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CYCLIC HOMOLOGY, A SURVEY *

JEAN-LOUIS LODAY

Strasbourg, France

Let $M_1, \ldots, M_n$ be n matrices and let $\mathrm{Tr}(M_1 \ldots M_n)$ be the trace of the product. It is well known that this product is not invariant under all permutations of the matrices. However, it is invariant under *cyclic* permutations because $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

In the last few years this simple phenomenon, i.e. invariance under cyclic permutation, appeared independently in at least three frameworks: trace of Fredholm operators and generalization of de Rham cohomology [C1], homology of Lie algebras [T], algebraic K-theory [LO].

It was extensively studied for the first time in Connes' work as a cohomology theory, now called *cyclic cohomology* and denoted by $HC^n(A)$, $n \geqslant 0$. These abelian groups are defined for any associative (and not necessarily commutative) algebra A over a commutative ring k.

In this survey I will mostly present the dual version: cyclic homology $HC_n(A)$.

The main features of this theory are the following. Firstly, it has strong connections with the so-called Hochschild homology $H_n(A, A)$ and with de Rham cohomology. Namely, there is a "periodicity exact sequence"

$$\ldots \to H_n(A, A) \to HC_n(A) \xrightarrow{S} HC_{n-2}(A) \to H_{n-1}(A, A) \to \ldots$$

from which we see immediately that $HC_n(k)$ is periodic of period 2, isomorphic to k when n is even and to 0 when n is odd. The map S permits the definition of $\varprojlim_S HC_n(A)$ which is a quotient of a periodic theory $HC_*^{\mathrm{per}}(A)$. If A is a ring of functions of a smooth algebraic variety V (or the ring of differentiable functions over a manifold V, in the topological framework) then

$$\varprojlim HC_*(A) = HC_*^{\mathrm{per}}(A) = H^*(V, C)$$

(de Rham cohomology of V).

* This paper is in final form and no version of it will be submitted for publication elsewhere.

In this sense $HC_*^{per}(A)$, which exists even if A is not commutative (for instance the C^*-algebra of a foliation), plays the role of de Rham cohomology. This enables Connes to define Chern classes for foliations and therefore to give an explicit formula for the index of an elliptic differential operator on a foliation.

Secondly, cyclic homology allows the computation of the homology of Lie algebras of matrices. The point is that $HC_{*-1}(A)$ is the primitive part of $H_*(\underline{gl}(A), k)$ when k is a characteristic zero field. A similar result is true for the orthogonal and symplectic Lie algebras but with cyclic homology replaced by skew-dihedral homology.

Thirdly, several points suggest a strong relationship with algebraic K-theory. In fact cyclic homology can be viewed as an "additive algebraic K-theory": the linear group being replaced by the Lie algebra of matrices, the determinant by the trace, the multiplicative formal group G_m by the additive one G_a, the groups K_n by HC_{n-1}, etc. Some results are known in this direction. For instance the algebraic K-theory of a simply-connected space X (in Waldhausen's sense) can be computed rationally from the cyclic homology of the minimal model of X. There are defined characteristic classes from algebraic K-theory to cyclic homology. However it seems that a lot remains to be done in the comparison between HC_* and K_*.

Finally, one should note that through different ways, namely the study of "anomalies", some quantification problems in theoretical physics can be formulated in terms of cyclic (co)-homology.

1. Cyclic homology of algebras [C3] [L–Q2]

Let k be a commutative ring with unity and let A be a (not necessarily commutative) associative k-algebra with unity. The *Hochschild homology* of A with coefficients in itself, denoted $H_n(A, A)$, is, by definition, the homology of the complex

$$\ldots \to A^{\otimes n+1} \xrightarrow{b} A^{\otimes n} \to \ldots \to A$$

where the boundary operator b is given by

$$b(a_0, \ldots, a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0, \ldots, a_i a_{i+1}, \ldots, a_n) + (-1)^n (a_n a_0, a_1, \ldots, a_{n-1}).$$

In particular if A is commutative $H_0(A, A) = A$ and $H_1(A, A) = \Omega^1_{A/k}$.

Let $t\colon A^{\otimes n+1} \to A^{\otimes n+1}$ be the cyclic operator

$$t(a_0, a_1, \ldots, a_n) = (-1)^n (a_n, a_0, \ldots, a_{n-1}).$$

Connes remarked [C1, C2] that if we divide $A^{\otimes n+1}$ by the action of t to get

$$C_n(A) = A^{\otimes n+1}/(1-t),$$

then b is still well defined and therefore $(C_*(A), b)$ is a complex.

1.0. Definition 0. If k contains $\mathbf{Q}$ the cyclic homology of A is

$$HC_n(A) = H_n(C_*(A), b).$$

Example. If $A = k$ then $HC_{2n}(k) = k$ and $HC_{2n+1}(k) = 0$. A generator of the even dimensional group is the class of $1 \otimes 1 \otimes \dots \otimes 1$ (odd number of 1s).

This definition does not behave well in characteristic different from zero. Therefore one introduces a new one.

The element $N = 1 + t + \dots + t^n$ operates on $A^{\otimes n+1}$ and there exists a bicomplex $\underset{=}{C}(A)$

$$\begin{tikzcd}
\downarrow & \downarrow & \downarrow & \\
A^{\otimes n+1} \arrow[d, "b"'] & A^{\otimes n+1} \arrow[l, "1-t"'] \arrow[d, "-b'"'] & A^{\otimes n+1} \arrow[l, "N"'] \arrow[d, "b"'] & \cdots \arrow[l, "1-t"'] \\
A^{\otimes n} & A^{\otimes n} \arrow[l] & A^{\otimes n} \arrow[l] & \cdots \arrow[l] \\
\downarrow & \downarrow & \downarrow &
\end{tikzcd}$$

where t, N and b are as above and

$$b'(a_0, \dots, a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0, \dots, a_i a_{i+1}, \dots, a_n).$$

1.1. Definition 1. The *cyclic homology* of A, denoted by $HC_n(A)$, is the homology of the total complex of the bicomplex $\underset{=}{C}(A)$.

For instance, if A is commutative

$$HC_0(A) = A, \qquad HC_1(A) = \Omega^1_{A/k}/dA.$$

The ground ring k does not appear in the notation $HC_n(A)$, however these groups do depend on k. For instance cyclic homology of $\mathbf{C}$ viewed as a $\mathbf{C}$-algebra is different from cyclic homology of $\mathbf{C}$ viewed as a $\mathbf{Q}$-algebra (try $n = 1$).

If k contains $\mathbf{Q}$ Definitions 0 and 1 are equivalent (via the projection map $A^{\otimes n+1} \to C_n(A)$).

There is another definition in terms of bicomplex which is useful for the comparison with de Rham cohomology.

