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Boolean powers as algebras of continuous functions

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Introduction

It has variously been noted that the Boolean powers, bounded and unbounded, of A. L. Foster may be described in terms of suitable continuous functions, i.e. that the bounded power of an algebra A by a Boolean algebra B is essentially the algebra of continuous functions on the Stone space ΩB of B with values in the discrete algebra A, operations defined pointwise, and that the Boolean power of A by a complete Boolean algebra B can be obtained somewhat analogously by using the continuous functions on dense open sets of ΩB with values in the discrete algebra A. For bounded Boolean powers, this seems first to have been remarked on by Jónsson [33], and both cases are treated by Ribenboim [44], but there only for sets rather than for algebras. For unbounded Boolean powers of arbitrary structures (involving relations as well as operations) this observation also occurs in Daigneault [17]. In addition, either case, or both, have been mentioned by Comer [16], Davey [19], Higgs [30], Quackenbush [43], Volger [51], and possibly others. Despite all this, these natural and convenient descriptions of Boolean powers have never actually been used as a basis for their study; for instance, Ribenboim, in characterizing certain lattice-ordered groups as Boolean powers of the integers in the second part of [44], employs their original definition by Foster, and Daigneault [17], who constructs elementary extensions by means of Boolean powers, treats the subject via polyadic algebras.

The object of this paper is to present a systematic study of Boolean powers in terms of their topological description. It is our belief that this approach leads to a substantial simplification of the known theory as well as interesting new results, and places the subject into a much more transparent conceptual framework.

The paper begins with a detailed exposition of the connection between the two different views of Boolean powers (Section 1). This is followed by a discussion of the functors which arise from associating with each topological space X and each algebra A either the algebra $C(X, A)$ of all continuous A-valued functions on X or the algebra $E(X, A)$ which is the quotient of the algebra of all A-valued functions of X continuous on some dense open set modulo coincidence on some dense open set (Section 2).

Among other things, we show, for any continuous map $f\colon X \to Y$ that the associated homomorphism $C(f, A)\colon C(Y, A) \to C(X, A)$ is a pure embedding whenever it is an embedding, thus establishing a similarity between these $C(f, A)$ and homomorphisms of Boolean algebras, and we characterize the contravariant functors, on the category of Boolean spaces, of the form $C(-, A)$. Next, we give a detailed account of the congruences on $C(X, A)$ and $E(X, A)$ which naturally arise from filters of open-closed, or regular open, sets in X, respectively (Section 3). In particular, we characterize the algebras for which all congruences are obtained in this fashion. As an application we obtain a wide class of algebras A for which some Boolean power fails to be embeddable into a power of A. In Section 4, we investigate to what extent the exponent law $(A^Y)^X \simeq A^{X \times Y}$, for sets X and Y and any algebra A, carries over to the functors $C(-, A)$, $E(-, A)$, and the Boolean filter powers $E_{\mathfrak{F}}(X, A)$, given by taking $E(X, A)$ modulo the congruence associated with any filter $\mathfrak{F}$ of regular open sets in X. In particular, we characterize, for any algebra A, the spaces X for which the exponent law holds for $E(-, A)$ with respect to any discrete Y, and obtain a restricted exponent law for Boolean ultrapowers.

Following this, we deal with the functor $C(-, A)$ in the special case of finite A, with particular attention to quasi-primal A, equationally compact algebras in the equational class generated by A, and primal A, including a short proof of the basic results of Hu [31], [32] for the latter (Section 5). In the next section, we first provide a Łoś type result concerning the first order properties of Boolean ultrapowers and then present a proof of Mansfield's [38] characterization of elementary equivalence in terms of the latter (Section 6). In the final section, we obtain the preservation of elementary equivalence and elementary embeddings by various different functors arising from $C(X, A)$ and $E(X, A)$ and characterize those Boolean spaces X for which $C(X, A)$ and $E(X, A)$ are elementarily equivalent for all A.

Concerning general background, we refer the reader to Bourbaki [10] for topology, Chang–Keisler [15] for model theory, Grätzer [28] for universal algebra, and MacLane [36] for category theory. Further, the basic facts concerning equational compactness and purity may be found in Mycielski [39] and Węglorz [53], respectively, or alternatively in Banaschewski and Nelson [7]. Also, we adopt the general convention that all algebras considered are of finitary type.

Finally, we wish to thank Paul Bankston for some help and advice, Walter Taylor for his encouraging interest, and the National Research Council of Canada for its generous financial support without which this work could not have been undertaken. Also, we gratefully acknowledge Stan Burris' paper on Boolean powers [12] as the main source of motivation for this study.

1. Fundamentals

For any topological space X, $\mathfrak{Q}X$ will be the Boolean algebra of open-closed subsets of X. Note that the correspondence $X \rightsquigarrow \mathfrak{Q}X$ is the object part of a contravariant functor $\mathfrak{Q}$ from the category TOP of all topological spaces and continuous maps to the category Boo of all Boolean algebras and their homomorphisms, $\mathfrak{Q}f$: $\mathfrak{Q}Y \to \mathfrak{Q}X$ given by $U \rightsquigarrow f^{-1}(U)$, for any continuous map f: $X \to Y$. On the other hand, one has the contravariant functor Ω: Boo $\to$ TOP which associates with each Boolean algebra A its ultrafilter, or Stone, space ΩA and with each homomorphism h: $A \to B$ of Boolean algebras the continuous map Ωh: $\Omega B \to \Omega A$, given by $P \rightsquigarrow h^{-1}(P)$. Here the points of ΩA are the ultrafiltres $P \subseteq A$ and the topology is given by the basic open sets $\{P | a \in P \in \Omega A\}$ for $a \in A$.

The contravariant functors $\mathfrak{Q}$: TOP $\to$ Boo and Ω: Boo $\to$ TOP are adjoint to each other on the right, and the spaces ΩA are exactly the Boolean (= zero-dimensional compact Hausdorff) spaces; moreover, restricted to the full subcategory BooS of TOP given by the latter, $\mathfrak{Q}$ and Ω provide the familiar dual equivalence due to M. H. Stone, i.e. Stone Duality. (Stone [47].) We shall not distinguish notationally between the full and the restricted versions of these functors.

Another Boolean algebra associated with any topological space X is $\mathfrak{R}X$, the Boolean algebra of all regular open subsets of X, i.e. those subsets U for which $U = I\Gamma U$, where I and Γ are the interior and closure operator, respectively, on the space X. We recall the following familiar facts concerning $\mathfrak{R}X$ (Sikorski [46], Halmos [29], Banaschewski–Bruns [5]): $\mathfrak{R}X$ is complete, the meet given by $\bigwedge U_i = I \bigcap U_i$ and the join by $\bigvee U_i = I\Gamma \bigcup U_i$. The complementation in $\mathfrak{R}X$ is $U \rightsquigarrow ICU$ (C for set complementation). For Boolean X, $\mathfrak{R}X$ is the completion of $\mathfrak{Q}X$ as well as the injective hull of $\mathfrak{Q}X$. Also, for Boolean X, $\mathfrak{R}X = \mathfrak{Q}X$ iff X is extremally disconnected, i.e. closures of open sets in X are open.

For a topological space X and an algebra A, $C(X, A)$ is the algebra of all functions from X into A continuous with respect to the discrete topology on A, with pointwise defined operations. It should be noted that, for any A, $C(X, A)$ only depends on the component space Y of A, i.e. $C(X, A) \simeq C(Y, A)$. Also, if X is discrete, then $C(X, A) \simeq A^X$.

In the special case where A is the two-element Boolean algebra $\mathbf{2}$, one evidently has $C(X, \mathbf{2}) \simeq \mathfrak{Q}X$ for all spaces X, and thus also, from Stone Duality, $C(\Omega B, \mathbf{2}) \simeq B$, for all Boolean algebras B.

For compact X, each $u \in C(X, A)$ has finite image, and so determines a function $\hat{u}$: $A \to \mathfrak{Q}X$, $\hat{u}(a) = u^{-1}\{a\}$, with the following properties:

(1) $\hat{u}$ has finite support, i.e., maps all but finitely many elements of A to $\emptyset$.

(2) $\bigvee Im\hat{u} = X$.

(3) For $a \neq b$ in A, $\hat{u}(a) \cap \hat{u}(b) = \emptyset$.

In view of the fact that every function from X into A which is constant on the blocks of a partition of X into open-closed sets is continuous, the map $u \rightsquigarrow \hat{u}$ produces a one-to-one correspondence between $C(X, A)$ and the set of all functions $A \to \Omega X$ with the above three properties.

Moreover, if λ is one of the fundamental operations of the algebra A (of arity n, say) and $u_1, \ldots, u_n \in C(X, A)$, then for each $a \in A$,

$$\begin{aligned} \lambda(u_1, \ldots, u_n)^{\wedge}(a) &= \lambda(u_1, \ldots, u_n)^{-1}\{a\} \\ &= \{x \in X \mid \lambda(u_1, \ldots, u_n)(x) = a\} \\ &= \{x \in X \mid \lambda(u_1(x), \ldots, u_n(x)) = a\} \\ &= \bigcup_{\lambda(a_1, \ldots, a_n) = a} \hat{u}_1(a_1) \cap \ldots \cap \hat{u}_n(a_n). \end{aligned}$$

Now recall (see Foster [23] or Grätzer [28]) that, for an algebra A and a Boolean algebra B, the *bounded Boolean power* (alias bounded Boolean extension) $A[B]^*$ of A by B is the algebra of all functions $u\colon A \to B$ of finite support such that $\bigvee \operatorname{Im} u = 1$ and $u(a) \wedge u(b) = 0$ for $a \neq b$ in A; with the operations given by the following formula, reminiscent of the definition of the convolution product in semi-group rings:

$$(*) \qquad \lambda(u_1, \ldots, u_n)(a) = \bigvee_{\lambda(a_1, \ldots, a_n) = a} u_1(a_1) \wedge \ldots \wedge u_n(a_n)$$

for each fundamental operation λ of arity n, and all $u_1, \ldots, u_n \in A[B]^*$, the join being meaningful since all the u_i have finite support.

