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Prediction problems

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CONTENTS

Introduction

The concept of prediction for strictly stationary sequences of random variables has been introduced in [12] by K. Urbanik who obtained an analogue of the Wold decomposition into deterministic and completely non-deterministic components. Moreover, he gave a representation of completely non-deterministic sequences by moving averages. The aim of the present paper is to extend some of Urbanik's results to stationary random fields. Chapter I is derated to the classical case of random fields indexed by the n-tuples of integers. In Chapter III we consider random fields defined on a countable lattice being also a group. In Chapter II a new concept of a Banach space of random variables is introduced. It generalizes the concept of L_0^p-spaces. The problem of prediction of a stationary-in-norm random field is discussed.

I. Prediction of strictly stationary random fields

Let $\{x_t\}$, $t = (t^1, t^2, \ldots, t^n) \in Z^n$, where Z^n is the set of n-vectors with integer components, be a strictly stationary random field, or, shortly, a stationary field, defined on a fixed probability space $(\Omega, \mathscr{F}, P)$. By the definition of a stationary field, for any $t, t_1, t_2, \ldots, t_m \in Z^n$, the multivariate distribution of the random variables $x_{t_1+t}, x_{t_2+t}, \ldots, x_{t_m+t}$ is independent of t.

Let $[x_t]$ denote the linear space generated by all random variables x_t, $t \in Z^n$, and closed with respect to the convergence in probability. In the sequel we shall identify the random variables which are equal almost surely. Of course, $[x_t]$ becomes a complete metric space under the Fréchet norm defined by

$$\|x\| = E\frac{|x|}{1+|x|}.$$

To each stationary field there corresponds a unique shift group $\{T_t\}$, $t \in Z^n$, of continuous linear operators in $[x_t]$ preserving the probability distribution and independence of random variables and $x_t = T_t x_0$ for all $t \in Z^n$. Conversely, every such group $\{T_t\}$, $t \in Z^n$, in connection with a random variable x defines a stationary field $x_t = T_t x$ ([2], p. 452, Chapter X).

Now we introduce some notations concerning the set Z^n. We shall write $t \leqslant s$ $(t < s)$, where $t, s \in Z^n$ and $t = (t^1, t^2, \ldots, t^n)$, $s = (s^1, s^2, \ldots, s^n)$, if the inequalities $t^1 \leqslant s^1$, $t^2 \leqslant s^2, \ldots, t^n \leqslant s^n$ $(t^1 < s^1, t^2 < s^2, \ldots, t^n < s^n)$ simultaneously hold. Clearly, $\leqslant$ is a partially ordered relation in Z^n. Further, if $t, s \in Z^n$, we define $\min(t, s)$ and $\max(t, s)$ as $\big(\min(t^1, s^1), \min(t^2, s^2), \ldots, \min(t^n, s^n)\big)$ and $\big(\max(t^1, s^1), \max(t^2, s^2), \ldots, \max(t^n, s^n)\big)$, respectively. Moreover, we shall write $t \to -\infty$, $t \in Z^n$, if $t^j \to -\infty$ for all $j = 1, 2, \ldots, n$. Let $r_1, r_2, \ldots, r_k$ be a system of numbers from the set $\{1, 2, \ldots, n\}$ such that $r_1 < r_2 < \ldots < r_k$. We define the projection in Z^n by

$$P_{r_1 r_2 \ldots r_k}(s) = (0, \ldots, 0, \underset{r_1\text{th place}}{s^{r_1}}, 0, \ldots, 0, \underset{r_2\text{th place}}{s^{r_2}}, 0, \ldots, 0, \underset{r_k\text{th place}}{s^{r_k}}, 0, \ldots, 0) \in Z^n$$

where $s = (s^1, s^2, \ldots, s^n) \in Z^n$. The unit vectors in Z^n will play an important role and we distinguish them by writing $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, $e_n = (0, 0, \ldots, 0, 1)$.

By $[x_t, t \leqslant s]$ we shall denote the closed linear subspace of $[x_t]$ spanned by the random variables x_t, $t \leqslant s$. We say that a stationary field $\{x_t\}$ *admits a prediction* if, for every $s \in Z^n$, there exists a continuous linear operator A_s from $[x_t]$ onto $[x_t, t \leqslant s]$ such that:

(i) $A_s x = x$ whenever $x \in [x_t, t \leqslant s]$.

(ii) For every $x \in [x_t]$ and $y \in [x_t, t \leqslant s]$ the random variables $x - A_s x$ and y are independent.

(iii) If for every $y \in [x_t, t \leqslant s]$ the random variables x and y are independent, then $A_s x = 0$.

(iv) For any $t, s \in Z^n$ the operators A_s and A_t satisfy the equation

$$A_t A_s = A_{\min(t,s)}.$$

The random variables $A_s x$ can be regarded as a linear prediction of x based on the full past of the random field $\{x_t\}$ up to the time $t = s$. The operator A_s will be called a *predictor* based on the full past of the random field $\{x_t\}$ up to the time $t = s$. It is easily seen that the conditions (i), (ii) and (iii) determine the predictor A_s uniquely. An optimality criterion is given by (iii). Moreover, (iv) is a condition of consistency of predictors. In the case $n = 1$ conditions (i), (ii) and (iii) imply that (iv) is always satisfied (see [12], Lemma 1). In general, however, this is not necessarily true, even if $\{x_t\}$ is a Gaussian stationary random field with zero mean. In this case, the concepts of independence and orthogonality are equivalent and square-convergence and convergence in probability are equivalent; hence the orthogonal projector A_s from $[x_t]$ onto $[x_t, t \leqslant s]$ always satisfies conditions (i), (ii), (iii). One can prove that the Gaussian stationary random field $\{x_t\}$, $t \in Z^n$, $n \geqslant 2$, with zero mean and the covariance function given by

$$R(t) = Ex_t x_0 = \begin{cases} 2 & t = 0, \\ 1 & t = (1, -1, 0, \ldots, 0) \quad \text{or} \\ & t = (-1, 1, 0, \ldots, 0), \\ 0 & \text{otherwise} \end{cases}$$

does not satisfy (iv).

Let $\{x_t\}$, $t \in Z^n$, be a strictly stationary random field admitting a prediction. Then the predictors A_s, $s \in Z^n$, and the shifts T_t, $t \in Z^n$, induced by $\{x_t\}$ satisfy the equation

$$A_{s+t} = T_s A_t T_{-s} \quad (s \in Z^n). \tag{I.1}$$

A stationary field $\{x_t\}$ admitting a prediction is called *deterministic* if $A_0 x = x$ for every $x \in [x_t]$. Further, it is called *completely non-deterministic* if $\lim_{t \to -\infty} A_{P_j(t)} x = 0$ for every $x \in [x_t]$ and $j = 1, 2, \ldots, n$.

Now we begin our study by proving the following lemma concerning a general vector field.

LEMMA I.1. *Suppose that $\mathscr{X}$ is a linear space and a_t, $t \in Z^n$, is a vector field in $\mathscr{X}$. Then, for every $s \in Z^n$ such that $s \leqslant 0$ the following equality holds:*

$$(\mathrm{I.2}) \qquad a_0 = \sum_{s \leqslant t \leqslant 0} a_t - \sum_{s \leqslant t \leqslant 0} \sum_{k=1}^{n} (-1)^{k+1} \sum_{\substack{r_1 < r_2 < \cdots < r_k \\ r_1, r_2, \ldots, r_k = 1, 2, \ldots, n}} a_{t - e_{r_1} - e_{r_2} - \ldots - e_{r_k}} + \\ + \sum_{k=1}^{n} (-1)^{k+1} \sum_{\substack{r_1 < r_2 < \cdots < r_k \\ r_1, r_2, \ldots, r_k = 1, 2, \ldots, n}} a_{P_{r_1 r_2 \ldots r_k}(s - (1, 1, \ldots, 1))}.$$

Proof (by induction). For $n = 1$, it is easy to see that equality (I.2) holds for all $s \leqslant 0$. On the other hand, for every $n = 1, 2, \ldots$, equality (I.2) holds for $s = (0, 0, \ldots, 0) \in Z^n$.

Suppose that (I.2) holds for all $p \in Z^k$ with $p \leqslant 0$ and $k < n$. Moreover, suppose that for a fixed $s \in Z^n$ such that $s \leqslant 0$ equality (I.2) holds. We shall prove that (I.2) is true if we replace s in (I.2) by $s - e_j$, $j = 1, 2, \ldots, n$. By the symmetry of the right-hand side of (I.2), it suffices to prove that (I.2) holds for $s - e_1$.

Denoting the right-hand side of (I.2) by $f(s)$ we have

$$f(s - e_1) = \sum_{s \leqslant t \leqslant 0} a_t - \sum_{s \leqslant t \leqslant 0} \sum_{k=1}^{n} (-1)^{k+1} \sum_{r_1 < r_2 < \ldots < r_k} a_{t - e_{r_1} - e_{r_2} - \ldots - e_{r_k}} + \\ + \sum_{\substack{t^1 = s^1 - 1 \\ s^2 \leqslant t^2 \leqslant 0 \\ \cdots\cdots \\ s^n \leqslant t^n \leqslant 0}} a_{(t^1, t^2, \ldots, t^n)} - \\ - \sum_{\substack{t^1 = s^1 - 1 \\ s^2 \leqslant t^2 \leqslant 0 \\ \cdots\cdots \\ s^n \leqslant t^n \leqslant 0}} \sum_{k=1}^{n} (-1)^{k+1} \sum_{r_1 < r_2 < \ldots < r_k} a_{t - e_{r_1} - e_{r_2} - \ldots - e_{r_k}} + \\ + \sum_{k=1}^{n} (-1)^{k+1} \sum_{r_1 = 1 < r_2 < \ldots < r_k} a_{(s^1 - 2, 0, \ldots, 0, \underset{r_2\text{th place}}{s^{r_2} - 1}, 0, \ldots, 0, \underset{r_k\text{th place}}{s^{r_k} - 1}, 0, \ldots, 0)} + \\ + \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 < r_1 < r_2 < \ldots < r_k} a_{(0, \ldots, 0, \underset{r_1\text{th place}}{s^{r_1} - 1}, 0, \ldots, 0, \underset{r_2\text{th place}}{s^{r_2} - 1}, 0, \ldots, 0, \underset{r_k\text{th place}}{s^{r_k} - 1})} \\ = \Big[\sum_{s \leqslant t \leqslant 0} a_t - \sum_{s \leqslant t \leqslant 0} \sum_{k=1}^{n} (-1)^{k+1} \sum_{r_1 < r_2 < \ldots < r_k} a_{t - e_{r_1} - \ldots - e_{r_k}} + \\ + \sum_{k=1}^{n} (-1)^{k+1} \sum_{r_1 < r_2 < \ldots < r_k} a_{P_{r_1 r_2 \ldots r_k}(s - (1, 1, \ldots, 1))} \Big] -$$

$$
\begin{aligned}
&- \sum_{k=1}^{n} (-1)^{k+1} \sum_{r_1<r_2<\ldots<r_k} a_{P_{r_1 r_2 \ldots r_k}(s-(1,1,\ldots,1))} + \\
&+ \sum_{\substack{t^1=s^1-1 \\ s^2\leqslant t^2\leqslant 0 \\ \cdots\cdots \\ s^n\leqslant t^n\leqslant 0}} a_{(s^1-1,t^2,\ldots,t^n)} - \\
&- \sum_{\substack{t^1=s^1-1 \\ s^2\leqslant t^2\leqslant 0 \\ \cdots\cdots \\ s^n\leqslant t^n\leqslant 0}} \sum_{k=2}^{n} (-1)^{k+1} \sum_{1<r_2<\ldots<r_k} a_{(s^1-2,t^2,\ldots,t^n)-e_{r_2}-\ldots-e_{r_k}} - \\
&- \sum_{\substack{s^2\leqslant t^2\leqslant 0 \\ \cdots\cdots \\ s^n\leqslant t^n\leqslant 0}} a_{(s^1-2,t^2,\ldots,t^n)} - \\
&- \sum_{\substack{t^1=s^1-1 \\ s^2\leqslant t^2\leqslant 0 \\ \cdots\cdots \\ s^n\leqslant t^n\leqslant 0}} \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{1<r_1<\ldots<r_k} a_{t-e_{r_1}-e_{r_2}-\ldots-e_{r_k}} + \\
&+ \sum_{k=1}^{n} (-1)^{k+1} \sum_{1<r_2<\ldots<r_k} a_{(s^1-2,0,\ldots,0,\underset{r_2\text{th place}}{s^{r_2}-1},0,\ldots,0,\underset{r_k\text{th place}}{s^{r_k}-1},0,\ldots,0)} + \\
&+ a_{(s^1-2,0,\ldots,0)} + \\
&+ \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{1<r_1<\ldots<r_k} a_{(0,\ldots,0,\underset{r_1\text{th place}}{s^{r_1}-1},0,\ldots,0\underset{r_2\text{th place}}{s^{r_2}-1},0,\ldots,0\underset{r_k\text{th place}}{s^{r_k}-1},0,\ldots,0)}.
\end{aligned}
$$

By the induction assumption, the expression in square brackets is equal to a_0. On the other hand, again by the induction assumption, the expression

$$
\begin{aligned}
&- \sum_{\substack{s^2\leqslant t^2\leqslant 0 \\ \cdots\cdots \\ s^n\leqslant t^n\leqslant 0}} a_{(s^1-2,t^2,\ldots,t^n)} - \\
&- \sum_{\substack{s^2\leqslant t^2\leqslant 0 \\ \cdots\cdots \\ s^n\leqslant t^n\leqslant 0}} \sum_{k=2}^{n} (-1)^{k+1} \sum_{1<r_2<\ldots<r_k} a_{(s^1-2,t^2,\ldots,t^n)-e_{r_2}-\ldots-e_{r_k}} + \\
&+ \sum_{k=2}^{n} (-1)^{k+1} \sum_{1<r_2<\ldots<r_k} a_{(s^1-2,0,\ldots,0,\underset{r_2\text{th place}}{s^{r_2}-1},0,\ldots,0,\underset{r_k\text{th place}}{s^{r_k}-1},0,\ldots,0)}
\end{aligned}
$$

is equal to $-a_{(s^1-2,0,\ldots,0)}$. Therefore,

$$
\begin{aligned}
f(s-e_1) = a_0 - & \\
& - \sum_{k=1}^{n} (-1)^{k+1} \sum_{r_1<r_2<\ldots<r_k} a_{(0,\ldots 0,s^{r_1}-1,0,\ldots,0,s^{r_k}-1,0,\ldots,0)} + \\
& + \Big[\sum_{\substack{s^2\leqslant t^2\leqslant 0 \\ \cdots\cdots \\ s^n\leqslant t^n\leqslant 0}} a_{(s^1-1,t^2,\ldots,t^n)} - \\
& - \sum_{\substack{t^1=s^1-1 \\ s^2\leqslant t^2\leqslant 0 \\ \cdots\cdots \\ s^n\leqslant t^n\leqslant 0}} \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{1<r_1<r_2<\ldots<r_k} a_{t-e_{r_1}-e_{r_2}-\ldots-e_{r_k}} + \\
& + \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{1<r_1<\ldots<r_k} a_{(s^1-1,0,\ldots,0,\underset{r_1\text{th place}}{s^{r_1}-1},0,\ldots,0,\underset{r_k\text{th place}}{s^{r_k}-1},0,\ldots,0)} \Big] - \\
& - \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{1<r_1<\cdots<r_k} a_{(s^1-1,0,\ldots,0,\underset{r_1\text{th place}}{s^{r_1}-1},0,\ldots,0,\underset{r_k\text{th place}}{s^{r_k}-1},0,\ldots,0)} + \\
& + \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{1<r_1<\ldots<r_k} a_{(0,\ldots,0,\underset{r_1\text{th place}}{s^{r_1}-1},0,\ldots,0,\underset{r_k\text{th place}}{s^{r_k}-1},0,\ldots,0)}.
\end{aligned}
$$

By virtue of the induction assumption the expression in the square brackets is equal to $a_{(s^1-1,0,\ldots,0)}$. Hence

$$
\begin{aligned}
f(s-e_1) = a_0 + a_{(s^1-1,0,\ldots,0)} - & \\
& - \sum_{k=1}^{n} (-1)^{k+1} \sum_{r_1<r_2<\ldots<r_k} a_{P_{r_1r_2\ldots r_k}(s-(1,1,\ldots,1))} - \\
& - \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{1<r_1<\ldots<r_k} a_{(s^1-1,0,\ldots,0,\underset{r_1\text{th place}}{s^{r_1}-1},0,\ldots,0,\underset{r_k\text{th place}}{s^{r_k}-1},0,\ldots,0)} \\
& + \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{1<r_1<\ldots<r_k} a_{P_{r_1r_2\ldots r_k}(s-(1,1,\ldots,1))} \\
= a_0 + a_{(s^1-1,0,\ldots,0)} - & \\
& - \sum_{k=1}^{n} (-1)^{k+1} \sum_{r_1=1<r_2<\ldots<r_k} a_{(s^1-1,0,\ldots,0,\underset{r_2\text{th place}}{s^{r_2}-1},0,\ldots,0,\underset{r_k\text{th place}}{s^{r_k}-1},0,\ldots,0)} - \\
& - \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{1<r_1<\ldots<r_k} a_{P_{r_1r_2\ldots r_k}(s-(1,1,\ldots,1))} -
\end{aligned}
$$

$$-\sum_{k=1}^{n-1}(-1)^{k+1}\sum_{1<r_1<\cdots<r_k} a_{(s^1-1,0,\ldots,0,\underset{r_1\text{th place}}{s^{r_1}-1},0,\ldots,0,\underset{r_k\text{th place}}{s^{r_k}-1},0,\ldots,0)}+$$

$$+\sum_{k=1}^{n-1}(-1)^{k+1}\sum_{1<r_1<\ldots<r_k} a_{P_{r_1r_2\ldots r_k}(s-(1,1,\ldots,1))}$$

$$= a_0+a_{(s^1-1,0,\ldots,0)}-$$

$$-\sum_{k=2}^{n}(-1)^{k+1}\sum_{1<r_2<\ldots<r_k} a_{(s^1-1,0,\ldots,0,\underset{r_2\text{th place}}{s^{r_2}-1},0,\ldots,0,\underset{r_k\text{th place}}{s^{r_k}-1},0,\ldots,0)}-$$

$$-a_{(s^1-1,0,\cdots,0)}-$$

$$-\sum_{k=1}^{n-1}(-1)^{k+1}\sum_{1<r_1<\cdots<r_k} a_{s^1-1,0,\ldots,0,\underset{r_1\text{th place}}{s^{r_1}-1},0,\ldots,0,\underset{r_k\text{th place}}{s^{r_k}-1},0,\ldots,0)}=a_0.$$

The lemma is thus fully proved.

LEMMA I.2. *0 is the only constant random variable belonging to the space* $[x_t]$.

Proof. Let c be a constant random variable belonging to $[x_t]$. For every positive number ε there exists a linear combination $\sum_{j=1}^{m} a_j x_{t_j}$ with real coefficients such that

$$\Big\|c-\sum_{j=1}^{m} a_j x_{t_j}\Big\|<\varepsilon.$$

Setting $q=\max(t_1,t_2,\ldots,t_m)$ and taking into account that c is invariant under the shift group $\{T_t\}$ induced by the stationary field $\{x_t\}$, we have the inequality

$$\Big\|T_{-q}c-\sum_{j=1}^{m} a_j T_{-q}x_{t_j}\Big\|=\Big\|c-\sum_{j=1}^{m} a_j\, x_{t_j-q}\Big\|<\varepsilon.$$

Since $\sum_{j=1}^{m} a_j x_{t_j-q}\in[x_t,\, t\leqslant 0]$ and ε was arbitrarily chosen, the relation $c\in[x_j,\, t\leqslant 0]$ is established. Thus, by (i), $A_0c=c$ and by (ii), $A_0c=0$. Consequently, $c=0$, which completes the proof.

Now we quote some results due to K. Urbanik ([12], Lemmas 3, 5, 6).

LEMMA I.3. *Let* $\{y_k\}$, $k=1,2,\ldots$, *be a sequence of independent random variables such that* 0 *is the only constant random variable belonging to* $[y_k]$. *If the series* $\sum_{k=1}^{\infty} y_k$ *converges with probability* 1 *when centred, then it converges with probability* 1, *regardless of the order of summation.*

LEMMA I.4. *Let* $\{v_t\}$, $t\in Z^n$, *be a random field of independent random variables such that* 0 *is the only constant random variable belonging to* $[v_t]$.

For every $x \in [v_t]$ there exists a system $\{a_t\}$, $t \in Z^n$, of real numbers such that

$$x = \sum_{t \in Z^n} a_t v_t,$$

where the series converges with probability 1, *regardless of the order of summation.*

LEMMA I.5. *Suppose that for every $k = 1, 2, \ldots$ the random variables x_k and y_k are independent. If* 0 *is the only constant random variable belonging to $[x_k]$, then the convergence $x_k + y_k \to 0$ in probability implies the convergence $x_k \to 0$ in probability.*

For any $s, l \in Z^n$ such that $l > 0$, we define an operator $A_{s,l}$ on $[x_t]$ by

$$A_{s,l} = I - \prod_{j=1}^{n} (I - A_{s-P_j(l)}) \tag{I.3}$$

where I denotes identity and the symbol $\prod_{j=1}^{n}$ denotes the composition of relevant operators.

In view of the condition of consistency of predictors the expression on the right-hand side of (I.3) can be written as

$$A_{s,l} = \sum_{r=1}^{n} A_{s-P_r(l)} - \sum_{\substack{r_1, r_2 = 1 \\ r_1 < r_2}}^{n} A_{s-P_{r_1 r_2}(l)} + \\ + \sum_{\substack{r_1, r_2, r_3 = 1 \\ r_1 < r_2 < r_3}}^{n} A_{s-P_{r_1 r_2 r_3}(l)} - \ldots + (-1)^{n+1} A_{s-l} \tag{I.4}$$

or, as

$$A_{s,l} = A_{s-P_1(l)} + A_{s-P_2(l)}(I - A_{s-P_1(l)}) + \\ + A_{s-P_3(l)}(I - A_{s-P_1(l)})(I - A_{s-P_2(l)}) + \ldots + \\ + A_{s-P_n(l)} \prod_{j=1}^{n-1} (I - A_{s-P_j(l)}). \tag{I.5}$$

Consequently, by (I.4), the following equation holds:

$$A_{s,l} = T_s A_{0,l} T_{-s} \quad (s \in Z^n). \tag{I.6}$$

The following lemma gives a characterization of the operator $A_{s,l}$.