Denote $A \otimes \bar{A}^{\otimes n}$ the quotient of $A^{\otimes n+1}$ by the submodule generated by $(a_0, \dots, a_n)$ where $a_i = 1$ for some i, $1 \leqslant i \leqslant n$. (Here $\bar{A} = A/k$). Put

$$B(a_0, \dots, a_n) = \sum_{i=0}^{n} (-1)^{in} (a_i, a_{i+1}, \dots, a_n, a_0, \dots, a_{i-1}).$$

Then there is a bicomplex $\underset{=}{B}(A)$.

$$\begin{tikzcd}
{} \arrow[d] & {} \arrow[d] & {} \arrow[d] \\
A \otimes \bar{A}^{\otimes 2} \arrow[d, "b"'] & A \otimes \bar{A} \arrow[l, "B"'] \arrow[d, "b"'] & A \arrow[l, "B"'] \\
A \otimes \bar{A} \arrow[d, "b"'] & A \arrow[l, "B"'] & \\
A & &
\end{tikzcd}$$

1.2. Definition 2. $HC_n(A)$ is the homology of the total complex of the bicomplex $\underline{\underline{B}}(A)$.

There is a map from $\mathrm{Tot}\,\underline{\underline{B}}(A)$ to $\mathrm{Tot}\,\underline{C}(A)$ inducing an isomorphism in homology, hence the equivalence of Definitions 1 and 2.

There is still another definition in terms of derived functors. The family $A^{\otimes n+1}$, $n \geqslant 0$ can be looked at as a ΔC^{op}-module $A^\natural$ where ΔC is a certain small category containing the simplicial category Δ and all the cyclic groups $C_n = \boldsymbol{Z}/n$ (ΔC is denoted Λ in [C3]) (see Section 12).

1.3. Definition 3. For any k-algebra A, $HC_n(A) = \mathrm{Tor}_n^{\Delta C}(k, A^\natural)$.

It can be shown [C3] [F–L] that this definition agrees with the others.

2. First properties [L–Q2]

It is well known that $H_0(k, k) = k$ and $H_n(k, k) = 0$ for $n > 0$. On the other hand one has

$$HC_{2n}(k) = k \quad \text{and} \quad HC_{2n+1}(k) = 0.$$

Therefore $HC_*(k)$ is periodic of period 2. This is not true for an arbitrary algebra A. However there is a kind of periodicity.

2.1. Theorem (Periodicity exact sequence). *There is an exact sequence*

$$\ldots \to H_n(A, A) \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{B} H_{n-1}(A, A) \to \ldots$$

First proved by Connes [C2] in the cohomological framework using Definition 0 this theorem is more easily proved using Definitions 1 or 2 [L–Q2] [T].

It is sometimes interesting to make cyclic homology periodic on the nose.

2.2 Definition. *Periodic cyclic homology* is the homology of the total complex of $\underline{C}^{per}(A)$ obtained from $\underline{C}(A)$ by continuing the rows on the left.

Another way of comparing cyclic homology to Hochschild homology is via a spectral sequence.

2.3. THEOREM. *There is a spectral sequence abutting to* $HC_n(A)$ *with* $E^1_{pq} = H_{q-p}(A)$ *and with* d^1: $H_{q-p}(A) \to H_{q-p+1}(A)$ *being* B.

Obviously from Definition 1 cyclic homology is related to the homology of the cyclic group $Z/n+1$ acting on $A^{\otimes n+1}$.

2.4. THEOREM. *There is a spectral sequence*

$$E^1_{pq} = H_p(Z/q+1, A^{\otimes q+1}) \Rightarrow HC_*(A).$$

2.5. Morita invariance. Two algebras A and A' are said to be *Morita equivalent* if there is an A–A'-bimodule P and an A'–A-bimodule Q such that $P \otimes_{A'} Q \simeq A$ as A-bimodules and $Q \otimes_A P \simeq A'$ as A'-bimodules.

Cyclic homology is Morita invariant; that means $HC_n(A) = HC_n(A')$. In particular cyclic homology of matrices over A is canonically isomorphic to cyclic homology of A. This equivalence is very helpful in the computations. Morita equivalence is still valid in the cohomological framework and also in the topological framework (see Sections 5 and 10, respectively). In particular, cyclic (co)-homology of the algebra of infinite matrices with rapid decay is isomorphic to cyclic (co-)homology of C.

3. Relationship to de Rham cohomology

In this section the algebra A is assumed to be commutative. Let $\Omega^1_A = \Omega^1_{A/k}$ be the A-module of Kähler differentials, generated by symbols dx for $x \in A$ with the relations $d(x+y) = dx+dy$, $d(xy) = xdy+ydx$ and $d(k) = 0$. The nth exterior power is denoted Ω^n_A.

The de Rham complex

$$A = \Omega^0_A \to \Omega^1_A \to \dots \to \Omega^n_A \xrightarrow{d} \Omega^{n+1}_A \to \dots$$

where $d(a_0 da_1 \dots da_n) = da_0 da_1 \dots da_n$ gives rise to de Rham cohomology $H^*_{DR}(A)$.

3.1. THEOREM [L–Q2]. *If A is smooth over k and k contains Q then*

$$HC_n(A) = \Omega^n_A/d\Omega^{n-1}_A \oplus H^{n-2}_{DR}(A) \oplus H^{n-4}_{DR}(A) \oplus \dots$$

This is proved using Definition 2 and a result by Hochschild, Kostant and Rosenberg: $H_n(A, A) = \Omega^n_A$ when A is smooth. This theorem was first proved by Connes [C2] in the framework of cohomology for the ring of differentiable functions on a manifold (cf. 10.1).

3.2. A similar comparison has been worked out in [F–T2] between periodic cyclic homology and crystalline homology in characteristic zero.

3.3. Deligne cohomology. In the case $k = C$, Deligne has defined a cohomology theory as follows. First one truncates the de Rham complex (at n, let say), then one shifts the complex by one and modify the right part by putting a map $Z \to \Omega_A^0$ (depending on n, in general). If we ignore this last modification then its homology is exactly cyclic homology for smooth A. The main advantage about Deligne cohomology (vs. de Rham) is that it bears a product, defined at the chain level and which is homotopy commutative. The advantage of using cyclic homology, namely Definition 0, is that this product becomes strictly commutative at the chain level in this framework (cf. [L–Q2]).

3.4. Of course, cyclic homology can be extended to algebraic varieties (and schemes) by sheafifying the complex $C_*(A)$ and taking the hyperhomology.

3.5. Positive characteristic. In characteristic $p > 0$ current research is trying to elucidate the relationship with de Rham–Witt cohomology.

4. Computations

4.1. Ground ring. We already showed that $HC_{2n}(k) = k$ and $HC_{2n+1}(k) = 0$.

4.2. Dual numbers, nilpotent ideal. Let $k[\varepsilon]$ be the ring of dual numbers ($\varepsilon^2 = 0$). Then [L–Q2]

$$HC_n(k(\varepsilon]) = HC_n(k) \oplus \bigoplus_{n=0}^{m} H_{n-m}(Z/m+1, k)$$

(homology of the cyclic group $Z/m+1$ with coefficients in k). In particular, if k is a characteristic zero field then $HC_{2n}(k[\varepsilon]) = HC_{2n}(k) \oplus k$, where a generator of this last summand is $\varepsilon \otimes \varepsilon \otimes \ldots \otimes \varepsilon$. This implies $HC_*^{\mathrm{per}}(k[\varepsilon]) = HC_*^{\mathrm{per}}(k)$.

More generally, if k contains Q, then [G1] $HC_*^{\mathrm{per}}(A) \to HC_*^{\mathrm{per}}(A/I)$ is an isomorphism when I is nilpotent.