The preceding discussion establishes the basic fact that, for any compact X, $C(X, A) \simeq A[\Omega X]^*$. Incidentally, this implies $C(X, A) \simeq C(\Omega\Omega X, A)$ for compact X, since $\Omega\Omega\Omega X \simeq \Omega X$.

There is another "algebra of continuous functions" associated with each topological space X and algebra A, namely $E(X, A) = D(X, A)/\sim$, where $D(X, A)$ is the algebra of all functions $X \to A$ continuous on some dense open subset of X, with pointwise operations, and $\sim$ is the equivalence relation on $D(X, A)$ identifying any two functions which agree on some dense open set. $D(X, A)$ is indeed a subalgebra of the set of all functions from X into A, and $\sim$ is a congruence, since all operations are finitary and finite intersections of dense open sets are dense open.

Alternatively, one has the direct limit (= updirected colimit) representation

$$E(X, A) = \varinjlim C(U, A) \qquad (U \text{ dense open in } X)$$

for the direct system determined by the restriction maps $C(U, A) \to C(V, A)$, for any dense open U and V such that $V \subseteq U$. Here the colimit maps $C(U, A) \to E(X, A)$ are given by first mapping $C(U, A)$ into $D(X, A)$ by arbitrarily extending the $u \in C(U, A)$ to functions on all of X and then passing to the quotient. It is clear that the resulting homomorphisms are embeddings, and $E(X, A)$ is the union of their images.

For the case $U = X$ this provides an embedding $C(X, A) \to E(X, A)$. We shall show later that this embedding is always pure, and elementary for extremally disconnected Boolean spaces X.

Note that, for discrete X and its Stone–Čech compactification βX, $E(\beta X, A) \simeq E(X, A) \simeq C(X, A) \simeq A^X$. Also, if A is finite and X extremally disconnected, then $C(X, A) = E(X, A)$ since any $u \in D(X, A)$ is $\sim$-congruent to $v \in C(X, A)$, where $v(x) = a$ iff $x \in \Gamma(u^{-1}\{a\} \cap Du)$.

If X is extremally disconnected Boolean, then $E(X, A)$ can also be described as an algebra consisting of continuous functions defined on all of X for infinite A in the following way: Let A^* be the extension of A obtained by adjoining a new element $*$, taken to be *absorbing* with respect to the operations, i.e. for any fundamental operation λ, of arity n, $\lambda(x_1, \ldots, x_n) = *$ whenever some $x_i = *$. Then, let $C_A(X, A^*)$ be the set of all those functions u: $X \to A^*$ which are continuous for the topology on A^* which makes $*$ the point at infinity for the one-point compactification of A viewed as a discrete space, and which are A-valued on some dense open set, i.e. $u^{-1}(A)$ is dense in X. Now, if λ is any fundamental operation of A, of arity n, and $u_1, \ldots, u_n \in C_A(X, A^*)$, then the function

$$x \rightsquigarrow \lambda(u_1(x), \ldots, u_n(x)), \qquad x \in W = u_1^{-1}(A) \cap \ldots \cap u_n^{-1}(A)$$

on the dense open set $W \subseteq X$ has an extension, necessarily unique, to a continuous function u: $X \to A^*$ since the topology on A^* is compact Hausdorff and $X = \beta W$, the latter by basic properties of extremally disconnected spaces (Gillman–Jerison [24], p. 86). Thus one may define the operation λ on $C_A(X, A^*)$ by putting $u = \lambda(u_1, \ldots, u_n)$. The resulting algebra $C_A(X, A^*)$ is isomorphic to $E(X, A)$, for if h_W: $C(W, A) \to C_A(X, A^*)$, $W \subseteq X$ dense open, is the map associating with each $u \in C(W, A)$ its continuous extension u^*: $X \to A^*$, then h_W is clearly a homomorphism, and these homomorphisms have the properties that characterize them as colimit maps for the diagram of the $C(W, A)$.

It should be noted that the operations of $C_A(X, A^*)$ are not defined pointwise, i.e. $C_A(X, A^*)$ is not a subalgebra of $(A^*)^X$. However, this will be the case for any A such that A^*, with the one-point compactification topology given above, is a *topological algebra*, and this holds whenever A satisfies the Finite Factorization Condition, i.e. for any $a \in A$ and any fundamental operation λ (arity n) of A, the set of all $x_1, \ldots, x_n$ such that $\lambda(x_1, \ldots, x_n) = a$ is finite (Banaschewski [4]).

For arbitrary X and A, the following considerations give a convenient description of the congruence $\sim$. For each function u: $X \to A$, let Du be the domain of continuity of u, i.e. $x \in Du$ iff u is continuous on some neighbourhood of x. Du is cleary an open subset of X, and $u \in D(X, A)$ iff Du is dense. Further, for $u, v \in D(X, A)$, $u \sim v$ iff $u|Du \cap Dv = v|Du \cap Dv$: the sufficiency is clear; on the other hand if $u|W = v|W$ for dense open W, then $u|Du \cap Dv \cap W = v|Du \cap Dv \cap W$ and so, in view of the fact that $Du \cap Dv \cap W$ is dense in $Du \cap Dv$, it follows that u and v coincide on $Du \cap Dv$.

Each function $u \in D(X, A)$ determines a function $\hat{u}$: $A \to \mathfrak{R}X$, such that $\bigvee Im\hat{u} = X$ and $\hat{u}(a) \cap \hat{u}(b) = \emptyset$ for $a \neq b$ in A; $\hat{u}$ is defined by $\hat{u}(a) = I\Gamma(u^{-1}\{a\} \cap Du)$. In view of the fact that for a function h: $A \to \mathfrak{R}X$, $\bigvee Imh = X$ iff $\bigcup Imh$ is dense, it follows that every function h: $A \to \mathfrak{R}X$ with $\bigvee Imh = X$ and $h(a) \cap h(b) = \emptyset$ for $a \neq b$ in A is equal to $\hat{u}$ for some $u \in D(X, A)$. Moreover, for u, $v \in D(X, A)$, $u \sim v$ iff $\hat{u} = \hat{v}$: if $u \sim v$, then $u|Du \cap Dv = v|Du \cap Dv$ and so for each $a \in A$, $\hat{u}(a) = I\Gamma(u^{-1}\{a\} \cap Du) = I\Gamma(u^{-1}\{a\} \cap Du \cap Dv) = I\Gamma(v^{-1}\{a\} \cap Du \cap Dv) = I\Gamma(v^{-1}\{a\} \cap Dv) = \hat{v}(a)$; on the other hand if $\hat{u} = \hat{v}$, then $I\Gamma(u^{-1}\{a\} \cap Du \cap Dv) = I\Gamma(v^{-1}\{a\} \cap \cap Du \cap Dv)$ for all $a \in A$ and consequently $u|Du \cap Dv = v|Du \cap Dv$.

If $u_1, \ldots, u_n \in D(X, A)$ and $U = Du_1 \cap \ldots \cap Du_n$, then for each fundamental operation λ of arity n,

$$\begin{aligned}\lambda(u_1, \ldots, u_n)\hat{}\,(a) &= I\Gamma\big(\lambda(u_1, \ldots, u_n)^{-1}\{a\} \cap U\big) \\ &= I\Gamma\{x \in U | \lambda(u_1(x), \ldots, u_n(x)) = a\} \\ &= I\Gamma \bigcup_{\lambda(a_1, \ldots, a_n) = a} u_1^{-1}\{a_1\} \cap \ldots \cap u_n^{-1}\{a_n\} \cap U \\ &= I\Gamma \bigcup_{\lambda(a_1, \ldots, a_n) = a} I\Gamma(u_1^{-1}\{a_1\} \cap \ldots \cap u_n^{-1}\{a_n\} \cap U) \\ &= I\Gamma \bigcup_{\lambda(a_1, \ldots, a_n) = a} I\Gamma(u_1^{-1}\{a_1\} \cap U) \cap \ldots \cap I\Gamma(u_n^{-1}\{a_n\} \cap U) \\ &= \bigvee_{\lambda(a_1, \ldots, a_n) = a} \hat{u}_1(a_1) \wedge \ldots \wedge \hat{u}_n(a_n).\end{aligned}$$

Now recall (Foster [23] or Grätzer [28]) that for an algebra A and a complete Boolean algebra B, the Boolean power of A by B, $A[B]$, is the set of all functions u: $A \to B$ with $\bigvee Imu = 1$ and $u(a) \wedge u(b) = 0$ for $a \neq b$ in A, with the fundamental operations defined by

$$\lambda(u_1, \ldots, u_n)(a) = \bigvee_{\lambda(a_1, \ldots, a_n) = a} u_1(a_1) \wedge \ldots \wedge u_n(a_n).$$

The preceding discussion has established that, for any topological space X, $E(X, A) = A[\mathfrak{R}X]$. In particular, if X is extremally disconnected and Boolean, then $E(X, A) = A[\mathfrak{Q}X]$ since $\mathfrak{Q}X = \mathfrak{R}X$ for such X. Moreover, for any X, there is an isomorphism $E(X, A) \to E(\Omega\mathfrak{R}X, A)$, on the basis of the natural isomorphism $\mathfrak{R}X \simeq \mathfrak{Q}\Omega\mathfrak{R}X = \mathfrak{R}\Omega\mathfrak{R}X$, explicitly given as follows: for any $u \in D(X, A)$, the open-closed sets $\Sigma_a \subseteq \Omega\mathfrak{R}X$, $a \in A$, which consist of all $\mathfrak{A}$ such that $\hat{u}(a) \in \mathfrak{A}$, are disjoint and have dense union, the latter by the meet continuity of Boolean algebras and the fact that $\bigvee \hat{u}(a)$ $(a \in A) = X$ in $\mathfrak{R}X$. Hence any u': $\Omega\mathfrak{R}X \to A$ such that $u'(\mathfrak{A}) = a$ for $\mathfrak{A} \in \Sigma_a$ belongs to $D(\Omega\mathfrak{R}X, A)$ and therefore determines an element $\bar{u} \in E(\Omega\mathfrak{R}X, A)$. The correspondence $u \rightsquigarrow \bar{u}$ is clearly an onto homomorphism; further, $u \sim v$ iff $\hat{u} \sim \hat{v}$ iff $u' \sim v'$ for any u, $v \in D(X, A)$, and thus $u \rightsquigarrow \bar{u}$ induces an isomorphism $E(X, A) \to E(\Omega\mathfrak{R}X, A)$.

