on homotopy classes of sections, since the fibres of $\mathbf{M}_0(\xi, \eta)$ are discrete.

2.8. *Application and example.* Take $X = S^1$, $F = C^2$ and $G = M(3)$. Then $\mathbf{M}_0(F, G) = \{(1, 1), (1, 2), (2, 1)\}$. There are, up to isomorphism, two C^*-bundles over S^1 with fibre C^2, since the group of these bundles is $\mathbf{Z}_2$ [St; 51: Theorem 18.5]. A similar argument yields that all C^*-bundles over S^1 with fibre $M(3)$ are trivial. So let ξ be a nontrivial C^*-bundle over S^1 with fibre C^2 and η the trivial bundle over S^1 with fibre $M(3)$. Then $\mathbf{M}_0(\xi, \eta)$ is the sum of a covering of S^1 by two circles, one of which is a double covering:



Since a section in $C^*U(\xi, \eta)$ gives a section in $\mathbf{M}_0(\xi, \eta)$, and there is only one section in $\mathbf{M}_0(\xi, \eta)$ – the one having its values in "the component of $(1, 1) \in \mathbf{M}_0(F, G)$" in the total space of $\mathbf{M}_0(\xi, \eta)$ – we conclude that every imbedding of ξ has fibrewise the multiplicity matrix $(1, 1)$. ∎

We now turn to our main application of multiplicity bundles in the definition of a new functor MH_R^1 to help us compute ΠH_R^1. We start with a proposition:

2.9. PROPOSITION. *For $i = 1, 2, 3$, let ξ_i, η_i, ζ_i be locally trivial C^*-bundles with finite dimensional fibres F_i, G_i and H_i over X_i, respectively. For $(i, j) = (2, 1)$ or $(3, 2)$, let $\varphi_{ij}\colon \xi_j \to \xi_i$, $\psi_{ij}\colon \eta_j \to \eta_i$, and $\theta_{ij}\colon \zeta_j \to \zeta_i$ be strong maps covering $f_{ij}\colon X_j \to X_i$ and let $\varphi_{31} = \varphi_{32} \circ \varphi_{21}$, $\psi_{31} = \psi_{32} \circ \psi_{21}$ and $f_{31} = f_{32} \circ f_{21}$. Then:*

(2.9.1) $$\mathbf{M}_0(\xi_i, \eta_i) \cong f_{ji}^* \mathbf{M}_0(\xi_j, \eta_j).$$

(2.9.2) *There are maps $\mathbf{M}_0^*(\varphi_{ij}, \psi_{ij})\colon \Gamma(\mathbf{M}_0(\xi_i, \eta_i)) \to \Gamma(\mathbf{M}_0(\xi_j, \eta_j))$ such that*

- (i) $\mathbf{M}_0^*(\varphi_{21}, \psi_{21}) \circ \mathbf{M}_0^*(\varphi_{32}, \psi_{32}) = \mathbf{M}_0^*(\varphi_{31}, \psi_{31})$;
- (ii) *The diagram*

$$\begin{array}{ccc} \Gamma(\mathbf{M}_0(\xi_2, \eta_2)) & \xrightarrow{M_0^*(\varphi_{21}, \psi_{21})} & \Gamma(\mathbf{M}_0(\xi_1, \eta_1)) \\ {\scriptstyle M^0(\xi_2, \eta_2)_*}\uparrow & & \uparrow{\scriptstyle M^0(\xi_1, \eta_1)_*} \\ \Gamma(C^*U(\xi_2, \eta_2)) & \xrightarrow[T]{} & \Gamma(C^*U(\xi_1, \eta_1)) \end{array}$$

commutes, where the bottom vertical arrow T represents the pull back via f_{21}.

(2.9.3) *Matrix multiplication $\mathbf{M}_0(F_i, G_i) \times \mathbf{M}_0(G_i, H_i) \to \mathbf{M}_0(F_i, H_i)$ induces an operation,*

$$\Gamma(\mathbf{M}_0(\xi_i, \eta_i)) \times \Gamma(\mathbf{M}_0(\eta_i, \zeta_i)) \to \Gamma(\mathbf{M}_0(\xi_i, \zeta_i))$$

such that the following diagrams commute:

(i)
$$\begin{array}{ccc} \Gamma(\mathbf{M}_0(\xi_2, \eta_2)) \times \Gamma(\mathbf{M}_0(\eta_2, \zeta_2)) & \to & \Gamma(\mathbf{M}_0(\xi_2, \zeta_2)) \\ {\scriptstyle \mathbf{M}_0^*(\varphi_{21}, \psi_{21}) \times \mathbf{M}_0^*(\psi_{21}, \theta_{21})}\downarrow & & \downarrow{\scriptstyle \mathbf{M}_0^*(\varphi_{21}, \theta_{21})} \\ \Gamma(\mathbf{M}_0(\xi_1, \eta_1)) \times \Gamma(\mathbf{M}_0(\eta_1, \zeta_1)) & \to & \Gamma(\mathbf{M}_0(\xi_1, \zeta_1)) \end{array}$$

(ii)
$$\begin{array}{ccc} \Gamma(\mathbf{M}_0(\xi_i, \eta_i)) \times \Gamma(\mathbf{M}_0(\eta_i, \zeta_i)) & \to & \Gamma(\mathbf{M}_0(\xi_i, \zeta_i)) \\ {\scriptstyle \mathbf{M}^0(\xi_i, \eta_i)_* \times \mathbf{M}^0(\eta_i, \zeta_i)_*}\uparrow & & \uparrow{\scriptstyle \mathbf{M}^0(\xi_i, \zeta_i)_*} \\ \Gamma(C^*U(\xi_i, \eta_i)) \times \Gamma(C^*U(\eta_i, \zeta_i)) & \to & \Gamma(C^*U(\xi_i, \zeta_i)) \end{array}$$

Proof. (2.9.1) Let (γ_{kl}) be the coordinate transformations of $\mathbf{M}_0(\xi_j, \eta_j)$, then both $\mathbf{M}_0(f_{ji}^* \xi_j, f_{ji}^* \eta_j)$ and $f_{ji}^* \mathbf{M}_0(\xi_j, \eta_j)$ have the same coordinate transformations $(\gamma_{kl} \circ f_{ji})$ and as $\xi_i \cong f_{ji}^* \xi_j$ and $\eta_i \cong f_{ji}^* \eta_j$,

$$\mathbf{M}_0(\xi_i, \eta_i) \cong \mathbf{M}_0(f_{ji}^* \xi_j, f_{ji}^* \eta_j) \cong f_{ji}^* \mathbf{M}_0(\xi_j, \eta_j).$$

(2.9.2) Let $\mathbf{M}_0(\varphi_{ij}, \psi_{ij})$ be the bundle map in the sense of [St; 51: 2.5], $\mathbf{M}_0(\varphi_{ij}, \psi_{ij})$: $\mathbf{M}_0(\xi_j, \eta_j) \to \mathbf{M}_0(\xi_i, \eta_i)$ which exists by 2.9.1, and $\mathbf{M}_0^*(\varphi_{ij}, \psi_{ij})$ the map assigning to $s \in \Gamma(\mathbf{M}_0(\xi_i, \eta_i))$ the unique $f_{ij}^* s \in \Gamma(M_0(\xi_j, \eta_j))$ such that

$$(*) \qquad \mathbf{M}_0(\varphi_{ij}, \psi_{ij})((f_{ij}^* s)(x)) = s(f_{ij}(x))$$

[St; 51: 2.11]. By [St; 51: 2.5 (17)]

$$\mathbf{M}_0(\varphi_{32}, \psi_{32}) \circ \mathbf{M}_0(\varphi_{21}, \psi_{21}) = \mathbf{M}(\varphi_{31}, \psi_{31})$$

and (i) follows from (*) and uniqueness in [St; 51: 2.11].

By looking at trivializations, we find that

$$\mathbf{M}_0(\varphi_{21}, \psi_{21}) \circ M^0(\xi_1, \eta_1) = M^0(\xi_2, \eta_2) \circ \mathrm{T}$$

and hence uniqueness in [St; 51: 2.11] implies (ii).

(2.9.3) We note that matrix multiplication defines the operations in question over trivial parts and the fact that the coordinate transformations of the multiplicity bundles are given by the multiplicity matrices of the coordinate transformations of the corresponding C^*-bundles, so that the operation

$$\Gamma(\mathbf{M}_0(\xi_i, \eta_i)) \times \Gamma(\mathbf{M}_0(\eta_i, \zeta_i)) \to \Gamma(\mathbf{M}_0(\xi_i, \zeta_i))$$

is well defined.

Again, using that (i) and (ii) hold fibrewise, hence over trivial parts of the bundles and using (2.1.3) for (ii), we conclude that (i) and (ii) are true. ∎

Let us from now on consider the full subcategory $S(FC_R^*)$ of $S(C_R^*)$ having as its objects those $\xi = (\mathbf{X}, (\xi_i)_D)$ such that each ξ_i is locally trivial with finite dimensional fibres. For $\xi = (\mathbf{X}, (\xi_i)_D)$ and $(V_i)_D \in N(\mathbf{X})$ we define

$$MC_R^0((V_i)_D, (\xi_i)_D) = \prod_{i \in D} \Gamma(M_0(\xi_i|_{V_i})) \quad \text{and}$$

$$MC_R^1((V_i)_D, (\xi_i)_D) = \prod_{(i,j) \in R} \Gamma(M_0(\xi_i|_{V_{ij}}, \xi_j|_{V_{ij}})).$$

Then (2.9.3) allows us to define

$$MZ_R^1((V_i)_D, (\xi_i)_D) = \{(s_{ij}) \in MC_R^1(V_i)_D, (\xi_i)_D): \ s_{ik} s_{kj} = s_{ij} \text{ over } V_{ijk}\}.$$

The group $MC_R^0((V_i)_D, (\xi_i)_D)$ acts on $MZ_R^1((V_i)_D, (\xi_i)_D)$ in an obvious way again by (2.9.3) and we define $MH_R^1((V_i)_D, (\xi_i)_D)$ to be the orbit set of that action. Finally, we let

$$MH_R^1(\xi) = \lim_{\rightarrow} MH_R^1((V_i)_D, (\xi_i)_D),$$

where $(V_i)_D$ runs through $N(\mathbf{X})$.

Now let f: $\mathbf{X} = (X, (A_i)_D, (U_i)_D) \to \mathbf{Y}$ and $\xi = (\mathbf{X}, (\xi_i)_D)$ and $\eta = (\mathbf{Y}, (\eta_i)_D)$ be objects of $S(FC_R^*)$ and $\Phi = (\Phi_i)_D$: $\xi \to \eta$ with Φ_i: $\xi_i \to \eta_i$ strong over $f|U_i$

and let $V = (V_i)_D \in N(Y)$. Again let $V^* = (f^{-1}(V_i) \cap U_i)_D \in N(\mathbf{X})$. By (2.9.3) (i), there is a map $MH^1_R(V, \Phi)$: $MH^1_R((V_i)_D, (\eta_i)_D) \to MH^1_R(V^*, (\xi_i)_D)$, which obviously commutes with "restriction" to smaller elements of $N(\mathbf{Y})$, thus defining $MH^1_R(\Phi)$: $MH^1_R(\eta) \to MH^1_R(\xi)$.

From (2.9.2) (i), we conclude that MH^1_R is a contravariant functor from $S(FC^*_R)$ to the category of sets.

By assigning multiplicities, we also obtain in a similar way a map $M(\xi)$: $\Pi H^1_R(\xi) \to MH^1_R(\xi)$, using the remark preceding 2.8 and Proposition 2.9, 2.9.3 (ii). From 2.9.2 (ii) we conclude that M: $\Pi H^1_R \to MH^1_R$ is a natural transformation.

3. Finite order C^*-bundles

If we want to classify finite order C^*-bundles, we have to assume that the index set D for the partition of the base space is finite. Then let

$$D_1 = \{\alpha \in D\colon \text{ for all } \beta \neq \alpha\colon (\beta, \alpha) \notin R\}$$

and for $k \geqslant 1$

$$D_{k+1} = \{\alpha \in D \setminus \bigcup_{l=1}^{k} D_l\colon \beta \in D \setminus \bigcup_{l=1}^{k} D_l, \beta \neq \alpha\colon (\beta, \alpha) \notin R\}$$

As D is finite, there is an $n \in N$ such that $D_n \neq \emptyset$ and $D_{n+1} = \emptyset$. Let $\tilde{D} = \{1, \ldots, n\}$, $\tilde{R} = \{(i, j)\colon i \leqslant j\}$, $\tilde{A}_i = \bigcup_{\alpha \in D_i} A_\alpha$ and $\tilde{U}_i = \bigcup_{\alpha \in D_i} U_\alpha$ for $1 \leqslant i \leqslant n$ and $\tilde{\mathbf{X}} = (X, (\tilde{A}_i)_{i=1}^n, (\tilde{U}_i)_{i=1}^n)$. Note that $U_\alpha \cap U_\beta = \emptyset$ if $\alpha, \beta \in D_i$, $\alpha \neq \beta$. Finally, let $\tilde{\xi} = (\tilde{\mathbf{X}}, (\tilde{\xi}_i)_{\tilde{D}})$ where $\tilde{\xi}_i$ is defined such that $\tilde{\xi}_i | U_\alpha = \xi_\alpha$ for $\alpha \in D_i$.

3.1. Lemma. *If for $\alpha \in D$ the set $\{\beta\colon (\alpha, \beta) \in R\}$ is empty, then*

$$U_\alpha = A_\alpha.$$

Proof. $A_\alpha \subseteq U_\alpha$, so we show $U_\alpha \subseteq A_\alpha$ or equivalently

$$U_\alpha \cap \bigcup_{\beta \neq \alpha} A_\beta = \emptyset.$$

If not, then for some β, $U_\alpha \cap A_\beta \neq \emptyset$, thus $U_\alpha \cap U_\beta \neq \emptyset$. Therefore, $(\beta, \alpha) \in R$, since by assumption $(\alpha, \beta) \notin R$. By definition of TOP_R, $U_\alpha \cap \bigcup \{A_\gamma\colon (\gamma, \alpha) \in R, \gamma \neq \alpha\} = \emptyset$, so $U_\alpha \cap A_\beta = \emptyset$, which is a contradiction. ∎

3.2. Proposition. *$\tilde{\mathbf{X}}$ is an object of $\mathrm{TOP}_{\tilde{R}}$ and $\xi \to \tilde{\xi}$ determines a functor $S(FC^*_R) \to S(FC^*_{\tilde{R}})$, where a family $(\varphi_\alpha)_D$ of strong maps gets mapped to $(\tilde{\varphi}_i)_{\tilde{D}}$ and $\tilde{\varphi}_i$ is defined by $\tilde{\varphi}_i | U_\alpha = \varphi_\alpha$ for $\alpha \in D_i$.*

Proof. Obviously, $\bigcup_{i=1}^{n} \tilde{A}_i = X$, $\tilde{A}_i \cap \tilde{A}_k = \emptyset$ if $i \neq k$ and $\tilde{A}_i \subseteq \tilde{U}_i$. By 3.1, $\tilde{A}_n = \tilde{U}_n$ is open, hence $\bigcup_{i<n} \tilde{A}_i$ closed and $(\bigcup_{i<n} \tilde{A}_i) \cap \tilde{U}_n = (\bigcup_{i<n} \tilde{A}_i) \cap \tilde{A}_n = \emptyset$. Now

suppose $\bigcup_{i\leqslant k} \tilde{A}_i$ is closed in X. Then again by 3.1, $\bigcup_{i<k} \tilde{A}_i$ is closed in $\bigcup_{i\leqslant k} \tilde{A}_i$, hence closed in X. Also

$$\bigcup_{i<k} \tilde{A}_i \cap \tilde{U}_k = (\bigcup_{i<k} \tilde{A}_i) \cap (\bigcup_{i\leqslant k} \tilde{A}_i \cap \tilde{U}_k) = \bigcup_{i<k} \tilde{A}_i \cap \tilde{A}_k = \emptyset.$$

Finally, the statement "$\tilde{U}_i \cap \tilde{U}_j \neq \emptyset$ implies $i \leqslant j$ or $j \leqslant i$" is trivial. Thus $\tilde{\mathbf{X}}$ is in $\mathrm{TOP}_{\tilde{R}}$ and this assignment is clearly functorial.

Similarly, $\xi = (\mathbf{X}, (\xi_\alpha)_D) \mapsto \tilde{\xi} = (\tilde{\mathbf{X}}, (\tilde{\xi}_i)_{i=1}^n)$ is functorial. ∎

3.3. Lemma. *The maps* $N(\mathbf{X}) \to N(\tilde{\mathbf{X}})$, $(V_\alpha)_D \mapsto (\tilde{V}_i)_{\tilde{D}}$ *where* $\tilde{V}_i = \bigcup_{\alpha\in D_i} V_\alpha$, $Z^1_R((V_\alpha)_D, (\xi_\alpha)_D) \to Z^1_{\tilde{R}}((\tilde{V}_i)_{\tilde{D}}, (\tilde{\xi}_i)_{\tilde{D}})$, $(\varphi_{\alpha\beta}) \mapsto (\tilde{\varphi}_{ij})$ *where* $\tilde{\varphi}_{ij} | V_{\alpha\beta} = \varphi_{\alpha\beta} =$ *and* $C^0_R((V_\alpha)_D, (\xi_\alpha)_D) \to C^0_{\tilde{R}}((\tilde{V}_i)_{\tilde{D}}, (\tilde{\xi}_i)_{\tilde{D}})$, $(\varphi_\alpha) \mapsto (\tilde{\varphi}_i)_D$, *where* $\tilde{\varphi}_i | V_\alpha = \varphi_\alpha$ *are bijections.*

Proof. The latter two maps are well defined since for $\alpha, \beta \in D_i$, $U_\alpha \cap U_\beta = \emptyset$ if $\alpha \neq \beta$. Using this fact, the proof is straightforward.

3.4. Proposition. *The map* $\varphi(\xi)$: $\Pi H^1_R(\xi) \to \Pi H^1_{\tilde{R}}(\tilde{\xi})$ *sending the class of* $(\varphi_{\alpha\beta})$ *to* $(\tilde{\varphi}_{ij})$ *defines a natural equivalence*

$$\varphi\colon \Pi H^1_R \to \Pi H^1_{\tilde{R}} \circ \sim .$$

Proof. Applying 3.3 to ξ and $\xi \times I$ gives that $\varphi(\xi)$ is a well defined bijection and naturality is straightforward. ∎

Hence we assume from now on without loss of generality that $D = \{1, \ldots, n\}$ and $R = \{(i, j)\colon i \leqslant j\}$.