4.3. Tensor algebra. Let V be a module over k and $T(V)$ its tensor algebra. Then [L–Q2]

$$HC_n(T(V)) = HC_n(k) \oplus \bigoplus_{m>0} H_n(Z/m, V^{\otimes m})$$

(Z/m acting by permutation on $V^{\otimes m}$).

4.4. Smooth algebra. We already mentionned that if A is smooth and k contains Q, then

$$HC_n(A) = \Omega_A^n / d\Omega_A^{n-1} \oplus H_{DR}^{n-2}(A) \oplus H_{DR}^{n-4}(A) \oplus \ldots$$

4.5. Group algebra. Let G be a group and $k[G]$ be its group algebra. Denote by $\langle G \rangle$ the set of conjugation classes of G and choose an element z in each class $\langle z \rangle \in \langle G \rangle$. Let $G_z = \{g \in G \mid gz = zg\}$ be the centralizer of z in G. There is defined a fibration

$$(*) \qquad S^1 \to BG_z \to X(G_z, z)$$

where BG_z is the classifying space of the discrete group G_z and such that the image of $1 \in \boldsymbol{Z} = \pi_1 S^1$ is $z \in G_z$. Then [B2]

$$HC_n(k[G]) = \bigoplus_{z \in \langle G \rangle} H_n(X(G_z, z), k)$$

(homology of the space $X(G_z, z)$). As Hochschild homology is

$$H_n(k[G], k[G]) = \bigoplus_{z \in \langle G \rangle} H_n(BG_z, k)$$

one can show that the periodicity sequence is the sum over $\langle G \rangle$ of the Gysin sequences of the fibrations $(*)$. If z is of infinite order then $X(G_z, z) = B(G_z/\{z\})$ and its homology is the homology of the discrete group $G_z/\{z\}$. On the other hand for $z = 1$, $X(G_1, 1) = BG \times BS^1$ and therefore:

(4.5)$'$. $HC_n(k[G])$ contains $\bigoplus_{l \geqslant 0} H_{n-2l}(G, k)$ as a direct factor (homology of the discrete group G) [K1]. As a corollary we see that if k contains $\boldsymbol{Q}$ then $HC_*(k[t, t^{-1}]) = HC_*(k) \oplus HC_{*-1}(k)$, $* > 0$. (Put $G = \boldsymbol{Z}$ in the above formula. Follows also from 4.4).

4.6. Tensor product [B3, Ka]. Let A and B be k-algebras and suppose that $\operatorname{char} k = 0$, B and $HC_*(B)$ are flat k-modules. Then there is an exact sequence

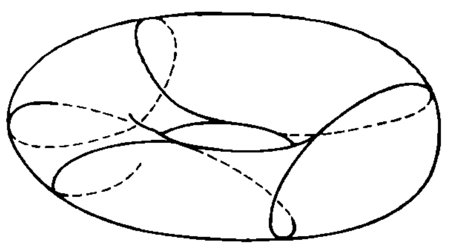
$$0 \to \Sigma \operatorname{Cotor}^{k[u]}(HC_*(A), HC_*(B)) \to HC_*(A \otimes B)$$
$$\to HC_*(A) \,\square_{k[u]}\, HC_*(B) \to 0,$$

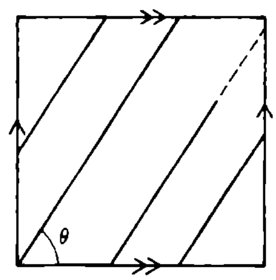
where $\square$ stands for the cotensor product of the two $k[u]$-comodules (see 6.1).

A similar statement holds in the framework of differential graded algebras [B1].

4.7. Differential graded algebra (see Section 9).

4.8. C^*-algebra of the Kronecker foliation [C2]. The Kronecker foliation of the torus is the foliation by lines of slope $\theta \in \boldsymbol{R}/\boldsymbol{Z}$





The associated C^*-algebra is Morita equivalent to the closure of the non-commutative algebra A_θ whose generic element is a formal sum

$\sum a_{n,m} U_1^n U_2^m$ where $a_{n,m} \in \boldsymbol{C}$, $n \in \boldsymbol{Z}$, $m \in \boldsymbol{Z}$, is a sequence of rapid decay and where $U_1 U_2 = \exp(2i\pi\theta) U_2 U_1$. Though Hochschild cohomology of A_θ may be infinite dimensional, $HC^*_{\text{per}}(A_\theta)$ is finite dimensional, of dimension 4, over $\boldsymbol{C}$. In particular $HC^{\text{ev}}_{\text{per}}(A_\theta)$ is of dimension 2 generated by the standard trace τ coming from HC^0 and φ coming from HC^2 and given by

$$\varphi(a_0, a_1, a_2) = a_0(\delta_1(a_1)\delta_2(a_2) - \delta_2(a_1)\delta_1(a_2))$$

(where δ_1 and δ_2 are the basic derivations of A_θ).

The pairing $\langle -,- \rangle$: $K_0(A_\theta) \times HC^{\text{ev}}_{\text{per}}(A_\theta) \to \boldsymbol{C}$ is given by

$$\langle [A_\theta], \tau \rangle = 1, \qquad \langle [\mathscr{S}], \tau \rangle = \theta, \qquad \langle [A_\theta], \varphi \rangle = 0, \qquad \langle [\mathscr{S}], \varphi \rangle = 1,$$

where the 2-dimensional group $K_0(A_\theta)$ is generated by the unit $[A_\theta]$ and the class of $\mathscr{S}$ = Schwarz space of the real line with the following module structure: $(\xi \cdot U_1)(s) = \xi(s+\theta)$, $(\xi \cdot U_2)(s) = e^{2i\pi s}\xi(s)$.

4.9. Generalized Jacobi matrices. A generalized Jacobi matrix is an infinite dimensional matrix (a_{ij}) $i \geqslant 0, j \geqslant 0$ which has only a finite number of non zero entries on each column and each row. The generalized Jacobi matrices form a ring Ck called the *cone* of k. It contains the ring of finite matrices $\tilde{k}$ as a two-sided ideal. The quotient $Sk = Ck/\tilde{k}$ is called the *suspension ring* of k. It is proved in [F–T1] that if k contains $\boldsymbol{Q}$ one has $HC_i(Ck) = 0$, $i > 0$ and

$$HC_{2i}(Sk) = 0, \qquad HC_{2i-1}(Sk) = k, \qquad i > 0.$$

In fact there is a morphism $k[t, t^{-1}] \to Sk$ inducing an isomorphism

$$HC_i(k[t, t^{-1}])/HC_i(k) \xrightarrow{\sim} HC_i(Sk), \qquad i > 0.$$

5. Cyclic cohomology [C1, C2, C3]

All the definitions of Section 1 can be dualized to give cyclic cohomology. In fact Definition 0 and the main theorems concerning it appeared first under this form in Connes' work.

We first deal with the analogue of Definition 0. Let k be a characteristic 0 field and A be a k-algebra. $C^n(A)$ is the vector space of $(n+1)$-linear functionals φ on A such that

$$\text{(t)} \qquad \varphi(a_0, \ldots, a_n) = (-1)^n \varphi(a_n, a_0, \ldots, a_{n-1}).$$

The Hochschild coboundary operator is given by the formula

$$(b\varphi)(a_0, \ldots, a_{n+1}) = \sum_{i=0}^{n} (-1)^i \varphi(a_0, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) + (-1)^{n+1} \varphi(a_{n+1} a_0, a_1, \ldots, a_n).$$

This defines a complex $(C^*(A), b)$ which we call *Connes' complex*. Remark that a cyclic cocycle is a Hochschild cocycle which satisfies (t).