We close this section with a comment on the significance of $E(X, A)$ in a special situation. For any Boolean algebra B, $\mathfrak{R}\Omega B \simeq C(\Omega\mathfrak{R}\Omega B, \mathbf{2}) \simeq E(\Omega\mathfrak{R}\Omega B, \mathbf{2}) \simeq E(\Omega B, \mathbf{2})$, and thus $E(\Omega B, \mathbf{2})$ represents the completion of $B \simeq \mathfrak{L}\Omega B$. This observation also follows from entirely different facts: by Brainerd–Lambek [9], the completion of B is the maximal ring of quotients of B, for B viewed as Boolean ring, and the construction of maximal rings of quotients given in Banaschewski [2], when applied to a Boolean ring B, exactly produces $E(\Omega B, \mathbf{2})$. Indeed, it follows from [2] that, more generally, $E(X, K)$ is the maximal ring of quotients of $C(X, K)$ for any Boolean space X and finite field K.

2. Functorial aspects

For any given algebra A, the correspondence $X \rightsquigarrow C(X, A)$ from topological spaces to algebras is actually the object part of a contravariant functor $C(-, A)$ from the category TOP to the category SP$\{A\}$ of all algebras embeddable into a power A^I of A and their homomorphisms, the algebra homomorphism $C(f, A)$: $C(Y, A) \to C(X, A)$ corresponding to a continuous map f: $X \to Y$ being $u \rightsquigarrow uf$. This functor $C(-, A)$ has an obvious adjoint on the right: for any algebra $B \in$ SP$\{A\}$, let $H(B, A)$ be the topological space whose points are the homomorphisms h: $B \to A$ and whose topology is obtained from the product topology of $A^{|B|}$, where A is given the discrete topology, $|B|$ being the underlying set of B. It is clear, then, that $H(B, A)$ is a *subspace* of $A^{|B|}$; in fact, this subspace is *closed*: since the operations of A are finitary, the condition that an $h \in A^{|B|}$ be a homomorphism can be described in terms of continuous maps from $A^{|B|} \to A$, and the latter is Hausdorff. Moreover, any homomorphism f: $B \to C$ determines a map $\bar{f}$: $H(C, A) \to H(B, A)$ such that $\bar{f}(h) = hf$ and $\bar{f}$ is in fact continuous since the maps $\bar{f}(h)(b) = h(f(b))$ on $H(C, A)$ are continuous for all $b \in B$. It is clear, then, that the correspondences $B \rightsquigarrow H(B, A)$ and $f \rightsquigarrow \bar{f}$ define a contravariant functor $H(-, A)$: SP$\{A\} \to$ TOP. Now, any continuous map f: $X \to H(B, A)$ determines a homomorphism f^*: $B \to C(X, A)$ such that $f^*(b)(x) = f(x)(b)$, since each $x \rightsquigarrow f(x)(b)$ is continuous, and each $b \rightsquigarrow f(x)(b)$ is a homomorphism. Analogously, any homomorphism h: $B \to C(X, A)$ determines a continuous map $h^\#$: $X \to H(B, A)$ such that $h^\#(x)(b) = h(b)(x)$ since each $b \rightsquigarrow h(b)(x)$ is a homomorphism, and each

$h(b)$ is continuous. It is obvious that $(f^*)^\# = f$ and $(g^\#)^* = g$ for all continuous $f\colon X \to H(B, A)$ and all homomorphisms $g\colon B \to C(X, A)$. Moreover, the required naturality conditions are obviously satisfied and hence the functor $H(-, A)$ is adjoint on the right to the functor $C(-, A)$.

In the following, we shall consistently assume that the algebra A is non-trivial for otherwise the functor $C(-, A)$ would be of no interest.

The fact that every embedding of Boolean algebras is pure is a special case of the following result:

PROPOSITION 1. *For a continuous $f\colon X \to Y$ the following are equivalent:*

(1) *$C(f, A)$ is a pure embedding,*

(2) *$C(f, A)$ is an embedding,*

(3) *$\mathrm{Im}(f)$ meets every open-closed subset of Y.*

Proof. (1)$\Rightarrow$(2) and (2)$\Rightarrow$(3) are trivial.

(3)$\Rightarrow$(1): Suppose S is a finite set of equations with constants $u_1, \ldots, u_n \in C(Y, A)$ and k variables, and that there are $v_1, \ldots, v_k \in C(X, A)$ with $p(u_1 f, \ldots, u_n f,\ v_1, \ldots, v_k) = q(u_1 f, \ldots, u_n f,\ v_1, \ldots, v_k)$ for all $(p, q) \in S$. Let $Y = \bigcup U_\alpha$ be a partition of Y into open blocks on which all the u_i are constant; then each U_α is open-closed and so there exists $x_\alpha \in X$ with $f(x_\alpha) \in U_\alpha$. But then $p(u_1, \ldots, u_n, \bar{v}_1, \ldots, \bar{v}_k) = q(u_1, \ldots, u_n, \bar{v}_1, \ldots, \bar{v}_k)$ for all $(p, q) \in S$, for the functions $\bar{v}_i\colon Y \to A$ which are constant on each U_α with value $v_i(x_\alpha)$, and this provides a solution of S in $C(Y, A)$, as required.

Since $E(X, A) = \varinjlim C(U, A)$ ($U \subseteq X$ dense open) and direct limits of pure embeddings are again such (Banaschewski [3]), we have the following

COROLLARY. *The natural embedding $C(X, A) \to E(X, A)$ is a pure embedding for any space X and any algebra A.*

Remark. Condition (3) of the above proposition is cleary equivalent with $C(f, \mathbf{2})$ being an embedding. This provides the first of many instances to be encountered in this paper, where $\mathbf{2}$ acts as a general test object, expressed by the slogan

What holds for $\mathbf{2}$ holds for all algebras.

J. Mycielski has informed us that this principle was already noticed some years ago by the Polish school of algebraists.

In view of the discussion in the previous section, the restriction of the functor $C(-, A)$ to the category BooS is of particular concern to us. For this, we first have the following simple observations:

(1) *$C(-, A)$ is faithful on* BooS. To any distinct maps $f, g\colon X \to Y$ in BooS, there exists an $x \in X$ such that $f(x) \neq g(x)$ and hence an open-closed set $U \subseteq X$ such that $f(x) \in U$ and $g(x) \notin U$; then any two-valued map k: $Y \to A$ constant on U and the complement CU of U is in $C(Y, A)$ and $C(f, A)(h) \neq C(g, A)(h)$.

(2) *For any map* $f\colon X \to Y$ *in* BooS, $C(f, A)$ *is an embedding iff* f *is onto.* This is an immediate consequence of Proposition 1.

(3) *For any map* $f\colon X \to Y$ *in* BooS, f *is an embedding iff* $C(f, A)$ *is onto.* The continuous maps $X \to A$ separate the points of X, but the continuous maps $X \xrightarrow{f} Y \to A$ only separate the points of X which f separates, and hence if f is not an embedding, then $C(f, A)$ is not onto. The converse results from the fact that the image of a continuous map $X \to A$ is finite and finite Boolean spaces are projective in Boo.

Next, we present a characterization of the functors $C(-, A)$ on BooS in purely categorical terms. In this we shall use the following terminology: If $X = \varinjlim X_\alpha$ is a limit in a category **K**, with limit maps $h_\alpha\colon X \to X_\alpha$, then a contravariant functor T on **K** is said to *copreserve* this limit iff $TX = \varinjlim TX_\alpha$ with colimit maps $Th_\alpha\colon TX_\alpha \to TX$. The copreservation of colimits is defined analogously.

PROPOSITION 2. *A contravariant functor T from* BooS *into the category of all algebras of given type is equivalent to a functor* $C(-, A)$ *iff* T *copreserves finite coproducts and projective limits.*

Proof. (⇒) For any X and Y in BooS, the map $C(X \amalg Y, A) \to C(X, A) \times \times C(Y, A)$ resulting from the coproduct maps $X, Y \to X \amalg Y$ is an isomorphism, and hence $C(-, A)$ copreserves finite coproducts. On the other hand if $X = \varprojlim X_\alpha$ is a projective limit, with limit maps $h_\alpha\colon X \to X_\alpha$ and diagram maps $h_{\alpha\beta}\colon X_\alpha \to X_\beta$, then $\Omega X = \bigcup h_\alpha^{-1}(\Omega X_\alpha)$ by Stone Duality, hence any open-closed partition of X is the inverse image of such a partition of some X_α by h_α, and thus any continuous map from X to a discrete space factors through some h_α. As a result, $C(X, A) = \bigcup \operatorname{Im} C(h_\alpha, A)$. Moreover, for each α, $\operatorname{Ker} C(h_\alpha, A) = \bigcup \operatorname{Ker} C(h_{\alpha\beta}, A)$ $(\beta \geqslant \alpha)$: if $uh_\alpha = vh_\alpha$ for u, v: $X_\alpha \to A$, then take $\beta \geqslant \alpha$ such that the partition of X by the sets $(uh_\alpha)^{-1}(a) = (vh_\alpha)^{-1}(a)$ $(a \in A)$ is the inverse image of a partition of X_β by h_β; then $uh_{\alpha\beta} = vh_{\alpha\beta}$. It now follows, by the familiar characterization of direct limits of algebras, that $\varinjlim C(X_\alpha, A) = C(X, A)$, with colimit maps $C(h_\alpha, A)$. Hence $C(-, A)$ copreserves projective limits.