We list some properties of $C^*U(\xi, \eta)$ for locally trivial C^*-bundles ξ and η.

3.5. Proposition. *Let ξ and η be locally trivial C^*-bundles with finite dimensional fibres over the paracompact Hausdorff space X, Y a metric space and $A \subseteq Y$ closed.*

Let E be the total space of $C^*U(\xi, \eta)$ and $p\colon E \to X$ the projection. Let $\bar{H}\colon Y \times I \to X$ a homotopy, $H'\colon A \times I \to E$ such that $pH'(x, t) = \bar{H}(x, t)$ for $x \in A$ and $h\colon Y \times \{0\} \to E$ such that $ph(x, 0) = \bar{H}(x, 0)$ and for $x \in A$, $h(x, 0) = H'(x, 0)$ be given. Then there is $H\colon Y \times I \to E$ such that $p \circ H = \bar{H}$, $H | A \times I = H'$ and $H | Y \times \{0\} = h$.

Proof. By [Du; 79: 3.5] $\bar{H}^* C^* U(\xi, \eta) \cong C^* U(\bar{H}^*\xi, \bar{H}^*\eta)$ has the NEP. Thus the section s over $A \times I \cup Y \times \{0\}$ defined by $s(x, t) = (x, t, H'(x, t))$ if $x \in A$ and $s(x, 0) = (x, 0, h(x, 0))$ extends to a neighborhood U of $A \times I$. Since I is compact, there is an open set $V \supseteq A$, so that $A \times I \subseteq V \times I \subseteq U$. Let $H''\colon V \times I \to E$ be defined by $H'' = \psi \circ s$, where $\psi\colon \bar{H}^* C^* U(\xi, \eta) \to C^* U(\xi, \eta)$ is the strong map covering $\bar{H}$. The metric d gives a function $\tau\colon X \to I$ such that $\tau^{-1}(0) = Y \backslash V$ and $\tau^{-1}(1) = A$, where $\tau(x) = d(x, Y \backslash V) \cdot (d(x, A) + d(x, Y \backslash V))^{-1}$. As $C^* U(\xi, \eta)$ is locally trivial, it has the CHP [Do; 63: Def. 4.1, Thm. 4.8] and the existence of H follows from the definition of the CHP.

3.6. COROLLARY. *With* ξ, η, E, P *as in* 3.5, X *a metric space and* π: $X\times I\to X$, $(x, t)\mapsto x$ *and* H: $X\times I\to E$ *such that* $p\circ H=\pi$, *suppose that* $A\subseteq X$ *is closed and that for* $x\in A$

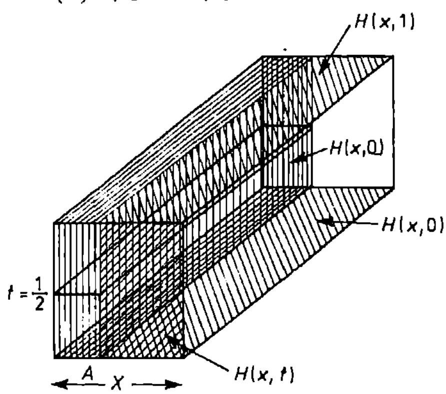
$$H(x, t)=H(x, 1-t).$$

Then $\mu_0^* H\simeq\mu_1^* H$ *rel* A.

Proof. Define G: $X\times I\times I\to E$ over π^1: $X\times I\times I\to X$, $\pi^1(x, t, s)=x$ in the following way

$$G(x, t, s)=\begin{cases} H(x, t) & \text{if } s=0 \text{ or } t\in\{0, 1\},\\ H\big(x, t(1-s)\big) & \text{if } x\in A \text{ and } t\in\left[0, \frac{1}{2}\right],\\ H\big(x, (1-t)(1-s)\big) & \text{if } x\in A \text{ and } t\in\left[\frac{1}{2}, 1\right]\end{cases}$$

(see the figure). G is defined over $X\times I\times\{0\}\cup A\times I\times I$ and extends to $X\times I\times I$ by 3.5. Then $G|X\times I\times\{1\}$: $\mu_0^* H\simeq\mu_1^* H$ rel A.



3.7. PROPOSITION. *Let* $F=M(p)$ *and* $G=M(q)$ *be finite dimensional* C^**-algebras,* $a\in\mathbf{M}_0(p, q)$ *and* $\bar{a}=(a, \delta a)$ *the extended matrix. Furthermore, let* X *be a paracompact Hausdorff space such that either* X *is contractible or* X *deformation retracts to* $\bigvee_{i=1}^{n} S^N=\underbrace{S^N\vee\ldots\vee S^N}_{n\text{-times}}$ *for some* $n, N\in\mathbf{N}$ *such that* $\pi_{N-1}(U(\bar{a}))=0$. *Then every map* φ: $X\to U(q)/U(\bar{a})\subseteq C^*U(F, G)$ *lifts to a map* $\tilde{\varphi}$: $X\to U(q)$, *that is* $\varphi(x)=\mathrm{Ad}[\tilde{\varphi}(x)]\circ\alpha_a$.

Proof. A lifting corresponds to a cross-section of $\varphi^*\mathbf{B}$, where $\mathbf{B}$ is the principal $U(\bar{a})$-bundle over $U(q)/U(\bar{a})$. $\mathbf{B}$ has total space $U(q)$ [St; 51: 7.4, 7.5, 8.15]. By [St; 51: 10.5], $\varphi^*\mathbf{B}$ is a principal $U(\bar{a})$-bundle. If X is contractible, $\varphi^*\mathbf{B}$ is trivial. Thus the cross-section of $\varphi^*\mathbf{B}$ — and hence $\tilde{\varphi}$ — exists.

Now suppose that X deformation retracts to $\bigvee_{i=1}^{N} S^n$ with $\pi_{N-1}(U(\bar{a}))=0$.

For $i = 1, \ldots, n$, let X_i be the ith copy of S^N in $\bigvee_{i=1}^{n} S^N \subseteq X$. Then $\pi_{N-1}(U(\bar{a})) = 0$ implies that $\mu_0^* \varphi^* \mathbf{B}|X_i$ is trivial [Hu; 75: 4.9.7] for all i and hence there is a section s_i of $\varphi^* \mathbf{B}|X_i$. Define a section s of $\varphi^* \mathbf{B}|\bigvee_{i=1}^{n} S^N$ inductively in the following way: $s|X_1 = s_1$. Suppose $s|X_1 \cup \ldots \cup X_k$ is defined. Then with $\{x_0\} = X_k \cap X_{k+1}$, $s(x_0) = s_{k+1}(x_0)\cdot(x_0, u_{k+1})$ for some $u_{k+1} \in U(\bar{a})$ (identifying $\varphi^* \mathbf{B}|X_{k+1}$ with the product bundle). We define for $x \in X_{k+1}$, $s(x) = s_{k+1}(x)\cdot(x, u_{k+1})$. Thus $\varphi^* \mathbf{B}|\bigvee_{i=1}^{n} S^N$ has a section and is hence trivial. As $\bigvee_{i=1}^{n} S^N$ is a deformation retract of X, $\varphi^* \mathbf{B}$ is equivalent to the trivial bundle [Hu; 75: 4.9.7, p. 50]. Thus $\varphi^* \mathbf{B}$ has a section and consequently, φ has a lifting $\tilde{\varphi}$. ∎

Now we are ready to introduce the inductive calculation of $\Pi \dot{H}_R^1$. So let $D = \{1, \ldots, n\}$, $R = \{(i, j) \in D \times D\colon i \leqslant j\}$, X a metric space, and $\mathbf{X} = (X, (A_i)_D, (U_i)_D) \in \mathrm{TOP}_R$. Also let $\xi = (\mathbf{X}, (\xi_i)_D) \in S(FC_R^*)$ and $(V_i)_D \in N(\mathbf{X})$. We assume without loss of generality that V_k is closed in $X \setminus (\bigcup_{i=1}^{k-1} A_i)$, abbreviate $E_{ij} = C^* U(\xi_i | V_{ij}, \xi_j | V_{ij})$ and define for $n \geqslant 2$, $l = 1, \ldots, n-1$.

$$ {}_lZ^1 = \{(\varphi_{ij}) \in \prod_{i-j \leqslant l} \Gamma(E_{ij})\colon j \leqslant k \leqslant i\colon \ \varphi_{kk} = \mathrm{id} \ \text{ and } $$
$$ \text{on } V_{ikj} \ \ \varphi_{ik} \circ \varphi_{kj} = \varphi_{ij}\}. $$

Similarly, let ${}_lZ^1 \times I$ be the set of all $(\varphi_{ij}) \in \prod_{i-j \in l} \Gamma(E_{ij} \times I)$ satisfying the same conditions as the elements of ${}_lZ^1$. Again $(\varphi_{ij}^0) \simeq (\varphi_{ij}^1)$ in ${}_lZ^1$, if there is a $(\varphi_{ij}) \in {}_lZ^1 \times I$ with $\varphi_{ij}^t = \mu_t^* \varphi_{ij}$ for $t = 0, 1$, and $\Pi_l Z^1$ denotes the set of equivalence classes. ΠC^0 acts on $\Pi_l Z^1$ in the usual way and we denote the orbit set by $\Pi_l H^1$ and denote by $[\![\varphi_{ij}]\!]_l$ the natural image of $(\varphi_{ij}) \in {}_lZ^1$ in $\Pi_l H^1$. For $l = 2, \ldots, n-1$, we get obvious maps $\Phi_l\colon \Pi_l H^1 \to \Pi_{l-1} H^1$ by ignoring those φ_{ij} for which $i - j = l$.

3.8. Let us determine the image of Φ_l. For $i = j + l$, let $V_{ij}^l = \bigcup_{j<k<i} V_{ikj}$. Then $[\![\varphi_{ij}]\!]_l \in \operatorname{im} \Phi_l$ if for all i, j such that $i = j + l$, the sections φ_{ij}' of $E_{ij}| V_{ij}^l$, defined by

$$ \varphi_{ij}' | V_{ikj} = \varphi_{ik} \circ \varphi_{kj} \tag{3.8.1} $$

extend to all of V_{ij}. Note that the φ_{ij}' are well defined because of the cocycle condition.

Proof. Clearly, if the extensions exist, they define together with $(\varphi_{ij}) \in {}_{l-1}Z^1$ — an element in ${}_lZ^1$ and hence in $\Pi_l H^1$, which gets mapped to $[\![\varphi_{ij}]\!]_{l-1} \in \Pi_{l-1} H^1$.

Conversely, suppose $[\![\varphi_{ij}]\!]_{l-1} \in \operatorname{im} \Phi_l$. Then there is a $[\![\psi_{ij}]\!]_l \in \Pi_l H^1$ such that $\Phi_l [\![\psi_{ij}]\!]_l = [\![\varphi_{ij}]\!]_{l-1}$. If for $i-j<l$, $\varphi_{ij} = \psi_i \psi_{ij} \psi_j^{-1}$ for some $(\psi_i) \in \Pi C^0$, then $\varphi_{ij} = \psi_i \psi_{ij} \psi_j^{-1}$ is the desired extension for $i=j+l$. If for $i-j<l$ there is $(H_{ij}) \in {}_{l-1}Z^1 \times I$ with $\mu_0^* H_{ij} = \psi_{ij}$ and $\mu_1^* H_{ij} = \varphi_{ij}$, then define for $i=j+l$

$$H_{ij}(x,t) = \begin{cases} \psi_{ij}(x) & \text{if } t=0, \\ H_{ik}(x,t) \circ H_{kj}(x,t) & \text{if } x \in V_{ikj},\ j \leqslant k \leqslant i \end{cases}$$

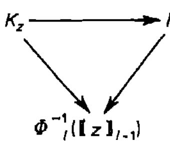
and extend H_{ij} to V_{ij} via 3.5. $\mu_1^* H_{ij}$ gives for $i=j+l$, the desired extension of φ'_{ij}. ∎

While 3.8 is somewhat obvious, a way to calculate the inverse image of $[\![\varphi_{ij}]\!]_{l-1}$ under Φ_l in general is not yet known. We will provide a method that allows for calculation at least for some applications. So for $z = (\varphi_{ij}) \in {}_{l-1}Z^1$, define K_z to be the set of $(n-l)$-tuples $[\psi_{ij}]$, $i=j+l$, of homotopy classes relative to V^l_{ij} of sections ψ_{ij} of E_{ij} for which $\psi_{ij} | V_{ikj} = \varphi_{ik} \circ \varphi_{kj}$, $j<k<i$.

3.9. Proposition. *There is a surjection*

$$K_z \to \Phi_l^{-1}([\![z]\!]_{l-1})$$

If $z' \in {}_{l-1}Z^1$ *and* $[\![z']\!]_{l-1} = [\![z]\!]_{l-1}$, *then there is a bijection* $K_z \to K_{z'}$ *such that*



commutes.

Proof. Let $z = (\varphi_{ij})$, $i-j<l$, such that $[\![z]\!]_{l-1} \in \operatorname{im} \Phi_l$ and $[\psi_{ij}] \in K_z$, $i=j+l$. Then the (ψ_{ij}) together with z define an element in ${}_lZ^1$ whose image in $\Pi_l H^1$ gets mapped to $[\![z]\!]_{l-1}$ under Φ_l. If for $i=j+l$, H_{ij} is a homotopy relative to V^l_{ij} between ψ^0_{ij} and ψ^1_{ij}, and we define for $i-j<l$, $H_{ij}(x,t) \equiv \varphi_{ij}(x)$, then we see that our map is well defined, and surjectivity follows from 3.8.

Now let $z' = (\varphi'_{ij}) \in {}_{l-1}Z^1$.

Case 1. If there is a $(\varphi_i) \in C^0$ such that $\varphi'_{ij} = \varphi_i \varphi_{ij} \varphi_j^{-1}$, then we define the map $K_z \to K_{z'}$ by $[\psi_{ij}] \mapsto [\varphi_i \psi_{ij} \varphi_j^{-1}]$.

Case 2. If there are $(H_{ij}) \in {}_{l-1}Z^1 \times I$ such that $\mu_0^* H_{ij} = \varphi_{ij}$ and $\mu_1^* H_{ij} = \varphi'_{ij}$ for $i-j<l$, then let, for $i=j+l$ and $(x,t) \in V_{ij} \times \{0\} \cup V^l_{ij} \times I$

$$H_{ij}(x,t) = \begin{cases} \psi_{ij}(x) & \text{if } t=0, \\ H_{ik}(x,t) \circ H_{kj}(x,t) & \text{if } x \in V_{ikj},\ j<k<1 \end{cases} \tag{3.9.1}$$

and extend it via 3.5 to V_{ij}. We then define the map $K_z \to K_{z'}$ by $[\psi_{ij}] \mapsto [\mu_1^* H_{ij}]$.

Now suppose that H_{ij}: $\psi^0_{ij} \simeq \psi^1_{ij}$ rel V^l_{ij}, $[\psi^t_{ij}] \in K_z$ for $t=0$, 1, and that

$$G_{ij}(x,t)=\begin{cases} H^0_{ij}(x,1-3t) & t\in\left[0,\frac{1}{3}\right], \\ H_{ij}(x,3t-1) & t\in\left[\frac{1}{3},\frac{2}{3}\right], \\ H^1_{ij}(x,3t-2) & t\in\left[\frac{2}{3},1\right] \end{cases}$$

H^t_{ij}, $t = 0, 1$, $i = j+l$ are homotopies so that $[\psi^t_{ij}]\mapsto[\mu_1^* H^t_{ij}]$ like before. Define then for $t = 0, 1$, $\mu_t^* G_{ij} = \mu_1^* H^t_{ij}$ and for $x\in V^l_{ij}$, $G_{ij}(x, t) = G_{ij}(x, 1-t)$. Hence 3.6 applies and yields that

$$\mu_1^* H^0_{ij} \simeq \mu_1^* H^1_{ij} \text{ rel } V^l_{ij}$$

or $K_z \to K_{z'}$ is well defined.

The inverse function is defined in the obvious way in Case 1. In Case 2, define for $i-j<l$, $H^*_{ij}(x,t) = H_{ij}(x, 1-t)$, and for $i-j=l$ and $(x,t) \in V^l_{ij}\times I \cup V_{ij}\times\{0\}$ and $[\psi'_{ij}]\in K_{z'}$

$$H^*_{ij}(x,t)=\begin{cases} \psi'_{ij}(x) & t=0, \\ H^*_{ik}(x,t)\circ H^*_{kj}(x,t) & \text{if } x\in V_{ikj},\ j<k<i \end{cases}$$

and extend to $V_{ij}\times I$ via 3.5.

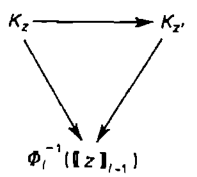
Let $K_{z'}\to K_z$ be defined by $[\psi'_{ij}]\mapsto[\mu_1^* H^*_{ij}]$. Now all that is left to show is that if $\psi'_{ij} = \mu_1^* H_{ij}$, then $\mu_1^* H^*_{ij} \simeq \mu_0^* H_{ij} = \psi_{ij}$ rel V^l_{ij}. Define

$$G_{ij}(x,t)=\begin{cases} H_{ij}(x,2t) & t\in\left[0,\frac{1}{2}\right], \\ H^*_{ij}(x,2t-1) & t\in\left[\frac{1}{2},1\right]. \end{cases}$$

Applying 3.6 yields that $\mu_1^* G_{ij} = \mu_1^* H^*_{ij} \simeq \mu_0^* G_{ij} = \psi_{ij}$. Note that the map $K_z\to K_{z'}$ depends on the cocycle homotopy $z\simeq z'$, but a calculation almost identical to the one for the inverse function shows that the map is independent of the extension of 3.9.1.

For general z, z', such that z is equivalent to z' in $\Pi_{l-1}H^1$, then the map $K_z\to K_{z'}$ is a composition of two maps according to Cases 1 and 2.

The commutativity of



is obvious. ■

3.10. *Remark.* In a similar way we can define on the level of multiplicity-cohomology the sets

$$MH^1_R = M_{n-1}H^1 \xrightarrow[\Psi_{n-1}]{} M_{n-2}H^1 \xrightarrow[\Psi_{n-2}]{} \dots \to M_2 H^1 \xrightarrow[\Psi_2]{} M_1 H^1$$

so that the diagrams

$$\begin{array}{ccc} \Pi_l H^1 & \xrightarrow{\Phi_l} & \Pi_{l-1} H^1 \\ {\scriptstyle M_{l*}} \downarrow & & \downarrow {\scriptstyle M_{l-1}} \\ M_l H^1 & \longrightarrow & M_{l-1} H^1 \end{array}$$

commute, where the M_l are the maps assigning the corresponding multiplicities.