5.1. DEFINITION 0. Let k be a characteristic zero field. *Cyclic cohomology* $HC^n(A)$ is the homology of the complex $(C^*(A), b)$ (Connes' notation is $H^n_\lambda(A)$ where λ stands for the cyclic permutation).

All the other definitions can be carried over in the cohomological framework and in particular:

DEFINITION 3. $HC^n(A) = \mathrm{Ext}^n_{\mathscr{K}}(k, (A)^\sim)^* =$ (cf. Section 12).

If in Definition 0 we ignore the cyclic condition (t) we get Hochschild cohomology $H^n(A, A^*)$ where A^* is the dual of A. All the properties of cyclic homology hold for cyclic cohomology: Morita invariance, periodicity exact sequence:

(5.2) $$\ldots H^{n+1}(A, A^*) \to HC^n(A) \xrightarrow{S} HC^{n+2}(A) \to H^{n+2}(A, A^*) \to \ldots,$$

computations, etc. In particular, we have

$$HC^{2n}(k) = k, \qquad HC^{2n+1}(k) = 0.$$

5.3. DEFINITION. *Periodic cyclic cohomology* is defined (in the smooth case) as

$$HC^n_{\mathrm{per}}(A) = \varinjlim_i HC^{n+2i}(A).$$

Connes has given another definition of cyclic cohomology which is quite convenient for concrete examples and when dealing with products. Let $\Omega = \bigoplus_{i=0}^{n} \Omega^i$ be a graded C-algebra, d a graded derivation of degree 1 such that $d^2 = 0$ and $\int: \Omega^n \to C$ a closed graded trace. This means that

$$\int d\omega = 0 \quad \text{for } \omega \in \Omega^{n-1},$$
$$\int \omega_2 \omega_1 = (-1)^{\deg\omega_1 \deg\omega_2} \int \omega_1 \omega_2 \quad \text{for } \deg\omega_1 + \deg\omega_2 = n.$$

Such a triple $(\Omega, d, \int)$ is called a *cycle of dimension n*. There is an obvious notion of sum of cycles and also a notion of product (take $\Omega \otimes \Omega'$). An *n-dimensional cycle over A* is a cycle like above plus a homomorphism $\varphi: A \to \Omega^0$.

The character of an n-dimensional cycle over A is the $(n+1)$-linear functional τ

$$\tau(a_0, \ldots, a_n) = \int \varrho(a_0) d\varrho(a_1) \ldots d\varrho(a_n).$$

It determines a cycle in $C^n(A)$ and any cycle can be realized that way.

A k-algebra R is said to be *flabby* if there exists a bimodule M finitely generated and projective as a right module such that $R \oplus M$ is isomorphic to

M as a bimodule. An n-dimensional cycle $(\Omega, d, \int, \varrho)$ in A such that Ω^0 is flabby gives a boundary in $C^n(A)$.

5.4. Definition 0′. The additive monoid of n-dimensional cycles over A modulo those such that Ω^0 is flabby is a group $HC^n(A)$.

By the above remarks it gives the same group as Definition 0 when k is a characteristic zero field.

From the product of cycles $(\Omega \otimes \Omega')$ cyclic cohomology inherits a product structure

$$HC^n(A) \times HC^p(B) \to HC^{n+p}(A \otimes B).$$

For $B = k$ the product by the standard generator of $HC^2(k)$ gives the map S: $HC^n(A) \to HC^{n+2}(A)$ appearing in the periodicity sequence.

One of the advantage of Definition 0′ is that many "concrete" situations give rise to an n-dimensional cycle. Here is an example (see Section 10 for some others).

5.5. n-summable Fredholm operators [C1]. Let A be a (not necessarily commutative) $\boldsymbol{Z}/2$-graded algebra over $\boldsymbol{C}$. An n-summable Fredholm operator F on the graded A-Hilbert space $H = H^+ \oplus H^-$ is an element in $L(H)$ (bounded operators in H) such that $F^2 = 1$, $F\varepsilon = -\varepsilon F$ (where $\varepsilon|_{H^+} = 1$, $\varepsilon|_{H^-} = -1$), $[F, a] \equiv Fa - (-1)^{\deg a} aF \in L^n(H)$ where $L^n(H)$ is the Schatten ideal $\{T \in L(H) | \operatorname{Tr}(|T|^n) < \infty\}$ and $a \in A$.

The trace extends by linearity to a linear functional on $L^1(H)$ and therefore the above data gives an element in $HC_0(A)$. For n greater than 1 there is defined an n-dimensional cycle over A as follows. Put $da = i[F, a]$ and let Ω^q be the linear span in $L^{n/q}(H)$ of the operators

$$(a_0 + \lambda 1)\, da_1 \ldots da_q, \qquad \lambda \in \boldsymbol{C}.$$

Put, for any ω, $d\omega = i[F, \omega]$ and for $\omega \in \Omega^n$, $\int \omega = \operatorname{Trace}(\varepsilon\omega)$. Then $(\Omega, d, \int)$ is an n-dimensional cycle over A defining an element in $HC^n(A)$.

Such a data comes naturally from elliptic operators on a manifold $V(A = C^\infty(V))$. In this case the Fredholm operator is p-summable for $p > \dim V$.

6. Operations

6.1. There is defined a product on cyclic cohomology [C2]

$$HC^n(A) \times HC^p(B) \to HC^{n+p}(A \otimes B).$$

It is constructed using Definition 0′ (see Section 5). It can be realized trough the Ext functors (cf. Definition 3) as a particular case of the product

$$\operatorname{Ext}^n_{\mathcal{K}}(M, N) \times \operatorname{Ext}^p_{\mathcal{K}}(M', N') \to \operatorname{Ext}^{n+p}_{\mathcal{K}}(M \otimes M', N \otimes N').$$

When $B = k$ the product by the generator $u \in HC^2(k)$ gives the periodicity map δ.

Translated into homology this gives a coproduct [B3, Ka]

$$HC_n(A \otimes B) \to \bigoplus_{p+q=n} HC_p(A) \otimes HC_q(B).$$

Hence, when $B = k$, $HC_*(A)$ becomes a comodule over $HC_*(k) = k[u]$.

There is also a slant product

$$HC^p(A \otimes B) \times HC_n(B) \to HC^{p-n}(A).$$

6.2. On cyclic homology there is defined a product with a shift of degree [L1, L–Q2] (of a different nature).

$$HC_n(A) \otimes HC_p(A') \xrightarrow{*} HC_{n+p+1}(A \otimes A').$$

With Definition 0 this product is defined by the formula

$$[x]*[y] = [x \cdot B(y)], \quad x \in C_n(A), \quad y \in C_p(A')$$

where the dot means shuffle product and the map B is the one described in Section 1.