LEMMA I.6. *For any $s, l \in Z^n$ such that $l > 0$, $A_{s,l}$ is a unique continuous linear operator from $[x_t]$ onto the closed subspace $\sum_{j=1}^{n} [x_t, t \leqslant s - P_j(l)]$*

where $\sum$ *denotes the complex sum of relevant subspaces, and such that*

(I) $A_{s,l}x = x$ *whenever* $x \in \sum_{j=1}^{n} [x_t, t \leqslant s - P_j(l)]$;

(II) *If for every* $y \in \sum_{j=1}^{n} [x_t, t \leqslant s - P_j(l)]$ *the random variables* x *and* y *are independent, then* $A_{s,l}x = 0$;

(III) *For every* $y \in \sum_{j=1}^{n} [x_t, t \leqslant s - P_j(l)]$ *and* $x \in [x_t]$ *the random variables* $x - A_{s,l}x$ *and* y *are independent.*

Proof. By (I.5) it follows that for every $x \in [x_t]$,

$$A_{s,l}x \in \sum_{j=1}^{n} [x_t, t \leqslant s - P_j(l)].$$

Suppose now that $x \in \sum_{j=1}^{n} [x_t, t \leqslant s - P_j(l)]$; then x can be represented as

$$x = x_1 + x_2 + \ldots + x_n \quad \text{where} \quad x_j \in [x_t, t \leqslant s - P_j(l)], \quad j = 1, 2, \ldots, n.$$

In view of (I.3),

$$A_{s,l}x = x - \prod_{j=1}^{n} (I - A_{s-P_j(l)})(x_1 + x_2 + \ldots + x_n)$$

$$= x - \sum_{r=1}^{n} \prod_{\substack{j=1 \\ j \neq r}}^{n} (I - A_{s-P_j(l)})(I - A_{s-P_r(l)})x_r = x,$$

which proves (I). Assertion (II) is easily obtained from (I.6). In order to prove (III) we suppose that

$$y \in \sum_{j=1}^{n} [x_t, t \leqslant s - P_j(l)] \quad \text{and} \quad x \in [x_t].$$

Putting

$$y_1 = A_{s-P_1(l)}y, \quad y_2 = A_{s-P_2(l)}(I - A_{s-P_1(l)})y, \ \ldots, \ y_n = A_{s-P_n(l)} \prod_{j=1}^{n-1} (I - A_{s-P_j(l)})y$$

we infer that the random variables $y_1, y_2, \ldots, y_n$ are independent and $y = y_1 + y_2 + \ldots + y_n$. On the other hand, since for $j = 1, 2, \ldots, n$ the random variables y_j and $x - A_{s,l}x$ are independent and (III) is thus proved. The uniqueness of the operator $A_{s,l}$ follows immediately from (I), (II) and (III). The lemma is thus proved.

In the sequel we shall be dealing with the following operators B_s and C_u defined by $B_s = A_{s,(1,1,\ldots,1)}$, $s \in Z^n$, and $C_u = A_{0,-u}$, $u \in Z^n$ and $u < 0$.

By the definition of operators $B_{s,l}$ we have the formulas

$$B_s = I - \prod_{j=1}^{n} (I - A_{s-e_j}) \quad (s \in Z^n) \tag{I.7}$$

and

$$C_u = I - \prod_{j=1}^{n} (I - A_{P_j(u)}) \quad (u \in Z^n \text{ and } u < 0). \tag{I.8}$$

LEMMA I.7. *For any $s, u \in Z^n$ the following equations hold:*

$$A_u B_s = B_s A_u = \begin{cases} B_s & \textit{whenever } s \leqslant u, \\ A_{\min(u,s)} & \textit{otherwise}. \end{cases} \tag{I.9}$$

Proof. Formula (I.7) implies that $B_s A_s = A_s B_s = B_s$ for every $s \in Z^n$. Suppose $u \leqslant s$ and $u \neq s$. There exists an index j such that $u^j \leqslant s^j - 1$ and by (I.7) we have the equations

$$\begin{aligned} A_u B_s &= A_u - \prod_{\substack{r=1 \\ r \neq j}}^{n} (I - A_{s-e_r})(I - A_{s-e_j}) A_u \\ &= A_u - \prod_{\substack{r=1 \\ r \neq j}}^{n} (I - A_{s-e_r})(A_u - A_{\min(s-e_j, u)}) = A_u. \end{aligned}$$

Thus, if $u \leqslant s$ and $u \neq s$ then $A_u B_s = B_s A_u = A_u$. In general, for any $u, s \in Z^n$ we have, by the foregoing arguments,

$$\begin{aligned} A_u B_s &= A_u (A_s B_s) = (A_u A_s) B_s = A_{\min(u,s)} B_s \\ &= \begin{cases} B_s & \text{whenever } \min(u, s) = s, \\ A_{\min(u,s)} & \text{otherwise}, \end{cases} \end{aligned}$$

which is equivalent to (I.9), and thus the lemma is proved.

LEMMA I.8. *For any $u \leqslant s \leqslant 0$ we have*

$$B_s C_{u-(1,1,\ldots,1)} = A_s C_{u-(1,1,\ldots,1)} \tag{I.10}$$

and

$$C_u B_0 = C_u A_0 = B_0 C_u = A_0 C_u = C_u \tag{I.11}$$

where $u < 0$.

Proof. For every $j = 1, 2, \ldots, n$ we have, by Lemma I.7 and (I.1), the equations

$$A_{P_j(u-(1,1,\ldots,1))} B_s = A_{\min(s, P_j(u-(1,1,\ldots,1)))} = A_s A_{P_j(u-(1,1,\ldots,1))}.$$

Consequently, for every $x \in [x_i]$ and $y \in [x_t, t \leqslant P_j(u-(1,1,\ldots,1))]$ the random variables $A_s x - B_s x$ and y are independent. Further, let y

be an arbitrary element in $\sum_{j=1}^{n} [x_t, t \leqslant P_j(u-(1,1,\ldots,1))]$. Then, by (I.5), y can be decomposed into y_j's such that

$$y = y_1 + y_2 + \ldots + y_n,$$

where

$$y_j = A_{P_j(u-(1,1,\ldots,1))} \prod_{r=1}^{j-1} (I - A_{P_r(u-(1,1,\ldots,1))}) y$$

and hence, $y_1, y_2, \ldots, y_n$ are independent. On the other hand,

$$y_j \in [x_t, t \leqslant P_j(u-(1,1,\ldots,1))], \quad j = 1, 2, \ldots, n.$$

By the foregoing arguments, for every $x \in [x_t]$ the random variables $A_s x - B_s x$ and y are independent. By virtue of Lemma I.6, we infer that

$$C_{u-(1,1,\ldots,1)}(A_s x - B_s x) = 0,$$

which implies equation (I.10). Equations (I.11) are simple consequences of (I.7) and (I.8). The lemma is thus fully proved.

For an arbitrary but fixed $y \in [x_t]$ we define $y_0 = x - B_0 x$, $y_s = A_s x - B_s x$, $s \leqslant 0$ and $s \neq 0$. By Lemma I.1 it follows that

$$\text{(I.12)} \qquad A_r y_s = \begin{cases} y_s & \text{whenever } s \leqslant r \ (r \in Z^n,\ s \in Z^n,\ s \leqslant 0), \\ 0 & \text{otherwise}. \end{cases}$$

Let $s_1, s_2, \ldots, s_m \leqslant 0$ and distinct ($s_j \in Z^n$, $j = 1, 2, \ldots, n$). Let $\lambda_{s_1}, \lambda_{s_2}, \ldots, \lambda_{s_m}$ be a system of real numbers. It is easily seen that there exists an index s from the set $\{s_1, s_2, \ldots, s_m\}$ such that the relation $s_j \leqslant s$ holds if and only if $s_j = s$. Consequently, by (I.12),

$$A_s\Big(\sum_{s_j \neq s} \lambda_{s_j} y_{s_j}\Big) = 0,$$

which implies that the random variables $\lambda_s y_s$ and $\sum_{s_j \neq s} \lambda_{s_j} y_{s_j}$ are independent. Further,

$$E\exp i\Big(\sum_{j=1}^{m} \lambda_{s_j} y_{s_j}\Big) = E\exp i(\lambda_s y_s) E\exp i\Big(\sum_{s_j \neq s} \lambda_{s_j} y_{s_j}\Big).$$

By induction, we have the equation

$$\text{(I.13)} \qquad E\exp i\Big(\sum_{j=1}^{m} \lambda_{s_j} y_{s_j}\Big) = \prod_{j=1}^{m} E\exp i(\lambda_{s_j} y_{s_j}),$$

which shows that the random variables y_s, $s \leqslant 0$, are independent.

Suppose now that $u \leqslant s \leqslant 0$, $u, s \in Z^n$. By (I.10) and (I.11) it follows that for every s such that $u \leqslant s \leqslant 0$ $C_{u-(1,1,\ldots,1)} y_s = 0$. Since the random variables y_s, $s \leqslant 0$, are independent, for every $z \in [x_t]$ the random

variables $C_{u-(1,1,\ldots,1)}z$ and y_s, $u \leqslant s \leqslant 0$, are independent. Thus, we have proved the following lemma.

LEMMA I.9. *If x is a fixed element of $[x_t]$ and $y_0 = x - B_0 x$, $y_s = A_s x - B_s x$, $s \leqslant 0$ and $s \neq 0$, then for every $u \leqslant 0$ and $z \in [x_t]$, the random variables $C_{u-(1,1,\ldots,1)}z$ and y_s, $u \leqslant s \leqslant 0$, are independent.*

LEMMA I.10. *There exists a continuous linear operator $C_{-\infty}$ on $[x_t]$ commuting with the shifts induced by $\{x_t\}$ and such that for every $x \in [x_t]$*

$$\lim_{u \to -\infty} C_u x = C_{-\infty} x. \tag{I.14}$$

Proof. Given $x \in [x_t]$, we define y_s, $s \leqslant 0$, in the same way as in Lemma I.9.

Putting $a_0 = I$, $a_t = A_t$, $t \in Z^n$ and $t \neq 0$, and taking into account Lemma I.1, we have the equation

$$a_0 = \sum_{s \leqslant t \leqslant 0} a_t - \sum_{s \leqslant t \leqslant 0} \sum_{k=1}^{n} (-1)^{k+1} \sum_{\substack{r_1, r_2, \ldots, r_k = 1 \\ r_1 < r_2 < \ldots < r_k}}^{n} a_{t - e_{r_1} - e_{r_2} - \ldots - e_{r_k}} +$$
$$+ \sum_{k=1}^{n} (-1)^{k+1} \sum_{\substack{r_1, r_2, \ldots, r_k = 1 \\ r_1 < r_2 < \ldots < r_k}}^{n} a_{P_{r_1 r_2 \ldots r_k}(s-(1,1,\ldots,1))}.$$

In view of (I.4), (I.7) and (I.8), we find that

$$a_0 = \sum_{s \leqslant t \leqslant 0} (a_t - B_t) + C_{s-(1,1,\ldots,1)}. \tag{I.15}$$

Consequently, for $x \in [x_t]$,

$$a_0(x) = x = \sum_{s \leqslant t \leqslant 0} (a_t - B_t) x + C_{s-(1,1,\ldots,1)} x, \tag{I.16}$$

which implies the equation

$$x = \sum_{s \leqslant t \leqslant 0} y_t + C_{s-(1,1,\ldots,1)} x. \tag{I.17}$$

By virtue of Lemma I.9, the random variables $C_{s-(1,1,\ldots,1)}x$ and y_t, $s \leqslant t \leqslant 0$, are independent and hence the series $\sum_{t \leqslant 0} y_t$ converges with probability 1 when centred (see [2], Theorem 2.8, p. 119). Since, by Lemma I.2, 0 is the only constant random variable belonging to $[x_t]$, the series $\sum_{t \leqslant 0} y_t$, according to Lemma I.3, converges with probability 1, or, equivalently, converges in probability. Hence and from (I.17) it follows that the limit in probability

$$C_{-\infty} x = \lim_{s \to -\infty} C_s x$$

exists. It is clear, by the Banach Theorem (see [1], Theorem 4, p. 23) that $C_{-\infty}$ is a continuous linear operator on $[x_t]$.

Now we shall prove that $C_{-\infty}$ commutes with the shift operators T_u, $u \in Z^n$. First, suppose that $u \leqslant 0$. By (I.16) it follows that

$$T_u x = T_u \sum_{\substack{s \leqslant t \leqslant 0 \\ t \neq 0}} (A_t - B_t)x + T_u(x - B_0 x) + T_u C_{s-(1,1,\ldots,1)} x$$

or, equivalently, by definition of A_t's and by (I.6), the following equation holds:

$$\text{(I.18)} \quad T_u x = \sum_{\substack{s \leqslant t \leqslant 0 \\ t \neq 0}} (A_{t+u} - B_{t+u}) T_u x + T_u x - B_u T_u x + T_u C_{s-(1,1,\ldots,1)} x.$$

On the other hand, by (I.15),

$$\text{(I.19)} \quad T_u x = \sum_{\substack{s \leqslant t \leqslant 0 \\ t \neq 0}} (A_t - B_t) T_u x + T_u x - B_0 T_u x + C_{s-(1,1,\ldots,1)} T_u x.$$

Therefore, by (I.18) and (I.19), we have the equation

$$\text{(I.20)} \quad C_{s-(1,1,\ldots,1)} T_u x - T_u C_{s-(1,1,\ldots,1)} x = \sum_{\substack{s \leqslant t \leqslant 0 \\ t \neq 0}} (A_{t+u} - B_{t+u}) T_u x + \\ + T_u x - B_u T_u x - \sum_{\substack{s \leqslant t \leqslant 0 \\ t \neq 0}} (A_t - B_t) T_u x - \big(T_u x - B_0(T_u x)\big).$$

Denoting the right-hand side and the left-hand side of (I.20) by z_s and g_s, respectively, we can easily see, by the assumption $u \leqslant 0$, that

$$z_s \in \sum_{j=1}^{n} \big[x_t,\, t \leqslant P_j\big(s - (1,1,\ldots,1)\big)\big].$$

On the other hand, by virtue of Lemma I.9 if $v \leqslant s + u - (1,1,\ldots,1)$ then, by (I.20), the random variables g_s and z_v are independent. Let $s \leqslant 0$ be fixed and $v \to -\infty$. Then, the random variables $C_{-\infty} T_u x - T_u C_{-\infty} x$ and g_s are independent for every $s \leqslant 0$. Letting $s \to -\infty$ and denoting $\lim_{s \to -\infty} g_s = g$, we infer that the random variables $C_{-\infty} T_u x - T_u C_{-\infty} x$ and g are independent. However, by (I.20), we have $g = C_{-\infty} T_u C_{-\infty} x$. Consequently, $C_{-\infty} T_u x - T_u C_{-\infty} x$ is a constant random variable. Since 0 is the only constant random variable belonging to $[x_t]$, we conclude that $C_{-\infty} T_u x - T_u C_{-\infty} x = 0$, which implies the equation

$$\text{(I.21)} \qquad C_{-\infty} T_u = T_u C_{-\infty} \quad (u \in Z^n \text{ and } u \leqslant 0).$$

Let s be an arbitrary element of the set Z^n. For a sufficiently large negative u the inequality $u + s \leqslant 0$ holds. Hence and by (I.21), $C_{-\infty} T_{u+s}$

$= T_{u+s}C_{-\infty}$ or, equivalently, $C_{-\infty}T_uT_s = T_uT_sC_{-\infty}$. Again by (I.21) and taking into account the fact that T_u is an 1-1 transformation, we infer that $C_{-\infty}T_s = T_sC_{-\infty}$ for every $s \in Z^n$, which completes the proof.

LEMMA I.11. *The stationary field $\{x_t\}$ is completely non-deterministic if and only if for every $x \in [x_t]$*

$$\lim_{u\to-\infty} C_u x = 0. \tag{I.22}$$

Proof. By (I.8) it is clear that if $\{x_t\}$ is completely non-deterministic then (I.22) holds. Conversely, suppose (I.22) holds. In view of (I.5) we infer that for every $j = 1, 2, \ldots, n$ and $x \in [x_t]$ the random variables $A_{P_j(t)}x$ and $C_t x$ are independent, which by (I.22) and Lemma I.5, implies that $\lim_{u\to-\infty} A_{P_j(t)}x = 0$ in probability. Consequently, $\{x_t\}$ field is completely non-deterministic. The lemma is thus proved.

THEOREM I.1. *Each stationary field admitting a prediction is the sum of two independent stationary fields admitting a prediction, one deterministic and the other completely non-deterministic. Moreover, if $x_t = x_t' + x_t''$ is such a decomposition, then $[x_t]$ is a direct sum of subspaces $[x_t']$ and $[x_t'']$.*

Proof. Let $\{x_t\}$ be a stationary field admitting a prediction. The limit operator $C_{-\infty}$ defined in Lemma I.10 satisfies, by virtue of (I.11), the equations

$$C_{-\infty}A_0 = A_0C_{-\infty} = C_{-\infty}$$

and by Lemma I.10, $C_{-\infty}$ commutes with the shifts; hence

$$C_{-\infty}A_t = C_{-\infty}T_tA_0T_{-t} = T_tC_{-\infty}A_0T_{-t} = T_tC_{-\infty}T_{-t} = C_{-\infty}, \tag{I.23}$$

which, by (I.4), implies that $C_{-\infty}C_u = C_{-\infty}$ for every $u < 0$. Now letting $u \to -\infty$, we get the equation

$$C_{-\infty}^2 = C_{-\infty} \tag{I.24}$$

and consequently

$$(I - C_{-\infty})^2 = I - C_{-\infty}, \tag{I.25}$$

where I denotes the unit operator.

For every $t \in Z^n$ we set

$$x_t' = C_{-\infty}x_t \quad \text{and} \quad x_t'' = (I - C_{-\infty})x_t. \tag{I.26}$$

It is clear that

$$x_t = x_t' + x_t'' \tag{I.27}$$

and, by (I.24) and (I.25),

$$[x_t'] = C_{-\infty}[x_t], \quad [x_t''] = (I - C_{-\infty})[x_t]. \tag{I.28}$$

Moreover

$$[x_t', t \leqslant 0] = C_{-\infty}[x_t, t \leqslant 0], \quad [x_t'', t \leqslant 0] = (I - C_{-\infty})[x_t, t \leqslant 0] \tag{I.29}$$

and

$$C_{-\infty}y' = y', \quad (I-C_{-\infty})y'' = y'' \tag{I.30}$$

whenever $y' \in [x_t']$ and $y'' \in [x_t'']$.

Since the operator $C_{-\infty}$ commutes with the shifts T_s induced by the stationary field $\{x_t\}$, we have

$$T_s x_0' = T_s C_{-\infty} x_0 = C_{-\infty} T_s x_0 = C_{-\infty} x_s = x_s'.$$

Consequently, by (I.27),

$$T_s x_0'' = T_s(x_0 - x_0') = x_s - x_s' = x_s''.$$

Thus, the random fields $\{x_t'\}$ and $\{x_t''\}$ are both stationary.

Let $y' \in [x_t']$ and $y'' \in [x_t'']$. In view of Lemma I.6, for every $u \in Z^n$ such that $u < 0$, the random variables $C_u y'$ and $(I-C_u)y''$ are independent, whence the independence of $C_{-\infty}y'$ and $(I-C_{-\infty})y''$ follows. Hence and by (I.34) the random variables y' and y'' are independent. In other words, the random fields $\{x_t'\}$ and $\{x_t''\}$ are independent.

It is easily seen that the operator A_0 restricted to $[x_t']$ and $[x_t'']$ is a predictor of $\{x_t'\}$ and $\{x_t''\}$ based on the full past up to the time $t = 0$ and hence, the random fields $\{x_t'\}$ and $\{x_t''\}$ admit a prediction. On the other hand, since $A_0 = I$ on $\{x_t'\}$, the field $\{x_t''\}$ is deterministic. Further, let $y'' \in [x_t'']$, we have, by (I.24), the equation

$$C_{-\infty}y = C_{-\infty}(I-C_{-\infty})y = 0,$$

which, by virtue of Lemma I.11, shows that the field $\{x_t''\}$ is completely non-deterministic.

Finally, since 0 is the only constant random variable belonging to $[x_t]$ and moreover, $\{x_t'\}$ and $\{x_t''\}$ are independent, the equation $[x_t] = [x_t'] + [x_t'']$ holds, which completes the proof.

The following theorem gives a characterization of a completely non-deterministic stationary field.

THEOREM I.2. *Let $\{x_t\}$ be a completely stationary non-deterministic field. Then, there exists a field $\{v_t\}$, $t \in Z^n$, of independent identically distributed random variables such that $[v_t, t \leqslant 0] = [x_t, t \leqslant 0]$ and x_s is a moving average*

$$x_s = \sum_{t \leqslant 0} a_t v_{t+s} \quad (s \in Z^n), \tag{I.31}$$

where the series converges in probability, regardless of the order of summation.

Conversely, if $\{v_t\}$ is a random field of independent identically distributed random variables such that 0 is the only constant random variable in $[v_t]$, then the moving average (I.31) *is a completely non-deterministic stationary random field if $[x_t, t \leqslant 0] = [v_t, t \leqslant 0]$.*

Proof. Let $\{x_t\}$, $t \in Z^n$, be a stationary completely non-deterministic random field. Put

$$v_t = x_t - B_t x_t \quad (t \in Z^n). \tag{I.32}$$

By Lemma I.9, the random variables v_t, $t \in Z^n$, are independent. Moreover, by (I.6), we have the equation

$$T_t v_0 = T_t(x_0 - B_0 x_0) = x_t - T_t B_0 T_{-t} x_t = x_t - B_t x_t = v_t, \tag{I.33}$$

which shows that the random variables v_t, $t \in Z^n$, are identically distributed.

Setting $y_0 = x_0 - B_0 x_0$, $y_t = A_t x_0 - B_t x_0$ for every $t \leqslant 0$ and $t \neq 0$, we have, by (I.17), the formula

$$x_0 = \sum_{s<t<0} y_t + C_{s-(1,1,\ldots,1)} x_0 \quad (s \leqslant 0). \tag{I.34}$$

Moreover,

$$y_t \in [x_u, u \leqslant t] \quad (t \leqslant 0). \tag{I.35}$$

Further, by Lemma I.7, $A_t B_t = B_t$, we find that

$$B_t y_t = 0 \quad (t \leqslant 0). \tag{I.36}$$

Now, by (I.35), there exists a sequence of linear combinations

$$\sum_{\substack{(-k,-k,\ldots,-k)<u<t \\ n \text{ times}}} a_u^{(k)} x_u, \quad k = 1, 2, \ldots, \ t \leqslant 0 \text{ and } t \neq 0,$$

tending to y_t in probability as $k \to \infty$. Replacing, in (I.32), x_t by $v_t + B_t x$ and denoting the expression

$$\sum_{\substack{(-k,-k,\ldots,-k)<u<t \\ n \text{ times } u \neq t}} a_u^{(k)} x_u + a_t^{(k)} B_t x_t$$

briefly by $z_t^{(k)}$, we get the convergence

$$a_t^{(k)} v_t + z_t^{(k)} \to y_t \quad (t \leqslant 0, \ t \neq 0) \tag{I.37}$$

in probability as $k \to \infty$. Since

$$z_t^{(k)} \in \sum_{j=1}^{n} [x_u, u \leqslant t - P_j(1, 1, \ldots, 1)]$$

we have, by (I.32),

$$z_t^{(k)} = B_t(a_t^{(k)} v_t + z_t^{(k)}),$$

which, by (I.36), implies that $z_t^{(k)} \to 0$ in probability as $k \to \infty$. Consequently, by (I.37), $a_t^{(k)} v_t \to y_t$ in probability as $k \to \infty$, which implies

that there exists a constant a_t such that $y_t = a_t v_t$, $t \leqslant 0$ and $t \neq 0$. Setting, in addition, $a_0 = 1$, we obtain from (I.34) the equation

$$x_0 = \sum_{s<t\leqslant 0} a_t v_t + C_{s-(1,1,\ldots,1)} x_0 \quad (s \leqslant 0).$$

Since $\{x_t\}$ field is completely non-deterministic and by Lemma I.11 the last equation yields

$$x_0 = \sum_{t\leqslant 0} a_t v_t,$$

where the series converges in probability regardless of the order of summation. Hence and from (I.33) formula (I.31) follows. Consequently, $[x_t, t \leqslant 0] \subset [v_t, t \leqslant 0]$, which together with the relation $v_t \in [x_t, t \leqslant 0]$ implies the identity $[x_t, t \leqslant 0] = [v_t, t \leqslant 0]$. The first part of the theorem is thus proved.