4. Third order C^*-bundles with finite dimensional fibres over cones over pairs of compact Riemannian manifolds

Throughout this section we assume that M is a compact Riemannian manifold with metric g, $N \subseteq M$ a compact submanifold possessing a tubular neighborhood in M. If M and N are connected and have the same dimensions, then we consider the union of a tubular neighborhood of the boundary of N with N to be a tubular neighborhood of N.

Let $X = CM$ the cone over M. We consider $M \times [0, 1)$ as a subset of X in the obvious way. Furthermore, we assume that all C^*-bundles are locally trivial with finite dimensional fibres.

Let $R = \{(i, j)\colon 1 \leqslant j \leqslant i \leqslant 3\}$, $A_1 = N \times \{0\}$, $A_2 = (M \backslash N) \times \{0\}$ and $A_3 = X \backslash (M \times \{0\}$ and let U_i, $i = 1, 2, 3$, be neighborhoods of A_i so that $\mathbf{X} = (X, (A_i), (U_i)) \in \mathrm{TOP}_R$. Furthermore, let $\xi = (\mathbf{X}, (\xi_i)) \in S(FC^*_R)$.

4.1. Lemma. *Let T be a closed tubular neighborhood of N in M. We can consider $T \subseteq B_1(TM)$, the unit disk bundle in TM and therefore have a norm function $\|\cdot\|\colon T \to [0, 1]$, $\|x\| = \sqrt{g(x, x)}$.*

Let $0 < c < 1$ and $F\colon T \to [0, c]$ so that $F^{-1}(c) = \partial T$ and $F^{-1}(0) = N$. Then there is an ε, $0 < \varepsilon < c$ and a function $f\colon [0, 1] \to [0, \varepsilon]$ such that

(i) $f^{-1}(\varepsilon) = \{1\}, f^{-1}(0) = \{0\}$,

(ii) *if $t < s$, then $f(t) < f(s)$,*

(iii) $f(\|x\|) < F(x)$ *for all* $x \in T \backslash N$.

Proof. For $n \in N$, define

$$f\left(\frac{1}{n}\right) = \frac{1}{n+1} \cdot \min\left\{F(x)\colon \frac{1}{n+1} \leqslant \|x\| \leqslant 1\right\}$$

and for $t \in \left[\frac{1}{n}, \frac{1}{n+1}\right]$,

$$f(t) = \left[\left(f\left(\frac{1}{n}\right) - f\left(\frac{1}{n+1}\right)\right)\left(t - \frac{1}{n+1}\right)\right] \cdot \left(\frac{1}{n} - \frac{1}{n+1}\right)^{-1} + f\left(\frac{1}{n+1}\right)$$

and $f(0) = 0$.

By construction, f has the desired properties. ■

4.2. PROPOSITION. *The sets* $(V_1, V_2, V_3) \in N(\mathbf{X})$ *of the form*

$$V_1 = T \times [0, \varepsilon],$$
$$V_2 = \{(x, s) \in (M \backslash N) \times [0, \varepsilon]: s \leqslant f(\|x\|) \text{ if } x \in T\} \quad \text{and}$$
$$V_3 = X \backslash (M \times \{0\})$$

where T *is a closed tubular neighborhod of* N *in* M, $0 < \varepsilon < 1$ *and* f: $[0, 1] \to [0, \varepsilon]$ *satisfies* 4.1 (i) *and* (ii) *are cofinal in* $N(\mathbf{X})$.

Proof. V_3 is clear and obviously, V_1 can be chosen of the form $T \times [0, d]$ for some closed tubular neighborhood T of N and $0 < d < 1$. Then, since $T \backslash \mathring{N}$ is compact, V_2 can be chosen of the form $\{(x, s) \in (M \backslash N) \times [0, c]: s \leqslant F(x)$ if $x \in T\}$ where F: $M \to [0, c]$ and $F^{-1}(c) = M \backslash \mathring{T}$ and $F^{-1}(0) = N$. Now we can apply 4.1 to obtain the desired result. ■

It is clear that we are trying here to obtain a result analogous to [Du; 79: 9.2]. The following lemma provides the main tool for this.

Until further notice we fix $(V_i) \in N(\mathbf{X})$, T, ε and f as in 4.2.

4.3. LEMMA. V_2, V_{12}, V_{23}, V_{13} *and* V_{123} *deformation retract strongly to* $(M \backslash \mathring{T}) \times \{\varepsilon\}$, $\partial T \times \{\varepsilon\}$, $(M \backslash \mathring{T}) \times \{\varepsilon\}$, $T \times \{\varepsilon\}$ *and* $\partial T \times \{\varepsilon\}$ *respectively. Furthermore, these retractions are given by the appropriate restrictions of* $\mu_0^* h$ *for some map* h: $[(M \times [0, \varepsilon]) \backslash (N \times \{0\})] \times [0, 1] \to (M \times [0, \varepsilon]) \backslash (N \times \{0\})$ *such that*

(i) $\mu_1^* h = \mathrm{id}$,

(ii) $h(V_\alpha \times [0, 1]) \subseteq V_\alpha$, $\alpha \in \{1, (1, 2), (2, 3), (1, 3), (1, 2, 3)\}$.

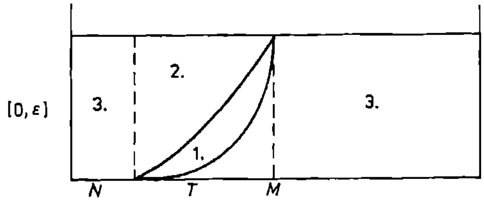
Proof. We define the function

$$h: \{(M \times [0, \varepsilon]) \backslash (N \times \{0\})\} \times I \to (M \times [0, \varepsilon]) \backslash (N \times \{0\})$$

in the following way: by 4.1 (ii), f has an inverse function f^{-1}. Define for $s \in [0, \varepsilon]$, $t \in [0, 1]$, $G(s, t) = 1 - t + t f^{-1}(s)$. Then we define for $(x, s, t) \in ((M \times [0, \varepsilon]) \backslash (N \times \{0\})) \times I$

$$(4.3.1) \qquad h(x, s, t) = \begin{cases} \left(\dfrac{x}{\|x\|} \cdot G(s, t), f(G(s, t))\right) & \text{if } x \in T \backslash \mathring{N} \text{ and } s \leqslant f(\|x\|) \leqslant f(G(s, t)), \\ \left(\dfrac{x}{f^{-1}(s)} \cdot G(s, t), f(G(s, t))\right) & \text{if } x \in T \backslash \mathring{N} \text{ and } f(\|x\|) \leqslant s, \\ (x, f(G(s, t))) & \text{otherwise.} \end{cases}$$

The following picture illustrates the situation



1. is the area where $s \leqslant f(\|x\|) \leqslant f(G(s, t))$. The top line is the set of (x, s) so that $f(\|x\|) = s$, the line below zone 1. is the set of (x, s), so that $\|x\| = G(s, t)$. An element (x, s) of zone 1. then gets first mapped to the element (y, s), where y is the unique vector on the line connecting $\frac{x}{\|x\|}$ and $\frac{x}{\|x\|} \cdot s$ such that $\|y\| = G(s, t)$, and then to $(y, f(\|y\|))$. In the area labelled 2, an element (x, s) gets mapped to $(x, f(G(s, t)))$ and then its norm increased so as to make h continuous.

It is routine to check that h is well defined and continuous, that $\mu_1^* h = \mathrm{id}$, that $h(x, \varepsilon, t) = (x, \varepsilon)$ and that restriction of $\mu_0^* h$ to the respective sets defines retractions onto the corresponding subsets.

Furthermore, for $(x, s) \in V_\alpha$, $\alpha \in \{2, (1, 2), (1, 3), (2, 3), (1, 2, 3)\}$, $h(x, s, t) \in V_\alpha$, which finishes the proof. ∎

Now we let $W_1 = T \times \{\varepsilon\}$, $W_2 = (M \backslash \mathring{T}) \times \{\varepsilon\}$ and $W_3 = X \backslash (M \times [0, \varepsilon))$. Although (W_1, W_2, W_3) is not an element of $N(\mathbf{X})$, the construction following 3.7 still works for (W_1, W_2, W_3) instead of (V_1, V_2, V_3). We denote the resulting sets and maps by $\Pi_2 H^1_W \xrightarrow{\varphi_W} \Pi_1 H^1_W$ and K^W_z for $z \in \operatorname{im} \varphi_W$.

4.4. Proposition. *Restriction to (W_1, W_2, W_3) defines bijections R_2: $\Pi_2 H^1 \to \Pi_2 H^1_W$, R_1: $\Pi_1 H^1 \to \Pi_1 H^1_W$ and R_z: $K_z \to K^W_{z|W}$ such that the diagrams*

(4.4.1)
$$\begin{array}{ccc} \Pi_2 H^1 & \xrightarrow{\varphi_2} & \Pi_1 H^1 \\ {\scriptstyle R_2}\downarrow & & \downarrow{\scriptstyle R_1} \\ \Pi_2 H^1_W & \xrightarrow[\varphi_W]{} & \Pi_1 H^1_W \end{array}$$

and

(4.4.2)
$$\begin{array}{ccc} K_z & \xrightarrow{R_z} & K^W_{z|W} \\ \downarrow & & \downarrow \\ \Phi^{-1}([\![z]\!]_1) & \to & \Phi_W^{-1}([\![z|W]\!]_1) \end{array}$$

commute.

Proof. R_1 is a bijection since X and W_3 are homotopy equivalent.

Now R_2 is well defined and onto, where we extend $\psi_{ij} \in \Gamma(E_{ji} | W_{ji})$ by defining $\varphi_{ij}(x, s) = \psi_{ij}(h(x, s, 0))$ where h is the map defined in 4.3.1.

R_2 is also injective. Suppose first that $(\varphi_{ij}^0), (\varphi_{ij}^1) \in Z_R^1((V_i), (\xi_i))$ and for $(i, j) \in \{(2, 1), (3, 2), (3, 1)\}$, $H_{ij} \in \Gamma(E_{ji} \times I | W_{ij} \times I)$ such that $H_{31} = H_{32} \circ H_{21}$ on $W_{321} \times I$ and $\mu_t^* H_{ij} = \varphi_{ij}^t | W_{ij}$ for $t = 0, 1$. Define for $(i, j) \in R$

$$G_{ij}(x, s, t) = \begin{cases} \varphi_{ij}^0(h(x, s, 1-3t)) & \text{if } t \in \left[0, \frac{1}{3}\right], \\ H_{ij}(h(x, s, 0), 3t-1) & \text{if } t \in \left[\frac{1}{3}, \frac{2}{3}\right], \\ \varphi_{ij}^1(h(x, s, 3t-2)) & \text{if } t \in \left[\frac{2}{3}, 1\right] \end{cases}$$

it is routine to check that $\mu_t^* G_{ij} = \varphi_{ij}^t$ for $t = 0$ or 1 and that on $V_{321} \times I$, $G_{31} = G_{32} \circ G_{21}$. Thus $[\![\varphi_{ij}^0]\!]_2 = [\![\varphi_{ij}^1]\!]_2$.

Now suppose that there are $\psi_i \in \mathrm{Aut}(\xi_i | W_i)$ such that $\varphi_{ij}^1 | W_{ij} = \psi_i \varphi_{ij}^0 \psi_j^{-1} | W_{ij}$. Then let $\varphi_1(x, s) = \psi_1(x, \varepsilon)$ and for $i = 2$ or 3, $\varphi_i(x, s) = \psi_i(h(x, s, 0))$. Obviously, $\varphi_{ij}^1 | W_{ij} = \varphi_i \varphi_{ij}^0 \varphi_j^{-1} | W_{ij}$, in particular $\varphi_{ij}^1 | W_{ij}$ and $\varphi_i \varphi_{ij}^0 \varphi_j^{-1} | W_{ij}$ are cocycle homotopic via constant homotopies, so the previous case applies or there is $(H_{ij}) \in Z_R^1((V_i \times I), (\xi_i \times I))$ such that $\mu_0^* H_{ij} = \varphi_i \varphi_{ij}^0 \varphi_j^{-1}$ and $\mu_1^* H_{ij} = \varphi_{ij}^1$.

We conclude that R_2 is injective. Now let $z = (\varphi_{32}, \varphi_{21}) \in {}_1Z^1$, so that $[\![z]\!]_1 \in \mathrm{im}\, \Phi_2$. Suppose $\psi_{31} \in \Gamma(W_{31}, E_{13} | W_{31})$, so that $\psi_{31} | W_{321} = \varphi_{32} \circ \varphi_{21} | W_{321}$. Define for $(x, s) \in V_{31}$, $t \in I$

$$H(x, s, t) = \begin{cases} \psi_{31}(h(x, s, 0)) & \text{if } t = 0, \\ \varphi_{32}(h(x, s, t)) \circ \varphi_{21}(h(x, s, t)) & \text{if } (x, s) \in V_{321} \end{cases}$$

and use 3.5 to extend H to all of $V_{31} \times I$. Let $\varphi_{31} = \mu_1^* H$. Then $R_z([\varphi_{31}]_{V_{321}}) = [\psi_{31}]_{W_{321}}$. Thus R_z is onto.

Now suppose $\varphi_{31}^t \in \Gamma(V_{31}, E_{13})$, $t = 0, 1$ such that $\varphi_{31}^t(x, s) = \varphi_{32}(x, s) \circ \varphi_{21}(x, s)$ for $(x, s) \in V_{321}$ and suppose furthermore that there is a homotopy H: $\varphi_{31}^0 | W_{31} \simeq \varphi_{31}^1 | W_{31}$ rel W_{321}. Define for $(x, s, t) \in V_{321} \times I$, $\tau \in I$

$$G(x, s, t, \tau) = \begin{cases} H(h(x, s, 0), t) & \text{if } \tau = 0, \\ \varphi_{32}(h(x, s, \tau)) \circ \varphi_{21}(h(x, s, \tau)) & \text{if } (x, s) \in V_{321}, \\ \varphi_{31}^t(h(x, s, \tau)) & \text{if } t = 0, 1 \end{cases}$$

and use 3.5 to extend G to $(V_{321} \times I) \times I$ and let $H_{31}(x, s, t) = G(x, s, t, 1)$. Then $\mu_t^* H_{31} = \varphi_{31}^t$, $H_{31} | V_{321} = \varphi_{32} \circ \varphi_{21}$ and H_{31}: $\varphi_{31}^0 \simeq \varphi_{31}^1$ rel V_{321} or R_z is injective.

Commutativity of 4.4.1 and 4.4.2 is obvious by construction. ∎

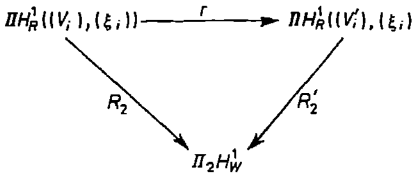
4.4 then allows us to compute ΠH_R^1 over "simpler" sets than whole neighborhoods of the A_i in this case.

4.5. Proposition. *If $(V_i') \subseteq (V_i) \in N(\mathbf{X})$ are given by tubular neighborhoods $T' \subseteq T$ of N, $0 < \varepsilon' \leqslant \varepsilon < 1$ and functions f' and f as in* 4.1, *respectively, then* *the restriction map*

$$\Pi H_R^1((V_i), (\xi_i)) \to \Pi H_R^1((V_i'), (\xi_i))$$

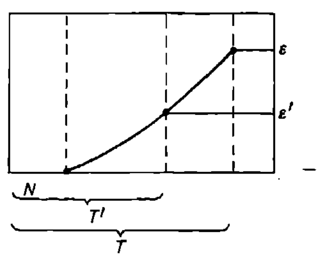
is a bijection.

Proof. First assume that $T' = T$ and $\varepsilon' = \varepsilon$, that is $f'(\|x\|) \leqslant f(\|x\|)$ for all $x \in T \backslash N$. Let again $W_1 = T \times \{\varepsilon\}$, $W_2 = (M \backslash \mathring{T}) \times \{\varepsilon\}$, $W_3 = X \backslash (M \times [0, \varepsilon))$ and $W = (W_i)_{i=1}^3$. Consider the commutative diagram



where all arrows are given by restriction. Since R_2 as well as R_2' are bijective, so is r.

In the general case we assume without loss of generality that $\|x\| \leqslant 1$ for all $x \in T \backslash N$ and $\|x\| \leqslant \varrho < 1$ for all $x \in T' \backslash N$. We construct an increasing function g: $[\varrho, 1] \to [\varepsilon', \varepsilon]$, so that $g(\varrho) = \varepsilon'$, $g(1) = \varepsilon$ and $g(t) \leqslant f(t)$ for $t \in [\varrho, 1]$ in the same way as we constructed f in 4.1. Thus, by the previous case, we can without loss of generality, assume that $f(t) = f'(t)$ if $t \in [0, \varrho]$.



Now there are strong deformation retractions r_{21}: $V_{21} \to V_{21}'$, r_{32}: $V_{32} \to V_{32}'$, r_{31}: $V_{31} \to (V_{31}' \cup V_{321})$ and r_i: $V_i \to V_i'$ for $i = 1, 2$. If $(\varphi_{ij}') \in Z_R^1((V_i'), (\xi_i))$, we use r_{21} and r_{32} to obtain extensions φ_{21} and φ_{32} of φ_{21}' and φ_{32}', respectively. Then we use r_{31} to extend the map defined to be φ_{31}' on V_{31}' and $\varphi_{32} \circ \varphi_{21}$ on V_{321} to obtain φ_{31}. Therefore, r is onto.

Now take (φ_{ij}^0), $(\varphi_{ij}^1) \in Z_R^1((V_i), (\xi_i))$. Suppose there are $(\varphi_i') \in C_R^0((V_i'), (\xi_i))$ and cocycle homotopies H_{ij}': $\varphi_i' \circ \varphi_{ij}^0 | V_{ij}' \circ \varphi_j'^{-1} \simeq \varphi_{ij}^1 | V_{ij}'$. We can use r_2 and r_1 to find extensions φ_2 and φ_1 of φ_2' and φ_1', respectively, and set $\varphi_3 = \varphi_3'$. Then clearly

$$H_{ij}': (\varphi_i \circ \varphi_{ij}^0 \circ \varphi_j^{-1}) | V_{ij}' \simeq \varphi_{ij}^1 | V_{ij}'$$

and as in the proof of 4.4 that R_2 is injective, this implies that there is a cocycle homotopy

$$H_{ij}\colon\ \varphi_i \circ \varphi_{ij}^0 \circ \varphi_j^{-1} \simeq \varphi_{ij}^1$$

or (φ_{ij}^0) and (φ_{ij}^1) are equivalent. Hence r is injective. ■

From now on we fix T and ε and let $W_1 = T \times \{\varepsilon\}$, $W_2 = (M \backslash \mathring{T}) \times \{\varepsilon\}$ and $W_3 = X \backslash (M \times [0, \varepsilon))$.