When A is commutative it provides $HC_{*-1}(A)$ with a structure of graded commutative algebra. This product is compatible with the product on Deligne cohomology (cf. 3.3). In particular when A is smooth then $x \in HC_n(A)$ decomposes as

$$x = x' + x'', \quad x' \in \Omega_A^n / d\Omega_A^{n-1}, \quad x'' \in H_{DR}^{n-2}(A) \oplus H_{DR}^{n-4}(A) \oplus \dots$$

(and similarly $y = y' + y''$).

Then

$$x*y = x' \wedge dy' \in \Omega_A^{n+p+1} / d\Omega_A^{n+p},$$

the other component (in H_{DR}^*) being 0.

6.3. The exterior product induces λ-operations on the homology of the Lie algebra of matrices. By restriction to the primitive part (see Section 7) it gives λ-operations on cyclic homology. These operations can be computed explicitly if one uses Definition 0.

THEOREM [L–P]. *The λ-operations on $HC_n(A)$ are induced by the formula*

$$\lambda^k(a_0, \dots, a_n) = (-1)^{k-1} \sum_{l=0}^{k-1} \binom{n+k-1-l}{n} \sum_{g \in S_{n,l}} \operatorname{sgn}(g)(a_0, a_{g(1)}, \dots, a_{g(n)})$$

where $S_{n,l}$ is the subset of the permutation group S_n acting on $\{1, \dots, n\}$ made of the permutations which have l descents.

(For instance $g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 5 & 4 \end{pmatrix}$ has 2 descents: $2 \to 1$ and $5 \to 4$).

Among the operations generated by the λ-structure (γ^k, ψ^k, etc.) the simplest one is the involution

$$(a_0, \ldots, a_n) \mapsto (-1)^{n(n+1)/2}(a_0, a_n, a_{n-1}, \ldots, a_1).$$

This involution is more generally defined when A is non-commutative provided that there is given an (anti)-involution on A (see Section 11).

The γ^k-operations induce the so-called γ-filtration on $HC_*(A)$. There are other filtrations coming from the bicomplexes $\underline{C}(A)$, $\underline{B}(A)$ and the rank coming from the comparison with the homology of the Lie algebra. The comparison of all these filtrations is an active field of research.

6.4. Let D be a derivation of A (over k). Then Goodwillie [G1] proved that D operates trivially on $HC_*^{\mathrm{per}}(A)$. This is a kind of "Poincaré lemma" for cyclic homology. This result is quite interesting in the framework of differential graded algebras (A, d) because it implies that $HC_*^{\mathrm{per}}(A, d)$ depends only on A_0/dA_1 if k contains $\boldsymbol{Q}$ (see also Section 9).

7. Lie algebras of matrices

In this section k is a characteristic zero field and A is a (not necessarily commutative) associative algebra over k. The $n \times n$-matrices with coefficients in A form a Lie algebra $\underline{gl}_n(A)$. There is a stabilization map $\underline{gl}_n(A) \hookrightarrow \underline{gl}_{n+1}(A)$ (put zeros on the last row and column) and $\underline{gl}(A) = \bigcup_n \underline{gl}_n(A)$.

The homology $H_*(\underline{gl}(A), k)$ of the Lie algebra of matrices is not only a coalgebra (thanks to the diagonal map) but also an algebra (thanks to the direct sum $\oplus$). It is in fact a Hopf algebra. Therefore, by a well-known theorem of Milnor and Moore, this Hopf algebra is the envelopping algebra of its primitive part (an element x is primitive iff $\Delta x = x \otimes 1 + 1 \otimes x$). Note that this primitive part is a Lie algebra. In the case at hand the primitive part is a commutative Lie algebra and therefore the Hopf algebra is the graded symmetric algebra over the primitive part.

7.1. THEOREM [L–Q1, L–Q2, T]. *If k is a characteristic zero field there is a canonical isomorphism*

$$HC_{n-1}(A) = \operatorname{Prim} H_n(\underline{gl}(A), k).$$

As a result this theorem enables us to compute the homology of $\underline{gl}(A)$ by combining it with the computations of cyclic homology.

This theorem has an obvious formulation in cohomology theory:

$$HC^{n-1}(A) = \operatorname{Indec} H^n(\underline{gl}(A), k).$$

The proof involves a kind of "plus-construction" and relies on classical invariant theory.

The same method gives information on the stability of the homology of the Lie algebra of matrices.

7.2. THEOREM [L Q1, L Q2]. *Let k be a characteristic zero field and A a commutative k-algebra. The stabilization map*

$$H_i(\underline{gl}_n(A), k) \to H_i(\underline{gl}_{n+1}(A), k)$$

is an isomorphism for $i \leqslant n$ and for $i = n+1$ its cokernel is $\Omega_A^{n-1}/d\Omega_A^{n-2}$.

The first theorem is analogous to $K_n(A)\otimes \mathbf{Q} = \operatorname{Prim} H_n(GL(A), \mathbf{Q})$, where $K_n(A)$ is algebraic K-theory of A. This is the reason why Tsygan [T] calls $HC_{n-1}(A) = K_n^+(A)$ additive K-theory of A (note the shift of index). With this new notation the product on cyclic homology defined in 6.2 becomes more natural:

$$K_n^+(A) \times K_p^+(A) \to K_{n+p}^+(A).$$

In fact it has the same properties as the product in algebraic K-theory. Namely, the role of Milnor K-theory (product of elements in K_1) is played here by $\Omega_A^n/d\Omega_A^{n-1} \subset HC_n(A)$. Hence, the second theorem is the additive analogue of a theorem by Suslin for K_n.

If one replaces the Lie algebra of matrices by the Lie algebra of orthogonal (resp. symplectic) matrices over a ring with involution one is led to a similar theorem about the primitive part of the homology. However cyclic homology has to be replaced by skew dihedral homology (see Section 11).

In characteristic different from zero only few results are known. In low dimension it is immediate that $H_1(\underline{gl}(A), k) = HC_0(A)$ (the trace invariant) and one can show [B] [K-L] that $H_2(\underline{sl}(A), k) = HC_1(A)$.

8. Algebraic K-theory

In [C2] Connes showed (in the cohomological framework) that there is a nice map from the Grothendieck group $K_0(A)$ to $HC_{2n}(A)$. It is given as follows. Let e be an idempotent in $k\times k$ matrices $\mathfrak{M}_k(A)$, i.e. $e^2 = e$. The isomorphism classes $[e]$ generate $K_0(A)$. The element $e\otimes e\otimes \dots \otimes e$ is a cycle in $C_{2n}(\mathfrak{M}_k(A))$ and therefore defines an element in $HC_{2n}(\mathfrak{M}_k(A)) = HC_{2n}(A)$ (Morita invariance). This construction gives a map

$$ch_{2n}\colon\ K_0(A) \to HC_{2n}(A).$$

For $n = 0$, $ch_0[e] = \operatorname{Tr}(e)$.