(⇐) For any functor T of the stated kind, let $A = T1$, 1 the one-element Boolean space; we show that T is equivalent to $C(-, A)$ on BooS. The first step towards this is to define the required natural isomorphism η_X: $TX \to C(X, A)$ for *finite* spaces K. For this, consider the following diagram

$$\begin{array}{ccc} T1 & \xrightarrow{\eta_1} & C(1, A) \\ {\scriptstyle Tj_x}\uparrow & & \uparrow{\scriptstyle j_x^*} \\ TK & \xrightarrow[\eta_K]{} & C(K, A) \end{array} \qquad (x \in K),$$

where $j_x\colon \mathbf{1} \to K$ is the map with value x, $j_x^* = C(j_x, A)$, η_1 is the obvious isomorphism, and the isomorphism η_K is uniquely determined by the requirement that the diagram commute for each $x \in K$, since both, Tj_x and j_x^* are product projections in view of the fact that the j_x are coproduct injections. The naturality of the η_K thus defined follows from the diagram

$$\begin{array}{ccc} T1 & \xrightarrow{\eta_1} & C(1,A) \\ {\scriptstyle Tj_x}\uparrow & & \uparrow{\scriptstyle j_x^*} \\ TK & \xrightarrow{\eta_K} & C(K,A) \\ {\scriptstyle Tf}\uparrow & & \uparrow{\scriptstyle f^*} \\ TL & \xrightarrow[\eta_L]{} & C(L,A) \end{array}$$

for any map $f\colon K \to L$, since $j_x^*(\eta_k Tf) = \eta_1 Tj_x Tf = \eta_1 T(fj_x) = (fj_x)^* \eta_L = j_x^*(f^* \eta_L)$ for all x. Now, for any $X \in \mathrm{BooS}$, let $\nu_R\colon X \to X/R$ be the quotient map for any closed equivalence relation R on X with finite quotient, and $\nu_{RS}\colon X/R \to X/S$ the induced map for any pair of such relations where $R \subseteq S$. Then, X is the projective limit of the X/R, with limit maps $\nu_R\colon X \to X/R$, and hence η_X can be defined by the following diagram

$$\begin{array}{ccc} TX & \xrightarrow{\eta_X} & C(X,A) \\ {\scriptstyle T\nu_R}\uparrow & & \uparrow{\scriptstyle \nu_R^*} \\ T(X/R) & \xrightarrow{\eta_{X/R}} & C(X/R,A) \\ {\scriptstyle T\nu_{RS}}\uparrow & & \uparrow{\scriptstyle \nu_{RS}^*} \\ T(X/S) & \xrightarrow[\eta_{X/S}]{} & C(X/S,A) \end{array}$$

where the left-hand side and the right-hand side are direct limit diagrams, and the bottom square commutes so that one has an isomorphism of diagrams: put $\eta_X = \varinjlim \eta_{X/R}$. To check naturality, consider the diagram

$$\begin{array}{ccc} TX & \xrightarrow{\eta_X} & C(X,A) \\ {\scriptstyle Tf}\uparrow & & \uparrow{\scriptstyle f^*} \\ TY & \xrightarrow[\eta_Y]{} & C(Y,A) \\ {\scriptstyle T\nu_R}\uparrow & & \uparrow{\scriptstyle \nu_R^*} \\ T(Y/R) & \xrightarrow[\eta_{Y/R}]{} & C(Y/R,A) \end{array}$$

for any continuous map $f\colon X \to Y$ in BooS, and any quotient map $\nu_R\colon Y \to Y/R$ modulo a closed equivalence relation R on Y with finite quotient. Since the latter represent Y as projective limit, the maps $T\nu_R$ are jointly onto, and thus the desired condition $\eta_X Tf = f^* \eta_Y$ will follow if it is shown that $\eta_X Tf T\nu_R = f^* \eta_Y T\nu_R$, i.e. $\eta_X T(\nu_R f) = (\nu_R f)^* \eta_{Y/R}$ for all R. This, however, amounts to considering just the case where Y is *finite*, for which one has the following diagram

$$\begin{tikzcd}
TX \arrow[rrr, "\eta_X"] & & & C(X, A) \\
& T(X/S) \arrow[ul, "Tv_S"'] \arrow[r, "\eta_{X/S}"] & C(X/S, A) \arrow[ur, "v_S^*"] & \\
TY \arrow[uu, "Tf"] \arrow[ur, "Tg"'] \arrow[rrr, "\eta_Y"'] & & & C(Y, A) \arrow[uu, "f^*"'] \arrow[ul, "g^*"]
\end{tikzcd}$$

where $S = \mathrm{Ker}(f)$ and $f = gv_S$ the decomposition $X \to X/S \to Y$ of f. Here, the two triangles commute trivially, the top square commutes by definition of η_X, and the bottom square by what has already been shown; hence the desired result.

Remark. The object part of the equivalence $T \simeq C(-, T1)$ is also contained in Gonshor [26] in its version for $\mathrm{Boo} \to A$. An application of this, as shown in [26], is that, for general linear groups, $GL_n(B) \simeq C(\Omega B, GL_n(2))$ for Boolean rings B. Analogously, $M_n(B) \simeq C(\Omega B, M_n(2))$ for matrix rings of degree n.

COROLLARY. *For each* $X \in \mathrm{BooS}$, $C(X, A) = \varinjlim A^{X/R}$ $(R \in \Sigma)$, *where* Σ *is the set of closed equivalence relations on* X *with finite quotient, and thus* $C(X, A)$ *is a direct limit of finite powers of* A.

We next give a characterization of the image of the functor $C(-, A)$ on BooS, essentially due to Foster [23], and Gould and Grätzer [27]. In the following, the pointwise quaternary discriminator on a power A^I is the quaternary operation Q on A^I for which $Q(t, u, v, w)$ is the function on I with value $v(i)$ if $t(i) = u(i)$ and value $w(i)$ otherwise.

PROPOSITION 3. *A subalgebra* $S \subseteq A^I$ *is isomorphic to* $C(X, A)$ *for some* $X \in \mathrm{BooS}$ *iff* S *contains the constant functions, each member of* S *has finite image, and* S *is closed under the pointwise quaternary discriminator.*

Proof. For any $X \in \mathrm{BooS}$, $C(X, A)$ clearly contains the constant maps, and each $u \in C(X, A)$ has finite image by the compactness of X. Now, if $t, u, v, w \in C(X, A)$, then there is an open-closed partition $U_1, \ldots, U_n$ of X such that t, u, v, and w are all constant on each of the U_i. But then the function $r\colon X \to A$ for which $r|X_i = v|X_i$ if $t|X_i = u|X_i$, and $r|X_i = w|X_i$ otherwise, is obviously continuous, and $r = Q(t, u, v, w)$.

On the other hand, if $S \subseteq A^I$ has the above three properties, then S is also closed under the pointwise $(2n+2)$-ary discriminator, for all $n \geqslant 1$, i.e. if $t_1, \ldots, t_n, u_1, \ldots, u_n, v, w \in S$, then $Q_n(t_1, \ldots, t_n, u_1, \ldots, u_n, v, w) \in S$, where

$$Q_n(t_1, \ldots, t_n, u_1, \ldots, u_n, v, w)(i) = \begin{cases} v(i) & \text{if } t_j(i) = u_j(i) \text{ for all } j \leqslant n, \\ w(i) & \text{otherwise.} \end{cases}$$

This follows from the fact that the Q_n can be defined inductively by $Q_1 = Q$, and

$$Q_{n+1}(t_1, \ldots, t_{n+1}, u_1, \ldots, u_{n+1}, v, w) = Q(t_{n+1}, u_{n+1}, Q_n(t_1, \ldots, t_n, u_1, \ldots, u_n, v, w), w)$$

for each $n \geqslant 1$. From this we conclude that any function $u\colon I \to A$ for which $\operatorname{Ker} u \supseteq \operatorname{Ker} u_1 \cap \ldots \cap \operatorname{Ker} u_n$ for some $u_1, \ldots, u_n \in S$ belongs to S: Since all u_i have finite image, there is a partition $I_1, \ldots, I_m$ of I such that, for each $i \leqslant n$ and $j \leqslant m$, $u_i|I_j$ is constant, with value b_{ij}, say, and consequently $u|I_j$ is constant, with value a_j, say. We may assume that $b_{ij} = b_{ik}$ for all $i \leqslant n$ implies $j = k$. Let $v_1 = Q_n(u_1, \ldots, u_n, b_{11}, \ldots, b_{n1}, a_1, u_1)$, and for each $j > 1$, $v_j = Q_n(u_1, \ldots, u_n, b_{1j}, \ldots, b_{nj}, a_j, v_{j-1})$; it follows inductively that $u = v_m$ belongs to S.

Now, for each finite subset $F \subseteq S$, $\Theta_F = \bigcap \{\operatorname{Ker} u \mid u \in F\}$ is an equivalence relation on I with finite quotient $I_F = I/\Theta_F$, and if $G \subseteq F$ there are natural maps $I_F \to I_G$ and $A^{I_G} \to A^{I_F}$ providing an inverse system of finite sets and a corresponding direct system of algebras; moreover, it follows from the preceding paragraph that $S = \varinjlim A^{I_F}$, hence we have, by Proposition 2, $C(X, A) \simeq \varinjlim C(I_F, A) = \varinjlim A^{I_F} = S$ for the Boolean space $X = \varprojlim I_F$.

Besides the functors $C(-, A)$, one also has functors of the form $C(X, -)$, for any topological space X: if $\mathscr{A}$ is any class of algebras (of the same type) closed under formation of products and subalgebras, then $C(X, A) \in \mathscr{A}$ whenever $A \in \mathscr{A}$ and for any homomorphism $h\colon A \to B$ one has the homomorphism $C(X, h)\colon C(X, A) \to C(X, B)$ such that $C(X, h)(u) = hu$.