4.6. Proposition. *If the ξ_i, $i = 1, 2, 3$ are trivial over the connected components of U_i, $i = 1, 2, 3$, if N is either closed or a tubular neighborhood of some closed manifold in M, then for each $(V_i) \in N(\mathbf{X})$ as in 4.2, the map*

$$\Pi H_R^1((V_i), (\xi_i)) \to MH_R^1((V_i), (\xi_i))$$

induced by assigning multiplicities, is onto. Consequently,

$$M(\xi)\colon\ \Pi H_R^1(\xi) \to MH_R^1(\xi)$$

is onto.

Proof. Since the ξ_i are trivial over the connected components of the U_i, we can consider all cocycles, bundle automorphisms and sections of the multiplicity bundles as maps from the connected components of the corresponding base spaces into the corresponding fibres. In particular, sections of the multiplicity bundles are constant maps over each connected component of the respective base spaces. If N is a tubular neighborhood of the closed manifold N', then so is any tubular neighborhood T of N. The arguments of the proof will then remain valid if we replace N by N'. If M or N are not connected, we can do the proof by applying our arguments to each connected component of M and N, respectively. Therefore we assume without loss of generality that N is closed and that M and N are connected.

Now let $(V_i) \in N(\mathbf{X})$ as in 4.2 and again $W_1 = T \times \{\varepsilon\}$, $W_2 = (M \backslash \mathring{T}) \times \{\varepsilon\}$ and $W_3 = X \backslash (M \times [0, \varepsilon))$. Consider the following diagram

$$\begin{array}{ccc} \Pi H_R^1((V_i), (\xi_i)) & \xrightarrow{\ R_2\ } & \Pi H_R^1((W_i), (\xi_i)) = \Pi_2 H_W^1 \\ & {\scriptstyle M}\searrow \quad \swarrow{\scriptstyle M \circ R_2^{-1}} & \\ & MH_R^1((V_i), (\xi_i)) & \end{array}$$

Since R_2 is a bijection, we only need to show that $M \circ R_2^{-1}$ is onto. So let $a \in MH_R^1((V_i), (\xi_i))$ be the canonical image of $(a_{31}, a_{32}, a_{21}) \in MZ_R^1((V_i), (\xi_i))$. Over each component of W_{ij}, $(i, j) = (3, 2)$ or $(2, 1)$, we let $\varphi_{ij} = \alpha_{a_{ij}}$. Then, with $\alpha_{ij} = \alpha_{a_{ij}}$, for $i, j \in \{1, 2, 3\}$, on W_{321} $\varphi_{32} \circ \varphi_{21} = \alpha_{32} \circ \alpha_{21} = \mathrm{Ad}(u) \circ \alpha_{31}$ by 2.2, where u is some locally constant map from W_{321} into the unitary group of the fibre of ξ_3.

Now if $\dim N < \dim M - 1$, then ∂T is connected and we extend $\varphi_{32} \circ \varphi_{21}$ to the constant map $\varphi_{31} \equiv \alpha_{32} \circ \alpha_{21}$.

If $\dim N = \dim M - 1$, then ∂T is an S^0-bundle over N. If $T \to N$ is not oriented, then ∂T is connected and we proceed as before. If $T \to N$ is oriented, then ∂T has two components ∂T_1 and ∂T_2. As the class of $(\varphi_{32}, \varphi_{21})$ in $\Pi_1 H_R^1$ is in the image of Φ_2, and T is connected, $\varphi_{32} \circ \varphi_{21}$ has constant multiplicity, i.e. $\varphi_{32} \circ \varphi_{21} = \mathrm{Ad}(u_1) \circ \alpha_{31}$ on ∂T_1 and $\varphi_{32} \circ \varphi_{21} = \mathrm{Ad}(u_2) \circ \alpha_{31}$ on ∂T_2.

Pick a trivialization $\{(U_\alpha), (f_\alpha)\}$ of $T \to N$ and a map u from the interval $[-1, 1]$ into the unitary group of the fibre of ξ_3 so that $u(-1) = u_1$ and $u(1) = u_2$.

If $f_\alpha(\partial T_1 | U_\alpha) = U_\alpha \times \{-1\}$, define for $v \in T | U_\alpha$, $f_\alpha(v) = (x, t)$,

$$\varphi_{31}(v) = \mathrm{Ad}(u(t)) \circ \alpha_{31}$$

and if $f_\alpha(\partial T_1 | U_\alpha) = U_\alpha \times \{+1\}$, define for $v \in T | U_\alpha$, $f_\alpha(v) = (x, t)$,

$$\varphi_{31}(v) = \mathrm{Ad}(u(1-t)) \circ \alpha_{31}.$$

As the group of the bundle $T \to N$ is $\mathbf{Z}_2 = \{-1, 1\}$ acting on $[-1, 1]$ by $t \mapsto zt$, $z \in \mathbf{Z}_2$, and $t \in [-1, 1]$, φ_{31} is well defined and extends $\varphi_{32} \circ \varphi_{21}$. ∎

The next proposition states conditions under which we can translate homotopy of the φ_{31}'s relative V_{321} into cocycle homotopy. Let us first introduce some notation for technical purposes.

Let F, G be finite dimensional C^*-algebras, $a \in \mathbf{M}_0(F, G)$ and $\alpha = \alpha_a$: $F \subsetneq G$. For a subset $V \subseteq U(F)$, $\alpha_a(V)$ is not contained in $U(G)$ if the defect δa of a is nonzero. We denote by $\alpha_a(V)_1$ the set of elements of $U(G)$ obtained by replacing all zeros in the diagonals of elements of $\alpha_a(V)$ that result from $\delta a \neq 0$ by ones, that is, replacing

$$(\ldots, \begin{pmatrix} U_k & 0 \\ 0 & 0 \end{pmatrix}, \ldots) \quad \text{by } (\ldots, \begin{pmatrix} U_k & 0 \\ 0 & \mathrm{Id} \end{pmatrix}, \ldots) \quad \text{or}$$

$\alpha_a(V)_1 = \{\alpha_a(v) + \mathrm{Id} - \alpha_a(\mathrm{Id}) |\ v \in V\}$.

4.7. PROPOSITION. *Let $(a_{31}, a_{32}, a_{21}) \in MZ_R^1((W_i), (\xi_i))$, $\alpha_{ij} = \alpha_{a_{ij}}$ for $(i, j) = (2, 1)$, $(3, 2)$ or $(3, 1)$ and $z^0 = (\varphi_{31}^0, \alpha_{32}, \alpha_{21})$ and $z^1 = (\varphi_{31}^1, \alpha_{32}, \alpha_{21}) \in Z_R^1((W_i), (\xi_i))$. Suppose that the connected components C_n, $n = 1, \ldots, k$ of $M \setminus \mathring{T}$ are contractible and each connected component D_j, $j = 1, \ldots, l$, of ∂T is either contractible or strongly deformation retracts to $\bigvee_{r=1}^{m(j)} S^{p(j)}$ for some $p(j) \in N$ such that $\pi_{p(j)-1}((U(\overline{a_{21}^j}))) = 0$, where $a_{21}^j = a_{21} | D_j$ and that the restrictions of ξ_1 to the connected components of $T \times \{\varepsilon\}$ are trivial. Then z^0 and z^1 are cocycle homotopic, if and only if there is a map*

$$\varphi\colon (W_3, (C_n), (D_j)) \to \big(U(F_3), (\alpha_{32}^n(U(F_2^n)))_1), (\alpha_{32}^{n(j)}(U(\overline{a_{21}^j}))_1)\big)$$

such that φ_{31}^0 is homotopic to $\mathrm{Ad}\,\varphi \circ \varphi_{31}^1$ relative ∂T. Here F_3 is the fibre of ξ_3, F_2^n is the fibre of $\xi_2 | C_n$, $\alpha_{32}^n = \alpha_{32} | C_n$, $n(j)$ is unique so that $D_j \subseteq C_{n(j)}$ and ∂T, C_n and $C_n \cap D_j$ are identified with W_{321}, $C_n \times \{\varepsilon\}$ and $(C_n \cap D_j) \times \{\varepsilon\} \subseteq X$, respec-

tively. (We will make this identification without notice whenever it is obvious that this identification has been made.)

Proof. Let (H_{31}, H_{32}, H_{21}) be a cocycle homotopy $z^0 \simeq z^1$. By 3.7, there is a lifting

$$\tilde{H}_{21}\colon \left((D_j)_{j=1}^l\right) \times I \to \left((U(F_2^{n(j)}))_{j=1}^l\right).$$

As $\mu_t^* H_{21} = \alpha_{21}$ for $t = 0$ or 1, we have that for $t \in \{0, 1\}$,

$$\mu_t^* \tilde{H}_{21}(D_j \times I) \subseteq U(\overline{a_{21}^{n(j)}}) \subseteq U(F_2^{n(j)}).$$

By multiplying $\tilde{H}_{21}$ from the right by the map $(x, t) \mapsto (\tilde{H}_{21}(x, 0))^*$, we assume without loss of generality that $\mu_0^* H_{21} | D_j = \mathrm{Id} \in U(F_2^{n(j)})$. Using a radial retraction of some tubular neighborhood U of ∂T in $M \setminus \mathring{T}$ we can extend $\tilde{H}_{21}$ to $U \times I$ so that the resulting extension still has constant value $\mathrm{Id} \in U(F_2^{n(j)})$ on $D_j \times \{0\}$. Using [Sp; 66: Chap. I, D. 1], we obtain a map

$$H_2\colon \left((C_n)_{n=1}^k\right) \times I \to \left((U(F_2^n))_{n=1}^k\right)$$

such that

(i) $H_{21} = \mathrm{Ad}\, H_2 \circ \alpha_{21}$,
(ii) $\mu_0^* H_2 | C_n \equiv \mathrm{Id} \in U(F_2^n)$, $\quad n = 1, \ldots, k \quad$ and,
(iii) $\mu_1^* H_2 | D_j \in U(\overline{a_{21}^j})$, $\quad j = 1, \ldots, l$.

As each C_n is contractible, $H_{32} \circ \mathrm{Ad}\, H_2$ lifts to a map

$$\tilde{H}_{32}\colon (M \setminus \mathring{T}) \times I \to U(F_3)$$

again by 3.7. As

$$\begin{aligned}\mathrm{Ad}\,(\mu_t^* \tilde{H}_{32}) \circ \alpha_{32} &= \mu_t^*(H_{32} \circ \mathrm{Ad}\, H_2)\\ &= \alpha_{32} \circ \mathrm{Ad}(\mu_t^* H_2)\\ &= \mathrm{Ad}\left(\alpha_{32}(\mu_t^* H_2)_1\right) \circ \alpha_{32}\end{aligned}$$

for $t = 0, 1$, where $\alpha_{32}^n\left(\mu_t^* H_2(x)\right)_1$ is the element of $\alpha_{32}^n\left(U(F_2^n)\right)_1$ corresponding to $\alpha_{32}^n\left(\mu_t^* H_2(x)\right) \in \alpha_{32}^n\left(U(F_2^n)\right)$ for $x \in C_n$.

Thus, for $t = 0, 1$,

$$[\alpha_{32}(\mu_t^* H_2)_1]^* \cdot (\mu_t^* \tilde{H}_{32}) = \delta_t\colon \left((C_n)_{n=1}^k\right) \to \left((U(\overline{a_{32}^n}))_{n=1}^k\right)$$

where $a_{32}^n = a_{32} | C_n$.

As $U(\overline{a_{32}^n})$ is connected and C_n is contractible for $n = 1, \ldots, k$, we obtain a map

$$\delta\colon \left((C_n)_{n=1}^k\right) \times I \to \left((U(\overline{a_{32}^n}))_{n=1}^k\right)$$

with $\mu_t^* \delta = \delta_t$, $t = 0, 1$. Replacing $\tilde{H}_{32}$ by $\tilde{H}_{32} \cdot \delta^*$, we may assume without loss of generality that

(i) $\mu_0^* \tilde{H}_{32} \equiv \mathrm{Id}$ and
(ii) $\mu_1^* \tilde{H}_{32} = \alpha_{32}(\mu_1^* H_2)_1$.

Using an argument as in the extension of $\tilde{H}_{21}$ to H_2, we can extend $\tilde{H}_{32}$ to a map

$$\tilde{H}_3\colon M \times I \to U(F_3).$$

Then the map

$$(W_3 \times \{0\}) \cup (M \times I) \to U(F_3)$$

which agrees with $\tilde{H}_3$ on $M \times I$ and has constant value Id on $W_3 \times \{0\}$ extends to $W_3 \times I$, since W_3 is homeomorphic to CM [Sp; 66: Chap. I, A. 3]. We call the resulting map H_3. Then

(i) $\mathrm{Ad}\, H_3 \circ \alpha_{32} = H_{32} \circ \mathrm{Ad}\, H_2$;
(ii) $\mu_0^* H_3 \equiv \mathrm{Id}$ and
(iii) $(\mu_1^* H_3)|(M \setminus \mathring{T}) = \alpha_{32}(\mu_1^* H_2)_1$.

Thus, replacing (H_{31}, H_{32}, H_{21}) by

$$(\mathrm{Ad}(H_3^*) \circ H_{31}, \mathrm{Ad}\, H_3^* \circ H_{32} \circ \mathrm{Ad}\, H_2, \mathrm{Ad}\, H_2^* \circ H_{21}) = (\mathrm{Ad}(H_3^*) H_{31}, \alpha_{32}, \alpha_{21})$$

where α_{ij} is identified with the map $(x, t) \mapsto \alpha_{ij}(x)$, $x \in W_{ij}$, and letting $\varphi = \mu_1(H_3)^*$, we have

$$\mathrm{Ad}(H_3) \circ H_{31}\colon \varphi_{31}^0 \simeq (\mathrm{Ad}\, \varphi) \circ \varphi_{31}^1 \ \text{rel}\, W_{321}.$$

Conversely, suppose that a φ exists and that $H\colon \varphi_{31}^0 \simeq (\mathrm{Ad}\, \varphi) \circ \varphi_{31}^1$ rel W_{321} is a homotopy. Let $F\colon W_3 \times I \to U(F_3)$ be a map so that $\mu_1^* F \equiv \mathrm{Id}$ and $\mu_0^* F = \varphi$ and let $\psi\colon ((C_n), (D_j)) \to ((U(F_2^n)), (U(\overline{a_{21}^j})))$ such that $\varphi | C_n = \alpha_{32}^n(\psi | C_n)_1$. Since each C_n is contractible, there is a map $G\colon ((C_n)_{n=1}^k) \times I \to (U(F_2^n))_{n=1}^k$ such that $\mu_0^* G = \psi$ and $\mu_1^* G | C_n = \mathrm{Id} \in U(F_2^n)$. Then $(H, \alpha_{32}, \alpha_{21})$ is a cocycle homotopy $(\varphi_{31}^0, \alpha_{32}, \alpha_{21}) \simeq (\mathrm{Ad}(\varphi) \circ \varphi_{31}^1, \alpha_{32}, \alpha_{21})$ and $(\mathrm{Ad}(F) \circ \varphi_{31}^1, \mathrm{Ad}\, F \circ \alpha_{32} \circ \mathrm{Ad}(G^*), \mathrm{Ad}(G) \circ \alpha_{21})$ is a cocycle homotopy

$$(\mathrm{Ad}\, \varphi \circ \varphi_{31}^1, \alpha_{32}, \alpha_{21}) \simeq (\varphi_{31}^1, \alpha_{32}, \alpha_{21})$$

and observing that cocycle homotopy is a transitive relation finishes the proof. ■

We are now ready to apply our results.

5. A remark on the continuity of the map $f\colon X \to S_A$ of I.5.3.3.

5.1. In what follows, ξ is a full C^*-semigroup bundle over the representation semigroup X satisfying I.5.3.1 and I.5.3.2. Let $A = \Gamma(\xi | \overline{\mathrm{Irr}(X)})$ and $B = \Gamma(\xi_0 | \overline{\mathrm{Irr}(X)})$ and $f\colon X \to S_A$ and $g\colon \overline{\mathrm{Irr}(X)} \to S_B$ the maps defined in I.5.3 and I.5.8. By I.4.6 and I.5.8, f is continuous if and only if g is. But g is continuous if and only if $g|(\overline{\mathrm{Irr}(X)\cap d^{-1}(n)})$ is continuous for all n. Let $Z_n = \overline{\mathrm{Irr}(X)\cap d^{-1}(n)}$. By I.5.10, $g|Z_n$ is continuous if and only if for all $x\in Z_n\backslash \mathrm{Irr}(X)$ and all nets $\{x_\alpha\}\subseteq Z_n\cap \mathrm{Irr}(X)$ such that $x=\lim x_\alpha$, $g(x) = \lim g(x_\alpha)$. Now fix n. Let $(A_i)_{i=1}^N$ and $(U_i)_{i=1}^N$ such that $\mathbf{X} = (Z_n; (A_i), (U_i)) \subseteq \mathrm{TOP}_R$ and let $(\xi_i)_{i=1}^N$, $\xi = (\mathbf{X}; (\xi_i)), (V_i)\in N(\mathbf{X})$ and $(\varphi_{ij})\in Z_R^1((V_i), (\xi_i))$ such that $\xi_0|Z_n$ is isomorphic to the bundle defined by (φ_{ij}). We let θ_i: $\xi_i|V_i < \xi_0|U_i$ be the imbeddings as in [Du; 79: § 6].

Now $A_N = U_N = V_N = Z_n\cap \mathrm{Irr}(X)$ and $\xi_0(x)\cong M(n)$ for all $x\in A_N$. Let $x\in A_i$, $i<N$. Let $W\subseteq V_i$ be a closed connected neighborhood of x and $C = \{b\in B\colon b(V_i)\subseteq \mathrm{im}(\theta_i)\}$. By [Du; 79: § 6] and [Di; 69: 10.1.12] there is for each $b\in B$ a $c\in C$ such that $b(x) = c(x)$. Thus, by an argument as the proof of I.5.8, g is continuous at x if and only if the map $k\colon Z_n\to S_C$ — defined again as in I.5.9 — is continuous at x.