In the odd case Connes performs a similar construction

$$ch_n\colon\ K_1(A) \to HC_{2n+1}(A)$$

as follows. Let u be an invertible matrix in A. Take

$$(u-1)\otimes(u^{-1}-1)\otimes \dots \otimes(u-1)\otimes(u^{-1}-1)\in C_{2n+1}(\mathfrak{M}_k(A)).$$

It is a cocycle and defines the element $ch_n[u]$ in $HC_{2n+1}(A)$. For $n=0$ and A commutative $HC_1(A)=\Omega_A^1/dA$ and

$$ch_0([u])=(\det u)^{-1}\,d(\det u)=d(\operatorname{Log}\det u).$$

To extend these maps to higher algebraic K-theory into maps

$$ch_n\colon\ K_i(A)\to HC_{i+2n}(A)$$

one uses the following composition [K2]

$$\begin{aligned} K_i(A)=\pi_i(BGL(A)^+)&\to H_i(GL(A),k)\\ &\to HC_{i+2n}(k[GL(A)])\to HC_{i+2n}(\mathfrak{M}(A))=HC_{i+2n}(A). \end{aligned}$$

The first map is the Hurewicz homomorphism, the second one is described in 4.5, the third one is induced by the natural map $k[GL(A)]\to\mathfrak{M}(A)$ and the last one is the Morita isomorphism.

In the cohomological framework this map from K-theory to cyclic homology defines for any $C\in HC^*(A)$ a map

$$\varphi_C\colon\ K_*(A)\to k.$$

As remarked above, $ch_0=K_1(A)\to HC_1(A)$ can be described using the logarithm (when it exists). This (together with the analogy Lie group – Lie algebra) suggests that for topological algebras (or more generally when there exists a logarithm) there is a strong relationship between $K_n(A)$ and $HC_{n-1}(A)$. This is illustrated by the following result announced by Goodwillie [G2]. Let A be a $\mathbf{Q}$-algebra and I a nilpotent ideal, then there is a rational isomorphism of relative theories

$$K_n(A,I)\otimes\mathbf{Q}\xrightarrow{\sim}HC_{n-1}(A,I).$$

In another situation, let A be a locally convex topological algebra, then there is a commutative diagram involving the periodicity exact sequence [C–K]

$$\begin{array}{ccccccccc} K^{\rm top}_{n+1}(A) & \to & K^{\rm rel}_n(A) & \to & K_n(A) & \to & K^{\rm top}_n(A) & \to & K^{\rm rel}_{n-1}(A)\\ \downarrow && \downarrow && \downarrow && \downarrow && \downarrow\\ HC_{n+1}(A) & \to & HC_{n-1}(A) & \to & H_n(A,A) & \to & HC_n(A) & \to & HC_{n-2}(A). \end{array}$$

As cyclic homology is "almost" Deligne cohomology one can also expect Chern classes from algebraic K-theory to Deligne cohomology. They do exist and give rise to the "regulator maps" [K3, K4] (see also C. Soulé, Régulateurs, Séminaire Bourbaki, Fév. 85, for a survey on regulators).

In the comparison between $K_n(A)$ and $HC_{n-1}(A)$ it seems that the logarithm is not sufficient. In [B] a new function appeared, the dilogarithm $(\sum_n z^n/n^2)$, which is mainly involved in K_3. Recent work of Beilinson indicates that polylogarithms $(\sum_n z^n/n^k)$ intervene (essentially in K_{2k-1}).

9. Differential graded algebras

As usual when we have a homology functor defined on the category of k-algebras, we can extend it to schemes (sheafification), to simplicial algebras and to differential graded algebras.

This last case has been worked out in detail by Burghelea in [B1] where it is shown that these new functors are related to the algebraic K-theory of spaces (in the sense of Waldhausen) and to equivariant cohomology. As a result it gives new computations of algebraic K-theory of spaces.

Let (A, d) be a differential graded algebra (DGA) over a characteristic zero field. Definition 0 of cyclic homology can be extended to a DGA by constructing a bicomplex $C_p(A_q)$ whose horizontal differential is of b-type (Hochschild) and vertical differential is induced by d. The homology of the total complex is $HC_*(A, d)$.

Let X be a simply connected simplicial set and let ΩX be the Kan simplicial group which models the loop space in the simplicial framework. Then $\boldsymbol{Q}[\Omega X]$ is a DGA.

On the other hand from X (or its geometric realisation) Waldhausen has defined the reduced algebraic K-theory $\tilde{K}(X)$.

The generalization of the comparison theorem between cyclic homology and the Lie algebra of matrices (see Section 7) gives the following.

9.1. THEOREM [B1]. *For any simply connected space X there is a rational isomorphism of reduced theories*

$$\tilde{K}_*(X)\otimes \boldsymbol{Q} = \tilde{HC}_{*-1}(\boldsymbol{Q}[\Omega X]).$$

This theorem together with the computation of cyclic homology for a tensor algebra (see 4.3) permits us to compute $\tilde{K}_*(X)\otimes\boldsymbol{Q}$ completely from the minimal model of X, a result originally found by W. C. Hsiang and R. Staffeldt [H–S].

We have seen in 4.5 that cyclic homology of a group algebra can be computed from the homology of some spaces. A similar computation was achieved by Goodwillie and by Burghelea for the $\boldsymbol{Q}$-differentiable graded algebra $C_*(\Omega X)$ for a connected space X in terms of the free loop space X^{S^1}.

9.2. THEOREM [B–F] [G1]. *For a connected space X*

$$HC_*(C_*(\Omega X)) = H_*(X^{S^1}\times_{S^1} ES^1, \boldsymbol{Q}).$$

The periodicity exact sequence identifies to the Gysin sequence of the fibration

$$X^{S^1} \to X^{S^1} \times_{S^1} ES^1 \to BS^1.$$

The combination of these two theorems gives a computation of the algebraic K-theory of simply connected spaces in terms of S^1-equivariant homology (thus avoiding any cyclic homology).

The articles [B–F] and [V–B] contain some explicit computations of cyclic homology for suspension, product of Eilenberg–MacLane spaces, projective spaces, etc.

9.3. THEOREM [G1]. *Rationally periodic cyclic homology depends only on the fundamental group of the space*:

$$HC_*^{\text{per}}(C_*(\Omega X))\otimes Q = HC_*^{\text{per}}(Z[\pi_1 X])\otimes Q.$$

This is because derivations act trivially on periodic cyclic homology (see Section 6).

10. C^*-algebras and foliations [C2, C4]

10.1. Continuous cyclic homology. When A is a C^*-algebra (or, more generally, a locally convex topological algebra) it would be desirable to take into account this topology in the definition of cyclic homology. This project is easier to perform in the cohomological framework as it suffices to take only those cocycles which are continuous. In this framework Connes is able to prove the theorems we already mentionned for "algebraic" cyclic (co)-homology: periodicity exact sequence, Morita invariance, etc. In particular, for a manifold we have

THEOREM. *Let V be a differentiable manifold. Then*

$$HC^*_{\text{per}}(C^\infty(V)) = H_*(V, C).$$

10.2. Foliations and action of discrete groups. To any manifold V supporting a foliation F there is associated a C^*-algebra $C^*(V, F)$ which is, in general, non-commutative. When the foliation comes from a submersion then $C^*(V, F)$ is equivalent to the study of the algebra of differentiable functions over the space of leaves V/F (this is a commutative algebra). Therefore the characteristic classes can be read off from the de Rham homology of V/F. When this is not the case unbounded cyclic cohomology of $C^*(V, F)$ replaces de Rham homology of the (bad) space V/F.

Similarly, let Γ be a discrete group acting on a manifold W. Let $C_0(W)$ be the C^*-algebra of continuous functions on W vanishing at infinity. The C^*-algebra crossed product is denoted $C_0(W)\rtimes\Gamma$. Connes has constructed a basic map

$$\mu\colon K_{*,\tau}(W\times_\Gamma E\Gamma)\to K_*(C_0(W)\rtimes\Gamma)$$

where $K_{*,\tau}$ is twisted topological K-homology. The main conjecture is:

μ is an isomorphism.