An interesting property of the functors $C(X, -)$ is as follows:

PROPOSITION 4. *For any space X, if $h\colon A \to B$ is a pure embedding, then $C(X, h)\colon C(X, A) \to C(X, B)$ is a pure embedding.*

Proof. $C(X, A)$ is the direct limit of the $C(X/R, A)$, where R ranges over the equivalence relations on X providing open partitions of X, and $C(X/R, A) = A^{X/R}$. Now, if $h\colon A \to B$ is pure, then all $h^{X/R}\colon A^{X/R} \to B^{X/R}$ are pure and $C(X, h)$ is the unique map making all diagrams

$$\begin{array}{ccc} A^{X/R} & \longrightarrow & C(X, A) \\ {\scriptstyle h^{X/R}} \uparrow & & \uparrow {\scriptstyle C(X,h)} \\ B^{X/R} & \longrightarrow & C(X, B) \end{array}$$

commute, i.e. $C(X, h) = \varinjlim h^{X/R}$, which implies $C(X, h)$ is also a pure embedding.

The discussion of the functors $C(-, A)$ leads one to consider what functorial properties the correspondence $X \to E(X, A)$ might have. Evidently, in order to associate a suitable homomorphism $E(Y, A) \to E(X, A)$ with a given continuous map $f\colon X \to Y$, it would be convenient to first have a way of obtaining a homomorphism $D(Y, A) \to D(X, A)$ from f, and for this it is natural to consider only those $f\colon X \to Y$ for which the inverse images of dense open sets are dense. Given this property, the correspondence $u \rightsquigarrow fu$ does indeed provide a homomorphism $D(Y, A) \to D(X, A)$, and the composite of this with the quotient map $D(X, A)$ clearly factors through $E(Y, A)$ to give a homomorphism $E(f, A)\colon E(Y, A) \to E(X, A)$. Since $E(X, A)$ is the direct limit of the $C(U, A)$ for the dense open $U \subseteq X$, it is evident that this determines a functor $E(-, A)\colon \mathrm{TOP}_{\#} \to \mathrm{Horn}\{A\}$, where $\mathrm{TOP}_{\#}$ is the category of all topological spaces and those continuous maps for which inverse images of dense open sets are dense, and $\mathrm{Horn}\{A\}$ is the universal Horn class generated by A, i.e. the smallest class of algebras which contains A and is hereditary, productive, and closed relative to direct limits. The fact that $E(X, A) \in \mathrm{Horn}\{A\}$ is also given in Burris [12]. We note that Grätzer [28], p. 147, seems to claim $E(X, A)$ belongs to $\mathrm{SP}\{A\}$; this is, of course, obvious for finite A in which case $\mathrm{SP}\{A\} = \mathrm{Horn}\{A\}$, but it does not hold for arbitrary A, as will be shown in the second remark following Proposition 8.

However, $E(X, A)$ does belong to a (possibly proper) subclass of $\mathrm{Horn}\{A\}$, namely the $\aleph_1$-implicational hull of A (Banaschewski–Herrlich [6]), i.e., the class of all algebras which satisfy the same universal Horn "sentences" with countably many premises as A does, the latter being of the form $\big(\bigwedge_{k\in\omega} p_k = q_k\big) \to p = q$, where $p_k, \ldots, q$ are elements of the absolutely free algebra of the type of A with countable basis. To see this, consider $u = (u_i) \in D(X, A)^{\omega}$, where X is Boolean, such that $p_k(u) \sim q_k(u)$ for all $k \in \omega$ and suppose that $p(u) \nsim q(u)$. Then there exists an open $W \subseteq X$ on which $p(u)$ and $q(u)$ are constant, with different values. On the other hand, $p_k(u)$ and $q_k(u)$ coincide on $\bigcap Du_i$ which is dense by the Baire category theorem (Bourbaki [10], p. 193). It follows that there exists $x \in W \cap \bigcap Du_i$ and then $p\big(u(x)\big) \neq q\big(u(x)\big)$ but $p_k\big(u(x)\big) = q_k\big(u(x)\big)$ for all $k \in \omega$ for $u(x) = u_i(x) \in A^{\omega}$. This shows that any sentence of the above form not satisfied by $E(X, A)$ also fails in A. As an example, we note that the abelian group $\mathbf{Q}$ of rational numbers does belong to $\mathrm{Horn}\{\mathbf{Z}\}$ but not to the $\aleph_1$-implicational hull of $\mathbf{Z}$, the latter since $\big(\bigwedge nx_n = x\big) \to x = 0$ holds for $\mathbf{Z}$ and not for $\mathbf{Q}$.

There remains the following problem: If $E(X, A) \notin \mathrm{SP}\{A\}$ it cannot belong to all $\aleph_\alpha$-implicational hulls of A since their intersections is, in fact, $\mathrm{SP}\{A\}$ (Banaschewski–Herrlich [6]); what, then, is the first α such that, for some X and A, $E(X, A)$ does not belong to the $\aleph_\alpha$-implicational hull of A?

Since $E(X, A) \simeq E(\Omega\mathfrak{K}X, A)$ one is particularly interested in the restriction of the functor $E(-, A)$ to the (full) subcategory of $\mathrm{TOP}_{\#}$ determined by the extremally disconnected Boolean spaces. Here, it turns out that a map $f\colon X \to Y$ between such spaces belongs to $\mathrm{TOP}_{\#}$ iff it is the Stone dual of a *complete* homomorphism from $\mathfrak{L}Y = \mathfrak{R}Y$ to $\mathfrak{L}X = \mathfrak{R}X$. Since a Boolean algebra A is complete iff ΩA is extremally disconnected (Halmos [29]), the subcategory of $\mathrm{TOP}_{\#}$ in question is the dual of the category CBoo of complete Boolean algebras and complete homomorphisms; we shall call this category CBooS. Note that Stone Duality now provides the functor $E(\Omega-, A)$: CBoo $\to$ Horn $\{A\}$.

The fact that $E(X, A) \cong E(\Omega\mathfrak{R}X, A)$ for all spaces X is actually part of a more general property of the functor $E(-, A)$. Indeed, there is a functor $\mathrm{TOP}_{\#} \to$ CBooS, extending the correspondence $X \rightsquigarrow \Omega\mathfrak{R}X$, such that for any $f\colon X \to Y$ in $\mathrm{TOP}_{\#}$, the diagram

$$\begin{array}{ccc} E(X, A) & \longrightarrow & E(\Omega\mathfrak{R}X, A) \\ \uparrow & & \uparrow \\ E(Y, A) & \longrightarrow & E(\Omega\mathfrak{R}Y, A) \end{array}$$

commutes, where the horizontal arrows are the natural isomorphisms and the upward ones correspond to f and the map $\Omega\mathfrak{R}f\colon \Omega\mathfrak{R}X \to \Omega\mathfrak{R}Y$ which results from f. The latter is obtained by the observation that, for $f\colon X \to Y$ in $\mathrm{TOP}_{\#}$, the correspondence $U \rightsquigarrow I\Gamma f^{-1}U$ provides a complete Boolean homomorphism from $\mathfrak{R}Y$ to $\mathfrak{R}X$; the less obvious part of this assertion, i.e. the preservation of arbitrary joins, results from the fact that, for any $U_\alpha \in \mathfrak{R}Y$, $W = \bigcup U_\alpha \cup C\Gamma \bigcup U_\alpha$ is dense open and

$$(I\Gamma \bigcup U_\alpha) \cap W = \bigcup U_\alpha,$$

so that

$$I\Gamma \bigcup f^{-1} U_\alpha = I\Gamma\big(f^{-1}(\Gamma \bigcup U_\alpha) \cap f^{-1}(W)\big) = I\Gamma f^{-1}(I\Gamma \bigcup U_\alpha)$$

since $f^{-1}(W)$ is dense by hypothesis. It follows that the functor $E(-, A)$ on $\mathrm{TOP}_{\#}$ is entirely determined by its restriction to CBooS. We note in addition that the above functor $\mathrm{TOP}_{\#} \to$ CBooS is a retraction, up to natural equivalence.

3. Congruences

In this section, we present some results concerning the congruence lattices of algebras of the form $C(X, A)$ and $E(X, A)$, with particular emphasis on bounded and arbitrary Boolean powers. We begin with a discussion of certain general aspects of congruence lattices.

For any type τ of algebras, there is a contravariant functor $\mathfrak{C}$: $\mathrm{Alg}(\tau) \to$ PoEns from the category $\mathrm{Alg}(\tau)$ of all algebras of that type and their homomorphisms to the category PoEns of all partially ordered sets and order preserving maps which results from the formation of congruence lattices: for any algebra $A \in \mathrm{Alg}(\tau)$, $\mathfrak{C}A$ is the (complete) partially ordered set of all congruences on A, partially ordered by set inclusion, and for any homomorphism $h\colon A \to B$ in $\mathrm{Alg}(\tau)$, $\mathfrak{C}h\colon \mathfrak{C}B \to \mathfrak{C}A$ is the map such that $\mathfrak{C}h(\theta) = (h \times h)^{-1}(\theta) = \mathrm{Ker}(\nu h)$, where $\nu\colon A \to A/\theta$ is the quotient map. It is obvious that this does, indeed, define a functor as claimed.

Recall that a direct limit $A = \varinjlim A_\alpha$ is *strict* iff all diagrams maps $A_\alpha \to A_\beta$ are embeddings, and that this implies that the colimit maps $A_\alpha \to A$ are embeddings. Then one has

LEMMA 1. *The functor* $\mathfrak{C}$: $\mathrm{Alg}(\tau) \to \mathrm{PoEns}$ *copreserves strict direct limits.*

Proof. It is enough to consider the case where $A = \bigcup A_\alpha$ for an up-directed family of subalgebras $A_\alpha \subseteq A$. Then, $\varprojlim \mathfrak{C}A_\alpha$ may be represented by all $(\theta_\alpha) \in \Pi\mathfrak{C}A_\alpha$ such that $\theta_\alpha = \theta_\beta | A_\alpha$ whenever $A_\alpha \subseteq A_\beta$. Clearly, for such a family, $\bigcup \theta_\alpha$ is a congruence on A, and the maps $(\theta_\alpha) \rightsquigarrow \bigcup \theta_\alpha$ and $\theta \rightsquigarrow (\theta | A_\alpha)$ are order preserving and inverse to each other. This proves that $\mathfrak{C}A = \varprojlim \mathfrak{C}A_\alpha$, the limit maps $\mathfrak{C}A \to \mathfrak{C}A_\alpha$ given by restriction.