By I.5.10 we have to show that for a net $\{x_\alpha\}\subseteq V_{iN}$ with $x=\lim x_\alpha$, $k(x) = \lim k(x_\alpha)$. We assume without loss of generality that $\{x_\alpha\}\subseteq W$. While for any x_α, point evaluation at x_α is an n-dimensional irreducible representation of B, this representation splits into a direct sum of irreducible representations once restricted to C. By definition of C the inequivalent irreducible representations of $\mathrm{ev}_{x_\alpha}|C$ are determined by the minimal ideals of the fibre of $\xi_i|W$ and the multiplicities are given by the multiplicity matrix of $\theta_i|W$. With [Va; 66: Prop. 3] and the above notation we obtain:

5.2. PROPOSITION. *The map k, and hence f, is continuous at x if and only if the multiplicities of the irreducible representations in $k(x)$ given by point evaluation at x and projection onto the minimal ideals of $\xi_i(x)$ appear in $k(x)$ with the same multiplicities with which the corresponding minimal ideal of the fibre of $\xi_i|(W\cap A_N)$ is imbedded into $\xi_0|(W\cap A_N)$ under θ_i.* ∎

III. Applications and open problems

1. Applications and final remarks

To classify all C^*-algebras in **BD** up to isomorphism, it is necessary to classify all representation semigroups, underlying C^*-bundles of proper C^*-semigroup bundles and possible structure maps. The following example shows that it is certainly not enough to just know the restriction of the underlying C^*-semigroup bundle to the closure of the irreducible elements of the base semigroup. The main tool for this is the following theorem of M. J. Dupré [Du; 79: 9.2].

1.1. THEOREM. *Let X be a smooth manifold and A a compact submanifold of X with tubular neighborhood T. Let ∂T and $\mathring{T}$ denote the boundary and interior of T, respectively. Let β be a C^*-bundle over T with fibre F and ξ a C^*-bundle over*

$X\backslash A$ with fibre G, $\dim G < \infty$ and $C^*U(F, G) \neq \emptyset$. The set of isomorphism classes of C^*-bundles ζ with $\zeta|A \cong \beta|A$ and $\zeta|(X\backslash A) \cong \xi$ is in bijective correspondence with the set

$$[\partial T, C^*U(\beta, \xi)]/([X\backslash \mathring{T}, \operatorname{Aut}\xi]|\partial T) \times ([T, \operatorname{Aut}\beta]|\partial T)_*. \ \blacksquare$$

1.2. *Notation.* Let D^+ and D^- denote the closed upper and lower hemispheres of S^2 respectively and identify $S^1 = D^+ \cap D^-$ and let E be a tubular neighborhood of S^1 in D^+ and $\mathring{D}^+$ the interior of D^+.

1.3. PROPOSITION. (i) *There are $n+1$ C^*-bundles over S^2 with fibre $M(n)$ over D^- and fibre $M(2n)$ over $\mathring{D}^+$.*

(ii) *There are two (resp. three) C^*-bundles over D^+ with fibre $M(n)$ over S^1 and $M(2n)$ over $\mathring{D}^+$ if n is odd (resp. even).*

Proof. (i) We use Theorem 1.1 with $X = S^2$, $A = D^-$ and $T = D^- \cup E$. Since both T and $X\backslash A$ are contractible, ξ and β are trivial.

$\mathbf{M}_0(M(n), M(2n)) = \{a = (1), b = (2)\}$. Then $\delta b = 0$ and $\delta a = 2n - a\cdot n = n$, so that $\bar{a} = (1, n)$ and $\bar{b} = b$. Thus $U(\bar{a}) = U(1) \times U(n)$ and $U(\bar{b}) = U(2)$. Therefore,

$$\begin{aligned}[\partial T, C^*U(\beta, \xi)] &= [\partial T, C^*U(M(n), M(2n))] \\ &\cong \pi_1(U(2n)/U(1) \times U(n)) \cup \pi_1(U(2n)/U(2)).\end{aligned}$$

By Chapter II, 2.5,

$$[\partial T, C^*U(\beta, \xi)] \cong \{0\} \cup \mathbf{Z}/n\mathbf{Z}.$$

The facts that $X\backslash \mathring{T}$ and T are contractible and $\mathbf{M}_0(M(n))$ and $\mathbf{M}_0(M(2n))$ are trivial, imply that

$$([X\backslash \mathring{T}, \operatorname{Aut}\xi]|\partial T) \times ([T, \operatorname{Aut}\beta]|\partial T)_* = \pi_0(PU(n)) \times \pi_0(PU(2n))_*$$

is trivial (II,2.4). (Here $PU(k) = U(k)/U(1)$ is the projective unitary group.) This finishes the proof of (i).

(ii) Here $X = D^+$, $A = S^1$ and $T = E$. Again, since $X\backslash A$ is contractible, ξ is trivial. Since C^*-bundles over S^1 with fibre $M(n)$ and group $PU(n)$ are classified by $\pi_0(PU(n)) = 0$ and since E strongly deformation retracts to S^1, β is trivial, too.

As before,

$$[\partial T, C^*U(\beta, \xi)] \cong \pi_1(U(2n)/(U(1) \times U(n))) \cup \pi_1(U(2n)/U(2)) = \{0\} \cup \mathbf{Z}/n\mathbf{Z},$$

and

$$[X\backslash \mathring{T}, \operatorname{Aut}\xi] = \pi_0(PU(2n)) = 0.$$

But $[T, \operatorname{Aut}\beta] \cong \pi_1(PU(n)) \cong \mathbf{Z}/n\mathbf{Z}$.

By II, 3.7, any map $\varphi\colon S^1 \to C^*U(M(n), M(2n))$ with multiplicity matrix (2) lifts to a map $u\colon S^1 \to U(2n)$ and any map $\delta\colon S^1 \to \operatorname{Aut}(M(n)) \cong PU(n)$ lifts to a map $v\colon S^1 \to U(n)$.

The action of $\pi_1(PU(n))$ on $\pi_1(U(2n)/U(2))$ is given by $([\delta], [\varphi]) \mapsto [\varphi \circ \delta^{-1}]$. With $a = (2)$ and $\alpha = \alpha_a$, $\varphi \circ \delta^{-1} = \mathrm{Ad}(u) \circ \alpha \circ \mathrm{Ad}(v^*) = \mathrm{Ad}(u \cdot \alpha(v)^*) \circ \alpha$. Thus by II, 2.6.3, $[\varphi \circ \delta^{-1}] = [\varphi] - [\mathrm{Ad}\,\alpha(v)]$. It is easy to check that if $[v] = k \in \mathbf{Z} \cong \pi_1(U(n))$, then $[\mathrm{Ad}\,\alpha(v)] = 2k \bmod n$.

Therefore, $\pi_1(PU(n))$ acts transitively on $\pi_1(U(2n)/U(2))$ if n is odd and there are exactly two bundles over D^+ with fibre $M(n)$ over S^1 and $M(2n)$ over $\mathring{D}^+$ corresponding to the multiplicity matrices (1) and (2) in this case. If n is even, however, there are three such bundles, namely one corresponding to the matrix (1) and one corresponding to (2) and $0 \in \mathbf{Z}/n\mathbf{Z} \cong \pi_1(U(2n)/U(n))$ and (2) and $1 \in \mathbf{Z}/n\mathbf{Z}$, respectively.

1.4. Example. Now let $B = C(D^2, M(n)) \times C(D^2, M(2n))$, where $D^2 = \{z \in \mathbf{C}\colon |z| \leqslant 1\}$ and $C(D^2, M(k))$ is the C^*-algebra of all continuous $M(k)$-valued functions on D^2 and suppose n is odd. For $i = 1, 2, \ldots, n$, let

$$A_i = \{(f, g) \in B\colon g = \mathrm{Ad}\left(\begin{bmatrix} u^i & 0 \\ 0 & \mathrm{Id} \end{bmatrix}\right) \alpha_{(2)}(f)\}$$

where $u\colon S^1 \to U(k)$ is the canonical generator of $\pi_1(U(k))$, i.e.

$$u(z) = \begin{bmatrix} z & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} \qquad \text{for } z \in S^1.$$

We can represent the A_i as the continuous sections of the n different C_1^*-bundles over S^2, so the A_i are pairwise nonisomorphic. Their representation semigroups, however, are isomorphic. In fact the sets $\overline{\mathrm{Irr}(S_{A_i})}$ are homeomorphic to two copies of D^2, one corresponding to the n-dimensional representations and one corresponding to the $2n$-dimensional representations. By 1.3 (ii), the $\xi(A_i)|\overline{\mathrm{Irr}(S_{A_i})}$ are isomorphic. ∎

6.5 of Chapter I, however, allows a complete classification in certain cases:

1.5. Proposition. *Let X be a representation semigroup with bounded dimension function satisfying*

5.1. *For all $y \in \overline{\mathrm{Irr}(X)}$ and $x_1, x_2 \in \mathrm{Core}(X)$, $y + x_1 \in \mathrm{Core}(X)$ and $y + x_2 \in \mathrm{Core}(X)$ implies $x_1 = x_2$ and*

5.2. *For all $y \in \overline{\mathrm{Irr}(X)}$ and $x \in X$, $y \in \mathrm{Core}(X)$ implies $x + y \notin \mathrm{Core}(X)$.*

The isomorphism classes of proper C^*-semigroup bundles ξ over X are in one-to-one correspondence with isomorphism classes of C^*-bundles β over $\mathrm{Core}(X)$ *with $\beta(x) \cong \xi(A)(x)$ where $A = \Gamma_b(\xi)$ (under the identification $X = S_A$).* ∎

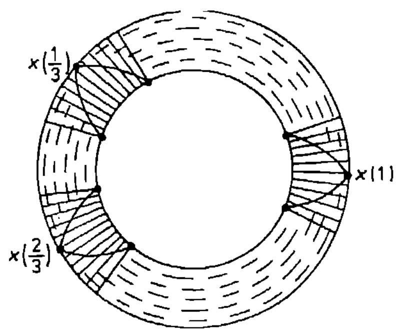
The following Theorem 1.10 provides the final means to work our main examples. Let us first describe the general setup and prove a few lemmas.

1.6. *Notation and remarks.* Let $X = \{x \in \mathbf{C}\colon |x| \leqslant 1\}$, $n \in \mathbf{N}$, $A_1 = \{x \in \mathbf{C}\colon x^n = 1\}$, $A_2 = S^1 \backslash A_1$ and $A_3 = X \backslash S^1$ (where $S^1 \subseteq X$ in the obvious way).

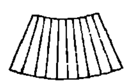
Furthermore, let $D = \{1, 2, 3\}$, $R = \{(i, j) \in D \times D \colon 1 \leqslant j \leqslant i \leqslant 3\}$ and U_1, U_2 and $U_3 = A_3$ be open sets, so that $\mathbf{X} = (X; (A_i)_{i=1}^3, (U_i)_{i=1}^3) \in \mathrm{TOP}_R$.

Moreover let $F_3, F_2^1, \ldots, F_2^n, F_1^1, \ldots, F_1^n$ be finite dmensional C^*-algebras and for $i = 1, 2, 3$, ξ_i a C^*-bundle over U_i, so that $\xi_3(x) \cong F_3$ for $|x| < 1$, $\xi_2(x(t)) \cong F_2^k$ for $\dfrac{k-1}{n} < t < \dfrac{k}{n}$, $k = 1, \ldots, n$ and $\xi_1\left(x\left(\dfrac{k}{n}\right)\right) \cong F_1^k$ for $k = 1, 2, \ldots, n$ where $x(t) = \exp(2\pi i\, t)$, $t \in [0, 1]$.

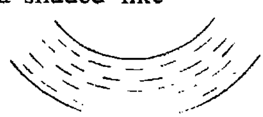
Now we let $\xi = (\mathbf{X}; (\xi_i)_{i=1}^3)$ and considering X as the cone over S^1, we are in the situation of Chapter II.4. We can choose $(V_i)_{i=1}^3 \in N(\mathbf{X})$ in such a way as described in the following picture for the case $n = 3$:



Here V_1 corresponds to the region shaded like

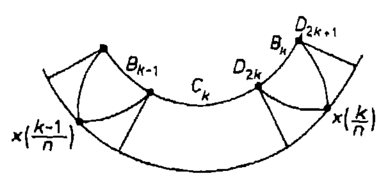


V_2 corresponds to the area shaded like



and V_3 is the interior of the disk.

The sets $(V_i)_{i=1}^3$ are of the form described in Chapter II.4.2 and we let $(W_i)_{i=1}^3$ be the sets introduced in Chater II preceding Proposition 4.4. Then both W_{31} and W_{32} are the disjoint union of n arcs each and W_{321} is a collection of $2n$ points. For $k = 1, 2, \ldots, n$, we let C_k be the component of W_{32} between $x\left(\dfrac{k-1}{n}\right)$ and $x\left(\dfrac{k}{n}\right)$ and B_k the component of W_{31} close to $x\left(\dfrac{k}{n}\right)$. Also we let $D_{2k} = B_k \cap C_k$ and $D_{2k+1} = C_{k+1} \cap B_k$ for $k = 0, \ldots, n$, where $B_0 = B_n$ and $C_{n+1} = C_1$.



Now let $(a_{31}, a_{32}, a_{21}) \in MZ^1_R((W_l), (\xi_i))$, $\alpha_{ij} = \alpha_{a_{ij}}$ and $z = (\alpha_{32}, \alpha_{21})$. Then the set K_z defined in Chapter II before Proposition 3.9 is in bijective correspondence with the set

$$(1.6.1) \qquad \prod_{k=1}^{n} \pi_1\big(C^*U(F_1^k, F_3), \alpha_{31} | B_k\big).$$

A bijection is given by identifying each B_k with $[0, 1]$ and sending $[\varphi_{31}] \in K_z$ to $[\mathrm{Ad}(\omega) \circ \varphi_{31}]$ for some map

$$\omega\colon (B_k)_{k=1}^n \to U(F_3)$$

with

$$(1.6.2) \qquad \mathrm{Ad}(\omega) \circ \alpha_{32} \circ \alpha_{21} = \alpha_{31}.$$

(II, 2.2.1) on W_{321}. This bijection depends only on the homotopy class of ω. If $a_{21}|B_k$ and $a_{32}|B_k$ are constant we will assume that so is $\omega|B_k$.

We note that Proposition II.4.7 applies to this setup. Let us denote by G_z the set of homotopy classes $[\varphi]$ of maps

$$(1.6.3) \qquad \varphi\colon \big(W_3, (C_k), (D_j)\big) \to \Big(U(F_3), \big(\alpha_{32}^k(U(F_2^k))_1\big), \big(\alpha_{32}^{n(j)}(U(\overline{a_{21}^j}))_1\big)\Big)$$

where we use the notation of II.4.7.

Let us briefly observe that if $M(p)$ and $M(q)$ are finite dimensional C^*-algebras, $a \in \mathrm{M}_0(p, q)$ and v is a loop in $U(p)$ with basepoint identity matrix, then

$$(1.6.4) \qquad [\alpha_a(v)_1] = a \cdot [v]$$

where $a\colon \pi_1(U(p)) \to \pi_1(U(q))$ is given by matrix multiplication after identifying both $\pi_1(U(p))$ and $\pi_1(U(q))$ with powers of Z. (1.6.4) can be seen immediately by looking at the canonical generators of $\pi_1(U(p))$ and the definition of α_a (II, 2.1.1).

We have

1.7. Lemma. *G_z is a group, isomorphic to the set*

$$\Big\{(r_1, \ldots, r_n, s_1, \ldots, s_n) \in \Big[\prod_{k=1}^{n} (a_{32}|C_k)\big(\pi_1(U(F_2^k))\big)\Big] \times \Big[\prod_{k=1}^{n} \pi_1(U(F_3))\Big]\colon r_1 + \ldots + r_n + s_1 + \ldots + s_n = 0\Big\}$$

and G_z acts on K_z by

$$([\varphi], [\mathrm{Ad}(\omega) \circ \varphi_{31}]) \mapsto [\mathrm{Ad}(\varphi) \circ \alpha_{31}] + [\mathrm{Ad}(\omega) \circ \varphi_{31}].$$

In particular, we have an isomorphism

$$\{[\varphi_{ij}] \in \Pi Z^1_R((W_i), (\xi_i))\colon \varphi_{32} = \alpha_{32}, \varphi_{21} = \alpha_{21}\} \cong Q_z = K_z/G_z.$$

Proof. Let us first observe that any element of G_z can be represented by a map φ with $\varphi|W_{321} \equiv \mathrm{Id}$. This is true, because we can connect for each j, $\varphi(D_j)$ to Id with a path in $\alpha_{32}^{n(j)}(U(\overline{a_{21}^j}))_1$ for arbitrary φ as in II.4.7, thus obtaining a map defined on $W_3 \times \{0\} \cup \bigcup_{j=1}^{2n} D_j \times [0, 1]$, which can be extended to a map on $W_3 \times \{0\} \cup (W_{32} \cup W_{21}) \times [0, 1]$ (with image of $C_k \times [0, 1]$ contained in $\alpha_{32}^k(U(F_2^k))_1$) using an appropriate retraction and then extending via another retraction to $W_3 \times [0, 1]$. This provides a homotopy between φ and a map with constant image $\{\mathrm{Id}\}$ on W_{321}. We will now assume without loss of generality that $\varphi|W_{321} \equiv \mathrm{Id}$. Then for each k, $\varphi|C_k$ defines a loop in $\alpha_{32}^k(U(F_2^k))_1$ and similarly $\varphi|B_k$ a loop in $U(F_3)$. The first claim of the lemma follows then directly from the fact that φ defines a null homotopy for the loop $\varphi|(W_{32} \cup W_{31})$ in $U(F_3)$.

For $(\varphi_{31}, \alpha_{32}, \alpha_{21})$ and $[\varphi] \in G_z$ we have by II, 2.6

$$[\mathrm{Ad}(\omega)\circ\mathrm{Ad}(\varphi)\circ\varphi_{31}] = [\mathrm{Ad}(\omega\cdot\varphi\cdot\omega^*)\circ\mathrm{Ad}(\omega)\circ\varphi_{31}]$$

$$= [\mathrm{Ad}(\varphi)\circ\alpha_{31}] + [\mathrm{Ad}(\omega)\circ\varphi_{31}]$$

and the isomorphism follows from II.4.7 and II.3.9. ∎

Now let $S_z = \{(a_i) \in MC_R^0((W_i), (\xi_i)): a_3 \circ a_{32} \circ a_2^{-1} = a_{32}$ and $a_2 \circ a_{21} \circ a_1^{-1} = a_{21}\}$. Then S_z is a subgroup of $MC_R^0((W_i), (\xi_i))$. For each $(a_i) \in S_z$, we obtain a map $K_z \to K_z$ by sending $r = (r_1, \ldots, r_n)$ to $a_3 \cdot r = (a_3 \cdot r_1, \ldots, a_3 \cdot r_n)$ (under the identification of K_z with the set of 1.6.1). By linearity of matrix multiplication, this action commutes with the action of G_z and thus yields a map $Q_z \to Q_z$.