In fact we are more interested in the injectivity of μ (rational injectivity of μ implies the Novikov conjecture on homotopy invariance of higher signatures). In order to test this injectivity one can use the maps

$$\varphi_C\colon K_*(C_0(W)\rtimes\Gamma)\to \mathbf{C}$$

induced by $C\in HC^*(C_0(W)\rtimes\Gamma)$ (cf. Section 8). Then there is a commutative diagram

$$\begin{tikzcd}
K_{*,\tau}(W\times_\Gamma E\Gamma) \arrow[r, "\mu"] \arrow[d, "ch_*"] & K_*(C_0(W)\rtimes\Gamma) \arrow[d, "\varphi_C"] \\
H_*(W\times_\Gamma E\Gamma, \mathbf{R}) \arrow[r, "\langle\omega_C, -\rangle"] & \mathbf{C}
\end{tikzcd}$$

where ch_* is the Chern character and $\omega_C\in H^*(W\times_\Gamma E\Gamma, \mathbf{R})$ is the element deduced from C.

The injectivity problem reduces to know if all $\omega\in H^*(W\times_\Gamma E\Gamma, \mathbf{R})$ come from cyclic cohomology (cf. [C4] for precise results).

In the foliation framework if D is a differential operator on V which is elliptic along the leaves there are defined a topological index $\mathrm{Ind}_t(D)$ and an analytic index $\mathrm{Ind}_a(D)$. For any $C\in HC^*(C^*(V, F))$ Connes [C4] proved the following index formula

$$\langle\omega_C, ch_*(\mathrm{Ind}_t D)\rangle = \varphi_C(\mathrm{Ind}_a D)$$

which reduces to the Atiyah–Singer index formula when the foliation comes from a submersion.

11. Dihedral and quaternion homologies [L2]

Let A be a k-algebra equipped with an involution $a\mapsto\bar a$ such that $\overline{ab}=\bar b\bar a$. Then $HC_n(A)$ inherits an involution. Therefore, when $1/2\in k$, HC_n splits into $HC_n^+\oplus HC_n^-$. This splitting is compatible with the product structure defined in 6.2.

Dealing with Definition 0 the involution y on $C_*(A)$ is induced by

$$y(a_0, a_1,\ldots, a_n) = (-1)^{n(n+1)/2}(\bar a_0, \bar a_n, \bar a_{n-1},\ldots, \bar a_1).$$

In fact, if we kill the action of t and y on the Hochschild complex then b is still well defined and the homology of the resulting complex is (when k contains $\mathbf{Q}$) denoted $HD_n(A)$ and called *dihedral homology*. The reason is that

t and y define an action of the dihedral group D_{n+1} on $A^{\otimes n+1}$ from which we took the coinvariants. It is immediate to verify that $HD_n(A) = HC_n^+(A)$ in this case.

In Definitions 1 and 2 for cyclic homology we took advantage of the fact that the trivial Z/n-module admits a periodic resolution. This is no longer true for D_n. However, it is true if we take the quaternion group Q_n whose homology is periodic of period 4. A particular resolution of the Q_n-module Z is thus used to give definitions of type 1 and 2 for *quaternion homology* $HQ_n(A)$ without any restriction on the characteristic.

As Q_n is a central extension of D_n with kernel $Z/2$, HQ_n and HD_n differ only by 2-torsion. In particular, if $1/2 \in k$, then

$$HQ_n = HD_n = HC_n^+ .$$

The existence of a small category ΔQ (resp. ΔD) containing Δ and the quaternion (resp. dihedral) groups mimicking ΔC (see Section 12) permits us to generalize Definition 3:

$$HQ_n(A) = \mathrm{Tor}_n^{\Delta Q}(k, A^\natural), \qquad HQ^n(A) = \mathrm{Ext}^n_{\Delta Q}(k, (A^\natural)^*),$$

$$HD_n(A) = \mathrm{Tor}_n^{\Delta D}(k, A^\natural), \qquad HD^n(A) = \mathrm{Ext}^n_{\Delta D}(k, (A^\natural)^*).$$

The 4-periodicity of the homology of Q_n gives rise to an exact sequence

$$\ldots \to HT_n(A) \to HQ_n(A) \to HQ_{n-4}(A) \to HT_{n-1}(A) \to \ldots$$

where the intermediate theory $HT_*(A)$ can be computed in terms of Hochschild homology and the involution on it. For instance, if $1/2 \in k$ there is an exact sequence

$$\ldots H_n^+(A, A) \to HT_n(A) \to H_{n-2}^-(A, A) \to H_{n-1}^+(A, A) \to \ldots$$

When $A = k$ (and of course trivial involution) $HQ_n(k)$ is periodic of period 4: $HQ_{4n}(k) = k$, $HQ_{4n+1} = k/2k$, $HQ_{4n+2} =$ 2-torsion of k, $HQ_{4n+3} = 0$.

If one replaces the action of y by $-y$, then one obtains *skew quaternion homology* ${}_{-1}HQ_*(A)$ and *skew dihedral homology* ${}_{-1}HD_*(A)$. If 2 is invertible there are both isomorphic to $HC_*^-(A)$.

Most of the properties and computations of cyclic homology can be carried over to dihedral and quaternion homology. One of the most interesting is the comparison with the homology of the Lie algebra of orthogonal (i.e. skew-symmetric) matrices $\underline{o}(A)$ and the Lie algebra of symplectic matrices $\underline{sp}(A)$.

THEOREM [J.-L. Loday and C. Procesi, also announced by J. Mc Iver]. *If k is a characteristic zero field and A a k-algebra with involution, then*

$${}_{-1}HD_n(A) = \mathrm{Prim}\, H_n(\underline{o}(A), k) = \mathrm{Prim}\, H_n(\underline{sp}(A), k).$$

Quaternion homology of group algebras decomposes as a sum of homology groups of spaces having only π_1 and π_2 (see [L2, § 4]).

Extension to DGA and comparison with hermitian K-theory is a current field of research.

12. Cyclic sets, Tor and Ext functors. Generalization

12.1. Cyclic sets. As mentionned in Sections 1 and 5 cyclic (co)-homology can be defined as a derived functor [C3]:

$$HC_n(A) = \mathrm{Tor}_n^{\Delta C}(k, A^\natural), \qquad HC^n(A) = \mathrm{Ext}^n_{\Delta C}(k, (A^\natural)^*).$$

The involved abelian category is the category of cyclic modules. A *cyclic module* is a functor from a small category ΔC^{op} (denoted Λ in [C3]) to the category of k-modules where the opposite category ΔC is characterized as follows:

(a) The objects of ΔC are $[n]$, $n \geqslant 0$.

(b) ΔC contains the simplicial category Δ (category of non decreasing maps $[n] = \{0 < 1 < \ldots < n\} \to [m] = \{0 < 1 < \ldots < m\}$) as a subcategory.

(c) Any morphism in ΔC can be written uniquely as a composite $\varphi \circ g$, $\varphi \in \mathrm{Hom}_\Delta([n], [m])$ and $g \in \mathrm{Aut}_{\Delta C}([n])$ for some m and n.