Suppose now that $\{v_t\}$, $t \in Z^n$, is a random field of independent identically distributed random variables such that 0 is the only constant random variable belonging to $[v_t]$. Let T_s be the shift transformation defined by means of the formula $T_s v_t = v_{t+s}$, $t, s \in Z^n$. Further, let $\{x_t\}$ be a random field of moving average (I.31) satisfying the condition $[x_t, t \leqslant 0] = [v_t, t \leqslant 0]$. Of course, $T_s x_t = x_{s+t}$, which shows that $\{x_t\}$ is a stationary random field. Moreover, $[x_t] = [v_t]$. Thus, by Lemma I.4, each element x in $[x_t]$ permits a representation

$$x = \sum_{t\in Z^n} \beta_t v_t, \tag{I.38}$$

which converges in probability regardless of the order of summation. It should be noted that this representation is unique except the trivial case $v_t = 0$, $t \in Z^n$. Since in this case, $\{x_t\}$ is obviously a completely non-deterministic stationary field, we shall assume in the sequel that $v_t \neq 0$.

For an element x having the expansion (I.38), we put

$$A_0 x = \sum_{t<0} \beta_t v_t. \tag{I.39}$$

We shall prove that A_0 is the predictor for $\{x_t\}$ based on the past up to the time $t = 0$. First of all, we note that the operator A_0 is a continuous linear operator from $[x_t]$ onto $[x_t, t \leqslant 0]$. Further, conditions (i) and (iii) are obvious, which, by the fact that 0 is the only constant random variable in $[x_t]$, implies that condition (ii) is also fulfilled. Finally, it is easily seen by (I.39) that the operators A_t defined by $A_t = T_t A_0 T_{-t}$, $t \in Z^n$, satisfy condition (iv). Consequently, A_t, $t \in Z^n$, constitute a family of predictors. Now, for x given by (I.38),

$$A_{P_j(s)} x = \sum_{t<P_j(s)} \beta_t v_t, \quad s \in Z^n,\ j = 1, 2, \ldots, n,$$

which implies that

$$\lim_{s\to-\infty} A_{P_j(s)}x = 0.$$

Thus, the field $\{x_t\}$ is completely non-deterministic, which completes the proof of the theorem.

The proof of the following theorem is similar to that of Theorem 3 in [12].

THEOREM I.3. *Let $\{x_t\}$, $t \in Z^n$, be a stationary random field admitting a prediction. Then there exists a norm $\|\cdot\|_0$ on $[x_t]$ invariant under the shifts induced by $\{x_t\}$ and such that convergence in the norm $\|\cdot\|_0$ is equivalent to convergence in probability. Moreover, for every x in $[x_t]$ and for every predictor A_s, $s \in Z^n$, the following formula holds:*

$$\|x - A_s x\|_0 = \inf\{\|x-y\|_0 \colon y \in [x_t, t \leqslant s]\}. \tag{I.40}$$

Now we are dealing with a stationary Gaussian random field $\{x_t\}$, $t \in Z^n$, with zero mean. It is well known that the correlation function $R(t) = Ex_{t+s}x_s$ is a positive definite function on Z^n and, by the theorem of Herglotz and Bochner, it has the following representation:

$$R(t) = \int_0^1\int_0^1 \ldots \int_0^1 e^{2\pi i(\lambda^1 t^1 + \ldots + \lambda^n t^n)} dv(\lambda^1, \lambda^2, \ldots, \lambda^n) \tag{I.41}$$

where $t = (t^1, t^2, \ldots, t^n)$ and v is a finite Borel measure on $[0,1]^n$, being the spectral measure of the random field $\{x_t\}$. Moreover, the random field $\{x_t\}$ can be represented as a random integral as follows:

$$x_t = \int_0^1\int_0^1 \ldots \int_0^1 e^{i\lambda t} Z(d\lambda) \qquad (t \in Z^n)$$

where Z is a set function with orthogonal increments on the Borel sets, on $[0,1]^n$, which is related to v by

$$EZ(\Lambda) = 0, \quad EZ(\Lambda_1)Z(\Lambda_2) = v(\Lambda_1 \cap \Lambda_2).$$

The following theorem is an analogue of Theorem 4.1 ([2], p. 569).

THEOREM I.4. *Let $\{x_t\}$, $t \in Z^n$, be a stationary Gaussian random field. Then it admits the linear least square prediction and it is completely non-deterministic if and only if the spectral measure v is absolutely continuous with respect to the Lebesgue measure on $[0,1]^n$ and the spectral density*

$$\frac{dv}{d\lambda} = |h(\lambda)|^2$$

satisfies the conditions

(1) $h(\lambda) = \sum_{t \geq 0} c_t e^{2\pi i(\lambda^1 t^1 + \lambda^2 t^2 + \ldots + \lambda^n t^n)}$,

(2) $h(\lambda) > 0$ *a.e. Lebesgue,*

where the series on the right-hand side of (1) *converges in* $L_2[0,1]^n$.

II. Prediction of stationary-in-norm fields in Banach spaces of random variables

§ 1. Banach spaces of random variables

Before introducing the conception of a Banach space of random variables we introduce the convention that the random variables we study in this chapter are defined on a fixed probability space $(\Omega, \mathscr{F}_\Omega, P)$ and the equality $x = y$ of random variables x and y means $x(\omega) = y(\omega)$ for almost all ω.

A Banach space B whose elements are random variables is said to be a *Banach space of random variables* if the following conditions are satisfied:

(i) For any non-zero independent random variables x and y the following inequality holds:

$$\|x+y\| > \max(\|x\|, \|y\|),$$

where $\|\cdot\|$ denotes the norm in B.

(ii) For any sequences $\{v_n\}$ and $\{u_n\}$ in B converging in norm $\|\cdot\|$ to v and u, respectively, if for every $n = 1, 2, \ldots$ the random variables v_n and u_n are independent, so are v and u.

(iii) Every series $\sum_{n=1}^{\infty} x_n$ of independent random variables is boundedly complete, i.e. if

$$\sup_k \Big\| \sum_{n=1}^{k} x_n \Big\| < \infty$$

then

$$\sum_{n=1}^{\infty} x_n \text{ converges.}$$

As can be seen, a relation between a probability property and a Banach space property is brought out. Namely, (i) expresses a strictness of the

norm $\|\cdot\|$ with respect to independent random variables, (ii) gives the behaviour of independent sequences in infinity and (iii) is a criterion of the convergence of series of independent random variables.

We now begin our study by mentioning the following results.

LEMMA 1.1. *0 is the only constant random variable belonging to a Banach space of random variables B.*

Proof. Let c be a constant random variable belonging to B. Then, by (i), $0 = \|c-c\| \geqslant \|c\|$. Consequently, $c = 0$, which completes the proof.

LEMMA 1.2. *Let $\{x_n\}$ and $\{y_n\}$ be sequences of random variables in a Banach space of random variables such that for every n, x_n and y_n are independent. Then, the equation*

$$\lim_{n\to\infty}\|x_n+y_n\| = 0 \quad \textit{implies} \quad \lim_{n\to\infty}\|x_n\| = 0.$$

Proof. This is an easy consequence of (i).

LEMMA 1.3. *Let $x_1, x_2, \ldots$ be a sequence of non-zero independent random variables in a Banach space of random variables B. Then $x_1, x_2, \ldots$ constitute a basis sequence.*

Proof. Given a system of real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ and an integer $m \leqslant n$. By condition (i) we have

$$\Big\|\sum_{i=1}^{m}\lambda_i x_i\Big\| \leqslant \Big\|\sum_{i=1}^{n}\lambda_i x_i\Big\|.$$

Consequently, by [9], $\{x_n\}$ is a basis sequence in B which completes the proof.

We now illustrate Banach spaces of random variables by the following examples.

EXAMPLE 1.1. Let $L_0^p(\Omega, \mathscr{F}_\Omega, P)$ be a space of random variables x with mean zero and $(E|x|^p)^{1/p} < \infty$ for a fixed p such that $1 \leqslant p \leqslant \infty$. Then $L_0^p(\Omega, \mathscr{F}_\Omega, P)$ is a Banach space of random variables under the norm defined by

$$\|x\|_{L_0^p} = (E|x|^p)^{1/p}$$

(see [8], p. 263, problem 2).

EXAMPLE 1.2 (Symmetric stable processes). Let S be a linear space of symmetric stable random variables of index a with $1 \leqslant a \leqslant 2$. By the definition of such a space, the characteristic function of a random variable x in S is of the form

$$(*) \qquad \text{ch.}\, f(x, u) = \exp[-|b|\,|u|^a], \quad (b \text{ real}, 1 \leqslant a \leqslant 2).$$

We shall assume that the space S is closed under convergence in probability. Following Schilder [11] we define the norm $\|\cdot\|_S$ by

$$\|x\|_S = |b|^{1/a}$$

whenever x has the characteristic function (*). Then, S becomes a Banach space of random variables.

EXAMPLE 1.3 (Finite-dimensional space of random variables). Let $x_1, x_2, \ldots, x_m$ be a system of linear independent and stochastic independent non-Gaussian random variables. Let L be a linear space generated by $x_1, x_2, \ldots, x_m$. We define the norm $\|\cdot\|_L$ by

$$\|y\|_L = \sum_{j=1}^{m} |a_j|$$

whenever $y = a_1x_1 + a_2x_2 + \ldots + a_mx_m$ $(a_j \in R)$. Then, as can easily be seen, L becomes a Banach space of random variables.

EXAMPLE 1.4 (Stochastic integrals). Let M be a symmetric atomless random measure defined on the σ-algebra of all Borel subsets of the unit interval $I = [0, 1]$ (see [14]). Let $L(M)$ be the set of all M-integrable functions, where those functions which are equal M almost everywhere will be identified. It is well known that $L(M)$ becomes an Orlicz space under a suitable norm determined by a non-decreasing continuous function Φ on the right half-line vanishing only at the origin. An interesting fact is that $L(M)$ is a Banach space if and only if the function Φ is equivalent to a convex function which is also non-decreasing continuous on the right half-line vanishing only at the origin.

In the sequel we shall identify the space $L(M)$ with the space of all random variables x of the form $x = \int_I f(s)M(ds)$ $(f \in L(M))$. Of course the correspondence $x \leftrightarrow f$ is 1-1. Moreover, the equation $\lim_{n\to\infty} x_n = 0$ (in probability) holds if and only if $\lim_{n\to\infty} f_n = 0$ (in Orlicz norm).

According to a suggestion made by prof. K. Urbanik the Luxemburg norm $\|\cdot\|$ defined by

$$\|x\| = \inf_{\lambda\in R}\left\{\int_I \Phi\left(\left|\frac{f(s)}{\lambda}\right|\right) \leqslant 1\right\}$$

whenever $x = \int_I f(s)M(ds)$, satisfies conditions (i) and (ii). Consequently, $L(M)$ is a Banach space of random variables.

§ 2. Prediction of stationary-in-norm sequences of random variables

A sequence of random variables x_n, $n \in Z$, where Z is the set of all integers, in a Banach space of random variables B is said to be *stationary-in-norm* if for any systems $a_1, a_2, \ldots, a_m$ of real numbers and $n, n_1, n_2, \ldots,$

$\ldots, n_m$ of integers the following equation holds:

$$\Big\|\sum_{j=1}^{m} a_j x_{n_j}\Big\| = \Big\|\sum_{j=1}^{m} a_j x_{n_j+n}\Big\|.$$

It is clear that for Hilbert spaces this definition is equivalent to the classical definition of stationary sequences in a wide sense.

By $[x_n]$ and $[x_n, n \leqslant k]$ we shall denote the subspaces of B spanned by the random variables x_n, $n \in Z$, and x_n, $n \leqslant k$, respectively.

Following K. Urbanik [12], we say that the sequence $\{x_n\}$ *admits a prediction* if there exist linear operators A_k, $k \in Z$, from $[x_n]$ onto $[x_n, n \leqslant k]$, respectively, such that for every x in $[x_n]$ and for every y in $[x_n, n \leqslant k]$ the random variables $x - A_k x$ and y are independent.

For a fixed $k \in Z$ the random variable $A_k x$ can be regarded as a linear prediction of x based on the full past of the sequence $\{x_n\}$ up to the time k. Then the operator A_k can be called a *predictor* based on the full past of the sequence $\{x_n\}$ up to the time k.

It should be noted, by the definitions of a Banach space of random variables and of a stationary-in-norm sequence admitting a prediction, that for every $x \in [x_n]$ the prediction $A_k x$ is the best approximation of x in the norm $\|\cdot\|$ by elements from the subspace $[x_n, n \leqslant k]$. Moreover, in a general Banach space of random variables, the predictor A_k satisfies the following conditions:

(2.1) $$A_k x = x \quad \text{whenever} \quad x \in [x_n, n \leqslant k];$$

(2.2) if for every $y \in [x_n, n \leqslant k]$ the random variables x and y are independent, then $A_k x = 0$, which can be inferred directly from the fact that 0 is the only constant random variable belonging to a Banach space of random variables.

To each stationary-in-norm sequence $\{x_n\}$ there corresponds a shift transformation $Tx_n = x_{n+1}$, $n \in Z$. Since

$$\Big\|T\Big(\sum_{j=1}^{n} a_j x_{n_j}\Big)\Big\| = \Big\|\sum_{j=1}^{n} a_j x_{n_j}\Big\|,$$

the transformation T can be extended to an isommetric transformation T on the whole $[x_n]$. Conversely, every isommetric transformation T in a Banach space of random variables B in connection with a random variable $x \in B$ defines a stationary-in-norm sequence $x_n = T^n x$. It can easily be shown that the predictor A_k and the shift transformation satisfy the equation

(2.3) $$A_k = T^k A_0 T^{-k}.$$

Furthermore, for $k \leqslant r$ the predictors satisfy the equation

$$A_k = A_k A_r = A_r A_k. \tag{2.4}$$

The proof of equations (2.4) is similar to the proof of Lemma 1 [11].

LEMMA 2.1. *Let A_k, $k \in Z$, be predictors for a stationary-in-norm sequence. There exists a bounded linear operator $A_{-\infty}$ on $[x_n]$ commuting with the shift T induced by $\{x_n\}$ and such that for every $x \in [x_n]$*

$$\lim_{k\to\infty} A_{-k}x = A_{-\infty}x \tag{2.5}$$

where the convergence is in norm $\|\cdot\|$.

Proof. Given an element $x \in [x_n]$, we put $y_1 = x - A_{-1}x$, $y_j = A_{1-j}x - A_{-j}x$ $(j = 2, 3, \ldots)$. Since, by (2.4), $y_j = A_{1-j}x - A_{-j}A_{1-j}$ $(j = 2, 3, \ldots)$, we infer, according to the definition of a stationary-in-norm sequence admitting a prediction, that for $k = 1, 2, \ldots$ the random variables $y_1, y_2, \ldots, y_k$ and $A_{-k}x$ are independent. Furthermore,

$$x = \sum_{j=1}^{k} y_j + A_{-k}x \quad (k = 1, 2, \ldots). \tag{2.6}$$

From the last equation and by (i) we deduce that

$$\sup_k \Big\| \sum_{j=1}^{k} y_j \Big\| \leqslant \|x\| < \infty. \tag{2.7}$$

Consequently, by the definition of a Banach space of random variables, the series $\sum_{j=1}^{\infty} y_j$ converges in the norm $\|\cdot\|$.

Hence and from (2.6) the following limit exists:

$$A_{-\infty}x = \lim_{k\to+\infty} A_{-k}x. \tag{2.8}$$

It is clear, by the Banach Theorem, the operator $A_{-\infty}$ defined by the last formula is a bounded linear one.

Finally, from the fact that $A_{-k}T = TA_{-k-1}$ it follows that $A_{-\infty}T = TA_{-\infty}$, which completes the proof.

We say that two sequences $\{x'_n\}$ and $\{x''_n\}$ of random variables are independent if the random variables y' and y'' are independent whenever $y' \in [x'_n]$ and $y'' \in [y''_n]$.

A stationary-in-norm sequence $\{x_n\}$ admitting a prediction is called *deterministic* if $A_0 x = x$ for every $x \in [x_n]$. Further, it is called *completely non-deterministic* if $\lim_{k\to-\infty} A_k x = 0$ whenever $x \in [x_n]$.

Given a stationary-in-norm sequence $\{x_n\}$ admitting a prediction with predictors A_k, $k \in Z$, we can define, according to Lemma 2.1, a bounded linear operator $A_{-\infty}$ on $[x_n]$. Put

$$x'_n = A_{-\infty}x_n, \quad x''_n = (I - A_{-\infty})x_n \quad (n \in Z), \tag{2.9}$$

where I denotes the unit operator.

It is easy to show that $\{x'_n\}$ and $\{x''_n\}$ are stationary-in-norm sequences admitting a prediction. Furthermore, $\{x'_n\}$ is deterministic and $\{x''_n\}$ is completely non-deterministic and they are independent.

We shall demonstrate these results by means of the following theorem, which is well known as Wold's decomposition theorem.

THEOREM 2.1. *Each stationary-in-norm sequence admitting a prediction is the sum of two independent stationary-in-norm sequences admitting a prediction, one deterministic and the other completely non-deterministic.*

Now, using the same method of K. Urbanik ([12], Theorem 2) and by above results we can prove the following theorem.

THEOREM 2.2. *Let $\{x_n\}$ be a stationary-in-norm completely non-deterministic sequence. Then, there exists a stationary-in-norm sequence $\{v_n\}$ of independent random variables such that $[v_n, n \leqslant 0] = [x_n, n \leqslant 0]$ and x_n is a moving average*

$$x_n = \sum_{j=-\infty}^{0} a_j v_{j+n} \quad (n \in Z), \tag{2.10}$$

where the series converges in norm $\|\cdot\|$, regardles of the order of summation. Conversely, if $\{v_n\}$ is a stationary-in-norm sequence of independent random variables, then the moving average (2.10) *is completely non-deterministic provided $[x_n, n \leqslant 0] = [v_n, n \leqslant 0]$.*

§ 3. Markov optimization property of stationary-in-norm sequences

Let $\{x_n\}$ be a stationary-in-norm sequence in a Banach space of random variables.

We say that the sequence $\{x_n\}$ has the *Markov optimization property* if for any $t, t_1, t_2, \ldots, t_n \in Z$ such that $t > t_n > \ldots > t_1$ the following equation holds:

$$\operatorname*{Inf}_{\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbf{R}} \Big\| x_t + \sum_{i=1}^{n} \lambda_i x_{t_i} \Big\| = \operatorname*{Inf}_{\lambda \in \mathbf{R}} \|x_t + \lambda x_{t_n}\|. \tag{3.1}$$

This definition says that the precise knowledge of the present (x_{t_n}) gives the best information about the future (x_t).

More generally, the sequence $\{x_n\}$ is said to have the *Markov optimization property of rank* $k \geqslant 1$ if, for any $t > u \geqslant u-k+1 > t_n > \ldots \ldots > t_1$, we have the equation

$$(3.2)\qquad \operatorname*{Inf}_{\substack{\lambda_1,\lambda_2,\ldots,\lambda_n \in R \\ \gamma_0,\gamma_1,\ldots,\gamma_{k-1}\in R}} \Big\| x_t + \sum_{i=0}^{k-1} \gamma_i x_{u-i} + \sum_{j=1}^{n} \lambda_j x_{t_j} \Big\| = \operatorname*{Inf}_{\gamma_0,\gamma_1,\ldots,\gamma_{k-1}} \Big\| x_t + \sum_{i=0}^{k-1} \gamma_i x_{u-i} \Big\|$$

and, in addition, for every s such that $0 \leqslant s < k$ there exist $t, u \in Z$ with $t > u$ such that

$$(3.3)\qquad \operatorname*{Inf}_{\gamma_0,\gamma_1,\ldots,\gamma_{k-1}} \Big\| x_t + \sum_{i=0}^{k-1} \gamma_i x_{u-i} \Big\| < \operatorname*{Inf}_{\gamma_0,\gamma_1,\ldots,\gamma_{s-1}} \Big\| x_t + \sum_{i=0}^{s-1} \gamma_i x_{u-i} \Big\|.$$

It should be noted that the rank of the sequence $\{x_n\}$ is uniquely determined.

In the sequel a sequence $\{x_n\}$ is said to be *non-trivial* if all x_n's are linear independent and, for the sake of simplicity, a non-trivial stationary-in-norm sequence admitting a prediction and having the Markov optimization property of rank k will be called a *stationary sequence admitting a k-dimensional prediction.*

LEMMA 3.1. *Let x_n be a stationary sequence admitting a k-dimensional prediction. Then, there exist unique sequences $\{f_t^{(i)}\}$, $t = 1, 2, \ldots,$ and $i = 1, 2, \ldots, k-1$, of real numbers such that for any $u < t$ the following equation holds:*

$$(3.4)\qquad A_u x_t = \sum_{i=0}^{k-1} f_{t-u}^{(i)} x_{u-i},$$

where $f_1^{(k-1)} \neq 0$.

Moreover, the sequences $\{f_t^{(i)}\}$ satisfy the equations

$$(3.5)\qquad f_t^{(0)} f_1^{(k-1)} = f_{t+1}^{(k-1)} \quad (t = 1, 2, \ldots)$$

and

$$(3.6)\qquad f_t^{(0)} f_1^{(i)} + f_t^{(i+1)} = f_{t+1}^{(i)} \quad (t = 1, 2, \ldots,\ i = 0, 1, \ldots, k-2).$$

Proof. Let the sequence $\{x_n\}$ satisfy the assumptions of the lemma. Then, by (3.2), we have the equations

$$\|X_t - A_0 x_t\| = \operatorname*{Inf}_{y \in [x_s, s \leqslant 0]} \|x_t - y\| = \operatorname*{Inf}_{\gamma_0,\gamma_1,\ldots,\gamma_{k-1}} \Big\| x_t + \sum_{i=0}^{k-1} \gamma_i x_{-i} \Big\|,$$

which imply the existence of unique real numbers $f_t^{(i)}$, $t = 1, 2, \ldots,$ and $i = 1, 2, \ldots, k-1$, such that

$$A_0 x_t = \sum_{i=0}^{k-1} f_t^{(i)} x_{-i} \quad (t = 1, 2, \ldots).$$

Next, from condition (3.3) it follows that $f_1^{(k-1)} \neq 0$.

Now, for any $t > u$ we have

$$A_u x_t = t_u A_0 x_{t-u} = T_u\Big(\sum_{i=0}^{k-1} f_{t-u}^{(i)} x_{-i}\Big) = \sum_{i=0}^{k-1} f_{t-u}^{(i)} x_{u-i}.$$

Consequently, equation (3.4) holds.

Let $t > 0$. We have, by the fact that $A_0 A_1 = A_0$ and by (3.4), the equations

$$\begin{aligned} A_0 x_{t+1} &= A_0 A_1 x_{t+1} \\ &= A_0\Big(\sum_{i=0}^{k-1} f_t^{(i)} x_{1-i}\Big) = f_t^{(0)} A_0 x_1 + \sum_{i=1}^{k-1} f^{(i)} x_{1-i} \\ &= f_t^{(0)} \sum_{i=0}^{k-1} f_1^{(i)} x_{-i} + \sum_{i=0}^{k-2} f_t^{(i+1)} x_{-i}, \end{aligned}$$

which together with (3.4) imply (3.5) and (3.6). The lemma is thus proved.