1.8. Lemma. *There is a map g: $S_z \to Q_z$ such that for all $a = (a_i)$, $b = (b_i) \in S_z$:*

(1.8.1) $$g(a \cdot b) = a_3 \cdot g(b) + g(a).$$

In particular, this map defines an action S_z on Q_z by

(1.8.2) $$(a, r) \mapsto a_3 r + g(a)$$

and this action is independent of the choice of the map ω used to identify K_z with the set of 1.6.1.

Proof. Let $a = (a_i)$, $b = (b_i) \in S_z$ and $\alpha_i = \alpha_{a_i}$ and $\beta_i = \alpha_{b_i}$. There are maps u_{ij}, v_{ij}, w_{ij} defined on W_{ij} with values in the appropriate unitary groups, so that

$$\mathrm{Ad}(u_{ij})\circ\alpha_{ij} = \alpha_i \circ \alpha_{ij} \circ \alpha_j^{-1}$$

and similarly for v_{ij} and (b_i) and w_{ij} and $(\alpha_i \circ \beta_i)$. Now let ω as in 1.6.2. On W_{321} we have:

$$\begin{aligned}\mathrm{Ad}(u_{31})\circ\alpha_{31} &= \alpha_3\circ\alpha_{31}\circ\alpha_1^{-1}\\ &= \alpha_3\circ\mathrm{Ad}(\omega)\circ\alpha_{32}\circ\alpha_{21}\circ\alpha_1^{-1}\\ &= \mathrm{Ad}(\alpha_3(\omega))\circ\alpha_3\circ\alpha_{32}\circ\alpha_2^{-1}\circ\alpha_2\circ\alpha_{21}\circ\alpha_1^{-1}\\ &= \mathrm{Ad}(\alpha_3(\omega))\circ\mathrm{Ad}(u_{32})\circ\alpha_{32}\circ\mathrm{Ad}(u_{21})\circ\alpha_{21}\\ &= \mathrm{Ad}(\alpha_3(\omega)\cdot u_{32}\cdot\alpha_{32}(u_{21})_1)\circ\alpha_{32}\,\alpha_{21}\\ &= \mathrm{Ad}(\alpha_3(\omega)\cdot u_{32}\cdot\alpha_{32}(u_{21})_1\cdot\omega^*)\circ\alpha_{31}.\end{aligned}$$

Thus, by modifying u_{31} if necessary,

$$u_{31} = \alpha_3(\omega)\cdot u_{32}\cdot\alpha_{32}(u_{21})_1\cdot\omega^*. \tag{1.8.3}$$

Let $\bar{u}_{32}$ and $\bar{u}_{21}$ be homotopies $u_{32}\simeq \mathrm{Id}$ and $u_{21}\simeq\mathrm{Id}$, respectively. We define a homotopy u in the following way: first let

$$u = \begin{cases}\bar{u}_{32}\cdot\alpha_{32}(\bar{u}_{21})_1 & \text{on } W_{321}\times[0,1],\\ \alpha_3(\omega^*)\cdot u_{31}\cdot\omega & \text{on } W_{31}\times\{0\}\end{cases} \tag{1.8.4}$$

and then extend to $W_{31}\times[0,1]$ via a retraction

$$W_{31}\times[0,1]\to(W_{31}\times\{0\})\cup(W_{321}\times[0,1]).$$

We let $v=\mu_1^* u$ and $\bar{u}_{31}=\alpha_3(\omega)\cdot u\cdot\omega^*$. Now let $(\varphi_{31},\alpha_{32},\alpha_{21})\in Z^1_R((W_i),(\xi_i))$. By II.3.7, there is a map $\bar{\varphi}_{31}\colon W_{31}\to U(F_3)$ so that $\varphi_{31}=\mathrm{Ad}(\bar{\varphi}_{31})\circ\alpha_{31}$ and without loss of generality, $\bar{\varphi}_{31}=\omega^*$ on W_{321}. Then

$$(H_{ij}) = \big(\mathrm{Ad}(\alpha_3(\bar{\varphi}_{31})\cdot\bar{u}_{31})\circ\alpha_{31},\ \mathrm{Ad}(\bar{u}_{32})\circ\alpha_{32},\ \mathrm{Ad}(\bar{u}_{21})\circ\alpha_{21}\big)$$

defines a cocycle homotopy

$$\begin{aligned}&(\alpha_3\circ\varphi_{31}\circ\alpha_1^{-1},\ \alpha_3\circ\alpha_{32}\circ\alpha_2^{-1},\ \alpha_2\circ\alpha_{21}\circ\alpha_1^{-1})\\ &\qquad\simeq\big(\mathrm{Ad}(\alpha_3(\bar{\varphi}_{31})\cdot\alpha_3(\omega)\cdot v\cdot\omega^*)\circ\alpha_{31},\ \alpha_{32},\ \alpha_{21}\big)\end{aligned}$$

Also in K_z,

$$\begin{aligned}&[\mathrm{Ad}(\omega\cdot\alpha_3(\bar{\varphi}_{31})\cdot\alpha_3(\omega)\cdot v\cdot\omega^*)\circ\alpha_{31}]\\ &\quad= [\mathrm{Ad}(\omega\cdot\alpha_3(\omega)^*\cdot\alpha_3(\omega\bar{\varphi}_{31})\cdot\alpha_3(\omega)\cdot\omega^*\cdot\omega\cdot v\cdot\omega^*)\circ\alpha_{31}]\\ &\quad= [\mathrm{Ad}(\omega\cdot\alpha_3(\omega)^*\cdot\alpha_3(\omega\bar{\varphi}_{31})\cdot\alpha_3(\omega)\cdot\omega^*)\circ\alpha_{31}]+[\mathrm{Ad}(\omega\cdot v\cdot\omega^*)\circ\alpha_{31}]\\ &\quad= [\mathrm{Ad}(\alpha_3(\omega\bar{\varphi}_{31}))\circ\alpha_{31}]+[\mathrm{Ad}(v)\circ\alpha_{31}]\\ &\quad= a_3\cdot[\varphi_{31}]+[\mathrm{Ad}(v)\circ\alpha_{31}]\end{aligned} \tag{1.8.5}$$

by II.2.6 and 1.6.4.

Using different homotopies instead of $\bar{u}_{31}$ and $\bar{u}_{21}$ yields a different loop v' instead of v. But then there is a cocycle homotopy

$$\begin{aligned}&\big(\mathrm{Ad}(\alpha_3(\bar{\varphi}_{31})\cdot\alpha_3(\omega)\cdot v'\cdot\omega^*)\circ\alpha_{31},\ \alpha_{32},\ \alpha_{21}\big)\\ &\qquad\simeq\big(\mathrm{Ad}(\alpha_3(\bar{\varphi}_{31})\cdot\alpha_3(\omega)\cdot v\cdot\omega^*)\circ\alpha_{31},\ \alpha_{32},\ \alpha_{21}\big)\end{aligned}$$

and hence by 1.7, an element $[\varphi] \in G_z$ so that

$$a_3 \cdot [\mathrm{Ad}(\omega) \circ \varphi_{31}] + [\mathrm{Ad}(v') \circ \alpha_{31}]$$
$$= a_3 \cdot [\mathrm{Ad}(\omega) \circ \varphi_{31}] + [\mathrm{Ad}(v) \circ \alpha_{31}] + [\mathrm{Ad}(\varphi) \circ \alpha_{31}].$$

We then define

(1.8.6) $$g(a) = [\mathrm{Ad}(v) \circ \alpha_{31}] (\mathrm{mod}\, G_z).$$

Let us now show that $g(a \cdot b) = a_3 g(b) + g(a)$, which will yield that $a_3 \cdot b_3 \cdot r + g(a \cdot b) = a_3 [b_3 \cdot r + g(b)] + g(a)$, i.e. S_z acts as a group on Q_z. So observe first that we can choose

$$w_{ij} = \alpha_i(v_{ij}) \cdot u_{ij},$$

because then

$$\begin{aligned} \mathrm{Ad}(w_{ij}) \circ \omega_{ij} &= \mathrm{Ad}(\alpha_i(v_{ij})) \circ \alpha_i \circ \alpha_{ij} \circ \alpha_i^{-1} \\ &= \alpha_i \circ \mathrm{Ad}(v_{ij}) \circ \alpha_{ij} \circ \alpha_j^{-1} \\ &= \alpha_i \circ \beta_i \circ \alpha_{ij} \circ \beta_j^{-1} \circ \alpha_j^{-1}. \end{aligned}$$

Let $\bar{u}_{ij}$ and $\bar{v}_{ij}$ be the homotopies defined in the definition of $g(a)$ and $g(b)$, respectively, and let $\bar{w}_{ij} = \alpha_i(\bar{v}_{ij}) \cdot \bar{u}_{ij}$. With $v'' = \mu_1^* \bar{w}_{31}$, $v' = \mu_1^* \bar{v}_{31}$ and $v = \mu_1^* \bar{u}_{31}$ we obtain that

$$\begin{aligned} [\mathrm{Ad}(v'') \circ \alpha_{31}] &= [\mathrm{Ad}(\alpha_3(v') \cdot v) \circ \alpha_{31}] \\ &= [\mathrm{Ad}(\alpha_3(v')) \circ \alpha_{31}] + [\mathrm{Ad}(v) \circ \alpha_{31}] \\ &= a_3 \cdot [\mathrm{Ad}(v') \circ \alpha_{31}] + [\mathrm{Ad}(v) \circ \alpha_{31}] \end{aligned}$$

or $g(ab) = a_3 g(b) + g(a)$.

Finally, let δ be a map $(W_{31}, W_{321}) \to (U(F_3), \{\mathrm{Id}\})$ and $s = [\mathrm{Ad}(\delta) \circ \alpha_{31}]$. Since $\delta | W_{321} \equiv \mathrm{Id}$, 1.8.3 still holds for $\omega' = \omega \cdot \delta$. We define in analogy with 1.8.4 $u' = \alpha_3(\omega')^* u_{31} \omega'$ and find that with $v' = \mu_1^* u'$, $v' = \alpha_3(\delta)^* \cdot v \cdot \delta$. We obtain again

$$\begin{aligned} &[\mathrm{Ad}(\omega' \cdot \alpha_3(\bar{\varphi}_{31}) \cdot \alpha_3(\omega')^* \cdot v' \cdot \omega'^*) \circ \alpha_{31}] \\ &\quad = [\mathrm{Ad}(\alpha_3(\omega' \bar{\varphi}_{31})) \circ \alpha_{31}] + [\mathrm{Ad}(v') \circ \alpha_{31}] \\ &\quad = [\mathrm{Ad}(\alpha_3(\omega \cdot \delta \cdot \bar{\varphi}_{31})) \circ \alpha_{31}] + [\mathrm{Ad}(\alpha_3(\delta)^* \cdot v \cdot \delta) \circ \alpha_{31}] \\ &\quad = a_3 \cdot s + [\mathrm{Ad}(\alpha_3(\omega \bar{\varphi}_{31}) \circ \alpha_{31}] - a_3 \cdot s + [\mathrm{Ad}(v) \circ \alpha_{31}] + s \\ &\quad = [Ad(\alpha_3(\omega \bar{\varphi}_{31})) \circ \alpha_{31}] + [\mathrm{Ad}(v) \circ \alpha_{31}] + s. \end{aligned}$$

As $[\mathrm{Ad}(\omega \cdot \delta) \circ \varphi_{31}] = [\mathrm{Ad}(\omega \cdot \delta \cdot \omega^* \cdot \omega) \circ \varphi_{31}] = [\mathrm{Ad}(\omega) \circ \varphi_{31}] + [\mathrm{Ad}(\omega \cdot \delta \cdot \omega^*) \circ \alpha_{31}] = [\mathrm{Ad}(\omega) \circ \varphi_{31}] + [\mathrm{Ad}(\delta) \circ \alpha_{31}]$ in general, the action of S_z on Q_z is in fact independent of the choice of ω. ∎

1.9. *Remark.* If the fibre type is constant on A_1 and A_2 and if a_{32} and a_{21} are constant, then ω can be chosen constant and a quick review of 1.8.4 and the construction of g show that $g \equiv 0$. ∎

Before we state the theorem, let us observe that we can use II.4.4, II.4.5 and some connectivity arguments to identify $\Pi H^1_R((W_i), (\xi_i))$ with $\Pi H^1_R(\xi)$ and similarly for MH^1_R.

1.10. THEOREM. *Let* $\mathbf{X} = (X; (A_i), (U_i))$ *and* $\xi = (\mathbf{X}; (\xi_i))$ *as in* 1.6. *The map*

$$M(\xi)\colon \Pi H^1_R(\xi) \to MH^1_R(\xi)$$

is surjective and for each $[a_{ij}] \in MH^1_R(\xi)$, $z = (\alpha_{32}, \alpha_{21})$

$$M(\xi)^{-1}([a_{ij}]) \cong Q_z/S_z. \tag{1.10.1}$$

The quotient Q_z/S_z *is with respect to the action defined in* 1.8.

Proof. Surjectivity of $M(\xi)$ follows from II.4.6. Now consider the commutative diagram of II.3.10.

$$\begin{array}{ccc} \Pi H^1_R(\xi) & \xrightarrow{\Phi_2} & \Pi_1 H^1_R(\xi) \\ {\scriptstyle M_2 = M(\xi)}\downarrow & & \downarrow{\scriptstyle M_1} \\ MH^1_R(\xi) & \xrightarrow[\Psi_2]{} & M_1 H^1_R(\xi) \end{array}$$

since all components of W_{32}, W_{21}, W_3, W_2 and W_1 are contractible, $\Pi C^0_R(\xi) \cong MC^0_R(\xi)$ and $\Pi_1 Z^1_R(\xi) \cong M_1 Z^1_R(\xi)$ under the assignment of multiplicity matrices. Therefore, M_1 is a bijection.

It is straightforward to check that Ψ_2 is injective and that $\mathrm{Im}(M_1 \circ \Phi_2) \subseteq \mathrm{Im}(\Psi_2)$. Surjectivity of $M(\xi)$ then implies that $\mathrm{Im}(\Psi_2) \subseteq \mathrm{Im}(M_1 \circ \Phi_2)$, so that we can identify $MH^1_R(\xi)$ and $\mathrm{Im}\,\Phi_2$.

Now let $(a_{ij}) \in MZ^1_R(\xi)$, $\alpha_{ij} = \alpha_{a_{ij}}$ and $z = (\alpha_{32}, \alpha_{21})$. By II.3.9, we have a surjection

$$K_z \to \Phi_2^{-1}([\![z]\!]_1) = M(\xi)^{-1}([a_{ij}])$$

where $[a_{ij}]$ is the class of (a_{ij}) in $MH^1_R(\xi)$ and we made the identification $\mathrm{Im}\,\Phi_2 = MH^1_R(\xi)$.

By 1.7, this map induces a surjection

$$Q_z \to M(\xi)^{-1}([a_{ij}]). \tag{1.10.2}$$

Now suppose that $(\varphi^0_{31}, \alpha_{32}, \alpha_{31})$ and $(\varphi^1_{31}, \alpha_{32}, \alpha_{31})$ satisfy

$$[\mathrm{Ad}(\omega) \circ \varphi^1_{31}] (\mathrm{mod}\, G_z) = a_3 \cdot [\mathrm{Ad}(\omega) \circ \varphi^0_{31}] (\mathrm{mod}\, G_z) + g(a)$$

for some $a = (a_i) \in MC^0_R(\xi)$. Let $\bar\varphi_{31}\colon W_{31} \to U(F_3)$ so that $\varphi^0_{31} = \mathrm{Ad}(\bar\varphi_{31}) \circ a_{31}$ and let u_{ij}, $\bar u_{ij}$ and ν as in the proof of 1.8.

With a calculation as in 1.8.5,

$$\begin{aligned} &a_3 \cdot [\mathrm{Ad}(\omega) \circ \varphi^0_{31}] (\mathrm{mod}\, G_z) + g(a) \\ &\qquad = [\mathrm{Ad}\,(\omega \cdot \alpha_3(\bar\varphi_{31}) \cdot \alpha_3(\omega) \cdot \nu \cdot \omega^*) \circ \alpha_{31}] (\mathrm{mod}\, G_z). \end{aligned} \tag{1.10.3}$$

But there is a cocycle homotopy (given by the $\bar u_{ij}$ and definition of the u_{ij})

$$\begin{aligned} &(\mathrm{Ad}\,(\alpha_3(\bar\varphi_{31}) \cdot \alpha(\omega) \cdot \nu \cdot \omega^*) \circ \alpha_{31}, \alpha_{32}, \alpha_{31}) \\ &\qquad \simeq (\alpha_3 \circ \varphi^0_{31} \circ \alpha_1^{-1}, \alpha_2 \circ \alpha_{32} \circ \alpha_2^{-1}, \dot\alpha_2 \circ \alpha_{21} \circ \alpha_1^{-1}) \end{aligned} \tag{1.10.4}$$

which is equivalent to $(\varphi_{31}^0, \alpha_{32}, \alpha_{21})$ in $\Pi H_R^1(\xi)$. By 1.10.3, 1.10.4 and 1.7, $(\varphi_{31}^1, \alpha_{32}, \alpha_{21})$ and $(\varphi_{31}^0, \alpha_{32}, \alpha_{21})$ are equivalent and hence the map of 1.10.2 induces a map

$$Q_z/S_z \to M(\xi)^{-1}([a_{ij}])$$

(which sends the class of $\mathrm{Ad}(\omega)\circ\varphi_{31}$ to the class of $(\varphi_{31}, \alpha_{32}, \alpha_{21})$!).

Now take for $t = 0, 1$, $(\varphi_{ij}^t) \in Z_R^1(\xi)$ with $\varphi_{32}^t = \alpha_{32}$ and $\varphi_{21}^t = \alpha_{21}$. Suppose there are $(\varphi_i) \in C_R^0(\xi)$ and $(H_{ij}) \in Z_R^1(\xi \times I)$ so that

$$\mu_0^* H_{ij} = \varphi_{ij}^0 \quad \text{and} \quad \mu_1^* H_{ij} = \varphi_i \circ \varphi_{ij}^1 \circ \varphi_j^{-1}.$$

As $\Pi C_R^0(\xi) \cong MC_R^0(\xi)$, we can assume without loss of generality that there is $a = (a_i) \in MC_R^0(\xi)$ with $\varphi_i = \alpha_i = \alpha_{a_i}$.

