NON-HILBERTIAN STRUCTURE OF THE WIENER MEASURE

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Let C be the space of all real-valued continuous functions on the interval $[0, 1]$ vanishing at the origin provided with the topology induced by the uniform convergence. Let μ be a Borel probability measure on C. We say that μ is *hilbertian* if C contains a μ-measurable subset H satisfying the following conditions:

(i) H is an inner product space under a μ-measurable inner product $(x, y)_H$;

(ii) the convergence induced by this inner product implies the uniform convergence in H;

(iii) $\mu(H) > 0$.

This notion was introduced by K. Urbanik.

Recently, S. Kwapień proved that *the Wiener measure w on C is not hilbertian* (an oral communication). His proof is based upon the Schwartz theory of radonifying operators. The aim of the present note is to give another simple proof of this result.

Contrary to our statement, let us suppose that the Wiener measure w is hilbertian. Let H be the inner product space satisfying conditions (i), (ii) and (iii). Then $w(H) = 1$ (see [3], p. 92). Moreover, the space H contains the space C' consisting of the absolutely continuous functions in C with square-integrable derivative (see [3], p. 91). Further, by Smolyanov's theorem (see [2], Theorem 4.1) we can assume that

$$(x, y)_H = \int_0^1 \int_0^1 K(t, u) \, dx(t) \, dy(u),$$

where the integral is taken in the sense of Ito and Wiener and the kernel K is square-integrable over $[0, 1] \times [0, 1]$. Moreover, for every pair $x, y \in C'$, we have the formula

$$(1) \qquad (x, y)_H = \int_0^1 \int_0^1 K(t, u) x'(t) y'(u) \, dt \, du.$$

Given $x \epsilon C'$ and $s \epsilon [0, 1]$, by $_s x$ we denote the function from C' defined by the condition $(_s x)'(t) = x'(t+s)$, where $t+s$ is taken mod 1. From (1) it follows that $(_s x, _s x)_H$ is a continuous function of s. Moreover,

$$(2) \qquad \int_0^1 (_s x, _s x)_H \, ds = \int_0^1 \int_0^1 T(t-u) x'(t) x'(u) \, dt \, du,$$

where $T(t-u) = \int_0^1 K(t-s, u-s) \, ds$ and $t-s$, $u-s$ are taken mod 1.

It is clear that the kernel T is positive-definite. By a theorem of F. Riesz (see [1], p. 209), the kernel is almost everywhere equal to a continuous function. Consequently, without loss of generality, we can assume that T is continuous itself. Setting

$$x'_n(t) = \begin{cases} n & \text{if } \dfrac{1}{2} - \dfrac{1}{n} < x \leqslant \dfrac{1}{2}, \\[2mm] -n & \text{if } \dfrac{1}{2} < x \leqslant \dfrac{1}{2} + \dfrac{1}{n}, \\[2mm] 0 & \text{otherwise}, \end{cases}$$

we get, by (2), the relation $\lim\limits_{n \to \infty} \int_0^1 (_s x_n, _s x_n)_H \, ds = 0$. Hence, it follows that, for almost all s, $\lim\limits_{n \to \infty} (_s x_n, _s x_n)_H = 0$.

But, on the other hand, for all s $(0 \leqslant s \leqslant 1)$,

$$\lim_{n \to \infty} \max_{0 \leqslant t \leqslant 1} |_s x_n(t)| = 1,$$

which contradicts the assumption that the convergence in H implies the uniform convergence. The theorem is thus proved.

REFERENCES

[1] M. Loève, *Probability theory*, New York 1955.
[2] O. Г. Смолянов, *Об измеримых полилинейных и степенных функционалах в некоторых линейных пространствах с мерой*, Доклады Академии Наук СССР 170 (1966), p. 526-529.
[3] Г. Е. Шилов и Фан Дык Тинь, *Интеграл, мера и производная на линейных пространствах*, Москва 1967.