LEMMA 3.2. *For a stationary sequence admitting a k-dimensional prediction we have the equations*

$$A_0\big(\mathrm{lin}(x_t, t > 0)\big) = A_0\big(\mathrm{lin}(x_1, x_2, \ldots, x_k)\big) = \mathrm{lin}(x_0, x_{-1}, \ldots, x_{-k+1}),$$

where $\mathrm{lin}(\cdot)$ *denotes the linear manifold generated by the random variables in brackets.*

Consequently, $A_0 x_1, A_0 x_2, \ldots, A_0 x_k$ *are linear independent.*

Proof. It suffices to show that for every $i = 0, 1, \ldots, k-1$ we have $x_{-i} \in Y = A_0\big(\mathrm{lin}(x_1, x_2, \ldots, x_k)\big)$. Contrary to this let us suppose that there exists an index r such that $1 \leqslant r \leqslant k$ and $x_{-k+r} \notin Y$. Then, as can easily be seen,

$$Y \subset \mathrm{lin}\{x_i, i = 0, -1, \ldots, -k+1 \text{ and } i \neq r\}$$

and consequently, by (3.4), $f_t^{(k-r)} = 0$ for every $t = 1, 2, \ldots, k$. Hence and by (3.6) we have $f_t^{(k-r+1)} = 0$ for every $t = 1, 2, \ldots, k-1$. In the same way we can prove that $f_t^{(k-r+2)} = 0$ for every $t = 1, 2, \ldots, k-2$. Proceeding successively, we infer by induction, that $f_t^{(k-1)} = 0$ for every $t = 1, 2, \ldots, k-r+1$, which contradicts the fact that $f_1^{(k-1)} \neq 0$. Hence, for every $i = 0, 1, \ldots, k-1$, $x_{-i} \in Y$, which completes the proof.

THEOREM 3.1. *Every stationary sequence $\{x_n\}$ admitting a k-dimensional prediction is completely non-deterministic.*

Proof. By virtue of Theorem 2.1, the sequence $\{x_t\}$ can be decomposed into independent components $\{x_t'\}$ and $\{x_t''\}$ such that the first one is deterministic and the second is completely non-deterministic. On the

other hand, by (3.4),

$$A_u x_t = A_u(x_t' + x_t'') = \sum_{i=0}^{k-1} f_{t-u}^{(i)} x_{u-i}' + \sum_{i=0}^{k-1} f_{t-u}^{(i)} x_{u-i}'' = x_t' + A_u x_t''.$$

Consequently, $A_u x_t'' = \sum_{i=0}^{k-1} f_{t-u}^{(i)} x_{u-i}''$ and since $\{x_t''\}$ is a completely non-deterministic sequence, we have, for a fixed $t \in Z$,

$$\lim_{u\to-\infty} A_u x_t'' = \lim_{u\to-\infty} \sum_{i=0}^{k-1} f_{t-u}^{(i)} x_{u-i}'' = 0,$$

which, by a simple reasoning, implies that for every $i = 0, 1, \ldots, k-1$, $\lim_{t\to\infty} f_t^{(i)} = 0$. Finally, the last equation implies that

$$\lim_{u\to-\infty} A_u x_t = \lim_{u\to-\infty} \sum_{i=0}^{k-1} f_{t-u}^{(i)} x_{u-i} = 0,$$

which means that the sequence $\{x_t\}$ is completely non-deterministic. Thus the theorem is fully proved.

LEMMA 3.3. *Let $\{x_n\}$ be a stationary sequence admitting a k-dimensional prediction. Then, the sequences $\{f_t^{(i)}\}$, $t = 1, 2, \ldots$ and $i = 0, 1, \ldots, k-1$ in Lemma 3.1 satisfy the following difference equations:*

$$\sum_{i=0}^{j} f_t^{(i)} f_{j-i}^{(0)} = f_{j+t}^{(0)} \quad (t = 1, 2, \ldots \text{ and } 0 \leqslant j \leqslant k-1) \tag{3.7}$$

and

$$\sum_{i=0}^{k-1} f_1^{(i)} f_{j-i}^{(0)} = f_{j+1}^{(0)} \quad (j = k, k+1, k+2, \ldots). \tag{3.8}$$

Moreover, let

$$x_t = \sum_{j=-\infty}^{0} a_j v_{j+t} \quad (t \in Z) \tag{3.9}$$

be the representation of $\{x_t\}$ as in Theorem 2.2; then for every $t = 0, 1, 2, \ldots$

$$f_t^{(0)} = a_{-t} \quad \text{with} \quad f_0^{(0)} = a_0 = 1. \tag{3.10}$$

Proof. From equation (3.9) we have

$$A_0 x_t = \sum_{j=-\infty}^{0} a_{j-t} v_j \quad (t > 0),$$

which together with the equations

$$A_0 x_t = \sum_{i=0}^{k-1} f^{(i)} x_{-i} = \sum_{i=0}^{k-1} f^{(i)} \sum_{j=-\infty}^{-i} a_{j+i} v_j$$

implies the equations

$$\sum_{i=0}^{j} f_i^{(i)} a_{i-j} = a_{-j-i} \quad \text{whenever} \quad 0 \leqslant j \leqslant k-1 \tag{3.11}$$

and

$$\sum_{i=0}^{k-1} f_i^{(i)} a_{-j+i} = a_{-j-i} \quad \text{whenever} \quad j \geqslant k. \tag{3.12}$$

In particular, putting $j = 0$ in (3.11) we get equation (3.10) and by (3.10), (3.11), (3.12) equations (3.7) and (3.8) hold. The lemma is thus proved.

LEMMA 3.4. *Let the sequences $\{f_i^{(i)}\}$ be the same as in Lemma 3.1. Then $\{f_i^{(0)}\}$ does not satisfy any difference equation of rank $s < k$ with real coefficients.*

Proof. Suppose that there exist real numbers $\lambda_0, \lambda_1, \ldots, \lambda_{s-1}$ with $s < k$ and for every $j \geqslant s$ we have

$$\sum_{i=0}^{s-1} \lambda_i f_{j-i}^{(0)} = f_{j+1}^{(0)},$$

which together with (3.7) implies

$$\sum_{r=0}^{s-1} \lambda_r \sum_{i=0}^{j} f_{i-r}^{(i)} f_{j-i}^{(0)} = \sum_{r=0}^{s-1} \lambda_r f_{i-r+j}^{(0)}$$

for $0 \leqslant j \leqslant k-1$.

Hence, if $0 \leqslant j \leqslant k-1$ and $t \geqslant s$, the following equation holds:

$$\sum_{i=0}^{j} \Big(\sum_{r=0}^{s-1} \lambda_r f_{i-r}^{(i)} \Big) f_{j-i}^{(0)} = f_{i+j+1}^{(0)}.$$

Hence and by (3.7) we have

$$\sum_{r=0}^{s-1} \lambda_r f_{i-r}^{(i)} = f_{i+1}^{(i)} \quad (i = 0, 1, \ldots, k-1 \text{ and } t \geqslant s).$$

By putting $\gamma_0 = \lambda_0$, $\gamma_1 = \lambda_1, \ldots, \gamma_{s-1} = \lambda_{s-1}$, $\gamma_s = 0, \ldots, \gamma_{k-1} = 0$ in the last equation we have

$$\sum_{r=0}^{k-1} \gamma_r f_{i-r}^{(i)} = f_{i+1}^{(i)} \quad (i = 0, 1, \ldots, k-1,\ t \geqslant k).$$

Taking into account the fact that the matrix $\{f_{k-r}^{(i)}\}_{i,r=0,1,\ldots,k-1}$ is non-singular (Lemma 3.2), we infer from the last equation and (3.8) that $f_1^{(r)} = \gamma_r$, $r = 0, 1, 2, \ldots, k-1$. Consequently, $\lambda_r = f_1^{(r)}$ for $r = 0, 1, \ldots, s-1$ and $f_1^{(s)} = 0$, $f_1^{(s+1)} = 0, \ldots, f_1^{(k-1)} = 0$, which, of course, contradicts the fact that $f_1^{(k-1)} \neq 0$. The lemma is thus proved.

In order to give a representation of a stationary sequence admitting a k-dimensional prediction it suffices to characterize the sequence $\{f_t^{(0)}\}$, which satisfies the difference equation (3.8) with coefficients $f_1^{(0)}, f_1^{(1)}, \ldots$ $\ldots, f_1^{(k-1)}$ and with the initial conditions $f_1^{(0)}, f_2^{(0)}, \ldots, f_k^{(0)}$ which are solutions of the equations (3.7).

Let us form the characteristic equation of the difference equation (3.8) as follows:

$$\lambda^k - \sum_{i=0}^{k-1} f^{(k-i-1)} \lambda^i = 0, \tag{3.13}$$

which we shall call the *characteristic equation of the* $\{x_t\}$ *sequence.*

Let $z_s = r_s(\cos\varphi_s + i\sin\varphi_s)$, where $r_s = |z_s|$ and $\varphi_s = \arg z_s$, $s = 1, 2, \ldots$ $\ldots, p \leqslant k$, be the solutions of equation (3.13). Suppose every solution z_k has multiplicity a_s with $a_1 + a_2 + \ldots + a_p = k$. Then, the solution of the difference equation (3.8) will be given by

$$f_t^{(0)} = \sum_{s=1}^{p} r_s^t \sum_{m=0}^{a_s-1} t^m (c_{s,m}^{(1)} \cos t\varphi_s + c_{s,m}^{(2)} \sin t\varphi_s) \tag{3.14}$$

$(t = 1, 2, \ldots)$ where $c_{s,m}^{(1)}, c_{s,m}^{(2)}$ $(m = 0, 1, \ldots, a_s - 1)$ are some real constants which are fully determined by the initial conditions $f_1^{(0)}, f_2^{(0)}, \ldots, f_k^{(0)}$ (see [5], Theorem 1, p. 59 and also p. 70).

LEMMA 3.5. *All solutions of the characteristic equation of a stationary sequence admitting a k-dimensional prediction have moduli less than* 1.

Proof. Let $\{x_t\}$ be a stationary sequence admitting a k-dimensional prediction. By virtue of Theorem 3.1 it is completely non-deterministic. Hence and by (3.4) we have $\lim_{t\to\infty} f_t^{(0)} = 0$. Suppose now that there exists a solution z_{s_0} of equation (3.13) such that $r_{s_0} = |z_{s_0}| = \max(r_s\colon s = 1, 2, \ldots$ $\ldots, p) > 1$. Dividing both sides of (3.14) by $r_{s_0}^t$ and letting $t \to \infty$, we obtain

$$\lim_{t\to\infty} \sum_{m=0}^{a_{s_0}-1} t^m (c_{s_0,m}^{(1)} \cos t\varphi_{s_0} + c_{s_0,m}^{(2)} \sin t\varphi_{s_0}) = 0.$$

Consequently, $c_{s_0,m}^{(1)} = c_{s_0,m}^{(2)} = 0$ $(m = 0, 1, \ldots, a_{s_0} - 1)$ and hence

$$f_t^{(0)} = \sum_{s\in\{1,2,\ldots,p\}\setminus\{s_0\}} r_s^t \sum_{m=0}^{a_s-1} t^m (c_{s,m}^{(1)} \cos t\varphi_s + c_{s,m}^{(2)} \sin t\varphi_s),$$

which, of course, satisfies a difference equation with real coefficients of rank $k_0 < k$. However, this is impossible because of Lemma 3.4 and we deduce that $|z_s| \leqslant 1$ for every $s = 1, 2, \ldots, p$.

Further, suppose that there exists a solution z_{s_0} $(1 \leqslant s_0 \leqslant p)$ of equation (3.13) such that $|z_{s_0}| = 1$. In view of (3.14) we infer that

$$\lim_{t\to\infty} \sum_{m=0}^{a_{s_0}-1} t^m (c^{(1)}_{s_0,m} \cos t\varphi_{s_0} + c^{(2)}_{s_0,m} \sin t\varphi_{s_0}) = 0,$$

which implies that $c^{(1)}_{s_0,m} = c^{(2)}_{s_0,m} = 0$ $(m = 0, 1, \ldots, a_{s_0}-1)$ and we have the same contradiction as above. Therefore, for every $s = 1, 2, \ldots, p$, $r_s = |z_s| < 1$, which completes the proof.

To formulate a representation theorem for a stationary sequence admitting a k-dimensional prediction we need the following definition. Let $\lambda_0, \lambda_1, \ldots, \lambda_{k-1}$ be a finite system of real numbers such that $\lambda_{k-1} \neq 0$ and every solution of the equation

$$x^k - \sum_{i=0}^{k-1} \lambda_i x^{k-i-1} = 0 \tag{3.15}$$

has moduli less than 1. Further, let f_t $(t = 1, 2, \ldots)$ be a solution of the difference equation

$$\sum_{i=0}^{k-1} \lambda_i f_{t-i} = f_{t+1} \quad (t = k,\ k+1,\ k+2, \ldots) \tag{3.16}$$

where the initial values $f_1, f_2, \ldots, f_k$ and the coefficients $\lambda_0, \lambda_1, \ldots, \lambda_{k-1}$ satisfy the equations

$$\sum_{i=0}^{j} \lambda_i f_{j-i} = f_{j+1} \quad (j = 0, 1, \ldots, k-1) \tag{3.17}$$

with $f_0 = 1$.

Then, the sequence $\{f_t\}_{t=0,1,2,\ldots}$ is called a *correlation sequence of rank k*. In particular, every correlation sequence of rank 1 is of the form $f_n = \varrho^n$ where $0 < |\varrho| < 1$. It should be noted that for every sequence $f_n = \varrho^n$, $n = 0, 1, 2, \ldots$, there exists a stationary Markovian sequence $\{x_n\}$ of random variables with normal distribution such that $Ex_n = 0$, $Ex_n^2 = 1$ and $Ex_j x_k = \varrho^{|k-j|}$ for which, of course, the sequence $f_n = \varrho^n$ is the sequence of correlation coefficients (see [2], p. 96). Hence the name "correlation sequence of rank k" given here has its origin in the classical case.

THEOREM 3.2 (Representation Theorem). *Let $\{x_t\}$ be a non-trivial stationary-in-norm sequence admitting a prediction and having the Markov optimization property of rank k. Then there exists a non-trivial stationary-in-norm sequence of independent random variables and a correlation sequence $\{f_t^{(0)}\}$ of rank k such that x_t is a moving average*

$$x_t = \sum_{j=-\infty}^{0} f_{-j}^{(0)} v_{j+t} \quad (t \in Z) \tag{3.18}$$

where the series converges in norm $\|\cdot\|$ regardless of the order of summation.

Conversely, let $\{v_t\}$ be a non-trivial stationary-in-norm sequence of independent random variables and let $\{f_t^{(0)}\}_{t=0,1,\ldots}$ be a correlation sequence of rank k. Then the moving average (3.18) *is a non-trivial stationary-in-norm sequence admitting a prediction and having the Markov optimization property of rank k.*

Proof. Suppose $\{x_t\}$ is a stationary sequence admitting a k-dimensional prediction. Then, by virtue of Theorem 3.1, it is completely non-deterministic and has a representation (3.9) with $a_{-t} = f_t^{(0)}$ $(t = 0, 1, 2, \ldots)$. Moreover, as has been shown, $\{f_t^{(0)}\}$ is a correlation sequence of rank k and equation (3.18) holds.

Conversely, if $\{v_n\}$ is a non-trivial stationary-in-norm of independent random variables and if $\{f_t^{(0)}\}$ is a correlation sequence of rank k given by (3.14), then, by the fact that the numbers r_s, $s = 1, 2, \ldots, p$ in (3.14) are less than 1, series (3.18) converges in norm $\|\cdot\|$ regardless of the order of summation.

Proceeding successively, we show that $[x_t, t \leqslant 0] = [v_t, t \leqslant 0]$ with x_t's given by (3.18). First, it is clear that $[x_t, t \leqslant 0] \subseteq [v_t, t \leqslant 0]$. On the other hand, by (3.16), (3.17) and (3.18), we have the equation

$$v_t = x_t - \sum_{i=0}^{k-1} \lambda_i x_{t-i-1} \quad (t \in Z),$$

which implies the inclusion $[v_t, t \leqslant 0] \subseteq [x_t, t \leqslant 0]$. Consequently, $[v_t, t \leqslant 0] = [x_t, t \leqslant 0]$. Finally, by virtue of Theorem 2.2, $\{x_t\}$ is a completely non-deterministic sequence. Moreover, let A_0 be the predictor based on the past up to time $t = 0$. Then, by (3.16), (3.17) and (3.18), it follows that $A_0 x_1 = \sum_{i=0}^{k-1} \lambda_i x_{-i}$, which, of course, is equivalent to the "Markov optimization property of rank k" of $\{x_t\}$ sequence. Thus, the theorem is fully proved.

Now we consider a particular case of our study. Suppose that $\{x_t\}$ is a symmetric stable sequence with index a, $1 \leqslant a \leqslant 2$ (see [11]). Let $[x_t]$ denote the linear space generated by all x_t and closed under convergence in probability. Then, by Example 1.2, $[x_t]$ is a Banach space of random variables.

THEOREM 3.3. *Let $\{x_t\}$, $t \in Z$, be a stationary symmetric stable sequence with index a $(1 < a \leqslant 2)$ admitting a prediction. Then it has the Markov optimization property if and only if for any $t > u > t_n > \ldots > t_1$ the following equation holds a.s.:*

$$E[x_t | x_u, x_{t_n}, \ldots, x_{t_1}] = E[x_t | x_u]. \tag{3.19}$$

Proof. Suppose $\{x_t\}$ admits a 1-dimensional prediction. There exists a real constant a such that, for any $t > u$, $A_u x_t = a^{t-u} x_u$. On the other hand, $A_u x_t = E[x_t | x_s\colon s \leqslant u]$. Consequently, equation (3.19) holds.

Conversely, if (3.19) holds then for any $t > u$ we have $A_u x_t = E[x_t | x_u]$. By virtue of the Kanter Theorem (see [3], Theorem 1.4) there exists a real constant $a_{t,u}$ such that $E[x_t | x_u] = a_{t,u} x_u$ a.s. Since $\{x_t\}$ is stationary, we infer that $a_{t,u} = a^{t-u}$ for a real constant a. Finally, $A_u x_t = a^{t-u} x_u$ for all $t > u$, which completes the proof.

THEOREM 3.4. *Let $\{x_t\}$, $t \in Z$, be a non-trivial stationary symmetric stable sequence with index α, $1 \leqslant \alpha \leqslant 2$. Then $\{x_t\}$ admits a prediction and has the Markov optimization property if and only if for every $n \geqslant 0$ the joint characteristic function of random variables $x_0, x_1, \ldots, x_n$ is of the form*

$$\text{(3.20)} \qquad \mathrm{ch.}f(x_n, x_{n-1}, \ldots, x_0; \lambda_n, \lambda_{n-1}, \ldots, \lambda_0) = \exp\Big[-\Big\{(1-|a|^\alpha)\sum_{k=0}^{n-1}\Big|\sum_{i=0}^{k}\lambda_{n-i}a^{k-i}\Big|^\alpha + \Big|\sum_{i=0}^{n}\lambda_i a^i\Big|^\alpha\cdot c$$

where $c = \|x_0\|^\alpha$ and $0 < |a| < 1$.

Proof. Suppose $\{x_t\}$ admits a 1-dimensional prediction. There exists a real number a, $0 < |a| < 1$, such that for any $t > u$ we have $A_u x_t = a^{t-u} x_u$. Hence and by Lemma 3.1 ([11]) it follows that

$$\begin{aligned}
-\log\mathrm{ch.}f(x_n, x_{n-1}, \ldots, x_0; \lambda_n, \lambda_{n-1}, \ldots, \lambda_0) &= \Big\|\sum_{i=0}^{n}\lambda_i x_i\Big\|^\alpha \\
&= |\lambda_n|^\alpha \|x_1 - a x_0\|^\alpha + \|(a\lambda_n + \lambda_{n-1})x_{n-1} - a(a\lambda_n + \lambda_{n-1})x_{n-2} + \\
&\quad + a(a\lambda_n + \lambda_{n-1})x_{n-2} + \lambda_{n-2}x_{n-2} + \ldots + \lambda_0 x_0\|^\alpha \\
&= |\lambda_n|^\alpha \|x_1 - a x_0\|^\alpha + |a\lambda_n + \lambda_{n-1}|^\alpha \|x_1 - a x_0\|^\alpha + \\
&\quad + \big\|a\big((a\lambda_n + \lambda_{n-1}) + \lambda_{n-2}\big)x_{n-2} + \lambda_{n-3}x_{n-3} + \ldots + \lambda_0 x_0\big\|^\alpha.
\end{aligned}$$

This yields by induction

$$\begin{aligned}
\Big\|\sum_{i=0}^{n}\lambda_i x_i\Big\|^\alpha = \|x_1 - a x_0\|^\alpha \sum_{k=0}^{n-1}\Big|a\big(a\ldots\big(a(a\lambda_n + \lambda_{n-1}) + \lambda_{n-2}\big) + \ldots\big) + \lambda_{n-k}\Big|^\alpha + \\
+ \Big|a\Big(a\big(\ldots\big(a(a\lambda_n + \lambda_{n-1}) + \lambda_{n-2}\big) + \ldots\big)\Big) + \lambda_0\Big|^\alpha \|x_0\|^\alpha,
\end{aligned}$$

which implies equation (3.20) because

$$\|x_1 - a x_0\|^\alpha = (1 - |a|^\alpha)\|x_0\|^\alpha.$$

Conversely, suppose that for every $n \geqslant 0$ the joint characteristic function of random variables $x_n, x_{n-1}, \ldots, x_0$ is given by (3.20). We define the operator A_0 by

$$\text{(3.21)} \qquad A_0 x_t = \begin{cases} a^t x_0 & \text{whenever } t \geqslant 0, \\ x_t & \text{whenever } t \leqslant 0. \end{cases}$$

Since the sequence $\{x_t\}$ is non-trivial, the operator A_0 can be extended to a linear operator on the whole $\operatorname{lin}(x_t, t \in Z)$. Moreover, as can easily be seen, A_0 can be extended to a linear continuous operator mapping $[x_t]$ onto $[x_t, t \leqslant 0]$ and satisfying the conditions of a predictor. Consequently, the sequence $\{x_t\}$ admits a prediction. Furthermore, by (3.21), we infer that the sequence $\{x_t\}$ has the Markov optimization property, which completes the proof.

§ 4. Definition of a stationary-in-norm random field admitting a prediction

Let B be a Banach space of random variables. A random field $\{x_t\}$, $t = (t^1, t^2, \ldots, t^n) \in \boldsymbol{R}^n$, where $\boldsymbol{R}^n$ is the n-dimensional Euclidean space and $x_t \in B$, is said to be *stationary-in-norm* if for any $\lambda_1, \lambda_2, \ldots, \lambda_m \in \boldsymbol{R}^1$ and $t, t_1, t_2, \ldots, t_m \in \boldsymbol{R}^n$ the following equation holds:

$$\Big\| \sum_{i=1}^{m} \lambda_i x_{t_i} \Big\| = \Big\| \sum_{i=1}^{m} \lambda_i x_{t_i+t} \Big\|, \tag{4.1}$$

where $\|\cdot\|$ is the norm in B.

In what follows we shall assume that $\{x_t\}$ is continuous in the norm $\|\cdot\|$, i.e. $\lim\limits_{t \to s} \|x_t - x_s\| = 0$ $(t, s \in \boldsymbol{R}^n)$, and, for simplicity, a continuous stationary-in-norm random field will be called stationary-in-norm.