Another functor which is relevant in the present context is the contravariant functor Φ: $\mathrm{Boo} \to \mathrm{PoEns}$ which associates with each Boolean algebra B the lattice ΦB of all filters in B, and with each homomorphism $h\colon B \to C$ in Boo the map $\Phi h\colon \Phi C \to \Phi B$ for which $\Phi h(F) = h^{-1}(F)$ $(F \in \Phi C)$. An argument similar to the proof of Lemma 1 then shows that Φ also copreserves strict direct limits. Actually, though, this could also be viewed as a special case of Lemma 1, by virtue of the familiar relationship between filters and congruences for Boolean algebras.

Now, for any space X and any algebra A, one has a natural map $\Phi\mathfrak{L}X \to \mathfrak{C}C(X, A)$, given by $\mathfrak{F} \rightsquigarrow \theta_{\mathfrak{F}}$, where $(u, v) \in \theta_{\mathfrak{F}}$ iff the set $\mathrm{Eq}(u, v) = \{x | u(x) = v(x)\}$ belongs to $\mathfrak{F}$. It is easily checked that for fixed A, these maps determine a natural transformation between the two functors $\Phi\mathfrak{L}$ and $\mathfrak{C}C(-, A)$ from TOP to PoEns. Note that if X is discrete, $\theta_{\mathfrak{F}}$ is the usual filter congruence determined by the filter $\mathfrak{F}$ in $\mathfrak{L}X = \mathfrak{P}X$, the power set of X, on $C(X, A) = A^X$. In general, one has the following basic fact:

PROPOSITION 5. *For any space X and any algebra A, the natural map $\Phi\mathfrak{L}X \to \mathfrak{C}C(X, A)$ is a complete lattice embedding.*

Proof. For any family $(\mathfrak{F}_\alpha)$ in $\Phi\mathfrak{L}X$ and any $u, v \in C(X, A)$, if $\mathfrak{F} = \bigcap \mathfrak{F}_\alpha$, then $(u, v) \in \theta_{\mathfrak{F}}$ iff $\mathrm{Eq}(u, v) \in \mathfrak{F}_\alpha$ for all α, and hence $\theta_{\mathfrak{F}} = \bigcap \theta_{\mathfrak{F}_\alpha}$, i.e. arbitrary meets are preserved. Regarding joins, we first note that a similar straightforward argument shows

$$\theta_{\bigcup \mathfrak{F}_\alpha} = \bigcup \theta_{\mathfrak{F}_\alpha}$$

for any up-directed family $(\mathfrak{F}_\alpha)$ in $\Phi\mathfrak{L}X$; the general result is then a consequence of the preservation of finite joins which is obtained as follows: If $(u, v) \in \theta_{\mathfrak{F} \vee \mathfrak{G}}$, then $\mathrm{Eq}(u, v) = U \cap V$ for some $U \in \mathfrak{F}$ and $V \in \mathfrak{G}$, and if $w \in C(X, A)$ is defined such that $w | U = u | U$ and $w | CU = v | CU$, then

$\mathrm{Eq}(u, w) \supseteq U$ and $\mathrm{Eq}(w, v) \supseteq V$ so that $(u, w) \in \theta_{\mathfrak{F}}$ and $(w, v) \in \theta_{\mathfrak{G}}$, and therefore $(u, v) \in \theta_{\mathfrak{F}} \circ \theta_{\mathfrak{G}} \subseteq \theta_{\mathfrak{F}} \vee \theta_{\mathfrak{G}}$. This shows $\theta_{\mathfrak{F} \vee \mathfrak{G}} \subseteq \theta_{\mathfrak{F}} \vee \theta_{\mathfrak{G}}$, and hence one has equality, the reverse inclusion being obvious. It remains to show that the map is an embedding, but this immediately results from the observation that $\mathfrak{F} = \{\mathrm{Eq}(u, v) | (u, v) \in \theta_{\mathfrak{F}}\}$ since any $U \in \mathfrak{L}X$ is of the form $\mathrm{Eq}(u, v)$, the algebra A being non-trivial.

If one associates with each closed subset of a space X, the collection of all its open-closed neighbourhoods one obtains an order-inverting map from the lattice $\mathfrak{F}X$ of all closed subsets of X into $\Phi\mathfrak{L}X$, i.e. a map $(\mathfrak{F}X)^* \to \Phi\mathfrak{L}X$ of partially ordered sets, where $(\ )^*$ indicates dual. Note that this preserves arbitrary meets but in general nothing else. However, for *Boolean* X the map is actually an *isomorphism* whose inverse is given by mapping each $\mathfrak{F} \in \Phi\mathfrak{L}X$ to its intersection: in this case, each $F \in \mathfrak{F}X$ is the intersection of its open-closed neighbourhoods, and $V \supseteq \bigcap \mathfrak{F}$ implies $V \in \mathfrak{F}$ for any $V \in \mathfrak{L}X$ and $\mathfrak{F} \in \Phi\mathfrak{L}X$.

The complete meet-homomorphism $(\mathfrak{F}X)^* \to \Phi\mathfrak{L}X$ in general determines a map $\Psi\colon (\mathfrak{F}X)^* \to \mathfrak{C}C(X, A)$ by composition with the natural map $\Phi\mathfrak{L}X \to \mathfrak{C}C(X, A)$, such that $(u, v) \in \Psi(F)$ iff $F \subseteq \mathrm{Eq}(u, v)$, i.e. $\Psi(F)$ is the kernel of the homomorphism $C(X, A) \to C(F, A)$ given by restriction $u \rightsquigarrow u|F$ to the closed subspace $F \subseteq X$. It therefore follows that $C(X, A)/\Psi(F) \lesssim C(F, A)$ for any closed subspace $F \subseteq X$. For *Boolean* X the embedding $C(X, A)/\Psi(F) \to C(F, A)$ is actually an isomorphism since any continuous $u\colon F \to A$ then has a continuous extension to X: The image of u is finite, and finite Boolean spaces are injective. (See also Burris [12].)

Combining these facts, we now have the following consequence of Proposition 5, also given in Burris [12]:

COROLLARY. *For Boolean X and any algebra A, $\Psi\colon (\mathfrak{F}X)^* \to \mathfrak{C}C(X, A)$ is a complete lattice embedding.*

It is natural to ask when the above Ψ is actually an isomorphism; which would mean that $C(X, A)$ does not have any "surprising" congruences. The following considerations are devoted to this question.

An algebra A will be called *powersimple* iff all congruences on powers of A are filter congruences, i.e. for any *discrete* space X, the natural map $\Phi\mathfrak{L}X \to \mathfrak{C}C(X, A)$ is an isomorphism. Note that this implies A is simple since $A \simeq C(1, A)$ and $\Phi\mathfrak{L}1$ is the two-element chain. Moreover, it implies that $\mathfrak{C}(A^n) \simeq 2^n$ for all natural numbers n – this being the special case for finite X. If the latter holds A will be called *finitely powersimple*.

We remark that powersimplicity is a categorical property of an algebra A in $\mathrm{HSP}\{A\}$ since it amounts to the condition that for any regular epimorphism $A^I \to B$ there exists a monomorphism $B \to A^{(\mathfrak{F})}$, the latter being a filter power of A for some filter $\mathfrak{F}$ on I, such that the composite $A^I \to B \to A^{(\mathfrak{F})}$ is the natural map.

Concerning the existence of powersimple algebras, we have the following:

EXAMPLE 1. *Any field is powersimple.* Because of the correspondence between ideals and congruences for rings this amounts to saying that, for a field K, any ideal J of a power K^I is given by a filter $\mathfrak{F}$ on I such that $u \in J$ iff the zero set $Z(u)$ belongs to $\mathfrak{F}$. To see this one notes that for any $u \in J$ one has a $v \in K^I$ such that $Z(u) = Z(uv)$ and uv is idempotent, and that for idempotent $e, f \in K^I$ $Z(e) \cap Z(f) = Z(e+f-ef)$. It follows then that the $Z(u)$, $u \in J$ form a filter $\mathfrak{F}$, and that any $f \in K^I$ such that $Z(f) \in \mathfrak{F}$ belongs to the ideal J.

EXAMPLE 2. $\mathbf{Z}$ *as lattice-ordered ring with unit is powersimple.* Analogously to the case of fields, it is enough to show that any l-ideal J of any power $\mathbf{Z}^I$ is determined by a filter $\mathfrak{F}$ on I such that $u \in J$ iff its zero set $Z(u)$ belongs to $\mathfrak{F}$. Since $Z(u) = Z(|u|)$, where $|u| = u \vee (-u)$, and $Z(v) \cap Z(w) = Z(v \vee w)$ for non-negative $v, w \in Z^I$, it follows that the sets $Z(u)$, $u \in J$, form a filter basis. Moreover, for any subset $S \subseteq I$ such that $S \supseteq Z(u)$ $(u \in J)$, $S = Z(wu)$ for the characteristic function w of the complement of S, and therefore $S \in \mathfrak{F}$ since $wu \in J$. Thus, $\mathfrak{F}$ is a filter. Finally, if $u \in J$ and w is the characteristic function of the complement of $Z(u)$, then $w = |w| \leqslant |u|$ and hence $w \in J$; it follows, for any $v \in Z^I$, that $v \in J$ whenever $Z(v) = Z(u)$ for some $u \in J$ since the latter implies that $v = vw$. In all, this shows $u \in J$ iff $Z(u) \in \mathfrak{F}$.