(d) $\mathrm{Aut}_{\Delta C}[n] = \mathbf{Z}/n+1$ for $n \geqslant 0$.

The paradigm of cyclic modules is $A^\natural$: $\Delta C^{\mathrm{op}} \to (k\text{–Mod})$ given by $[n] \mapsto A^{\otimes n+1}$. The action of the face (resp. degeneracy) operator δ_i (resp. σ_j) of Δ is:

$$(\delta_i)^*(a_0, \ldots, a_n) = (a_0, \ldots, a_i a_{i+1}, \ldots, a_n) \quad \text{for } 0 \leqslant i < n,$$

$$(\delta_n)^*(a_0, \ldots, a_n) = (a_n a_0, a_1, \ldots, a_{n-1}),$$

$$(\sigma_j)^*(a_0, \ldots, a_n) = (a_0, \ldots, a_j, 1, a_{j+1}, \ldots, a_n) \quad \text{for } 0 \leqslant j \leqslant n.$$

The action of the cyclic generator t of $\mathrm{Aut}_{\Delta C}([n])$ is

$$t(a_0, \ldots, a_n) = (a_n, a_0, \ldots, a_{n-1}).$$

The category of cyclic modules is abelian and has enough projectives [Ka]. The derived functors of $\mathrm{Hom}_{\Delta C}$ (resp. $\otimes_{\Delta C}$) are the functors

$$\mathrm{Ext}^n_{\Delta C} \text{ (resp. } \mathrm{Tor}_n^{\Delta C}).$$

The proof of the equivalence between Definitions 2 and 3 (of Section 1) results from the choice of a particular biresolution of the trivial cyclic module k. This biresolution is a mixture of a periodic resolution for the cyclic groups and a classical resolution for the category Δ [C3].

The category ΔC leads naturally to the notion of cyclic set, i.e. a functor X: $\Delta^{\mathrm{op}} \to$ (Sets). There is a well defined geometric realization for such a functor, denoted $|X|^{\Delta C}$. If we restrict X to Δ^{op} this gives a simplicial set whose geometric realization is denoted $|X|$ instead of $|X|^\Delta$.

THEOREM [C3] [B–F]. *Any cyclic set* $X: \Delta C^{op} \to (\text{sets})$ *determines a homotopy fibration* $|X| \to |X|^{\natural C} \to BS^1$.

The main point is that the classifying space of the category ΔC is homotopy equivalent to $BS^1 = B(SO(2))$.

When A is a group algebra the cyclic module $A^{\natural}$ is (as a functor) the composite of a cyclic set and of the functor which assigns to a set a free k-module. This permits us to prove that, in this case, $HC_*(A)$ is the homology of the classifying space of the cyclic set [B–F] (cf. 4.5).

THEOREM [B3] [G1]. *Let* X *be a cyclic set and* $|X|^{\natural C}$ *its geometric realization. There is an action of* S^1 *on* $|X|$ *and the homology of* $|X|^{\natural C}$ *is the equivariant homology of the free loop space* $|X|^{S^1}$ *over* $|X|$:

$$H_*(|X|^{\natural C}) = H_*(|X|^{S^1} \times_{S^1} ES^1).$$

More generally in terms of homotopy theory cyclic sets form a good model for S^1-spaces.

THEOREM [DHK]. *The category of cyclic sets is a closed model category in the sense of Quillen and is equivalent to the closed model category of* S^1-*spaces.*

12.2. Crossed simplicial groups [F–L]. Most of the properties of cyclic sets and cyclic modules depends on the properties (a), (b) and (c) of the category ΔC. It is therefore quite reasonable to make the following

DEFINITION. A sequence of groups $\{G_n\}$ $n \geqslant 0$ is a *crossed simplicial group* if it is equipped with the following structure. There is a small category ΔG such that

(a) Objects of ΔG are $[n]$, $n \geqslant 0$.

(b) ΔG contains Δ as a subcategory.

(c) Any morphism in ΔG can be uniquely written as a composite $\varphi \circ g$ where $\varphi \in \mathrm{Hom}_\Delta([m], [n])$ and $g \in \mathrm{Aut}_{\Delta G}[n]$.

(d) $\mathrm{Aut}_{\Delta G}[n] = G_n^{op}$ (opposite group of G_n) $n \geqslant 0$.

Then $\{G_n\}$ is a simplicial set but not necessarily a simplicial group.

EXAMPLE 1. Take $G_n = \{1\}$, then $\Delta G = \Delta$.

EXAMPLE 2. Let G be a simplicial group. Then $\{G_n\}$ is a crossed simplicial group, the composition in ΔG being

$$(\varphi \circ g) \circ (\psi \circ h) = (\psi^*(g)\, h) \circ (\varphi \circ \psi).$$

EXAMPLE 3. Take $G_n = \boldsymbol{Z}/n+1$, then $\Delta G = \Delta C$. This category is described in [C3] in terms of homotopy classes of maps from S^1 to S^1 sending $\boldsymbol{Z}/n+1$ to itself. Here is a presentation of ΔC^{op} by generators and relations. The generators are those of Δ^{op}, i.e. d_i and s_j, and $t \in \boldsymbol{Z}/n+1$ (one for each n). The relations are the standard relations of Δ, the relation $t^{n+1} = 1$ (of $Z/n+1$) and

(1) $$d_i t = t d_{i-1}, \quad 1 \leqslant i \leqslant n,$$

(2) $$s_j t = t s_{j-1}, \quad 1 \leqslant j \leqslant n.$$

One can show that this implies

$$d_0 t = d_n, \quad s_0 t = t^2 s_n.$$

The geometric realization of the simplicial set $\{Z/n+1\}$ is homotopy equivalent to $SO(2) = S^1$.

EXAMPLE 4. Take $G_n = D_{n+1}$ (resp. Q_{n+1}). Then there exists a small category ΔD (resp. ΔQ) which makes $\{G_n\}$ into a crossed simplicial group. The category ΔQ^{op} is presented by the generators of Δ and the generators t, y of Q_{n+1}. The relations are those of Δ, those of Q_{n+1} (i.e. $t^{n+1} = y^2$, $yty^{-1} = t^{-1}$), the relations (1) and (2) above and

$$d_i y = y d_{n-i}, \; 0 \leqslant i \leqslant n,$$

$$s_j y = y s_{n-j}, \; 0 \leqslant j \leqslant n.$$

ΔD^{op} has a similar presentation.

The geometric realization of the simplicial set $\{D_{n+1}\}$ (resp. $\{Q_{n+1}\}$) is homotopy equivalent to the group $O(2)$ (resp. the normalizer of S^1 in the quaternion group $SU(2) = S^3$).

EXAMPLE 5. The family of symmetric groups $\{S_{n+1}\}$ (resp. braid groups $\{B_{n+1}\}$) is a crossed simplicial group [F–L].

Most of the properties of cyclic sets can be generalized to ΔG^{op}-sets. In particular this is true for the three theorems quoted above. Also Tor and Ext functors can be defined for ΔG^{op}-modules. As $A^{\natural}$ is a ΔD^{op}-module (and therefore ΔQ^{op}-module) the derived functors give dihedral and quaternion (co)-homology of A (see Section 11).

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