We now introduce some notations concerning the space $\boldsymbol{R}^n$. For any $t, s \in \boldsymbol{R}^n$ with $t = (t^1, t^2, \ldots, t^n)$ and $s = (s^1, s^2, \ldots, s^n)$ we shall write $t \leqslant s$ $(t < s)$ if the inequalities $t^1 \leqslant s^1, t^2 \leqslant s^2, \ldots, t^n \leqslant s^n$ $(t^1 < s^1, t^2 < s^2, \ldots, t^n < s^n)$ simultaneously hold. Of course, $\leqslant$ is a partially ordered relation. The notions $\max(t, s)$ and $\min(t, s)$ are obviously defined by $\leqslant$. Given a system of $r_1, r_2, \ldots, r_k$ of numbers from the set $\{1, 2, \ldots, n\}$, we define a projector $p_{r_1 r_2 \ldots r_n}$ in $\boldsymbol{R}^n$ by

$$p_{r_1 r_2 \ldots r_n}(t) = (0, \ldots, 0, \underset{r_1\text{th place}}{\underset{\downarrow}{t^{r_1}}}, 0, \ldots, 0, \underset{r_2\text{th place}}{\underset{\downarrow}{t^{r_2}}}, 0, \ldots, 0, \underset{r_k\text{th place}}{\underset{\downarrow}{t^{r_k}}}, 0, \ldots, 0) \in \boldsymbol{R}^n \quad (t \in \boldsymbol{R}^n).$$

Further, we shall write $t \to -\infty$ $(t \in \boldsymbol{R}^n)$ if, for every $j = 1, 2, \ldots, n$, $t^j \to \infty$. Finally, let $\mathscr{R}_0$ denote the ring of all n-dimensional bounded semiclosed intervals in $\boldsymbol{R}^n$, i.e. the intervals of the form $I_{s,l} = \{t \in \boldsymbol{R}^n\colon\ s - l < t \leqslant s\}$ with $s, l \in \boldsymbol{R}^n$ and $l > 0$. Sometimes, for the convenience of writing, if $s_1, s_2 \in \boldsymbol{R}^n$ with $s_2 < s_1$ we shall use the symbol $(s_2, s_1]$ to denote the semiclosed interval $I_{s_1, s_1 - s_2}$.

By $[x_t]$ we shall denote the subspace of B spanned by all random variables x_t, $t \in \boldsymbol{R}^n$. Further, for every $a \in \boldsymbol{R}^n$ let $[x_t, t \leqslant a]$ denote the subspace of $[x_t]$ spanned by the random variables x_t, $t \leqslant a$.

We say that a stationary-in-norm random field $\{x_t\}$ *admits a prediction* if, for every $a \in \boldsymbol{R}^n$, there exists a linear operator A_a from $[x_t]$ into $[x_t, t \leqslant a]$ such that:

(4.i) For every $x \in [x_t]$ and $y \in [x_t, t \leqslant a]$ the random variables $x - A_a x$ and y are independent.

(4.ii) For any $a, b \in \boldsymbol{R}^n$

$$A_a A_b = A_{\min(a,b)}.$$

It should be noted that this definition is a modification of one given by K. Urbanik [13] for a strictly stationary process.

The random variables $A_a x$ can be regarded as a linear prediction of x based on the full past of the random field $\{x_t\}$ up to the time $t = a$ (with respect to the relation $\leqslant$) and the operator A_a will be called a predictor based on the full past of $\{x_t\}$ up to the time $t = a$. This implies by (4.i) that for every $x \in [x_t]$ and $a \in \boldsymbol{R}^n$, $A_a x$ is a unique element in $[x_t, t \leqslant a]$ such that

$$\|x - A_a x\| = \operatorname*{Inf}_{y \in [x_t, t \leqslant a]} \|x - y\|. \tag{4.2}$$

Moreover, (4.ii) is a condition of consistency of the random field, which immediately implies that the operators A_a, $a \in \boldsymbol{R}^n$, commute. It is very easy to show that, in a general Banach space of random variables B, if $n = 1$ then condition (4.ii) can be derived from (4.i). Generally, however, it is not necessarily true even if $\{x_t\}$ is a Gaussian stationary random field with mean zero. In this case, convergence is determined by the squares-norm and the concepts of independence and orthogonality are equivalent. Therefore, an admissible predictor A_a is simply the orthogonal projector from $[x_t]$ onto $[x_t, t \leqslant a]$ and consequently, an admissible prediction is the least-squares prediction. For instance, let $\{x_t\}$, $t \in \boldsymbol{R}^n$ and $n \geqslant 2$, be a Gaussian stationary random field with mean zero and the correlation function $R(t)$, $t \in \boldsymbol{R}^n$, given by

$$R(t) = \boldsymbol{E} x_s x_{s+t} = \begin{cases} 2 & t = 0, \\ 1 & t = (1, -1, 0, \ldots, 0) \quad \text{or} \\ & t = (-1, 1, 0, \ldots, 0), \\ 0 & \text{otherwise}. \end{cases}$$

One can show, by a simple calculation, that this random field does not satisfy (4.ii).

Proceeding successively, let us note that in the definition of a stationary-in-norm random field admitting a prediction we do not assume the continuity of the operators A_a, $a \in \boldsymbol{R}^n$, but, in fact, this property can be derived from (4.i). Further properties of A_a's can be derived from the following lemma.

LEMMA 4.1. *Let $\mathscr{X}$ be a subspace of $[x_t]$ and let A be a linear operator from $[x_t]$ into $\mathscr{X}$ such that for every $x \in [x_t]$ and $y \in \mathscr{X}$ the random variables $x - Ax$ and y are independent. Then, A is a continuous operator from $[x_t]$ onto $\mathscr{X}$ and*

(α) *$Ax = x$ whenever $x \in \mathscr{X}$,*

(β) *if for every $y \in \mathscr{X}$ the random variables x and y are independent, then $Ax = 0$.*

Proof. Given $x \in [x_t]$, we have $x = x - Ax + Ax$, and since the random variables $x - Ax$ and Ax are independent, we have, by (1.i), $\|Ax\| \leqslant \|x\|$, which shows that A is bounded. Next, suppose $x \in \mathscr{X}$; we have $x - Ax \in \mathscr{X}$ and hence $x - Ax$ is independent of itself. Therefore, $x - Ax$ is a constant random variables and by Lemma 1.1 we conclude that $Ax = x$, which proves (α).

Now using again the equation $x = x - Ax + Ax$, we infer that $x - Ax$ and Ax are independent. Hence Ax is a constant random variable and consequently $Ax = 0$. The lemma is fully proved.

Let $\{x_t\}$, $t \in \boldsymbol{R}^n$, be a stationary-in-norm random field. Then there exists a unique strongly continuous shift group $\{T_t\}$, $t \in \boldsymbol{R}^n$, of linear isommetric operators in $[x_t]$ such that $T_t x_0 = x_t$ for every $t \in \boldsymbol{R}^n$. Conversely, given an element $x \in B$ and a strongly continuous group $\{T_t\}$, $t \in \boldsymbol{R}^n$, of linear isommetric operators in B, we can define a stationary-in-norm random field $\{x_t\}$ by setting $x_t = T_t x$.

For every stationary-in-norm random field $\{x_t\}$, $t \in \boldsymbol{R}^n$, admitting a prediction, predictors A_t, $t \in \boldsymbol{R}^n$, and shift operators T_t, $t \in \boldsymbol{R}^n$, satisfy by (4.2) the equation

$$A_t = T_t A_0 T_{-t} \quad (t \in \boldsymbol{R}^n). \tag{4.3}$$

A stationary-in-norm random field $\{x_t\}$, $t \in \boldsymbol{R}^n$, admitting a predictions is called *deterministic* if $A_0 x = x$ for every $x \in [x_n]$. Further, it is called *completely non-deterministic* if $\lim_{t \to -\infty} A_{p_j(t)} x = 0$ $(t \in \boldsymbol{R}^n)$ for every $x \in [x_t]$ and $j = 1, 2, \ldots, n$.

§ 5. Decomposition Theorem

Suppose $\{x_t\}$, $t \in \boldsymbol{R}^n$, is a stationary-in-norm random field admitting a prediction with predictors A_t, $t \in \boldsymbol{R}^n$.

For any $s, l \in \boldsymbol{R}^n$ such that $l > 0$ we define an operator $A_{s,l}$ acting on $[x_t]$ by

$$A_{s,l} = I - \prod_{j=1}^{n} (I - A_{s-p_j(l)}) \tag{5.1}$$

where I denotes the identity and the symbol $\prod$ denotes the composition of relevant operators.

In view of the condition of consistency (4.ii) the expression on the right-hand side of (5.1) can be written as

$$A_{s,l} = \sum_{k=1}^{n} (-1)^{k+1} \sum_{\substack{r_1, r_2, \ldots, r_k = 1 \\ r_1 < r_2 < \ldots < r_k}}^{n} A_{s-p_{r_1 r_2 \ldots r_k}(l)} \tag{5.2}$$

or, as

$$\begin{aligned} A_{s,l} &= A_{s-p_1(l)} + A_{s-p_2(l)}(I - A_{s-p_1(l)}) + \\ &+ A_{s-p_3(l)}(I - A_{s-p_1(l)})(I - A_{s-p_2(l)}) + \ldots + A_{s-p_n(l)} \prod_{j=1}^{n-1} (I - A_{s-p_j(l)}). \end{aligned} \tag{5.3}$$

Consequently, by (4.3), the following equation holds:

$$A_{s,l} = T_s A_{0,l} T_{-s}. \tag{5.4}$$

The following lemma gives a characterization of the operator $A_{s,l}$.

LEMMA 5.1. *For any $s, l \in R^n$ such that $l > 0$, $A_{s,l}$ is a unique continuous linear operator from $[x_t]$ onto the closed subspace $\sum_{j=1}^{n} [x_t, t \leqslant s - p_j(l)]$, where $\sum$ denotes the complex sum of relevant spaces and such that*

(5.i) $A_{s,l} x = x$ *whenever* $x \in \sum_{j=1}^{n} [x_t, t \leqslant s - p_j(l)]$.

(5.ii) *If for every* $y \in \sum_{j=1}^{n} [x_t, t \leqslant s - p_j(l)]$ *the random variables x and y are independent, then* $A_{s,l} = 0$.

(5.iii) *For every* $y \in \sum_{j=1}^{n} [x_t, t \leqslant s - p_j(l)]$ *and* $x \in [x_t]$ *the random variables $x - A_{s,l} x$ and y are independent.*

Proof. From (5.3) it follows that $A_{s,l}$ maps $[x_t]$ into $\sum_{j=1}^{n} [x_t, t \leqslant s - p_j(l)]$. By virtue of Lemma 4.1 it suffices to prove (5.iii).

Accordingly, suppose $y \in \sum_{j=1}^{n} [x_t, t \leqslant s - p_j(l)]$ and $x \in [x_t]$. Setting $y_1 = A_{s-p_1(l)} y$, $y_2 = A_{s-p_2(l)}(I - A_{s-p_1(l)}) y, \ldots, y_n = A_{s-p_n(l)} \prod_{j=1}^{n-1} (I - A_{s-p_j(l)}) y$, we infer that the random variables $y_1, y_2, \ldots, y_n$ are independent and, by (5.3), $y = y_1 + y_2 + \ldots + y_n$. On the other hand, since for every $j = 1, 2, \ldots, n$ the random variables $y_j \in [x_t, t \leqslant s - p_j(l)]$ and $x - A_{s-l} x$ are independent, the random variables $x - A_{s,l} x$ and y are independent. The uniqueness of the operator $A_{s,l}$ follows immediately from (5.i), (5.ii) and (5.iii). The lemma is thus proved.

For any bounded semiclosed interval $I_{s,l} \in \mathscr{R}_0$ we put

$$A(I_{s,l}) = A_s - A_{s,l} \tag{5.5}$$

and, in addition, $A(\emptyset) = 0$.

It is clear that

$$A(I_{s,l}) = T_s A(I_{0,l}) T_{-s} \tag{5.6}$$

and

$$T_t A(I_{s,l}) = A(I_{s,l}+t) T_t \quad (t \in R^n). \tag{5.7}$$

In the sequel we need the following remark. Let I_{s_1,l_1} and I_{s_2,l_2} be intervals from $\mathscr{R}_0$. Then, $I_{s_1,l_1} \cap I_{s_2,l_2} \neq \emptyset$ if and only if

$$\min(s_1, s_2) > \max(s_1 - l_1, s_2 - l_2). \tag{5.8}$$

Moreover, in this case

$$I_{s_1-l_1} \cap I_{s_2,l_2} = I_{\min(s_1,s_2),\min(s_1,s_2)-\max(s_1-l_1,s_2-l_2)}. \tag{5.9}$$

LEMMA 5.2. *For any $l_1, l_2 \in R^n$ such that $l_1, l_2 > 0$ the following equation holds:*

$$A_{0,l_1} A_{0,l_2} = A_{0,l_1} + A_{0,l_2} - A_{0,\min(l_1,l_2)}. \tag{5.10}$$

Proof. By the definition of operators $A_{s,l}$ we have

$$\begin{aligned} A_{0,l_1} A_{0,l_2} &= \Big[I - \prod_{j=1}^{n} (I - A_{-p_j(l_1)})\Big]\Big[I - \prod_{j=1}^{n} (I - A_{-p_j(l_2)})\Big] \\ &= I - \prod_{j=1}^{n} (I - A_{-p_j(l_1)}) - \prod_{j=1}^{n} (I - A_{-p_j(l_2)}) + \prod_{j=1}^{n} (I - A_{-p_j(l_1)})(I - A_{-p_j(l_2)}) \\ &= A_{0,l_1} + A_{0,l_2} - I + \prod_{j=1}^{n} (I - A_{-p_j(l_1)})(I - A_{-p_j(l_2)}). \end{aligned}$$

On the other hand,

$$\begin{aligned} (I - A_{-p_j(l_1)})(I - A_{-p_j(l_2)}) &= I - A_{-p_j(l_1)} - A_{-p_j(l_2)} + A_{-p_j(l_1)} A_{-p_j(l_2)} \\ &= I - A_{-p_j(l_1)} - A_{-p_j(l_2)} + A_{-\max(p_j(l_1),p_j(l_2))} \\ &= I - A_{-p_j(\min(l_1,l_2))}. \end{aligned}$$

Consequently, by the foregoing results we get the equation

$$A_{0,l_1} A_{0,l_2} = A_{0,l_1} + A_{0,l_2} - I + \prod_{j=1}^{n} (I - A_{-p_j(\min(l_1,l_2))}),$$

which, by (5.1), completes the proof.

LEMMA 5.3. *For any $u, s, l \in R^n$ such that $l > 0$ we have*

$$(5.11)\quad A_u A_{s,l} = \begin{cases} A_{\min(u,s),\min(u,s)-s+l} & \textit{whenever } \min(u,s)-s+l > 0, \\ A_{\min(u,s)} & \textit{otherwise}. \end{cases}$$

Proof. First, we suppose that $s \geqslant u > s-l$. Then, by (5.2), we have

$$(5.12)\quad \begin{aligned} A_u A_{s,l} &= A_u\Big(\sum_{k=1}^{n}(-1)^k \sum_{\substack{r_1,r_2,\ldots,r_k=1\\ r_1<r_2<\cdots<r_k}}^{n} A_{s-p_{r_1r_2\ldots r_k}(l)}\Big) \\ &= \sum_{k=1}^{n}(-1)^{k+1} \sum_{\substack{r_1,r_2,\ldots,r_k=1\\ r_1<r_2<\ldots<r_k}}^{n} A_{\min(u,s-p_{r_1r_2\ldots r_k}(l))} \\ &= \sum_{k=1}^{n}(-1)^{k+1} \sum_{\substack{r_1,r_2,\ldots,r_k=1\\ r_1<r_2<\cdots<r_k}}^{n} A_{u-p_{r_1r_2\ldots r_k}(u-s-l)} \\ &= A_{u,u-s+l}. \end{aligned}$$

Next, we suppose that $s \geqslant u$ and the relation $u > s-l$ does not hold. In this case there exists an index j such that $u^j \leqslant s^j - l^j$. Consequently,

$$(5.13)\quad \begin{aligned} A_u A_{s,l} &= A_u\Big(I - \prod_{k=1}^{n}(I - A_{s-p_k(l)})\Big) \\ &= A_u - \prod_{\substack{k=1\\ k\neq j}}^{n}(I - A_{s-p_k(l)})(A_u - A_u A_{s-p_j(l)}) \\ &= A_u - \prod_{\substack{k=1\\ k\neq j}}^{n}(I - A_{s-p_k(l)})(A_u - A_u) = A_u. \end{aligned}$$

In general, let $s, u, l \in R^n$ be arbitrary with $l > 0$. Since $A_s A_{s,l} = A_{s,l}$ we have the equation

$$A_u A_{s,l} = A_u A_s A_{s,l} = A_{\min(u,s)} A_{s,l},$$

which together with equations (5.12), (5.13) completes the proof.

LEMMA 5.4. *Let $s_1, s_2, l_1, l_2 \in R^n$ such that $l_1 > 0$, $l_2 > 0$ and* $\min(s_1, s_2) > \max(s_1 - l_1, s_2 - l_2)$. *Then*

$$(5.14)\quad \begin{aligned} A_{s_1,l_1} A_{s_2,l_2} = {} & A_{\min(s_1,s_2),\min(s_1,s_2)-s_1+l_1} + \\ & + A_{\min(s_1,s_2),\min(s_1,s_2)-s_2+l_2} - \\ & - A_{\min(s_1,s_2),\min(s_1,s_2)-\max(s_1-l_1,s_2-l_2)}. \end{aligned}$$

Proof. Since $A_{s_1}A_{s_1,l_1} = A_{s_1,l_1}$ and $A_{s_2}A_{s_2,l_2} = A_{s_2,l_2}$, we have, by the condition of consistency (4.ii), the equation

$$\begin{aligned} A_{s_1,l_1}A_{s_2,l_2} &= A_{s_1}A_{s_1}A_{s_1,l_1}A_{s_2}A_{s_2}A_{s_2,l_2} \\ &= A_{\min(s_1,s_2)}A_{s_1,l_1}A_{\min(s_1,s_2)}A_{s_2,l_2}; \end{aligned}$$

hence and by Lemma 5.3 the following equations hold:

$$\begin{aligned} A_{s_1,l_1}A_{s_2,l_2} &= A_{\min(s_1,s_2),\min(s_1,s_2)-s_1+l_1}A_{\min(s_1,s_2),\min(s_1,s_2)-s_2+l_2} \\ &= T_{\min(s_1,s_2)}A_{0,\min(s_1,s_2)-s_1+l_1}A_{0,\min(s_1,s_2)-s_2+l_2}T_{-\min(s_1,s_2)}. \end{aligned}$$

Consequently, by Lemma 5.2, it follows that

$$\begin{aligned} A_{s_1,l_1}A_{s_2,l_2} &= T_{\min(s_1,s_2)}[A_{0,\min(s_1,s_2)-s_1+l_1}+A_{0,\min(s_1,s_2)-s_2+l_2}- \\ &\qquad - A_{0,\min\{\min(s_1+s_2)-s_1+l_1,\,\min(s_1,s_2-s_2+l_2)\}}]T_{-\min(s_1,s_2)} \\ &= T_{\min(s_1,s_2)}[A_{0,\min(s_1,s_2)-s_1-l_1}+A_{0,\min(s_1,s_2)-s_2+l_2}- \\ &\qquad - A_{0,\min(s_1,s_2)-\max(s_1-l_1,s_2-l_2)}]T_{-\min(s_1,s_2)}, \end{aligned}$$

which, by (5.4), implies (5.14). Thus, the lemma is proved.

LEMMA 5.5. *For any* $s_1, s_2, l_1, l_2 \in R^n$ *such that* $l_1, l_2 > 0$

(a) *if* $\min(s_1, s_2) > s_1 - l_1$ *and the relation* $\min(s_1, s_2) > s_2 - l_2$ *does not hold, then*

$$A_{s_1,l_1}A_{s_2,l_2} = A_{\min(s_1,s_2),\min(s_1,s_2)-s_1+l_1}; \tag{5.15}$$

(b) *if neither of the relations* $\min(s_1, s_2) > s_1 - l_1$ *and* $\min(s_1, s_2) > s_2 - l_2$ *holds, then*

$$A_{s_1,l_1}A_{s_2,l_2} = A_{\min(s_1,s_2)}. \tag{5.16}$$

Proof. Suppose that condition (a) holds. By virtue of Lemma 5.3 it follows that

$$\begin{aligned} A_{s_1,l_1}A_{s_2,l_2} &= A_{\min(s_1,s_2)}A_{s_1,l_1}A_{\min(s_1,s_2)}A_{s_2,l_2} \\ &= A_{\min(s_1,s_2),\min(s_1,s_2)-s_1+l_1}A_{\min(s_1,s_2)} \\ &= A_{\min(s_1,s_2),\min(s_1,s_2)-s_1+l_1}, \end{aligned}$$

which proves (5.15).

Now suppose that condition (b) is satisfied, we have, by Lemma 5.3, the equation

$$\begin{aligned} A_{s_1,l_1}A_{s_2,l_2} &= A_{\min(s_1,s_2)}A_{s_1,l_1}A_{\min(s_1,s_2)}A_{s_2,l_2} \\ &= A_{\min(s_1,s_2)}A_{\min(s_1,s_2)} = A_{\min(s_1,s_2)}, \end{aligned}$$

which proves (5.16). The lemma is thus proved.

LEMMA 5.6. *For any bounded semiclosed intervals* I_{s_1,l_1} *and* I_{s_2,l_2} *in* R^n *the following equation holds*:

$$A(I_{s_1,l_1})A(I_{s_2,l_2}) = A(I_{s_1,l_1}\cap I_{s_2,l_2}). \tag{5.17}$$

Proof. Suppose $I_{s_1,l_1}\cap I_{s_2,l_2}\neq\emptyset$; we have, by (5.5), (5.8), (5.9) and Lemmas 5.3 and 5.4, the equations

$$\begin{aligned}
A(I_{s_1,l_1})A(I_{s_2,l_2}) &= (A_{s_1}-A_{s_1,l_1})(A_{s_2}-A_{s_2,l_2})\\
&= A_{s_1}A_{s_2}-A_{s_1}A_{s_2,l_2}-A_{s_2}A_{s_1,l_1}+A_{s_1,l_1}A_{s_2,l_2}\\
&= A_{\min(s_1,s_2)}-A_{\min(s_1,s_2),\min(s_1,s_2)-s_2+l_2}-\\
&\quad -A_{\min(s_1,s_2),\min(s_1,s_2)-s_1+l_1}+A_{\min(s_1,s_2),\min(s_1,s_2)-s_1+l_1}+\\
&\quad +A_{\min(s_1,s_2),\min(s_1,s_2)-s_2+l_2}-A_{\min(s_1,s_2),\min(s_1,s_2)-\max(s_1-l_1,s_2-l_2)}\\
&= A_{\min(s_1,s_2)}-A_{\min(s_1,s_2),\min(s_1,s_2)-\max(s_1-l_1,s_2-l_2)}\\
&= A(I_{\min(s_1,s_2),\min(s_1,s_2)-\max(s_1-l_1,s_2-l_2)})\\
&= A(I_{s_1,l_1}\cap I_{s_2,l_2}),
\end{aligned}$$

which proves (5.17) for the case $I_{s_1,l_1}\cap I_{s_2,l_2}\neq\emptyset$.

Suppose now that $I_{s_1,l_1}\cap I_{s_2,l_2}=\emptyset$. Then, by (5.8), it follows that one of the following cases must occur:

(a) $\min(s_1,s_2)>s_1-l_1$ and the relation $\min(s_1,s_2)>s_2-l_2$ does not hold or

(b) $\min(s_1,s_2)>s_2-l_2$ and the relation $\min(s_1,s_2)>s_1-l_1$ does not hold or

(c) neither of the relations $\min(s_1,s_2)>s_1-l_1$ and $\min(s_1,s_2)>s_2-l_2$ holds.