EXAMPLE 3. *Any (possibly infinite) quasi-primal algebra is powersimple.* Such an algebra A has the quaternary discriminator as derived operation, and hence also the $(2n+2)$-ary discriminator Q_n which has been used in the proof of Proposition 3. Now for any congruence θ on an arbitrary subalgebra $B \subseteq A^I$, let $u, v \in B$ be such that $\mathrm{Eq}(u, v) \supseteq \bigcap \mathrm{Eq}(u_i, v_i)$ for finitely many $(u_i, v_i) \in \theta$. Then

$$(Q_n(u_1, \ldots, u_n, u_1, \ldots, u_n, u, v), Q_n(u_1, \ldots, u_n, v_1, \ldots, v_n, u, v)) \in \theta,$$

but by the pointwise properties of Q_n and the hypothesis on u and v this is just (u, v). It follows that $\theta = \theta_{\mathfrak{F}}$ for the filter $\mathfrak{F}$ on I generated by the sets $\mathrm{Eq}(u, v)$ for $(u, v) \in \theta$. Note that this proves more than the powersimplicity of A: Since $B/\theta \subseteq A^{(\mathfrak{F})}$, and the latter algebra belongs to Horn $\{A\}$, we have HSP $\{A\}$ = Horn $\{A\}$ *for quasi-primal* A. In addition, if A is finite, Horn $\{A\}$ = SP $\{A\}$, and hence HSP $\{A\}$ = SP $\{A\}$ for finite quasi-primal A, a result originally due to Pixley [40].

EXAMPLE 4. *Any simple algebra in a congruence-factorable equational class is finitely powersimple*, where congruence factorability means that all congruences on a product $A \times B$ of algebras in the class are of the form

$$\{((a, b), (a', b')) \mid (a, a') \in \theta, (b, b') \in A\}$$

for some $\Theta \in \mathfrak{C}A$ and $\Lambda \in \mathfrak{C}B$. Regarding the usefulness of this observation, we note that an equational class is congruence – factorable whenever all congruence lattices occurring are distributive, and that any non-trivial equational class contains simple algebras (Magari [37]).

We now return to the question raised above, providing the following answer:

PROPOSITION 6. *For any algebra A the natural map $\Phi\mathfrak{Q}X \to \mathfrak{C}C(X, A)$ is an isomorphism for all spaces X iff A is powersimple.*

Proof. In one direction this is obvious, and in the other we have to show the map in question is onto. From the strict direct limit representation $C(X, A) = \varinjlim C(X/R, A) = \varinjlim A^{X/R}$, R ranging over the equivalence relations on X with open blocks, we obtain the projective limit representation $\mathfrak{C}C(X, A) = \lim \mathfrak{C}(A^{X/R})$. Further, the quotient maps $\nu_R \colon X \to X/R$ determine a strict direct limit representation $\mathfrak{P}(X/R) \to \mathfrak{Q}X$ and therefore a projective limit representation $\mu_R \colon \Phi\mathfrak{Q}X \to \Phi\mathfrak{P}(X/R)$. Hence one has the following commuting square of partially ordered sets

$$\begin{array}{ccc} \Phi\mathfrak{Q}X & \xrightarrow{\mu_R} & \Phi\mathfrak{P}(X/R) \\ \uparrow & & \uparrow \\ \mathfrak{C}C(X, A) & \longrightarrow & \mathfrak{C}(A^{X/R}) \end{array}$$

where the vertical maps are the natural ones under discussion, the bottom one is the limit map, and the map on the right is an isomorphism, for any R, by hypothesis. It follows that the map on the left is an isomorphism.

Note that if A is the (powersimple!) two-element Boolean algebra **2**, then the isomorphism $\Phi\mathfrak{Q}X \to \mathfrak{C}C(X, A)$ is just the known isomorphism between filter and congruence lattices for Boolean algebras, and hence Proposition 6 states that what is true for **2** regarding $\mathfrak{C}C(X, -)$ holds for any powersimple A.

If the algebra A is merely finitely powersimple, the above argument still works for compact X: compactness permits one to consider only those R for which X/R is finite. Hence:

COROLLARY 1. *If A is finitely powersimple, then the natural map $\Phi\mathfrak{Q}X \to$ $\to \mathfrak{C}C(X, A)$ is an isomorphism for any compact X.*

A further consequence of this is

COROLLARY 2. *Any finite, finitely powersimple algebra is powersimple.*

Proof. For such algebras A, the natural map $\Phi\mathfrak{Q}\beta X \to \mathfrak{C}C(\beta X, A)$ is an isomorphism for any discrete X by the preceding corollary; now, in addition, one also has the isomorphism $\Phi\mathfrak{P}X \to \Phi\mathfrak{Q}\beta X$ and $C(X, A) \to$ $\to C(\beta X, A)$, the latter given by extending the $X \to A$ continuously to βX (A finite!), and the natural map $\Phi\mathfrak{P}X \to \mathfrak{C}(A^X)$ is then a composite of isomorphisms.

Note that the converse of Corollary 2 does not hold: any simple modular lattice M containing an infinite chain, with the property that for all $x \leqslant y$ in M, the length $d(x, y)$ of a maximal chain from x to y is finite, provides a counterexample: M, being a simple algebra in a congruence distributive equational class, is finitely powersimple. The modularity of M implies that $d(x \wedge z, y \wedge z) \leqslant d(x, y)$ and $d(x \vee z, y \vee z) \leqslant d(x, y)$ for any $x \leqslant y$ and $z \in M$; consequently, any principal congruence θ on M^{ω} generated by a pair (u, v) with $u \leqslant v$ and $\sup\{d(u(n), v(n)) \mid n \in \omega\}$ finite has the property that $\sup\{d(s(n), t(n)) \mid n \in \omega\}$ is finite for any $(s, t) \in \theta$ with $s \leqslant t$. Thus the congruence on M^{ω} generated by any pair (u, v) of constant functions with $u < v$ is not a filter congruence, which proves the point.

Now, putting Corollary 1 together with the above observations concerning the map $(\mathfrak{F}X)^* \to \Phi\mathfrak{L}X$, we obtain the following result which also appears in Burris [12], and, for more restricted A, in Davey [19].

COROLLARY 3. *If A is finitely powersimple and X Boolean, then Ψ: $(\mathfrak{F}X)^* \to \mathfrak{C}C(X, A)$ is an isomorphism.*

The following consequence of the preceding corollary is a counterpart to a result in Burris [12]:

COROLLARY 4. *If A is finitely powersimple, then, for any Boolean spaces X and Y, $X \simeq Y$ whenever $C(X, A) \simeq C(Y, A)$.*

Proof. The hypothesis provides a lattice isomorphism $\mathfrak{F}X \simeq \mathfrak{F}Y$, and for Boolean spaces, as actually for much more general spaces, this implies $X \simeq Y$ (for compact T_1-spaces for instance, see Bourbaki [10], p. 148).

In this corollary, nothing is said as to how the given isomorphism $C(X, A) \to C(Y, A)$ might be related to the isomorphism which results from the homeomorphism $X \to Y$ obtained from it. Somewhat related to this point is the following

COROLLARY 5. *For finitely powersimple A, the functor $C(-, A)$ provides a dual equivalence between* BooS *and the subcategory of* SP$\{A\}$ *given by all $C(X, A)$ and all constant preserving homomorphisms between them.*

Proof. Since it has already been noted that $C(-, A)$ is faithful on BooS, and since all homomorphisms of the form $C(f, A)$ clearly preserve the constants, it remains to show that each constant preserving homomorphism h: $C(X, A) \to C(Y, A)$ results from a continuous map $Y \to X$. Since A is simple, the evaluation homomorphism e_z: $C(Y, A) \to A$, for any $z \in Y$, determines a unique $x \in X$ such that $\operatorname{Ker}(e_z h) = \Psi\{x\}$, by Corollary 3. If f: $Y \to X$ is the resulting *set* map, and $u \in C(X, A)$, $z \in Y$, and $a = u(f(z))$, then $(u, \mathbf{a}) \in \Psi\{f(z)\}$ and hence $(h(u), \mathbf{a}) \in \operatorname{Ker}(e_z)$ so that $h(u)(z) = u(f(z))$. This shows $h(u) = uf$ for all $u \in C(X, A)$, which first establishes the continuity of f, since the topology of X is the weak topology relative to $C(X, A)$ by Booleanness, and then $h = C(f, A)$.

A special case of this corollary, in its variant formulation for Boo instead of BooS is treated in Ribenboim [44], where A is the lattice-ordered abelian group $\mathbf{Z}$ of integers; there, an internal characterization is given of the range of the functor $\dot{C}(-, \mathbf{Z})$.

The final consequence of Proposition 6 in this sequence concerns equational compactness. We are in particular interested when the functor $C(\Omega -, A)$ reflects this property, which means, by Stone Duality, when the equational compactness of $C(X, A)$ implies that $X \in \mathrm{BooS}$ is extremally disconnected. Evidently, it requires conditions on A for this to hold, for if we take the algebra with empty type (i.e. just sets) then *all* $C(X, A)$ are equationally compact. But we have:

COROLLARY 6. *If A is finitely powersimple, then $C(\Omega -, A)$ reflects equational compactness.*

Proof. Suppose $C(X, A)$ is equationally compact, and let $p \colon \tilde{X} \to X$ be the projective cover of X in BooS. Then $C(p, A) \colon C(X, A) \to C(\tilde{X}, A)$ is pure by Proposition 1, and hence it has a left inverse $h \colon C(\tilde{X}, A) \to C(X, A)$. Now, by the above Corollary 3, there is a closed set $F \subseteq \tilde{X}$ such that $h(u) = h(v)$ iff $u|F = v|F$ for all $u, v \in C(\tilde{X}, A)$. Suppose $F \neq \tilde{X}$, and hence $p(F) \neq X$, since $p \colon \tilde{X} \to X$ is the projective cover; then, there is an open-closed $U \subseteq X$ not meeting $p(F)$, and if $u, v \colon X \to A$ are constant on U and CU with equal value on the latter and different values on the former, then u and v are continuous such that $up|F = vp|F$, and hence $u = h(up) = h(vp) = v$, a contradiction. This shows $F = \tilde{X}$, thus $\mathrm{Ker}(h)$ is trivial, and therefore $C(p, A)$ is an isomorphism; this implies p is a homeomorphism, and hence X is extremally disconnected.