Let $\bar{\varphi}_{31}\colon W_{31} \to U(F_3)$ with $\mathrm{Ad}(\bar{\varphi}_{31})\circ\alpha_{31} = \varphi_{31}^1$. There is then a cocycle homotopy

$$(\alpha_3\circ\varphi_{31}^1\circ\alpha_1^{-1}, \alpha_3\circ\alpha_{32}\circ\alpha_2^{-1}, \alpha_2\circ\alpha_{21}\circ\alpha_1^{-1}) \simeq \big(\mathrm{Ad}(\alpha_3(\bar{\varphi}_{31})\cdot\alpha_3(\omega)\cdot v\cdot\omega^*)\circ\alpha_{31}, \alpha_{32}, \alpha_{21}\big)$$

and hence a cocycle homotopy

$$(\varphi_{31}^0, \alpha_{32}, \alpha_{21}) \simeq \big(\mathrm{Ad}(\alpha_3(\bar{\varphi}_{31})\cdot\alpha_3(\omega)\cdot v\cdot\omega^*)\circ\alpha_{31}, \alpha_{32}, \alpha_{21}\big).$$

By 1.7 and 1.8,

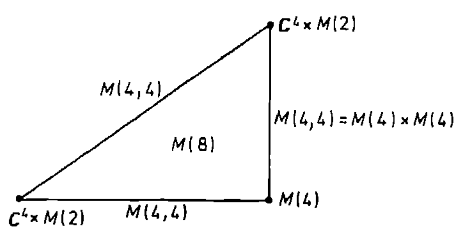
$$[\mathrm{Ad}(\omega)\circ\varphi_{31}^0] = (a_3[\mathrm{Ad}(\omega)\circ\varphi_{31}^1] + [\mathrm{Ad}(v)\circ\alpha_{31}]) \bmod G_z$$

or the map in 1.10.5 is injective. ■

2. Applications

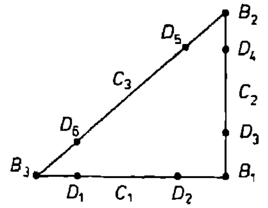
2.1. *The group C*-algebra of the group p4gm.* Our main application is to give a new and complete description of the group C^*-algebra of a group called *p4gm*, which had been computed in Raeburn by [RA; 82]. The group *p4gm* is the group of motions on $\boldsymbol{R}^2$ generated by the 4 maps that send $(x, y) \in \boldsymbol{R}^2$ to $(x+1, y)$, $(x, y+1)$, $(-y, x)$ and $(\frac{1}{2}-x, \frac{1}{2}+y)$.

By [Ra; 82: Prop. 8], $C^*(p4gm)$ is isomorphic to the algebra of sections of a C^*-bundle $\xi(p4gm)$ over a triangle with fibres assigned in the following picture:



Choosing an appropriate homeomorphism between the triangle and a two-disk, we are in the setup of 1.6 in the case $n = 3$.

Let us identify the boundary of the triangle with $W_{32} \cup W_{31}$ in the following way:



In this situation, each a_{32} is the multiplicity matrix of an imbedding $M(4, 4) \subsetneq M(8)$ and therefore $a_{32} \equiv (1, 1)$. Hence every element (a_{ij}) of $MZ^1_R(\xi)$ is uniquely determined by a_{21}.

Notice that by II.2.5.1 and II.2.5.2

$$\pi_1\big(C^*U(C^4 \times M(2), M(8)), \alpha_a\big) = 0$$

independent of

$$a \in \mathbf{M}_0((1, 1, 1, 1, 2), 8) \quad \text{and} \quad \pi_1\big(C^*U(M(4), M(8)), \alpha_{(1)}\big) = 0.$$

Therefore $K_z = 0$ for $z = (\alpha_{32}, \alpha_{21})$, $\alpha_{ij} = \alpha_{a_{ij}}$ and $(a_{ij}) \in MZ^1_R(\xi)$ unless $a_{31}|B_1 \equiv (2)$. (Of course (1) and (2) are the only possible values for $a_{31}|B_1$.)

Now suppose $(a_{ij}) \in MZ^1_R(\xi)$ with $a_{31}|B_1 = (2)$ and again $z = (\alpha_{32}, \alpha_{21})$ where $\alpha_{ij} = \alpha_{a_{ij}}$. By II.2.5 and 1.6.1,

$$K_z \cong \pi_1\big(C^*U(M(4), M(8)), \alpha_{31}|B_1\big) \times \pi_1\big(C^*U(C^4 \times M(2), M(8)), \alpha_{31}|B_2\big)$$
$$\times \pi_1\big(C^*U(C^4 \times M(2), M(8)), \alpha_{31}|B_3\big) \cong Z/4Z \times \{0\} \times \{0\} \cong Z/4Z.$$

Suppose $[\mathrm{Ad}(\omega) \circ \varphi_{31}] = r (\mathrm{mod}\, 4)$. Le $u\colon C_2 \to U(4) \times U(4)$ be a loop so that $[u] = (r, 1) \in \pi_1(U(4) \times U(4))$ and $v\colon B_1 \to U(8)$ a loop so that $[v] = -r \in \pi_1(U(8))$. Then the map φ from the boundary of the triangle into $U(8)$ defined to be identical with v on B_1, with $\alpha_{32}(v)$ on C_2 and with Id on the rest of the boundary is a null homotopic loop by construction and extends to the whole triangle. Denoting this extension by φ too, we see that $[\mathrm{Ad}(\omega) \circ \varphi_{31}] + [\mathrm{Ad}(\varphi) \circ \alpha_{31}] = 0 \ (\mathrm{mod}\, 4)$. So by 1.7, G_z acts transitively on K_z, or $Q_z = 0$. We have thus proved a major part of

2.2. THEOREM. *For C^*-bundles over a triangle X, which have fibre $C^4 \times M(2)$ over two of the vertices, $M(4)$ over the third vertex, $M(4) \times M(4)$ over the rest of the boundary and $M(8)$ over the interior of X, the natural transformation*

$$M\colon \Pi H^1_R \to MH^1_R$$

defines a bijection. In particular, $\xi(p4gm)$ *is characterized by the assignment* $D_i \mapsto a(i)$ *of multiplicity matrices* (see 2.1.1), *given by the table*

i	1	2	3	4	5	6
$a(i)$	$\begin{bmatrix}11001\\00111\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}11001\\00111\end{bmatrix}$	$\begin{bmatrix}01101\\10011\end{bmatrix}$	$\begin{bmatrix}10011\\01101\end{bmatrix}$

Furthermore, if ξ' *is given by the assignment* $D_i \mapsto a'(i)$, *then* ξ' *and* $\xi(p4gm)$ *are isomorphic if and only if the* $a'(i)$ *can be obtained by one or more of the following operations*:

(1) *permuting simultaneously the first four columns of each of the two pairs*
  (i) $a(4)$, $a(5)$;
  (ii) $a(6)$, $a(1)$;

(2) *permuting simultaneously the rows of each of the three pairs*
  (i) $a(3)$, $a(4)$;
  (ii) $a(5)$, $a(6)$;
  (iii) $a(1)$, $a(2)$.

Proof. The fact that M defines a bijection in this case follows directly from the remarks preceding 2.2 and from 1.10.

The table for the assignment $D_i \mapsto a(i)$ defines a multiplicity map a_{21} and hence an element $(a_{ij}) \in MZ^1_{\mathbf{R}}(\xi)$ by the previous remark. The fact that $\xi(p4gm)$ is characterized by this assignment is a direct translation of the conditions listed in [Ra; 82: Prop. 8].

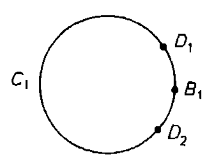
Since $M(\xi)$ is a bijection, the only equivalences of cocycles are given by elements $(a_i) \in MC^0_{\mathbf{R}}(\xi)$. The row and column permutations of 2.2 are exactly the action of $MC^0_{\mathbf{R}}(\xi)$ on the cocycle (a_{ij}) defined by the assignment $D_i \mapsto a(i)$. ∎

The next example shows that $M(\xi)$ need not always be injective:

2.3. Example. With the notation of 1.6, let $n = 1$, $F_1 = M(k)$ for some $k \in N$, $F_2 = M(2k)$ and $F_3 = M(4k)$. Clearly, $MC^0_{\mathbf{R}}(\xi)$ is trivial in this setup.

If $(a_{ij}) \in MZ^0_{\mathbf{R}}(\xi)$, both a_{32} and a_{21} can only take values (1) and (2) and as W_{31} is connected, $a_{31} = a_{32} \cdot a_{21}$ can only take values (1), (2), or (4) and since W_{32} is connected, a_{32} is constant and thus so is a_{21} in this example.

If either $a_{32} = (1)$ or $a_{21} = (1)$, $K_z = 0$. If $a_{21} = a_{32} = (2)$, then $K_z \cong \pi_1(C^* U(M(k), M(4k)), \alpha_{(4)}) = Z/kZ$



Let $u\colon B_1 \to U(4k)$ and $v\colon C_1 \to U(2k)$ be loops. Then a map $\varphi\colon B_1 \cup C_1 \to U(4k)$ defined by

$$\varphi|B_1 = u, \ \varphi|C_1 = \alpha_{32}(v) = \begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix}$$

extends to the whole disk iff

$$[u] = -2\cdot[v] \quad \text{in } \pi_1(U(4k)) \cong \boldsymbol{Z}, \quad [v] \in \pi_1(U(2k)) \cong \boldsymbol{Z}.$$

Thus by 1.7,

$$Q_z = K_z/G_z = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ \boldsymbol{Z}_2 & \text{if } k \text{ is even.} \end{cases}$$

2.4. PROPOSITION. *There are four (resp. five) C^*-bundles over the two disk with fibre $M(k)$ over one boundary point, $M(2k)$ over the rest of the boundary circle and $M(4k)$ over the interior of the disk, if k is odd (resp. even). They are completely characterized by the values of pairs of multiplicity matrices (a_{32}, a_{21}) given by* ((1), (1)), ((1), (2)), ((2), (1)) *and* ((2), (2)) *if k is odd. If k is even, there are two bundles in the case of the multiplicity matrices being* ((2), (2)).

2.5. *Remark.* In [Ev; 79] Bruce Evans calculated the crossed product C^*-algebra associated with the action of the permutation group S_3 acting on an equilateral triangle as its symmetry group. Using Figure 2.1.1, the fibres are $C^2 \times M(2)$ over the vertex in B_2, $M(3) \times M(3)$ over the sides corresponding to C_3 and C_2 including the vertices in B_1 and B_3 and fibre $M(6)$ over the rest of the triangle. Again, 1.10 applies for the case $n = 3$. Since the fibre over C_1 is $M(6)$ it can be seen as in 2.1 that the action of G_z is transitive on K_z for all z and hence $M(\xi)\colon \Pi H^1(\xi)_R \to MH^1(\xi)_R$ is a bijection. In this case $MH^1(\xi)_R$ has 22 elements and the C^*-algebra of the action of S_3 on the triangle corresponds to the representative of the multiplicity cocycle with

$$a_{21}|D_4 \cup D_5 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}. \ \blacksquare$$

Our last example about certain bundles over a three ball shows that the methods are not necessarily limited to the setting of 1.6, but that additional assumptions might have to be made to make the K_z computable.

2.6. PROPOSITION. *Let p, r, s be positive integers, r and s relatively prime, p odd and $k = p(r+s)$. There are exactly two C^*-bundles over the 3-ball X with fibres $M(pr, ps)$ over the north pole, $M(k)$ over the rest of the boundary sphere and fibre $M(2k)$ over the interior which have a continuous identity field.*

Proof. Again we can consider X as the cone over its boundary sphere S^2. We identify W_3 with a closed three ball, W_1 with the closed upper hemisphere of W_3 and W_2 with the lower closed hemisphere of W_3. Then $\Pi C^0_R(\xi)$ is trivial in this case, since W_i is contractible for $i = 1, 2, 3$ and the automorphism groups of $M(2k)$, $M(k)$ and $M(pr, ps)$ are connected as r and s are relatively prime. Similarly, all W_{ij} are connected and hence the multiplicity matrices a_{ij} are constant. Therefore,

$$MZ^1_R(\xi) = \{(a_{31}, a_{32}, a_{21})\colon\ a_{31} = a_{32}\cdot a_{21},\ a_{21} = (1, 1),$$
$$a_{32} = (1) \text{ or } (2)\}.$$

Since we are only interested in bundles with continuous identity section, we consider only cocycles with multiplicity matrices ((2, 2), (2), (1, 1)). For simplicity we will use the same notation for ΠZ^1_R, ΠH^1_R, etc.

The fact that $MC^0_R(\xi) = \Pi C^0_R(\xi)$ is trivial implies that

$$\Pi_1 H^1_R(\xi) = [W_{32}, C^* U(M(k), M(2k))] \times [W_{21}, C^* U(M(pr, ps), M(k))]$$
$$\cong \pi_1(C^* U(M(pr, ps), M(k)), \alpha_{(1,1)})$$

since W_{32} is contractible and $W_{21} \cong S^1$.

From 2.5.1 and the long exact homotopy sequence

$$\pi_1(U(1) \times U(1)) \to \pi_1(U(k) \to \pi_1(C^* U(M(pr, ps), M(k)), \alpha_{(1,1)}) \to 0,$$
$$(n, m) \mapsto p(rn + sm)$$

we obtain

$$\pi_1(C^* U(M(pr, ps), M(k)), \alpha_{(1,1)}) \cong \mathbf{Z}/p\mathbf{Z}.$$

Thus each element of $\Pi_1 H^1_R(\xi)$ can be represented by a pair $(\varphi_{32}, \varphi_{21})$ of bundle imbeddings with

$$\varphi_{32} \equiv \alpha_{(2)} \quad \text{and} \quad \varphi_{21} \equiv \mathrm{Ad}(\omega) \circ \alpha_{(1,1)}$$

for some loop ω in $U(k)$, (II.3.7). Then

$$\begin{aligned} \varphi_{32} \circ \varphi_{21} &= \alpha_{(2)} \circ \mathrm{Ad}(\omega) \circ \alpha_{(1,1)} \\ &= \mathrm{Ad}(\alpha_{(2)}(\omega)) \circ \alpha_{(2)} \circ \alpha_{(1,1)} \\ &= \mathrm{Ad}(\alpha_{(2)}(\omega) \cdot u) \circ \alpha_{(2,2)} \end{aligned}$$

for some constant map $u\colon W_{321} \cong S^1 \to U(2k)$ (II.2.2.1). This imbedding extends to W_{31} iff $\alpha_{(2)}(\omega)$ is null homotopic, i.e.

$$[\alpha_{(2)}(\omega)] = 2[\omega] = 0 \pmod p.$$

Since p is odd, $[\omega] = 0$ or in the notation of (II.3.8), $\operatorname{im} \Phi_2 = \{(\alpha_{(2)}, \alpha_{(1,1)})\}$. With $z = (\alpha_{(2)}, \alpha_{(1,1)})$, we have $K_z \cong \{[\varphi]_{S^1}\colon \varphi | S^1 = \alpha_{(2)} \circ \alpha_{(1,1)} = \mathrm{Ad}(u) \circ \alpha_{(2,2)}\}$ $\cong \pi_2(C^* U(M(pr, ps), M(2k)), \alpha_{(2,2)})$ via the bijection $[\varphi]_{S^1} \mapsto [\mathrm{Ad}(u^*) \circ \varphi]$, where u is a constant map which exists by II.2.2.

By the long exact homotopy sequence and with $\theta_1, \theta_2, \theta_3$ being the maps induced by $\alpha_{(2)}$ on the homotopy groups, we obtain a diagram:

$$\begin{array}{ccccccc}
\pi_2(U(k)) \to & \pi_2(E_1, \alpha_{(1,1)}) & \xrightarrow{\partial} & \pi_1(U(1)\times U(1)) & \xrightarrow{i_*} & \pi_1(U(k)) \\
& \downarrow\theta_1 & & \downarrow\theta_2 & & \downarrow\theta_3 \\
\pi_2(U(2k)) \to & \pi_2(E_2, \alpha_{(1,1)}) & \xrightarrow[\partial]{} & \pi_1(U(2)\times U(2)) & \xrightarrow[i_*]{} & \pi_1(U(2k))
\end{array}$$

where $E_i = C^* U(M(pr, ps), M(ik))$. Since $\pi_2(U(l)) = 0$ for all l, we have the diagram with exact rows

$$\begin{array}{ccccccc}
0 \to & Z & \xrightarrow{\partial} & Z\times Z & \xrightarrow{i_*} & Z \\
& \downarrow\theta_1 & & \downarrow\theta_2 & & \downarrow\theta_3 \\
0 \to & Z & \xrightarrow[\partial]{} & Z\times Z & \xrightarrow[i_*]{} & Z
\end{array}$$

(Here $\pi_2(C^* U(M(pr, ps), M(lk)), \alpha_{(l,l)}) = \operatorname{im}\partial \cong \ker i_* = \{(-ns, nr):\ n \in Z\} \cong Z$ for $l = 1, 2$.) θ_2 and θ_3 are given by multiplication with 2. It is easy to convince oneself that the diagram is commutative and hence θ_1 is given by multiplication with 2 as well.

Again by II.4.7, $(\varphi_{31}^0, \alpha_{(2)}, \alpha_{(1,1)})$ and $(\varphi_{31}^1, \alpha_{(2)}, \alpha_{(1,1)})$ are cocycle homotopic iff $\operatorname{Ad}\varphi \circ \varphi_{31}^1$ is homotopic to φ_{31}^0 rel S^1 for some φ: $(W_3, W_{32}, W_{21}) \to (U(2k), \alpha_{(2)}(U(k)), \alpha_{(2)}(U(1, 1)))$. The set of homotopy classes of the restrictions of such maps to W_{31} can easily been seen to agree with the image of $\pi_2(C^* U(M(pr, ps), M(k)), \alpha_{(1,1)})$ in $\pi_2(C^* U(M(pr, ps), M\times(2k)), \alpha_{(2,2)})$ under θ_1 or $K_z \cong Z_2$.

Since W_3 and W_{32} are contractible and $\pi_0(\operatorname{Aut}(M(pr, ps)) = \pi_0(PU(pr)\times PU(ps)) = 0$, II.4.6 applies to yield that $M(\xi)$ is onto. This finishes the proof. ∎

3. Open problems and final remarks

For the computation of group C^*-algebras of Moore groups and their complete description, we first need to compute the representation theory of the group. This can be done — at least theoretically — by using the Mackey-machine [Ma; 52] and is certainly one of the most complicated parts of solving this problem. Once this is done, however, the representation semigroup determines the fibres of the bundle (I.5.3.2) and the multiplicity cocycle (II.5.2; [Du; 2]. The examples we know seem to indicate that actually the whole cocycle is determined by the knowledge of the space of representations.

As mentioned in 1, the classification of possible structure maps of C^*-semigroup bundles ξ over a representation semigroup X with $\xi_0|\overline{\operatorname{Irr}(X)}$ fixed remains a major problem.