In case (a) we have, by virtue of Lemma 5.3 and (5.15), the equation

$$\begin{aligned}
A(I_{s_1,l_1})A(I_{s_2,l_2}) &= (A_{s_1}-A_{s_1,l_1})(A_{s_2}-A_{s_2,l_2})\\
&= A_{\min(s_1,s_2)}-A_{s_1}A_{s_2,l_2}-A_{s_2}A_{s_1,l_1}+A_{s_1,l_1}A_{s_2,l_2}\\
&= A_{\min(s_1,s_2)}-A_{\min(s_1,s_2)}-A_{\min(s_1,s_2),\min(s_1,s_2)-s_1+l_1}+\\
&\qquad +A_{\min(s_1,s_2),\min(s_1,s_2)-s_1+l_1}=0.
\end{aligned}$$

Further, it is exactly the same to prove that if (b) holds then

$$A(I_{s_1,l_1})A(I_{s_2,l_2})=0.$$

Finally, let us suppose that (c) holds. Using again Lemma 5.3 and equation (5.16), we have

$$\begin{aligned}
A(I_{s_1,l_1})A(I_{s_2,l_2}) &= A_{\min(s_1,s_2)}-A_{s_1}A_{s_2,l_2}-A_{s_2}A_{s_1,l_1}+A_{s_1,l_1}A_{s_2,l_2}\\
&= A_{\min(s_1,s_2)}-A_{\min(s_1,s_2)}-A_{\min(s_1,s_2)}+A_{\min(s_1,s_2)}=0,
\end{aligned}$$

which completes the proof of the lemma.

LEMMA 5.7. *Let* $s_1, s_2, s_3, l_1, l_2, l_3 \in R^n$ *such that* $l_1, l_2, l_3 > 0$. *If there exists an index* i, $1 \leqslant i \leqslant n$, *such that*

$$(5.18)\qquad \begin{cases} s_2 = s_1 - p_i(l_1), \\ l_2^j = l_1^j \quad \textit{whenever } j \neq i,\ j = 1, 2, \ldots, n, \\ l_3 = l_1 + p_i(l_2), \\ s_1 = s_3, \end{cases}$$

then the following equation holds:

$$(5.19)\qquad A_{s_2} = A_{s_1,l_1} + A_{s_2,l_2} - A_{s_3,l_3}.$$

Proof. Denote the right-hand side of (5.19) by Δ; we have, by (5.1) and (5.18), the equations

$$\begin{aligned} \Delta - A_{s_2,l_2} &= \prod_{j=1}^{n} (I - A_{s_3 - p_j(l_2)}) - \prod_{j=1}^{n} (I - A_{s_1 - p_j(l_1)}) \\ &= \prod_{\substack{j=1 \\ j \neq i}}^{n} (I - A_{s_1 - p_j(l_1)}) [A_{s_1 - p_i(l_1)} - A_{s_1 - p_i(l_2)}] \\ &= [A_{s_2} - A_{s_2 - p_i(l_2)}] \prod_{j=1}^{n} (I - A_{s_1 - p_j(l_1)}). \end{aligned}$$

Consequently, for every $x \in [x_t]$, $\Delta x - A_{s_2,l_2} x \in [x_t, t \leqslant s_2]$. Hence the operator Δ maps $[x_t]$ into $[x_t, t \leqslant s_2]$. On the other hand, by virtue of Lemma 5.3 we have

$$\Delta = A_{s_2}\Delta = A_{s_2}A_{s_1,l_1} = A_{s_2}A_{s_2,l_2} - A_{s_2}A_{s_3,l_3} = A_{s_2} + A_{s_2,l_2} - A_{s_2,l_2} = A_{s_2},$$

which completes the proof.

LEMMA 5.8. *For any intervals* I_{s_1,l_1}, I_{s_2,l_2} *and* I_{s_3,l_3} *in* R^n *such that* $I_{s_1,l_1} \cap I_{s_2,l_2} = \emptyset$ *and* $I_{s_3,l_3} = I_{s_1,l_1} \cup I_{s_2,l_2}$ *the following equation holds:*

$$(5.20)\qquad A(I_{s_3,l_3}) = A(I_{s_1,l_1}) + A(I_{s_2,l_2}).$$

Proof. From the assumptions of the lemma it follows that either $s_1 \leqslant s_2$ or $s_2 \leqslant s_1$. Let us suppose $s_2 \leqslant s_1$. Then, there exists an index i, $1 \leqslant i \leqslant n$, such that relations (5.18) are valid. Hence and by (5.19)

$$A_{s_1} - A_{s_3,l_3} = A_{s_1} - A_{s_1,l_1} + A_{s_2} - A_{s_2,l_2},$$

or, equivalently,

$$A(I_{s_3,l_3}) = A(I_{s_1,l_1}) + A(I_{s_2,l_2}),$$

which completes the proof.

Let us denote the unbounded interval $\{t \in R^n\colon t \leqslant s\}$ by $I_{s,\infty}$ and put $A(I_{s,\infty}) = A_s$. It is clear that

$$I_{s_1,\infty} \cap I_{s_2,\infty} = I_{\min(s_1,s_2),\infty} \tag{5.21}$$

and

(5.22)

$$I_{s_1,\infty} \cap I_{s_2,l_2} = \begin{cases} I_{\min(s_1,s_2),\min(s_1,s_2)-s_2+l_2} & \text{whenever } \min(s_1,s_2)-s_2+l_2 > 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

LEMMA 5.9. $A(I_{s_1,\infty} \cap I_{s_2,l_2}) = A(I_{s_1,\infty})A(I_{s_2,l_2})$.

Proof. The lemma follows immediately from (5.22) and Lemma 5.3.

LEMMA 5.10. *For any disjoint intervals in R^n $I_{s_1,l_1}, I_{s_2,l_2}, \ldots, I_{s_m,l_m}, I_{s_{m+1},\infty}$ and for any $y_1, y_2, \ldots, y_{m+1} \in [x_i]$ the random variables $A(I_{s_1,l_1})y_1$, $A(I_{s_2,l_2})y_2, \ldots, A(I_{s_m,l_m})y_m$, $A(I_{s_{m+1},\infty})y_{m+1}$ are independent.*

Proof. Let $a_1, a_2, \ldots, a_m, a_{m+1}$ be a system of real numbers. Then, by Lemma 5.9, we infer that

$$A(I_{s_{m+1},\infty})\Big(\sum_{j=1}^{m} A(I_{s_j,l_j})y_j\Big) = A_{s_{m+1}}\Big(\sum_{j=1}^{m} A(I_{s_j,l_j})y_j\Big) = 0.$$

Consequently the random variables $a_{m+1}A(I_{s_{m+1},\infty})y_{m+1}$ and $\sum_{j=1}^{m} a_j A(I_{s_j,l_j})y_j$ are independent. In the language of the characteristic function we can write

$$\begin{aligned} E\exp i\Big(\sum_{j=1}^{m} a_j A(I_{s_j,l_j})y_j + a_{m+1}A(I_{s_{m+1},\infty})y_{m+1}\Big) \\ = E\exp i a_{m+1}A(I_{s_{m+1},\infty})y_{m+1} \cdot E\exp i \sum_{j=1}^{m} a_j A(I_{s_j,l_j})y_j. \end{aligned} \tag{5.23}$$

Further, there is an index r, $1 \leqslant r \leqslant m$ such that for every $j = 1, 2, \ldots$ $\ldots, m$ and $j \neq r$

$$I_{s_j,l_j} \subset R^n | I_{s_r,\infty}, \quad \text{i.e.,} \quad I_{s_j,l_j} \cap I_{s_r,\infty} = \emptyset.$$

Of course we may assume that $r = m$. Then, since

$$A_{s_m}\big(a_m A(I_{s_m,l_m})y_m\big) = a_m A(I_{s_m,l_m})y_m \quad \text{and} \quad A_{s_m}\Big(\sum_{j=1}^{m-1} a_j A(I_{s_j,l_j})y_j\Big) = 0,$$

the random variables $a_m A(I_{s_m,l_m})$ and $\sum_{j=1}^{m-1} a_j A(I_{s_j,l_j})y_j$ are independent. Consequently,

$$E\exp i\Big(\sum_{j=1}^{m} a_j A(I_{s_j,l_j})y_j\Big) = E\exp i a_m A(I_{s_m,l_m})y_m \cdot E\exp i\Big(\sum_{j=1}^{m-1} a_j A(I_{s_j,l_j})y_j\Big).$$

By induction we have

$$E\exp i\Big(\sum_{j=1}^{m} a_j A(I_{s_j,t_j})y_j\Big) = \prod_{j=1}^{m} E\exp i a_j A(I_{s_j,t_j})y_j,$$

which together with equation (5.23) completes the proof.

LEMMA 5.11. *There exists a bounded linear operator A on $[x_t]$ commuting with shift operators T_t, $t \in \boldsymbol{R}^n$, and such that for every $x \in [x_t]$*

$$\lim_{l\to\infty} A_{0,l}x = Ax.$$

Proof. Given an element $x \in [x_t]$ and a sequence $0 = t_0 > t_1 > t_2 > \ldots$ of elements of $\boldsymbol{R}^n$ tending to $-\infty$, we put, for every $s = (s^1, s^2, \ldots, s^n)$ with non-negative integer components $s^1, s^2, \ldots, s^n$,

$$I_s = \{t = (t^1, t^2, \ldots, t^n) \in \boldsymbol{R}^n\colon\ t^j_{s^j-1} < t^j \leqslant t^j_{s^j},\ \ j = 1, 2, \ldots, n\}.$$

It is clear that the intervals I_s are disjoint. Hence and by Lemma 5.10 the random variables $z_s = A(I_s)x$, $s \in Z^n$ and $s \geqslant 0$, are independent. Let k be an arbitrary positive integer. Then, by Lemma 5.10, the random variables z_s, $0 \leqslant s \leqslant (\underbrace{k, k, \ldots, k}_{n\text{-times}})$ and $A_{0,-t_k}x$ are independent. Moreover, from Lemma 5.8 it follows that

$$\sum_{0\leqslant s\leqslant(k,k,\ldots,k)} z_s = \sum_{0\leqslant s\leqslant(k,k,\ldots,k)} A(I_s)x = A(I_{0,-t_k})x,$$

which, by virtue of (5.5), implies the equation

$$A_0 x = \sum_{0\leqslant s\leqslant(k,k,\ldots,k)} z_s + A_{0,-t_k}x. \tag{5.24}$$

Hence and by the definition of a Banach space of random variables we have the relation

$$\sup_k \Big\| \sum_{\substack{0\leqslant s\leqslant(k,k,\ldots,k)\\ n\text{-times}}} z_s \Big\| < \infty.$$

Consequently, by the definition of a Banach space of random variables, the series $\sum_{\substack{s\in Z^n\\ s\geqslant 0}} z_s$ converges in the norm $\|\cdot\|$. By virtue of equation (5.24) the limit

$$Ax = \lim_{k\to\infty} A_{0,t_k}x \tag{5.25}$$

exists for every $x \in [x_t]$.

Now from the fact that $\|A_{0,-t_k}\| = 1$, $k = 1, 2, \ldots$, and by the Banach Theorem, the operator A defined by (5.25) is a linear bounded one.

Proceeding successively, we shall show that A commutes with the shift operations T_u, $u \in \boldsymbol{R}^n$.

First, let us suppose that $u \leqslant 0$. By virtue of (5.24) and (5.7) we have the equation

$$A_u T_u x = \sum_{\substack{0 \leqslant s \leqslant (k,k,\ldots,k) \\ n\text{-times}}} A(I_s+u)T_u x + T_u A_{0,-t_k} x. \tag{5.26}$$

On the other hand, replacing x in (5.24) by $T_u x$ we get the equation

$$A_0(T_u x) = \sum_{0 \leqslant s \leqslant (k,k,\ldots,k)} A(I_s)T_u x + A_{0,-t_k} T_u x. \tag{5.27}$$

From (5.26) and (5.27) it follows that

$$\begin{aligned} A_{0,-t_k}(T_u x) - T_u A_{0,-t_k} x \\ = A_0(T_u x) - A_u(T_u x) + \sum_{0 \leqslant s \leqslant (k,k,\ldots,k)} [A(I_s+u)T_u x - A(I_s)T_u x]. \end{aligned} \tag{5.28}$$

Denoting the right-hand side of (5.28) by g_k and the left-hand side by h_k, we infer, by virtue of the assumption $u \leqslant 0$ that $g_k \in \sum_{j=1}^{n} [x_t, t \leqslant p_j(t_k)]$. Hence for every $r = 1, 2, \ldots$ such that $t_r \leqslant t_k + u$ the random variables g_r and h_k are independent. Let $k = 1, 2, \ldots$ be fixed and $r \to \infty$. Then, the random variables $AT_u x - T_u Ax$ and h_k are independent. Letting $k \to \infty$ and denoting $\lim_{k\to\infty} h_k = h$, we infer that the random variables $AT_u x - T_u Ax$ and h are independent. However, by (5.28), we have $h = AT_u x - T_u Ax$. Consequently $AT_u x - T_u Ax$ is a constant random variable. Since 0 is the only constant random variable belonging to $[x_t]$, we conclude that $AT_u x - T_u Ax = 0$, which implies the equation

$$AT_u = T_u A \quad (u \leqslant 0). \tag{5.29}$$

Now for any $s \in R^n$ and a sufficiently large negative $u \in R^n$ the inequality $s+u \leqslant 0$ holds. Hence and by (5.29) $AT_{s+u} = T_{s+u}A$, or equivalently $AT_s T_u = T_s T_u A$, which, by the fact that T_u is an 1-1 transformation, implies the equation $AT_s = T_s A$ for every $s \in R^n$. The lemma is thus proved.

LEMMA 5.12. *A stationary-in-norm random field* $\{x_t\}$, $t \in R^n$, *is completely non-deterministic if and only if for every* $x \in [x_t]$

$$\lim_{l\to\infty} A_{0,l} x = Ax = 0. \tag{5.30}$$

Proof. It is clear by (5.2) that if $\{x_t\}$ is completely non-deterministic then (5.30) is valid. Conversely, suppose that (5.30) is true. In view of (5.3) we infer that for every $j = 1, 2, \ldots, n$ and $x \in [x_t]$ the random variables $A_{p_j(t)} x$ and $A_{0,-t} x - A_{p_j(-t)} x$ are independent. Hence and by Lemma 1.2 we have $\lim_{t\to-\infty} A_{p_j(t)} x = 0$, which completes the proof.

We say that two random fields $\{x_t'\}$ and $\{x_t''\}$ are *independent* if the random variables y' and y'' are independent whenever $y' \in [x_t']$ and $y'' \in [x_t'']$.

We now formulate the main result of this section.

THEOREM 5.1. *Each stationary-in-norm random field admitting a prediction is the sum of two independent stationary-in-norm random fields admitting a prediction, one deterministic and the other completely non-deterministic.*

Proof. Let $\{x_t\}$, $t \in \boldsymbol{R}^n$, be a stationary-in-norm random field admitting a prediction. The limit operator A defined in Lemma 5.11 satisfies, by virtue of (5.11), the equation

$$A_u A = A A_u = A \quad (u \in \boldsymbol{R}^n)$$

and hence

$$A^2 = A, \quad (I-A)^2 = I-A, \tag{5.31}$$

where I denotes the unit operator.

Putting $x_t' = Ax_t$ and $x_t'' = (I-A)x_t$ for every $t \in \boldsymbol{R}^n$, we have the formula

$$x_t = x_t' + x_t''. \tag{5.32}$$

Moreover, it can easily be seen that the random fields $\{x_t'\}$ and $\{x_t''\}$ are independent. Now, using the fact that the operator A commutes with the shift operators T_s, $s \in \boldsymbol{R}^n$, induced by the stationary-in-norm field $\{x_t\}$, we infer that the random fields $\{x_t'\}$ and $\{x_t''\}$ are both stationary-in-norm.

This implies that the operator A_a ($a \in \boldsymbol{R}^n$) restricted to $[x_t']$ and $[x_t'']$ is a predictor of $\{x_t'\}$ and $\{x_t''\}$, respectively, based on the full past up to the time $t = a$. Since the condition of consistency is obviously satisfied, we conclude that $\{x_t'\}$ and $\{x_t''\}$ admit a prediction. Finally, for every $a \in \boldsymbol{R}^n$, $A_a = I$ on $[x_t']$, and hence $\{x_t'\}$ is deterministic and because of the fact that $Ay'' = A(I-A)y'' = 0$ $\{x_t''\}$ is completely non-deterministic. Thus the theorem is fully proved.

§ 6. Stochastic measures

Let B be a Banach space of random variables and let $\mathscr{R}$ denote the ring of all bounded Borel subsets of $\boldsymbol{R}^n$.

A set function $M\colon \mathscr{R} \to B$ is said to be a *stochastic measure* if

(6.i) For any $A_1, A_2, \ldots, A_m \in \mathscr{R}$ such that $A_i \cap A_j = \emptyset$ for $i \neq j$, the random variables $M(A_1), M(A_2), \ldots, M(A_m)$ are independent,

(6.ii) For any sequence $\{A_m\}$, $m = 1, 2, \ldots$, of disjoint sets in $\mathscr{R}$ such that $\bigcup_{m=1}^{\infty} A_m \in \mathscr{R}$ we have

$$M\Big(\bigcup_{m=1}^{\infty} A_m\Big) = \sum_{m=1}^{\infty} M(A_m),$$

where the series on the right-hand side converges in the norm $\|\cdot\|$.

From the definitions of a stochastic measure and a Banach space of random variables it follows that for any $A, B \in \mathscr{R}$ such that $A \subset B$ we have

$$\|M(A)\| \leqslant \|M(B)\|, \tag{6.1}$$

which, according to ([1], Lemma 4, p. 320 and Lemma 5, p. 321), implies the existence of a non-negative Borel measure μ on $\mathscr{R}$ such that

$$\mu(E) \leqslant \|M(E)\| \quad (E \in \mathscr{R}) \tag{6.2}$$

and

$$\|M(E)\| \to 0 \tag{6.3}$$

whenever the sets E are bounded in common and $\mu(E) \to 0$.

In the sequel every countable union of subsets of sets $E \in \mathscr{R}$ with $M(E) = 0$ will be called an *M-null set*. Of course, this is the same as a μ-null set. Consequently, saying "M-almost everywhere" is the same as saying "μ-almost everywhere".

Modifying the well-known definition of integration with respect to a vector-valued measure ([3], p. 239), K. Urbanik [13] has defined a stochastic integral of functions on $\boldsymbol{R}^1$, which, by changing the symbols, can be generalized to the integral of real functions on the space $\boldsymbol{R}^n$ with respect to a B-valued stochastic measure defined on the ring $\mathscr{R}$. Namely, we put

$$\int_E f(u)M(du) = \sum_{j=1}^{m} \lambda_j M(E \cap E_j) \tag{6.4}$$

whenever f is a simple function of the form $f = \sum_{j=1}^{m} \lambda_j X_{E_j}$ where $E, E_j \in \mathscr{R}$ and E_j's are disjoint.

A real function f is said to be *integrable over $\boldsymbol{R}^n$ with respect to M* if there exists a sequence $\{f_m\}$ of simple functions convergent to f M-almost everywhere and such that for every increasing sequence $E_1 \subseteq E_2 \subseteq$ of sets in $\mathscr{R}$ the sequence $\{\int_{E_m} f_m(u)M(du)\}$ converges in the norm. Putting $E = \bigcup_{m=1}^{\infty} E_m$ we define

$$\int_E f(u)M(du) = \lim_{m\to\infty} \int_{E_m} f_m(u)M(du). \tag{6.5}$$

Let us denote by $L(M)$ the space of all real functions integrable over R^n with respect to the stochastic measure M, where the functions in $L(M)$ which are equal M-almost everywhere will be identified. From the definition of a Banach space of random variables and (6.5) it follows that for any disjoint sets $E_1, E_2, \ldots, E_m \in \mathscr{R}$ and for any $f_1, f_2, \ldots, f_m \in L(M)$ the random variables

$$\int_{E_1} f_1(u)M(du), \quad \int_{E_2} f_2(u)M(du), \quad \ldots, \quad \int_{E_m} f_m(u)M(du)$$

are independent.

Let $[M]$ denote the subspace of B spanned by all random variables $M(E)$, $E \in \mathscr{R}$. Using the same method of K. Urbanik ([13], Theorem 2.1), we can derive the following theorem.

THEOREM 6.1. *For every stochastic measure M the equation*

$$[M] = \Big\{ \int_{R^n} f(u)M(du) \colon f \in L(M) \Big\}$$

holds. Moreover, each element of $[M]$ is uniquely representable as an integral

$$\int_{R^n} f(u)\, M(du).$$

Now, let $\{x_t\}_{t \in R^n} \subset B$ be a stationary-in-norm random field admitting a prediction. Then the operator-valued interval function $A(I_{s,l})$ defined by (5.5) in conjunction with a random variable $x \in [x_t]$ defines a $[x_t]$-valued stochastic interval function N on the set $\mathscr{R}_0$ of all semiclosed bounded intervals in R^n. Namely,

$$N(I_{s,l}) = A(I_{s,l})x \quad (I_{s,l} \in \mathscr{R}_0). \tag{6.6}$$

Of course, for every independent disjoint intervals I_{s_j,l_j}, $j = 1, 2, \ldots, m$, the random variables $N(I_{s_j,l_j})$, $j = 1, 2, \ldots, m$, are independent. On the other hand, by (5.20), $N(I_{s_1,l_1} \cup I_{s_2,l_2}) = N(I_{s_1,l_1}) + N(I_{s_2,l_2})$ whenever I_{s_j,l_j} are disjoint with $I_{s_1,l_1} \cup I_{s_2,l_2} \in \mathscr{R}_0$. Moreover

$$\lim_{\substack{c \to b \\ c > b}} N\big((a, c]\big) = N\big((a, b]\big) \tag{6.7}$$

for all intervals $(a, b] \in \mathscr{R}_0$.

LEMMA 6.1. *Let $I_{s_1,l_1}, I_{s_2,l_2}, \ldots$ be a sequence of disjoint bounded semiclosed intervals contained in a bounded semiclosed interval I_{s_0,l_0}. Then the series $\sum_{k=1}^{\infty} N(I_{s_k,l_k})$ converges in the norm $\|\cdot\|$.*

Proof. It suffices to note the relation

$$\sup_m \Big\| \sum_{k=1}^{m} N(I_{s_k,l_k}) \Big\| \leqslant \|N(I_{s_0,l_0})\| < \infty. \tag{6.8}$$

THEOREM 6.2. *The $[x_t]$-valued stochastic interval function $N(\cdot)$ defined by* (6.6) *can be extended to a unique $[x_t]$-valued stochastic measure M on $\mathscr{R}$ such that $M(I) = N(I)$ whenever $I \in \mathscr{R}_0$.*

Proof. Let us denote the ring of all finite unions of intervals from $\mathbf{R}_0$ by $\mathscr{R}_*$ and set $N\big(\bigcup_{j=1}^{m} I_j\big) = \sum_{j=1}^{n} N(I_j)$ for disjoint intervals $I_1, I_2, \ldots$ $\ldots, I_m \in \mathscr{R}_0$. We remark that the stochastic set function $N(\cdot)$ defined in such a way satisfies

$$\|N(E_1)\| \leqslant \|N(E_2)\| \quad \text{for} \quad E_1 \subset E_2 \tag{6.9}$$

and is countably additive on $\mathscr{R}_*$. In fact, inequality (6.9) is obvious. On the other hand, for any linear functional F in $[x_t]$ a scalar-valued set function $N_F(\cdot)$ on $\mathscr{R}_*$ defined by $N_F(E) = F\big(N(E)\big)$ is finitely additive on $\mathscr{R}_*$ and, by (6.9) is bounded on every interval from $\mathscr{R}_0$. Moreover, by (6.7), $\lim_{\substack{c\to b\\ c>b}} N_F\big((a, c]\big) = N_F\big((a, b]\big)$ and from ([1], p. 97) it follows that $N_F(\cdot)$ is countable additive on $\mathscr{R}_*$. Now, for any sequence $E_1, E_2, \ldots$ of disjoint sets from $\mathscr{R}_*$ with $\bigcup_{k=1}^{\infty} E_k \in \mathscr{R}_*$ the series $\sum_{k=1}^{\infty} N(E_k)$, by Lemma 6.1, converges in norm and for any linear functional F in $[x_t]$ we have

$$F\Big(\sum_{k=1}^{\infty} N(E_k)\Big) = \sum_{k=1}^{\infty} N_F(E_k) = N_F\Big(\bigcup_{k=1}^{\infty} E_k\Big) = F\Big(N\big(\bigcup_{k=1}^{\infty} E_k\big)\Big)$$

Consequently,

$$\sum_{k=1}^{\infty} N(E_k) = N\Big(\bigcup_{k=1}^{\infty} E_k\Big),$$

which shows that $N(\cdot)$ is countably additive on $\mathscr{R}_*$. Hence and by the Kas extension theorem (see [6]) there is a unique vector-valued set function M on $\mathscr{R}$ such that $M(E) = N(E)$ whenever $E \in \mathscr{R}_*$. Finally, using the method of Prékopa [10], we can show that M is a stochastic set function on $\mathscr{R}$. The theorem is thus proved.

§ 7. Completely non-deterministic random fields

In this section we are dealing with a non-zero completely non-deterministic random field $\{x_t\}$, which in the sequel will be called non-trivial. For such a random field we have, by the equation $x_0 = A(I_{0,l})x_0 + A_{0,l}x_0$ and Lemma 5.12, the relation

$$x_0 \in \big[A(I)x\colon\ I \in \mathscr{R}_0,\ x \in [x_t]\big].$$

Consequently,

$$[x_t] = \big[A(I)x\colon\ I \in \mathscr{R}_0,\ x \in [x_t]\big]. \tag{7.1}$$

LEMMA 7.1. *Let F be a bounded linear functional in $[x_t]$, g a scalar continuous function defined on an interval $(-a, a] \subset R^n$ with $g(0) \neq 0$, and y an element of $[x_t]$. If for all $I, J \in \mathcal{R}_0$ contained in the interval $(-a, a]$ we have the equation*

$$F\Big(A(I)\int_{-a}^{a} g(t)T_t A(J)y\,dt\Big) = 0, \tag{7.2}$$

then

$$F\big(A((0, a])y\big) = 0,$$

where the symbol $\int_{-a}^{a}$ means $\int_{-a^1}^{a^1}\ \int_{-a^n}^{a^n}$ and the integral in (7.2) is taken in the Bochner sense.

Proof. Given $m = 1, 2, 3, \ldots$ we divide every 1-dimensional interval $(0, a^j]$ on the jth axes, $j = 1, 2, \ldots, n$, into m disjoint intervals

$$\Big(0, \frac{a^j}{m}\Big], \Big(\frac{a^j}{m}, \frac{2a^j}{m}\Big], \ldots, \Big(\frac{(m-1)a^j}{m}, \frac{ma^j}{m}\Big].$$

It is clear that such a decomposition induces a decomposition of the interval $(0, a] \subset R^n$ into m^n disjoint intervals of the form

$$I_{\left(\frac{r_1 a^1}{m}, \frac{r_2 a^2}{m}, \ldots, \frac{r_n a^n}{m}\right), \frac{a}{m}} \quad \text{with} \quad r_1, r_2, \ldots, r_n \in \{1, 2, \ldots, n\},$$

which, for simplicity, we denote by $I_1, I_2, \ldots, I_{m^n}$. Moreover, we may assume that $a/m + h < a$, where $h \in R^n$ is supposed to be given and $h > 0$. Put

$$J_j = I_{b_j+h, 2h+\frac{a}{m}} \cap I_{a,a} \quad \text{whenever} \quad I_j = I_{b_j, \frac{a}{m}}.$$

Then, for every $t \in R^n$ such that $t \notin \Big(-\frac{a}{m} - h, \frac{a}{m} + h\Big]$, we have $I_j \cap (J_j + t) = \emptyset$ and consequently $A(I_j)T_t A(J_j) = 0$ whenever $t \notin \Big(-\frac{a}{m} - h, \frac{a}{m} + h\Big]$.

Therefore the following equation holds:

$$A(I_j) \int_{(-a,a]\setminus\left(-\frac{a}{m}-h, \frac{a}{m}+h\right]} g(t)T_t A(J_j)y\,dt = 0. \tag{7.3}$$

Hence

$$\begin{aligned} (7.4)\quad & A(I_j)\int_{-a}^{a} g(t)\,T_t A(J_j)y\,dt \\ &= A(I_j)\int_{-h}^{h} g(t)T_t A(J_j)y\,dt + A(I_j)\int_{\left(-\frac{a}{m}-h, \frac{a}{m}+h\right]\setminus(-h,h]} g(t)T_t A(J_j)y\,dt. \end{aligned}$$

Now, for every $t \in (-h, h]$, we have

$$I_j \cap \big((t, a+t] \setminus (a_j - h + t, b_j + h + t)\big) = \emptyset,$$

which yields the equation

$$A(I_j) T_t A\big((0, a] \setminus J_j\big) = 0 \quad \text{whenever} \quad t \in (-h, h]$$

and hence

$$(7.5) \qquad A(I_j) \int_{(-h,h]} g(t) T_t A\big((0, a]\big) y\, dt = A(I_j) \int_{-h}^{h} g(t) T_t A(J_j) y\, dt.$$

Consequently, by (7.4), we have the formula

$$(7.6) \qquad A(I_{a,a}) \int_{(-h,h]} g(t) T_t A\big((0, a]\big) y\, dt$$

$$= \sum_{j=1}^{m^n} A(I_j) \int_{(-h,h]} g(t) T_t A(J_j) y\, dt$$

$$= \sum_{j=1}^{m^n} A(I_j) \int_{-a}^{a} g(t) T_t A(J_j) y\, dt -$$

$$- \sum_{j=1}^{m^n} A(I_j) \int_{\left(-\frac{a}{m}-h, \frac{a}{m}+h\right] \setminus (-h,h]} g(t) T_t A(J_j) y\, dt.$$

Furthermore, since the intervals $I_j \cap (J_j + t)$, $j = 1, 2, \ldots, m^n$, are disjoint, we have

$$\sum_j A(I_j) \int_{\left(-\frac{a}{m}-h, \frac{a}{m}+h\right] \setminus (-h,h]} g(t) T_t A(J_j) y\, dt$$

$$= \int_{\left(-\frac{a}{m}-h, \frac{a}{m}+h\right] \setminus (-h,h]} g(t) A\Big(\sum_{j=1}^{m^n} \big(I_j \cap (J_j + t)\big)\Big) T_t y\, dt.$$

Consequently, by virtue of (7.6),

$$F\Big(A(I_{a,a}) \int_{-h}^{h} g(t) T_t A(I_{a,a}) y\, dt\Big)$$

$$= -F\Big(\int_{\left(-\frac{a}{m}-h, \frac{a}{m}+h\right] \setminus (-h,h]} g(t) A\Big(\sum_{j=1}^{m^n} \big(I_j \cap (J_j + t)\big)\Big) T_t y\, dt\Big)$$

and, hence

$$\left\|F\left(A(I_{a,a})\int_{-h}^{h} g(t)T_t A(I_{a,a})y\,dt\right)\right\|$$
$$\leqslant \|F\|\cdot q\cdot \|A(I_{a,a})\|\,\|y\|\left|\left(-\frac{a}{m}-h, \frac{a}{m}+h\right]\setminus(-h, h]\right|,$$

where $q = \max_{t\in I_{a,2a}}|g(t)|$ and $|E|$ denotes the Lebesgue measure of the set $E \subset R^n$.

Since m can be arbitrarily chosen, we have the equation

$$F\left(A(I_{a,a})\int_{-h}^{h} g(t)T_t A(I_{a,a})y\,dt\right) = 0,$$

which, after dividing by $2^n h^1 h^2 \ldots h^n$ and passing to the limit as $h \to 0$, implies, in view of the assumption $g(0) \neq 0$, the equation $F\left(A(I_{a,a})y\right) = 0$. The lemma is thus proved.

LEMMA 7.2. *Suppose that the random field $\{x_t\}$ is non-trivial. The $[x_t]$-valued stochastic interval function M_0 defined on $\mathcal{R}_0$ by the formula*

$$M_0\left((a, b]\right) = A\left((a, b]\right)\int_a^{\infty} e^{-t}\,T_t x_0\,dt, \tag{7.7}$$

where $a, b \in R^n$, $a < b$ and the symbol e^{-t} denotes $e^{-(t^1+t^2+\ldots+t^n)}$ whenever $t = (t^1, t^2, \ldots, t^n)$, can be extended to an $[x_t]$-valued stochastic measure on $\mathcal{R}$. The class of M_0-null sets coincides with the class of Lebesgue null sets. Moreover, for any interval $I \in \mathcal{R}_0$, we have the equation

$$[M_0(J)\colon J \in \mathcal{R}_0, J \subset I] = A(I)[x_t] \tag{7.8}$$

and for $E \in \mathcal{R}$, $f \in L(M_0)$

$$A(I)\int_E f(u)M_0(du) = \int_{E\cap I} f(u)M_0(du). \tag{7.9}$$

Proof. It should be noted that for every $t \leqslant a$ we have

$$A\left((a, b]\right)x_t = 0.$$

Consequently, for all $c \leqslant a$,

$$M_0\left((a, b]\right) = A\left((a, b]\right)\int_c^{\infty} e^{-t}T_t x_0\,dt. \tag{7.10}$$

Hence, in particular, it follows that M_0 is really a stochastic interval function, since it is defined by (6.6), which by Theorem 6.2 can be exten-

ded to an $[x_t]$-valued stochastic measure on $\mathscr{R}$. Moreover, this extension is unique. Further, from (5.7) and (5.17) we get the equations

$$A(I)M_0(J) = M_0(I\cap J), \quad T_tM_0(I) = e^tM_0(I+t) \quad \text{for all } I, J \in \mathscr{R}_0.$$

Hence, taking into account the uniqueness of the extension of M_0 onto $\mathscr{R}$, we obtain the equations

$$A(I)M_0(E) = M_0(I\cap E), \quad T_tM_0(E) = e^tM_0(E+t) \tag{7.11}$$

($I \in \mathscr{R}_0$, $E \in \mathscr{R}$); as a consequence of the first equation we get formula (7.9) for simple functions. The general case can be obtained by an approximation of M_0-integrable functions by simple ones.

Now we shall prove the relation

$$A\big((a,b]\big)\int_a^\infty e^{-t}T_tx\,dt \in [M_0] \tag{7.12}$$

for all $x \in [x_t, t \leqslant 0]$ and $a < b$ with $a, b \in \boldsymbol{R}^n$.

This implies that the set of elements x satisfying (7.12) is a subspace of the space $[x_t]$. If $h \leqslant 0$ and $h \in \boldsymbol{R}^n$, then by (7.7)

$$\begin{aligned} A\big((a,b]\big)\int_a^\infty e^{-t}T_tT_hx_0\,dt &= e^hA\big((a,b]\big)\int_{a+h}^\infty e^{-t}T_tx_0\,dt \\ &= e^hM_0\big((a,b]\big) \end{aligned} \tag{7.13}$$

and, consequently, all elements T_hx_0 with $h \leqslant 0$ satisfy (7.12), which shows that (7.12) is true.

The inclusion $[M_0(J)\colon J \subset \mathscr{R}_0,\ J \subset I] \subset A(I)[x_t]$ is obvious. By the second equation of (7.11) it suffices to prove the converse inclusion for intervals I of the form $(0, a]$ ($a \in \boldsymbol{R}^n$ and $a > 0$). Suppose that there exists an element y in $A\big((0,a]\big)[x_t]$ which does not belong to $[M_0(J)\colon J \in \mathscr{R}_0,\ J \subset (0,a]]$. There is then a linear functional F on $[x_t]$ vanishing on $[M_0(J)\colon J \in \mathscr{R}_0,\ J \subset (0,a]]$ and such that $F(y) = 1$. Given two intervals $I, J \in \mathscr{R}_0$ contained in $(0, a]$, we put

$$z(I,J) = e^aA(I)\int_0^\infty e^{-t}T_tT_{-a}A(J)y\,dt. \tag{7.14}$$

Since $T_{-a}A(J)y \in [x_t, t \leqslant 0]$, we have, by (7.11) and (7.12), the relation

$$z(I,J) \in \big[M_0(U)\colon\ U \in \mathscr{R}_0,\ U \subset (0,a]\big]. \tag{7.15}$$

Thus

$$F\big(z(I,J)\big) = 0. \tag{7.16}$$

Further, from the equation $A\big((0,a]\big)T_tA(J) = 0$ for $t \geqslant a$ we get the formula

$$z(I,J) = A(I)\int_{-a}^a e^{-t}T_tA(J)y\,dt. \tag{7.17}$$

Hence and by (7.16) and Lemma 7.1 we get the equation

$$F\big(A\big((0,a]\big)y\big) = 0.$$

However, $A\big((0,a]\big)y = y$, and hence $F(y) = 0$, which contradicts the equation $F(y) = 1$ and thus (7.8) is proved. From (7.1) and (7.8) it follows that the stochastic measure M_0 is not identically equal to zero. Consequently, the non-negative measure μ_0 associated with M_0 does not vanish identically and, by (7.11), the class of M_0-null sets, like the class of μ_0-null sets, is invariant under translations. Thus, it coincides with the class of Lebesgue null sets, which completes the proof.

Let $\{T_t\}_{t\in R^n}$ be a strongly continuous group of linear isommetric operators in B. A B-valued stochastic measure M is said to be *$\{T_t\}$-homogeneous* if for each set $E \in \mathscr{R}$ we have the equation

$$T_t M(E) = M(E+t), \qquad t \in R^n.$$

Here $E+t$ denotes the set $\{u+t\colon u \in E\}$.

LEMMA 7.3. *For each completely non-deterministic random field $\{x_t\}$ there exists an $[x_t]$-valued $\{T_t\}$-homogeneous stochastic measure M such that for any interval $I \in \mathscr{R}_0$*

$$[M(J)\colon J \in \mathscr{R}_0,\ J \subset I] = A(I)[x_t]. \tag{7.18}$$

Proof. For a trivial random field $\{x_t\}$ the trivial measure M satisfies the assertion of the theorem. Suppose the random field $\{x_t\}$ is non-trivial. First we shall prove that there exists an element y_0 belonging to $A\big((0,e]\big)[x_t]$ such that

$$A\big((0,e]\big)\int_{-e}^{e} T_t y_0\,dt \neq 0 \tag{7.19}$$

where $e = (1,1,\ldots,1) \in R^n$. Contrary to this let us suppose that

$$A\big((0,e]\big)\int_{-e}^{e} T_t z\,dt = 0 \quad \text{for all } z \in A\big((0,e]\big)[x_t].$$

Given an element $y \neq 0$ in $A\big((0,e]\big)[x_t]$, there exists a linear functional F on $[x_t]$ with $F(y) = 1$. Since, for any interval $J \in \mathscr{R}_0$ contained in $(0,e]$

$$A\big((0,e]\big)\int_{-e}^{e} T_t A(J) y\,dt = 0,$$

we have, by (5.17),

$$A(I)\int_{-e}^{e} T_t A(J) y\,dt = 0$$

for all intervals $I \in \mathscr{R}_0$ contained in $(0, e]$. Hence and by Lemma 7.1

$$F\big(A\big((0, e]\big)y\big) = 0,$$

which, in view of the formula $A\big((0, e]\big)y = y$, contradicts the equation $F(y) = 1$. Thus (7.19) holds for an element y_0 in $A\big((0, e]\big)[x_t]$.

Putting

$$M\big((a, b]\big) = A\big((a, b]\big) \int_{a-e}^{b} T_t y_0 dt \tag{7.20}$$

and taking into account the formula $A\big((0, e]\big)y_0 = y_0$, we get the equation

$$M\big((a, b]\big) = A\big((a, b]\big) \int_{c}^{d} T_t y_0 dt \quad \text{for} \quad d \geqslant b \quad \text{and} \quad c \leqslant a-e. \tag{7.21}$$

Hence M is a stochastic interval function on $\mathscr{R}_0$. Moreover,

$$A(I)M(J) = M(I \cap J) \tag{7.22}$$

for all $I, J \in \mathscr{R}_0$. By Theorem 6.2, the function M can be extended to a stochastic measure on $\mathscr{R}$ and this extension is unique. Furthermore, $T_t M(I) = M(I+t)$ for all $I \in \mathscr{R}_0$. Taking into account the uniqueness of the extension, we have the formula $T_t M(E) = M(E+t)$ for all $E \in \mathscr{R}$. Thus, the stochastic measure M is $\{T_t\}$-homogeneous.

Now let us note, by (7.22), that

$$[M(J)\colon J \in \mathscr{R}_0, J \subset I] \subset A(I)[x_t]. \tag{7.23}$$

Let M_0 be the stochastic measure defined by formula (7.7). Then, there exists a function $g \in L(M_0)$ satisfying the equation

$$M(E) = \int_E g(u) M_0(du) \quad (E \in \mathscr{R}). \tag{7.24}$$

Since $M\big((0, 1]\big) \neq 0$ the class of M-null sets is the class of Lebesgue measure zero and, consequently, coincides with the class of M_0-null sets. Thus, the function g differs from zero almost everywhere with respect to each of the measures M and M_0. From (7.24) we get the formula

$$\int_E f(u) M(du) = \int_E f(u) g(u) M_0(du) \tag{7.25}$$

for all sets $E \in \mathscr{R}$ and all simple functions f. Let $\{f_n\}$ be a sequence of simple functions such that $|f_n(u)| \leqslant |g(u)|^{-1}$ and $\lim_{n\to\infty} f_n(u) = g(u)^{-1}$ M_0-almost everywhere. Then, by the theorem on dominated convergence and formula (7.25), we get the relation

$$M_0(I) = \lim_{n\to\infty} \int_I f_n(u) g(u) M_0(du) \in [M(I)\colon J \in \mathscr{R}_0,\ J \subset I], \tag{7.26}$$

which together with (7.23) implies (7.18). The lemma is thus proved.

Now we shall prove a representation theorem for non-trivial completely non-deterministic random fields.

THEOREM 7.1. *Let $\{x_t\}$ be a non-trivial completely non-deterministic random field. Then there exists an $[x_t]$-valued non-trivial $\{T_t\}$-homogeneous stochastic measure M and a function f belonging to $L(M)$ such that*

$$[M(J)\colon J \in \mathscr{B}_0,\ J \subset \{t \in \boldsymbol{R}^n\colon t \leqslant 0\}] = [x_t, t \leqslant 0] \tag{7.27}$$

and

$$x_t = \int_{-\infty}^{t} f(u-t)M(du) \quad (t \in \boldsymbol{R}^n), \tag{7.28}$$

where the symbol $\int_{-\infty}^{t}$ *with* $t \in \boldsymbol{R}^n$ *denotes* $\int_{-\infty}^{t_1}\int_{-\infty}^{t_2} \quad \int_{-\infty}^{t_n}$ *whenever* $t = (t^1, t^2, \ldots, t^n)$.

Conversely, if M is a non-trivial $\{T_t\}$-homogeneous stochastic measure and $f \in L(M)$, then the random field (7.28) *is non-trivial and completely non-deterministic provided* (7.27) *holds.*

Proof. Given a non-trivial completely non-deterministic random field $\{x_t\}$, there exists, by Lemma 7.3, a non-trivial $[x_t]$-valued $\{T_t\}$-homogeneous stochastic measure M satisfying the condition (7.27). Hence and by Theorem 6.1 there exists a function $f \in L(M)$ such that $x_0 = \int_{-\infty}^{0} f(u)M(du)$. Consequently, by the translation property of $\{T_t\}$-homogeneous measures

$$x_t = T_t x_0 = \int_{-\infty}^{t} f(u-t)M(du) \quad (t \in \boldsymbol{R}^n). \tag{7.29}$$

Now suppose that M is a non-trivial $\{T_t\}$-homogeneous stochastic measure, that $f \in L(M)$ and that the random field $\{x_t\}$ defined by (7.28) satisfies (7.27). Then, of course $[x_t] = [M]$ and, by Theorem 6.1, each element $x \in [x_t]$ has an integral representation $x = \int_{\boldsymbol{R}^n} g(u)M(du)$ where $g \in L(M)$. Put, for $s \in \boldsymbol{R}^n$,

$$A_s x = \int_{-\infty}^{s} f(u)M(du). \tag{7.30}$$

Then, by (7.27), the linear operator A_s transforms $[x_t]$ onto $[x_t, t \leqslant s]$. Further, it is clear that for every $x \in [x_t]$ and $y \in [x_t,\ t \leqslant s]$ the random variables $x - A_s x$ and y are independent. Next, if $s, t \in \boldsymbol{R}^n$ we have, by (7.30),

$$A_t A_s x = \int_{-\infty}^{t} \chi_{(-\infty,s]}(u) f(u) M(du) = \int_{-\infty}^{\min(t,s)} f(u)M(du) = A_{\min(t,s)} x.$$

Therefore the condition of consistency is also satisfied. Finally, by (7.30),

$\lim_{t\to-\infty} A_{p_j(t)}x = 0$ for every $x \in [x_t]$ and $j = 1, 2, \ldots, n$. Thus the random field (7.28) is completely non-deterministic. Obviously, it is non-trivial. The theorem is thus proved.

III. Prediction of strictly stationary random fields on groups

Let $(G, +)$ stand for a countable additive Abelian group. Let $\leqslant$ be a partial ordering on G such that:

(1) For any $t, s, u \in G$ the relation $s \leqslant t$ implies $s+u \leqslant t+u$.

(2) For $t, s \in G$ the elements $t \wedge s$ (minimal) and $t \vee s$ (maximal) exist and $u \wedge (t \vee s) = (u \wedge t) \vee (u \wedge s)$.

(3) For arbitrary $t, s \in G$ the set $\{u \in G\colon s \leqslant u \leqslant t\}$ is finite.

For every $t \in G$ let U_t denote the down-neighbourhood of t with respect to the relation $\leqslant$. This means that $U_t = \{u \in G\colon u \leqslant t$ and $u \neq t$ and for $s \in G$ the relation $u \leqslant s \leqslant t$ implies either $s = u$ or $s = t\}$. It is clear by (3) that U_t is non-empty and $U_t = t + U_0$, where 0 denotes the zero element of the group G and $t + U_0 = \{t+u\colon u \in U_0\}$. Consequently, $\operatorname{card}(U_t) =$ constant for $t \in G$. Since G is countable, we can write $U_0 = \{e_1, e_2, \ldots\}$ with $\operatorname{card}(U_0) \leqslant \infty$. Moreover, from the definition of the set U_0 we infer that $e_i \wedge e_j = e_i + e_j$ for $i \neq j$. Consequently, U_0 is a generator of G and hence either $G \cong Z^n$ or $G \cong Z_*^N$, where Z^n is the set of all n-vectors with integer components and Z_*^N is the set of all finite sequences of integers, depending upon whether $\operatorname{card}(U_0) = n < \infty$ or $\operatorname{card}(U_0) = \infty$. In the case $G \cong Z^n$ the prediction problems for strictly stationary random fields have been solved. Therefore, in the sequel, we suppose that $\operatorname{card}(U_0) = \infty$. Then G can be identified with Z_*^N and the element $e_j \in U_0$ can be identified with $(0, \ldots, 0, \underset{j\text{th place}}{-1}, 0, \ldots) \in Z_*^N$.