Another question asks, is it possible to extend the Duality Theorem II.5.14 to C^*-algebras that have only finite dimensional irreducible representations without an upper bound on these dimensions? This requires us to identify an appropriate subalgebra of compatible sections which corresponds to the sections vanishing at infinity in the usual C^*-bundles over locally compact Hausdorff spaces. It is easy to see, however, that the algebra of compatible sections that vanish at infinity consists only of the zero section since, for an irreducible element x_0 in the base representation semigroup of dimension $n \in N$ and any compatible section s with $s(x_0) \neq 0$, $\|s(k \cdot x_0)\| = \|s(x_0)\|$ for all $k \in N$. Therefore, $\|s(x)\|$ cannot be made smaller than $\|s(x_0)\|$ on the complement of the compact set of those elements of the base semigroup which have dimension less than a given natural number.

The most difficult problem, however, remains the computation of ΠH^1_R. The computation of MH^1_R can already become extremely involved once the fibres have many minimal ideals. It is conceivable, however, that MH^1_R could be computed with the help of a computer. Beyond these problems, we needed relatively restrictive hypotheses for I.4.6 and II.4.7. At this stage we have no idea of how much can be said under more general hypotheses. We needed that the components of W_{12} and W_{32} are contractible to be able to homotope any cocycle to one of the form $(\varphi_{31}, \alpha_{32}, \alpha_{21})$ and we do not know how to obtain a result as in Lemma 1.7 without being able to obtain that homotopy. In fact, in 2.6, we needed the extra condition that p is odd to make up for the lack of contractibility of W_{21}.

Another unsolved problem is to express the map g: $S_z \to Q_z$ in 1.8, which arises by the action of $\Pi C^0_R(\xi)$ on $\Pi Z^1_R(\xi)$, completely algebraically. 1.8 shows that g is "well behaved" in the sense that the given action commutes with all the other relevant actions on K_z, but we were not able to prove a first conjecture that g might be trivial.

Clearly, it would be necessary to extend the techniques of calculating ΠH^1_R to the case where R is the usual order on sets D of natural numbers with more than 3 elements. Here seem to be some possibilities, using the concept of "abstract stratification" [Ve; 84] and systems of "controlled" tubular neighborhoods.

Given the difficulty in developing methods for a complete computation of ΠH^1_R, it seems worthwhile studying possible connections with other functors like K-theory of the section algebra. This might yield coarser invariants which should be more accessible to computation.

Appendix

A simple proof of Dupré's classification Theorem II.1.1 for a restricted class of bundles

Throughout we assume that M is a compact Riemannian manifold, $N \subseteq M$ a compact submanifold with tubular neighbourhood T and $X = CM$.

Also, we assume that each component of $M \setminus \mathring{T}$ and T is contractible and that each component of ∂T is either contractible or strongly deformation retracts to S^1. We let $A_1 = N \times \{0\}$, $A_2 = (M \backslash N) \times \{0\}$ and $A_3 = X \backslash (M \times \{0\}) \subseteq X$ and $U_1 \supseteq A_1$, $U_2 \supseteq A_2$ and $U_3 = A_3$ of the same form as the V_i in II.4.2, so that $\mathbf{X} = (X_1, (A_i)_{i=1}^3, (U_i)_{i=1}^3) \in \mathrm{TOR}_R$, where $R = \{(i,j) : 1 \leqslant i \leqslant j \leqslant 3\}$.

Furthermore, we let ξ_i be locally trivial C^*-bundles over U_i $(i = 1, 2, 3)$ with finite dimensional fibres and $\xi = (\mathbf{X}, (\xi_i)_{i=1}^3)$. Finally, we assume that any cocycle $(\varphi_{ij}) \in Z_R^1((V_i), (\xi_i))$ is equivalent to one of the form (ψ_{ij}) where $\psi_{i+1,i} = \alpha_{a_{i+1,i}}$ where $i = 1, 2$ and a_{ij} is the multiplicity matrix of φ_{ij}. (This assumption is satisfied if all components of ∂T are contractible by II.3.9.)

We prove Dupré's main classification Theorem [Du; 2] in this particular setup.

THEOREM. *The set $B_R(\xi)$ of isomorphism classs of C^*-bundles β over X with $\beta | A_i = \xi_i | A_i$, $i = 1, 2, 3$, is in one-to-one correspondence with the set $\Pi H_R^1(\xi)$. This correspondence is given by the bundle pasting map* [Du; 79: § 6]

$$b\colon \Pi H_R^1(\xi) \to B_R(\xi).$$

We split the proof into several lemmas:

1. LEMMA. *For any bundle β over X (over $X \times I$) with $\beta | A_i \cong \xi_i | A_i (\beta | (A_i \times I) \cong (\xi_i \times I) | (A_i \times I)$, there is a $(V_i)_{i=1}^3 \in N(\mathbf{X})$ and a cocycle*

$$z = (\varphi_{ij}) \in Z_R^1((V_i)_{i=1}^3, (\xi_i)_{i=1}^3)$$
$$\big(z = (H_{ij}) \in Z_R^1((V_i \times I)_{i=1}^3, (\xi_i \times I)_{i=1}^3)\big)$$

so that $\beta \cong b(z)$.

Proof. By restricting ourselves to connected components, we assume without loss of generality that M and N are connected.

Case 1. β is over X.

Since A_1 is contractible, $\xi | A_1$ is trivial and there are "matrix unit sections" $e_{ij}^k \in \Gamma(\xi | A_1)$, i.e.

1.1. $e_{ij}^{k*} = e_{ji}^k$ and

1.2. $e_{ij}^k e_{mn}^l = \delta_{kl} \cdot \delta_{jm} \cdot e_{in}^k$.

By [Du; 79: 3.4], the e_{ij}^k extend to a closed neighbourhood $U \subseteq U_1$ of A_1 in $A_1 \cup A_2$ still satisfying 1.1 and 1.2 and still denoted e_{ij}^k.

Similarly, there are $f_{rs}^t \in \Gamma(\xi | A_2)$ satisfying similar conditions as 1.1 and 1.2. Again by [Du; 79: 3.4] they extend to some closed neighbourhood $V \subseteq U_2$ (closed in U_2). Then there are functions $A_{ijt}^{krs}\colon U \to \mathbf{C}$ such that for $x \in U$

$$e_{ij}^k(x) = \sum_{r,s,t} A_{ijt}^{krs}(x) f_{rs}^t(x).$$

By 1.1 and 1.2, these functions satisfy

1.1'. $A_{jit}^{krs} = \overline{A_{ijt}^{ksr}}$ and

1.2'. $\delta_{kl} \delta_{jm} A_{int}^{krs} = \sum_b A_{ijt}^{krb} A_{mnt}^{lbs}$.

We can assume without loss of generality that V is so "small" that there is a retraction $\varrho\colon V \to A_2$. For $x \in V \cap U$ we define

$$e_{ij}^k(x) = \sum_{r,s,t} A_{ijt}^{krs}(\varrho(x))\, f_{rs}^t(x)$$

Then these e_{ij}^k satisfy 1.1 and 1.2 by 1.1′ and 1.2′. Thus we have matrix unit sections over $A_1 \cup (V \cap U)$, which we extend via [Du; 79: 3.4] to a neighbourhood W of $A_1 \cup (V \cap U)$. We let $V_1 = W$, $V_2 = V$ and $V_3 = A_3$, then $(V_i)_{i=1}^3 \in N(\mathbf{X})$.

The e_{ij}^k generate a trivial bundle $\beta_1 \subseteq \xi | V_1$ over V_1 and the f_{rs}^t generate a trivial bundle $\beta_2 \subseteq \xi | V_2$. By construction, $\beta_1 | V_1 \cap V_2 \subseteq \beta_2 | V_1 \cap V_2$. Let $\beta_3 = \xi | V_3$. Then obviously $\beta_i \cong \xi_i | V_i$ and these isomorphisms together with the natural inclusions yield the cocycle z and by [Du; 79: 6.1], $\xi \cong b(z)$.

Case 2. β is over $X \times I$. The proof of case 1 carries over almost literally.

2. Lemma. *Let* $\mathbf{Y} = \mathbf{X}$ *or* $\mathbf{X} \times I$, $W_i = V_i$ *or* $V_i \times I$ *so that* $(W_i)_{i=1}^3 \in N(\mathbf{Y})$ *and* $\eta_i = \xi_i$ *or* $\xi_i \times I$. *If* $z^1, z^0 \in Z_{\mathbf{R}}^1((W_i), (\eta_i))$ *and* $z \in C_{\mathbf{R}}^0((W_i), (\eta_i))$ *such that* $z^1 = z \cdot z^0$, *then*

$$b(z^1) \cong b(z^0)$$

(*where* z *acts on* z^0 *as usual*).

Proof. [Du; 79] Theorem 6.1. ∎

3. Lemma. *Let* β *be a* C^*-*bundle over* $X \times I$ *satisfying* $\beta | (A_i \times I) \cong (\xi_i \times A_i) | (A_i \times I)$, $i = 1, 2, 3$, *and* z *a cocycle so that* $\beta \cong b(z)$. *If for all* $t \in [0, 1]$, $\mu_t^*(z) = \mu_0^*(z)$, *then*

$$\beta \cong \mu_0^*(\beta) \times I.$$

Proof. Let $(V_i) \in N(X)$, so that $z = (\varphi_{ij}) \in Z_{\mathbf{R}}^1((V_i \times I), (\xi_i \times I))$ and $\theta_i\colon (\xi_i \times I) | (V_i \times I) < (\mu_0^* b(z) \times I) | (V_i \times I)$ and $\theta_i^0\colon \xi_i | V_i < \mu_0^* b(z) | V_i$, so that $\theta_i(b, t) = (\theta_i^0(b), t)$. Then by assumption $\varphi_{ij}(b, t) \equiv \varphi_{ij}(b, 0)$ and hence $(\theta_i \circ \varphi_{ij})(b, t) = (\theta_i^0(\varphi_{ij}(b, 0)), t) = (\theta_j^0(b), t) = \theta_j(b, t)$ and by [Du; 79: 6.1], $\mu_0^* b(z) \times I \cong b(z) \cong \beta$. ∎

4. Lemma. *Let* β *be a* C^*-*bundle over* $X \times I$ *with* $z = (H_{31}, H_{32}, H_{21}) \in Z_R^1((V_i \times I), (\xi_i \times I))$ *so that* $\beta \cong b(z)$. *Suppose that the* H_{ij} *have multiplicity matrices* a_{ij} *and that* $H_{32} \equiv \alpha_{a_{32}}$ *and* $H_{21} \equiv \alpha_{a_{21}}$ *and the* V_i *have the form of* II.4.2. *Then*

$$\beta \cong \mu_0^* \beta \times I.$$

Proof. By restricting our attention to each component of V_{31} separately, we assume without loss of generality that V_{31} is connected. In agreement with the proof of [Du; 79: 7.4] we let Y be the total space of the bundle $\operatorname{Aut}(\xi_3 | V_{31})$, Z the total space of $C^*U(\xi_1 | V_{31}, \xi_3 | V_{31})$ and $R_0\colon Y \to Z$, $R_0(\psi) = \psi \circ (\mu_0^* H_{31})$. Also we let $Z_0 = R_0(Y) \subseteq Z$. By [Du; 79: Proof of 7.4], Z_0 is open and closed in Z and R_0: $Y \to Z_0$ is locally trivial. Now H_{31} defines a unique homotopy f: $V_{31} \times I \to Z$ with $f(V_{31} \times I) \subseteq Z_0$. We let $W = (V_{31} \times \{0\}) \cup (V_{321} \times I)$, $\bar{f}_0$: $W \to Y, f_0 = \mathrm{id}_{(\xi_3 \times I)|W}$. As in the proof of II.3.5, replacing there X by Z_0, Y by V_{31}, A by V_{321}, E by Y, p by R_0, $\bar{H}$ by f, H' by $\bar{f}_0|(V_{321} \times I)$ and h by $\bar{f}_0|(V_{31} \times \{0\})$, there is an $\bar{f}$: $V_{31} \times I \to Y$ such that:

(1) $\bar{f}|W = \mathrm{id}_{(\xi_3 \times I)|W}$,

(2) $R_0 \circ \bar{f} = f$.

Thus there is a unique section ψ of $\mathrm{Aut}((\xi_3 \times I)|(V_{31} \times I))$ such that $\psi|((\xi_3 \times I)|W) = \mathrm{id}_{(\xi_3 \times I)|W}$ and $H_{31}(x, t) = \psi(x, t) \circ (\mu_0^* H_{31} \times \mathrm{id}_I)$. We define $\bar{\varphi}_3$ to be $\psi(x, t)^{-1}$ on $V_{31} \times I$ and $\mathrm{id}_{(\xi_3 \times I)|((V_{32} \times I) \cup (V_3 \times \{0\}))}$ on $(V_{32} \times I) \cup (V_3 \times \{0\})$ and use [Sp; 66: D1, pg. 57] and the CHP again to obtain $\varphi_3 \in \mathrm{Aut}(\xi_3 \times I)$, so that

(1) $\varphi_3 \circ H_{31} = \mu_0^* H_{31} \times \mathrm{id}_I$ on $V_{31} \times I$

(2) $\varphi_3 \circ H_{32} = H_{32}$ on $V_{32} \times I$.

We let $\varphi_2 = \mathrm{id}_{(\xi_2 \times I)}$ and $\varphi_1 = \mathrm{id}_{(\xi_1 \times I)}$ and

$$z' = (\varphi_3 H_{31} \varphi_1^{-1}, \varphi_3 H_{32} \varphi_2^{-1}, \varphi_2 H_{21} \varphi_1^{-1}).$$

Then

$$\begin{aligned} \beta &\cong b(z) \\ &\cong b(z') && \text{by Lemma 2} \\ &\cong \mu_0 b(z') \times I && \text{by Lemma 3} \\ &\cong \mu_0^* b(z) \times I && \text{by Lemma 2} \\ &\cong \mu_0^* \beta \times I && \text{by assumption. } \blacksquare \end{aligned}$$

5. Lemma. *Let β be a C^*-bundle over $X \times I$ satisfying $\beta|(A_i \times I) \cong (\xi_i \times I)|(A_i \times I)$ and $z = (H_{31}, H_{32}, H_{21})$ a cocycle so that $\beta \cong b(z)$ and such that for $t = 0, 1$*

$$\mu_t^* H_{32} \equiv \alpha_{a_{32}} \quad \text{and} \quad \mu_t^* H_{21} \equiv \alpha_{a_{21}}$$

where a_{ij} is again the multiplicity matrix of H_{ij}. Then

$$\beta \cong \mu_0^*(\beta) \times I.$$

Proof. A careful analysis of the proof of II.4.7 shows that we can replace the W_i there by our V_i and that there are maps H_i from $V_i \times I$ into the unitary groups of the corresponding fibres such that

$$\mathrm{Ad}(H_1) \equiv \mathrm{id}_{\xi_1 \times I},$$

$$\mathrm{Ad}(H_3^*) \circ H_{32} \circ \mathrm{Ad}(H_2) \equiv \alpha_{a_{32}} \quad \text{and}$$

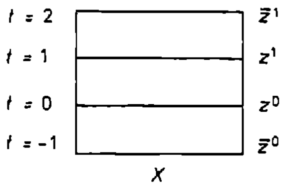
$$\mathrm{Ad}(H_2^*) \circ H_{21} \equiv \alpha_{a_{21}}.$$

The rest follows then from Lemmas 2 and 4. ■

6. Proposition. *Let β be a C^*-bundle over $X \times I$ so that $\beta|(A_i \times I) \cong (\xi_i \times I)|(A_i \times I)$. Then*

$$\beta \cong (\mu_0^* \beta) \times I.$$

Proof. Let $z = (H_{31}, H_{32}, H_{21})$ be a cocycle for β and $z^t = \mu_t^* z$ for $t = 0, 1$. By assumption z^t is homotopic to a cocycle $\bar{z}^t = (\varphi_{ij}^t)$ such that $\varphi_{i+1,i}^t = \alpha_{a_{i+1,i}}$, $i = 1, 2$, where a_{ij} is the multiplicity matrix of H_{ij}. The claim follows then from Lemma 5 by transitivity of the homotopy relation applied to the bundle over $X \times [-1, 2]$ with cocycle given by (H_{ij}) on $X \times I$, and the homotopies $z^t \simeq \bar{z}^t$ over $X \times [-1, 0]$ and $X \times [1, 2]$, respectively.



7. Lemma. *Let* β *be a* C^**-bundle over* X *such that* $\beta | A_i \cong \xi_i | A_i$, $(V_i)_{i=1}^3 \in N(\mathbf{X})$ *and let* $\theta_i^t\colon \xi_i | V_i \subsetneq \beta | V_i$ *for* $t = 0, 1$, $i = 1, 2, 3$ *such that* $\operatorname{im}(\theta_1^t | V_{12}) \subseteq \operatorname{im}(\theta_2^t | V_{12})$. *Then there are* $(V_i')_{i=1}^3 \in N(\mathbf{X})$, $V_i' \subseteq V_i$, $i = 1, 2, 3$ *and* $\theta_i\colon (\xi_i | V_i') \times I \subsetneq (\beta | V_i') \times I$ *such that*

(1) $\operatorname{im}(\theta_1 | (V_{12}' \times I)) \subseteq \operatorname{im}(\theta_2 | (V_{12}' \times I))$ *and*

(2) $\operatorname{im}(\mu_t^* \theta_i) \subseteq \operatorname{im}(\theta_i^t | V_i')$ *for* $t = 0, 1$, $i = 1, 2, 3$.

Proof. This is the obvious generalization of [Du; 79: 8.2] and the proof is almost completely analogous, except that we use the ideas of the proof of Lemma 1 instead of [Du; 79: 3.5] to obtain condition (1).

Proof of theorem. By Proposition 6, the map b is well defined on $\Pi H_R^1(\xi)$ and has its image contained in $B_R(\xi)$ after identifying $B_R(\xi)$ with the set $\Pi B_R(\xi)$ of equivalence classes of bundles β over X with $\beta | A_i \cong \xi_i | A_i$ (β^0 is equivalent to β^1 if and only if there is a bundle β over $X \times I$ with $\beta | (A_i \times I) \cong (\xi_i | A_i) \times I$ such that $\beta^t \cong \mu_t^* \beta$ for $t = 0, 1$). This latter identification is again possible because of Proposition 6. By Lemma 1, the map b is surjective.

The proof of injectivity of b is now a straightforward generalization of the proof of the main theorem in [Du; 79], using Lemma 7 instead of [Du; 79: 8.2]. ∎

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