In the sequel we need the following lemma.

LEMMA III. 1. *Suppose $\mathscr{X}$ is a linear space and a_t, $t \in G$, is a vector field with $a_t \in \mathscr{X}$ for all t. Then, for any $n, m = 1, 2, \ldots$ with $n \leqslant m$ and for every system $s^1, s^2, \ldots, s^n$ of positive integers the following equality holds:*

$$\text{(III.1)}\quad a_0 = \sum_{(s^1e_1)\wedge(s^2e_2)\wedge\ldots\wedge(s^ne_n)\leqslant t\leqslant 0} a_t - \sum_{(s^1e_1)\wedge(s^2e_2)\wedge\ldots\wedge(s^ne_n)\leqslant t\leqslant 0} \sum_{k=1}^{m}(-1)^{k+1} \sum_{r_1<r_2<\ldots<r_k} a_{t+e_{r_1}\wedge e_{r_2}\wedge\ldots\wedge e_{r_k}} + \sum_{k=1}^{m}(-1)^{k+1} \sum_{\substack{r_1<r_2<\ldots<r_k\\ r_1,r_2,\ldots,r_k=1,2,\ldots,m}} a_{(\gamma_{r_1}e_{r_1})\wedge(\gamma_{r_2}e_{r_2})\wedge\ldots\wedge(\gamma_{r_k}e_{r_k})},$$

where

$$\gamma_j = \begin{cases} s^j+1 & \text{for} \quad j \leqslant n, \\ 1 & \text{for} \quad j > n. \end{cases}$$

Proof (by induction). In the case $m = n$ equality (III.1) holds by Lemma I.1. Let us denote the right-hand side of (III.1) by P_m and suppose that for a fixed $m \geqslant n$ $P_m = a_0$. We shall prove that $P_{m+1} = a_0$.

Accordingly, we have

$$\begin{aligned} P_{m+1} = P_m - & \\ & - \sum_{(s^1e_1)\wedge(s^2e_2)\wedge\ldots\wedge(s^ne_n)<t<0} \sum_{k=2}^{m+1} (-1)^{k+1} \sum_{r_1<r_2<\ldots<r_{k-1}\leqslant m} a_{t+e_{r_1}\wedge\ldots\wedge e_{r_{k-1}}\wedge e_{m+1}} + \\ & + \sum_{k=2}^{m+1} (-1)^{k+1} \sum_{r_1<r_2<\ldots<r_{k-1}\leqslant m} a_{(\gamma_{r_1}e_{r_1})\wedge\ldots\wedge(\gamma_{r_{k-1}}e_{r_{k-1}})\wedge e_{m+1}} - \\ & - \sum_{(s^1e_1)\wedge\ldots\wedge(s^ne_n)<t<0} a_{t+e_{m+1}} + a_{e_{m+1}} \\ = P_m + \Big[& \sum_{(s^1e_1)\wedge\ldots\wedge(s^ne_n)<t<0} \sum_{k=1}^{m} (-1)^{k+1} \sum_{\substack{r_1<r_2<\ldots<r_k \\ r_1,r_2,\ldots,r_k=1,2,\ldots,m}} a_{t+e_{r_1}\wedge\ldots\wedge e_{r_k}\wedge e_{m+1}} - \\ & - \sum_{k=1}^{m} (-1)^{k+1} \sum_{\substack{r_1<r_2<\ldots<r_k \\ r_1,r_2,\ldots,r_k=1,2,\ldots,m}} a_{(\gamma_{r_1}e_{r_1})\wedge\ldots\wedge(\gamma_{r_k}e_{r_k})\wedge e_{m+1}} - \\ & - \sum_{(s^1e_1)\wedge\ldots\wedge(s^ne_n)<t<0} a_{t+e_{m+1}} \Big] + a_{e_{m+1}}. \end{aligned}$$

By the induction assumption the expression in the square brackets is equal to $-a_{e_{m+1}}$. Hence and by the last equation we have $P_{m+1} = P_m = a_0$ and we are done.

Let $\{x_t\}_{t\in G}$ be a strictly stationary random field, or, shortly, a stationary field. By the definition of a stationary field for any $t, t_1, t_2, \ldots, t_m \in G$ the multivariate distribution of the random variables $x_{t_1+t}, x_{t_2+t}, \ldots, x_{t_m+t}$ is independent of t.

Let $[x_t]$ denote the linear space generated by all random variables x_t, $t \in G$, and closed with respect to convergence in probability. In the sequel we shall identify the random variables which are equal almost surely. Of course, $[x_t]$ becomes a complete metric space under the Fréchet norm defined by $\|x\| = E\dfrac{|x|}{1+|x|}$.

To each stationary field there corresponds a unique shift group $\{T_t\}_{t\in G}$ of continuous linear operators in $[x_t]$ preserving the probability

distribution and the independence of random variables and $x_t = T_t x_0$ for all $t \in G$. Conversely, every such group $\{T_t\}$ in conjunction with a random variable x defines a stationary field $x_t = T_t x$ (see [2], p. 452, Chapter X).

By $[x_t, t \leqslant s]$ we shall denote the closed linear subspace of $[x_t]$ spanned by the random variables x_t, $t \leqslant s$. We say that $\{x_t\}$ field *admits a prediction* if for every $s \in G$ there exists a continuous linear operator A_s from x_t onto $[x_t, t \leqslant s]$ such that

(i) $A_s x = x$ whenever $x \in [x_t, t \leqslant s]$.

(ii) For every $x \in [x_t]$ and $y \in [x_t, t \leqslant s]$ the random variables $x - A_s x$ and y are independent.

(iii) If for every $y \in [x_t, t \leqslant s]$ the random variables x and y are independent, then $A_s x = 0$.

(iv) For any $t, s \in G$

$$A_t A_s = A_s A_t = A_{t \wedge s}.$$

The random variable $A_s x$ can be regarded as a linear prediction of x based on the full past of the random field $\{x_t\}$ up to the time $t = s$ (with respect to the relation $\leqslant$). Hence the operator A_s can be called a *predictor based on the full past of the random field* $\{x_t\}$ *up to the time* $t = s$.

Let $\{x_t\}_{t \in G}$ be a stationary field admitting a prediction. Then the predictors A_s, $s \in G$, and the shift operations T_t, $t \in G$, induced by $\{x_t\}$ satisfy the equation

$$A_s = T_s A_0 T_{-s} \quad (s \in G). \tag{III.2}$$

Moreover, 0 is the only constant random variable belonging to the space $[x_t]$.

LEMMA III.2. *Let $k_1, k_2, \ldots$ be an arbitrary sequence of positive integers. Then there exists a unique operator $A_{\{k_m\}}$ from $[x_t]$ onto a closed subspace $X_{\{k_m\}}$ of $[x_t]$ such that:*

(I) $A_{\{k_m\}} x = x$ *whenever* $x \in X_{\{k_m\}}$.

(II) *For any* $x \in [x_t]$ *and* $y \in X_{\{k_m\}}$ *the random variables* $x - A_{\{k_m\}} x$ *and* y *are independent.*

(III) *If for every* $y \in X_{\{k_m\}}$ *the random variables* x *and* y *are independent, then* $A_{\{k_m\}} x = 0$.

Moreover, for every $x \in [x_t]$

$$A_{\{k_m\}} x = \lim_{n \to \infty} \Big(I - \prod_{j=1}^{n} (I - A_{k_j e_j})\Big) x \tag{III.3}$$

where I denotes the identity.

Proof. For every $n = 1, 2, \ldots$ we define an operator $A_{\{k_m\}}^{(n)}$ via the formula

$$A_{\{k_m\}}^{(n)} = I - \prod_{j=1}^{n} (I - A_{k_j e_j}). \tag{III.4}$$

It is clear by condition (iv) that

$$A_{\{k_m\}}^{(n)} = \sum_{p=1}^{n} (-1)^{k+1} \sum_{\substack{r_1 < r_2 < \ldots < r_p \\ r_1, r_2, \ldots, r_p = 1, 2, \ldots, n}} A_{(k_{r_1} e_{r_1}) \wedge (k_{r_2} e_{r_2}) \wedge \ldots \wedge (k_{r_p} e_{r_p})} \tag{III.5}$$

and

$$\begin{aligned} A_{\{k_m\}}^{(n)} = {} & A_{k_2 e_2}(I - A_{k_1 e_1}) + \\ & + A_{k_3 e_3}(I - A_{k_1 e_1})(I - A_{k_2 e_2}) + \ldots + A_{k_n e_n} \prod_{j=1}^{n-1} (I - A_{k_j e_j}). \end{aligned} \tag{III.6}$$

Consequently, for every $x \in [x_t]$ the random variables

$$x_1 = A_{k_1 e_1} x, \quad x_2 = A_{k_2 e_2}(I - A_{k_1 e_1}) x, \quad \ldots, \quad x_n = A_{k_n e_n} \prod_{j=1}^{n-1} (I - A_{k_j e_j}) x$$

are independent. Moreover, $A_{\{k_m\}}^{(n)}$ is a unique operator from $[x_t]$ onto the subspace $\sum_{j=1}^{n} [x_t, t \leqslant k_j e_j]$, where $\sum$ denotes the complex sum of relevant spaces, such that for every $x \in [x_t]$ and $y \in \sum_{j=1}^{n} [x_t, t \leqslant k_j e_j]$ the random variables $x - A_{\{k_m\}}^{(n)} x$ and y are independent. Hence, for every $x \in [x_t]$ the random variables $x_1, x_2, \ldots, x_n$ and $\prod_{j=1}^{n} (I - A_{k_j e_j}) x$ are independent. On the other hand, from equation (III.6) we get the formula

$$x = \sum_{j=1}^{n} x_j + \prod_{j=1}^{n} (I - A_{k_j e_j}) x, \tag{III.7}$$

which, by Theorem 2.8 ([2], p. 119), implies that the series $\sum_{j=1}^{n} x_j$ converges with probability 1 when centred. Since 0 is the only constant random variable belonging to $[x_t]$, the series $\sum_{j=1}^{n} x_j$, according to Lemma 3 [12], converges with probability 1. Hence and from (III.7) the limit in probability

$$A_{\{k_m\}} x = \lim_{n \to \infty} \Big(I - \prod_{j=1}^{n} (I - A_{k_j e_j}) \Big) x$$

exists, and it is clear that the operator $A_{\{k_m\}}$ defined by the last formula is linear continuous.

We now prove (I), (II) and (III). It is easily seen by the fact that 0 is the only constant random variable belonging to $[x_t]$ that (II) implies (I) and (III). To prove (II) let us take an arbitrary $x \in [x_t]$ and $y \in X_{\{k_m\}}$ where $X_{\{k_m\}} = A_{\{k_m\}}([x_t])$. By virtue of the foregoing arguments the random variables $A^{(n)}_{\{k_m\}}y$ and $x - A^{(n)}_{\{k_m\}}x$ are independent. Letting $n \to \infty$, we get the independence of the random variables $y = \lim A^{(n)}_{\{k_m\}}y$ and $x - A_{\{k_m\}}x$, which completes the proof.

In the sequel we shall be dealing with the following operators $B_0 = A_{\{k_m\}}$ with $k_m = 1$ for every $m = 1, 2, \ldots$ and $C_n = A_{\{k_m\}}$ with $k_1 = n$, $k_2 = n$, $\ldots, k_n = n$ and $k_j = 1$ for $j > n$. Furthermore, we put

$$B_t = T_t B_0 T_{-t} \quad (t \in G). \tag{III.8}$$

LEMMA III.3. *For any $s, u \in G$*

$$A_u B_s = B_s A_u = \begin{cases} B_s & \textit{whenever } s \leqslant u, \\ A_{u \wedge s} & \textit{otherwise}. \end{cases}$$

Proof. First let us note that $A_s B_s = B_s A_s = B_s$ for every $s \in G$. Suppose now that $u \leqslant s$ and $u \neq s$. Then there exists a $j, j = 1, 2, \ldots$, such that $u \leqslant s + e_j$ and by (III.3) we have the equation

$$A_u B_s = \lim_{n \to \infty} \Big(A_u - A_u \prod_{k=1}^{n} (I - A_{s+e_k})\Big)$$

$$= \lim_{n \to \infty} \Big(A_u - (A_u - A_{u(s+e_j)}) \prod_{\substack{k=1 \\ k \neq j}}^{n} (I - A_{s+e_k})\Big) = A_u.$$

In general, let u, s be arbitrary. By the foregoing arguments we have

$$A_u B_s = A_u(A_s B_s) = (A_u A_s) B_s = A_{u \wedge s} B_s$$

$$= \begin{cases} B_s & \text{whenever } u \wedge s = s, \\ A_{u \wedge s} & \text{otherwise}, \end{cases}$$

which completes the proof.

LEMMA III.4. *For any s such that $(ne_1) \wedge (ne_2) \wedge \ldots \wedge (ne_n) \leqslant s \leqslant 0$ we have*

$$B_s C_{n+1} = A_s C_{n+1}$$

and

$$C_n B_0 = C_n A_0 = C_n$$

for all $n = 1, 2, \ldots$

Proof. The lemma follows directly from the definitions of the operators A_s, B_s and C_k.

LEMMA III.5. *If x is a fixed element of $[x_t]$ and $y_0 = x - B_0 x$, $y_s = A_s x - B_s x$ for $s \leqslant 0$ and $s \neq 0$, then for every $n = 1, 2, \ldots$ and $z \in [x_t]$ the random variables $C_{n+1} z$ and y_s, $(ne_1) \wedge (ne_2) \wedge \ldots \wedge (ne_n) \leqslant s \leqslant 0$, are independent.*

Proof. Let λ_s, λ_{n+1} $(s \leqslant 0)$ be a system of real numbers. By Lemmas III.2 and III.4 it follows that the random variables

$$\lambda_{n+1} C_{n+1} z \quad \text{and} \quad \sum_{(ne_1)\wedge\ldots\wedge(ne_n)\leqslant s\leqslant 0} \lambda_s y_s$$

are independent. On the other hand, by Lemma III.3, it can easily be seen that the random variables $\lambda_s y_s$ are independent. Consequently, we have

$$E\exp i\Big(\sum_{(ne_1)\wedge\ldots\wedge(ne_n)\leqslant s\leqslant 0} \lambda_s y_s + \lambda_{n+1} C_{n+1} z\Big)$$
$$= E\exp i(\lambda_{n+1} C_{n+1} z) \cdot \prod_{(ne_1)\wedge\ldots\wedge(ne_n)\leqslant s\leqslant 0} E\exp i(\lambda_s y_s),$$

which completes the proof.

LEMMA III.6. *There exists a continuous linear operator C_∞ on $[x_t]$ commuting with the shifts $\{T_t\}_{t\in G}$ induced by $\{x_t^*\}$ and such that for every x in $[x_t]$*

$$\lim_{n\to\infty} C_n x = C_\infty x.$$

Proof. Given $x \in [x_t]$, we define y_s, $s \leqslant 0$, in the same way as in Lemma III.5.

Let us fix $m \geqslant n$. Putting $a_0 = I$, $a_t = A_t$ for $t \in G$ and $t \neq 0$, and taking into account Lemma III.1, we have the equation

$$a_0 = \sum_{(ne_1)\wedge\ldots\wedge(ne_n)\leqslant t\leqslant 0} \Big(a_t - \sum_{k=1}^{m} (-1)^{k+1} \sum_{\substack{r_1, r_2, \ldots, r_k = 1 \\ r_1 < r_2 < \ldots < r_k}}^{m} a_{t+e_{r_1}\wedge\ldots\wedge e_{r_k}}\Big) +$$
$$+ \sum_{k=1}^{m} (-1)^{k+1} \sum_{\substack{r_1, r_2, \ldots, r_k = 1 \\ r_1 < r_2 < \ldots < r_k}}^{m} a_{(\gamma_{r_1} e_{r_1})\wedge\ldots\wedge(\gamma_{r_k} e_{r_k})},$$

where $\gamma_j = n+1$ if $j \leqslant n$ and $\gamma_j = 1$ if $j > n$.

Letting $m \to \infty$, we have, by the definition of the operators B_s^* and C_n, the equation

$$\text{(III.9)} \qquad I = \sum_{(ne_1)\wedge\ldots\wedge(ne_n)\leqslant t\leqslant 0} (a_t - B_t) + C_{n+1}.$$

Consequently, for $x \in [x_t]$

$$\text{(III.10)} \qquad x = \sum_{(ne_1)\wedge\ldots\wedge(ne_n)\leqslant t\leqslant 0} y_t + C_{n+1} x.$$

By virtue of Lemma III.5 and ([2], Theorem 2.8, p. 119) the series $\sum_{t<0} y_t$ converges with probability 1 when centred. Since 0 is the only constant random variable belonging to $[x_t]$, the series $\sum_{t<0} y_t$ converges with probability 1. Hence and from (III.10) the limit in probability

$$C_\infty x = \lim_{n\to\infty} C_{n+1} x$$

exists. It is clear that C_∞ is a linear continuous operator.

Now by the same technique as in the proof of Lemma I.10 we can prove that C_∞ commutes with the shifts induced by $\{x_t\}$ field. The lemma is thus proved.

Let $\{x_t\}_{t\in G}$ be a stationary field admitting a prediction. Then it is called *deterministic* if $A_0 x = x$ for every $x \in [x_t]$. Furthermore, it is called *completely non-deterministic* if for every $x \in [x_t]$

$$C_\infty x = \lim_{n\to\infty} C_n x = 0.$$

Next we say that two random fields $\{x_t'\}$ and $\{x_t''\}$ are *independent* if arbitrary $x' \in [x_t']$ and $x'' \in [x_t'']$ are independent.

THEOREM III.1. *Each stationary field $\{x_t\}_{t\in G}$ admitting a prediction is the sum of two independent stationary fields admitting a prediction, one deterministic and the other completely non-deterministic.*

Proof. Let $\{x_t\}_{t\in G}$ be a stationary field admitting a prediction. Then the stationary fields $x_t' = C_\infty x_t$ and $x_t' = (I - C_\infty) x_t$, where C_∞ is the limit operator in Lemma III.6, admit a prediction with predictors A_t, $t \in G$, restricted to the subspaces $[x_t']$ and $[x_t'']$, respectively. Moreover, since $C_\infty A_0 = C_\infty$ and $C_\infty(I - C_\infty) = 0$ the random field $\{x_t'\}$ is deterministic and $\{x_t''\}$ completely non-deterministic. The theorem is thus proved.

Now we shall prove a representation theorem of a completely non-deterministic stationary field.

THEOREM III.2. *Let $\{x_t\}_{t\in G}$ be a completely non-deterministic stationary field. Then there exists a random field $\{v_t\}_{t\in G}$ of independent identically distributed random variables such that $[v_t, t \leqslant 0] = [x_t, t \leqslant 0]$ and x_s is a moving average*

$$x_s = \sum_{\substack{t<0\\ t\in G}} a_t v_{t+s} \quad (s \in G) \tag{III.11}$$

where a_t's are real and the series converges in probability, regardless of the order of summation.

Conversely, if $\{v_t\}_{t\in G}$ is a random field of identically distributed random variables such that 0 is the only constant random variable in $[v_t]$, then the

moving average (III.11) *is a completely non-deterministic stationary field provided* $[x_t, t \leqslant 0] = [v_t, t \leqslant 0]$.

Proof. Let $\{x_t\}_{t\in G}$ be a stationary completely non-deterministic random field. Put

$$v_t = x_t - B_t x_t \quad (t \in G), \tag{III.12}$$

where B_t is the operator defined by (III.8). This implies, by Lemma III.5, that the random variables v_t, $t \in G$, are independent. Moreover, by (III.8) we have the equations

$$T_t v_0 = T_t(x_0 - B_0 x_0) = x_t - T_t B_0 T_{-t} x_t = x_t,$$

which shows that the random variables are identically distributed.

Now setting $y_0 = x_0 - B_0 x_0$, $y_t = A_t x_0 - B_t x_0$ for every $t \leqslant 0$ and $t \neq 0$, we have, by (III.10), the formula

$$x_0 = \sum_{(ne_1)\wedge\ldots\wedge(ne_n)\leqslant t\leqslant 0} y_t + C_{n+1} x_0 \quad (n = 1, 2, \ldots). \tag{III.13}$$

Using the same method as in the proof of Theorem 2 in [12], we infer that there exists a sequence of real numbers $\{a_t\}_{t\leqslant 0}$ such that $y_t = a_t v_t$ for every $t \leqslant 0$. Then equation (III.13) yields

$$x_0 = \sum_{(ne_1)\wedge\ldots\wedge(ne_n)\leqslant t\leqslant 0} a_t v_t + C_{n+1} x_0,$$

which, by letting $n \to \infty$, implies (III.11). The first part is thus proved.

The second part is clear and, for details, the reader is referred to [12], Theorem 2.

We now consider a stationary Gaussian random field $\{x_t\}_{t\in G}$ with mean zero. It is well known that the correlation function $R(t) = Ex_{t+s}x_0$ is a positive definite function on $G \cong Z_*^N$ and, by the theorem of Herglotz and Bochner, it is easy to show that $R(t)$ has the representation

$$R(t) = \int_T e^{2\pi i\lambda t} \nu(d\lambda)$$

where T is the infinite product $[0,1]\times[0,1]\times\ldots$ and for $\lambda = (\lambda^1, \lambda^2, \ldots) \in T$ and $t = (t^1, t^2, \ldots, t^m, 0, \ldots) \in Z_*^N$, $\lambda t = \lambda^1 t^1 + \lambda^2 t^2 + \ldots + \lambda^m t^m$ and $\nu(d\lambda)$ is a finite Borel measure on T, being the spectral measure of the random field $\{x_t\}$. Furthermore, $\{x_t\}$ can be represented as a random integral as follows:

$$x_t = \int_T e^{2\pi i\lambda t} Z(d\lambda) \quad (t \in Z_*^N)$$

where Z is a set function with orthogonal increments on the Borel sets on T which is related to ν by

$$EZ(\Lambda) = 0, \quad EZ(\Lambda_1)Z(\Lambda_2) = v(\Lambda_1 \cap \Lambda_2).$$

Since for a Gaussian random field $\{x_t\}$ with mean zero the concepts of independence and orthogonality are equivalent, the orthogonal projector A_s from $[x_t]$ onto $[x_t, t \leqslant s]$ always satisfies conditions (i), (ii) and (iii). Consequently, the random field $\{x_t\}$ admits a prediction if and only if, for any $t, s \in G$, $A_t A_s = A_{t \wedge s}$. Finally, using the classical method for a stationary completely non-deterministic Gaussian sequence x_n, $n = 0, \pm 1, \pm 2,$ and by Theorem III.2, we obtain the following theorem.

THEOREM III.3. *Let $\{x_t\}_{t \in Z_+^N}$ be a stationary Gaussian random field with mean zero. Then it admits a prediction and it is completely non-deterministic if and only if the spectral measure ν is absolutely continuous with respect to the Lebesgue product measure $d\lambda$ on T and the spectral density*

$$\frac{d\nu}{d\lambda} = |h(\lambda)|^2$$

satisfies the conditions

$$(\alpha) \qquad h(\lambda) = \sum_{\substack{t \geqslant 0 \\ t \in Z_+^N}} c_t e^{2\pi i \lambda t} \qquad (c_t \in \boldsymbol{R}),$$

$$(\beta) \qquad h(\lambda) > 0 \quad \textit{a.e. Lebesgue},$$

where the series on the right-hand side of (α) converges in $L_2^{(T)}$.

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