We now turn to the congruence on the algebras $E(X, A)$. In this setting one has, for any space X and any algebra A, a natural map $\Phi\mathfrak{R}X \to \mathfrak{C}D(X, A)$, given by $\mathfrak{F} \rightsquigarrow \theta_{\mathfrak{F}}$, where $(u, v) \in \theta_{\mathfrak{F}}$ iff there is a dense open $U \subseteq D(u) \cap D(v)$ and a $V \in \mathfrak{F}$ such that $u|U \cap V = v|U \cap V$, or, equivalently, $u|D(u) \cap D(v) \cap V = v|D(u) \cap D(v) \cap V$, i.e. $\mathrm{Eq}(u, v) \supseteq D(u) \cap D(v) \cap V$ for some $V \in \mathfrak{F}$. One readily sees that, for fixed A, these maps provide a natural transformation between the funtors $\Phi\mathfrak{R}$ and $\mathfrak{C}D(-, A)$ from $\mathrm{TOP}_{\#}$ to PoEns. Moreover, the congruences $\theta_{\mathfrak{F}}$ on any $D(X, A)$ obviously contain $\sim$ and hence we also have $\Phi\mathfrak{R}X \to \mathfrak{C}E(X, A)$.

PROPOSITION 7. *For any space X and any algebra A, the natural map $\Phi\mathfrak{R}X \to \mathfrak{C}E(X, A)$ is a complete lattice embedding.*

Proof. It is clear that one only has to consider $\Phi\mathfrak{R}X \to \mathfrak{C}D(X, A)$ since the sublattice of the latter consisting of all the congruences above $\sim$ is isomorphic to $\mathfrak{C}E(X, A)$. That arbitrary meets and up-directed joins are preserved is easily checked in essentially the same way as this was done in the proof of Proposition 5. The preservation of joins is seen as follows: Given that $\mathrm{Eq}(u, v) \supseteq W \cap U \cap V$ for some dense open $V \subseteq D(u) \cap D(v)$, $U \in \mathfrak{F}$ and $V \in \mathfrak{G}$, restrict to the dense open set $W_0 = W \cap (U \cup ICU) \cap$

$\cap (V \cup ICV)$ so that $W_0 \cap U$ and $W_0 \cap V$ become open-closed in W_0, and then use the same argument as in the proof of Proposition 5 on the space W_0. Finally, that the map is an embedding results from the fact that, for any $U \in \mathfrak{R}X$, any function constant on U and on ICU belongs to $D(X, A)$.

Now, if one associates with any closed subset of the space X the collection of its regular open neighbourhoods one obtains, analogously to the previously considered $(\mathfrak{F}X)^* \to \Phi\mathfrak{Q}X$, an order-preserving map $(\mathfrak{F}X)^* \to \Phi\mathfrak{R}X$. For $X \in$ CBooS, this is a complete lattice isomorphism, simply because $\mathfrak{R}X = \mathfrak{Q}X$ in that case.

Hence, we have, by composition with the natural map $\Phi\mathfrak{R}X \to \mathfrak{C}E(X, A)$ (see also Burris [12]):

COROLLARY. *For $X \in$ CBooS and any algebra A, the map Λ: $(\mathfrak{F}X)^* \to \mathfrak{C}E(X, A)$ is a complete lattice embedding.*

Note that, for any closed $F \subseteq X$, the congruence $\Lambda(F)$ is induced by all the pairs (u, v) in $D(X, A)$ such that $u|D(u) \cap D(v) \cap V = v|D(u) \cap D(v) \cap V$ for some regular open neighbourhood V of F.

Incidentally, if one is *given* that the map $(\mathfrak{F}X)^* \to \Phi\mathfrak{R}X$ by taking regular open neighbourhoods is a complete lattice embedding, then one can prove that X is extremally disconnected, compact, normal, and T_1 provided it is T_0; thus, for T_0-spaces X, if this map is a complete lattice embedding, then $X \in$ CBooS. Hence, the same conclusion follows if Λ is a complete lattice embedding, providing the converse of the corollary, for T_0-spaces X. We omit the details, but make the following observation:

If a space X has the property that the map Λ: $(\mathfrak{F}X)^* \to \mathfrak{C}E(X, A)$ is a complete lattice embedding for $A = \mathbf{2}$, the two-element Boolean algebra, then it has this property for *all* algebras A: in view of the isomorphism $\mathfrak{R}X \to E(X, \mathbf{2})$ and the correspondence between congruences and filters for Boolean algebras, if $(\mathfrak{F}X)^* \to \mathfrak{C}E(X, \mathbf{2})$ is a complete lattice embedding, then the same holds for the map $(\mathfrak{F}X)^* \to \Phi\mathfrak{R}X$ which, as was noted, implies that $X \in$ CBooS (for T_0-spaces X), and then the above corollary applies. This provides another instance where $\mathbf{2}$ acts as a general test object.

As in the case of $C(X, A)$, we now consider the question when the map from filters to congruences is an isomorphism.

PROPOSITION 8. *For any algebra A, the natural map $\Phi\mathfrak{R}X \to \mathfrak{C}E(X, A)$ is an isomorphism for all $X \in$ CBooS iff A is powersimple.*

Proof. In one direction this is obvious since the case $X = \beta Y$ for discrete Y just amounts to the condition of powersimplicity. For any powersimple A and $X \in$ CBooS, consider the following diagram

$$\begin{array}{ccc} \Phi\mathfrak{R}X & \xrightarrow{\sigma_U} & \Phi\mathfrak{Q}U \\ \downarrow & & \downarrow \\ E(X, A) & \longrightarrow & C(U, A) \end{array}$$

where the unnamed maps are the obvious ones and σ_U maps any $\mathfrak{F} \in \Phi\mathfrak{R}X$ to the filter $\{W \mid I\Gamma W \in \mathfrak{F}\}$ in $\mathfrak{Q}U$. The congruence on $C(U, A)$ which the latter determines is then the set of all (u, v) such that $\mathrm{Eq}(u, v) \supseteq W$ for some $W \in \mathfrak{Q}U$, where $I\Gamma W \in \mathfrak{F}$, but this is obviously the same as the set of all (u, v) such that $\mathrm{Eq}(u, v) \supseteq U \cap V$ for some $V \in \mathfrak{F}$ since $\mathrm{Eq}(u, v) \in \mathfrak{Q}U$, and hence the above square commutes.

Now, notice that $\mathfrak{R}X = \varinjlim \mathfrak{Q}U$ (U ranging over the dense open sets of X) is a strict direct limit in the category Boo, the colimit maps being $W \rightsquigarrow I\Gamma W$, and hence $\Phi\mathfrak{R}X = \varprojlim \Phi\mathfrak{Q}U$, by the remarks following Lemma 1, the limit maps being the above σ_U. This now implies that $\Phi\mathfrak{R}X \to \mathfrak{C}E(X, A)$ is an isomorphism since all $\Phi\mathfrak{Q}U \to \mathfrak{C}C(U, A)$ are isomorphisms by Proposition 6.

We now have the following analogue to Corollary 4 of Proposition 6, in view of the basic isomorphism $X \to \Omega\mathfrak{Q}X$ for Boolean spaces:

Corollary 1. *If A is powersimple, then, for any X and Y in* CBooS, *$X \simeq Y$ whenever $E(X, A) \simeq E(Y, A)$.*

A noteworthy consequence concerning the embedding i: $C(X, A) \to E(X, A)$ is the following

Corollary 2. *If A is powersimple and $X \in$* CBooS, *then the map $\mathfrak{C}i$: $\mathfrak{C}E(X, A) \to \mathfrak{C}C(X, A)$ is an isomorphism.*

Proof. This results from the commuting triangle

$$\begin{array}{ccc} & \Phi\mathfrak{R}X = \Phi\mathfrak{Q}X & \\ \swarrow & & \searrow \\ E(X, A) & \longrightarrow & C(X, A) \end{array}$$

whose downward maps are isomorphisms by Propositions 6 and 8.

Incidentally, it should be noted that, for powersimple A, the map $\mathfrak{C}E(X, A) \to \mathfrak{C}C(X, A)$ is onto for any space X, although, obviously, there may be many congruences on $E(X, A)$ having the same restriction to $C(X, A)$.

Remark 1. Proposition 8 actually holds for *any* space X, rather than for the $X \in$ CBooS only. To see this, we consider the following square of complete lattice homomorphisms

$$\begin{array}{ccc} \Phi\mathfrak{R}X & \longrightarrow & \mathfrak{C}E(X, A) \\ \downarrow & & \downarrow \\ \Phi\mathfrak{R}\Omega\mathfrak{R}X & \longrightarrow & \mathfrak{C}E(\Omega\mathfrak{R}X, A) \end{array}$$

where the horizontal maps are the natural ones, and the vertical maps are the isomorphisms which result from the natural isomorphisms $\mathfrak{R}X \to \mathfrak{Q}\Omega\mathfrak{R}X = \mathfrak{R}\Omega\mathfrak{R}X$ and $E(X, A) \to E(\Omega\mathfrak{R}X, A)$ of Boolean algebras and algebras, respectively. Evidently, by Proposition 8 for $E(\Omega\mathfrak{R}X, A)$, the map $\Phi\mathfrak{R}X \to \mathfrak{C}E(X, A)$ will be an isomorphism, for any powersimple A, if this square commutes. To see the latter, we recall that the isomorphism $E(X, A) \to E(\Omega\mathfrak{K}X, A)$ may be described by associating with each $u \in C(W, A)$, $W \subseteq X$ dense open, the function $u^{\#}: W^{\#} \to A$, where $W^{\#} \subseteq \Omega\mathfrak{K}X$ is the (dense open!) set consisting of all $\mathfrak{U}$ which contain some $I\Gamma u^{-1}(a)$ $(a \in A)$, and $u^{\#}(\mathfrak{U}) = a$ for these $\mathfrak{U}$. Then, for any filter $\mathfrak{F}$ in $\mathfrak{K}X$, the congruence on $E(X, A)$ associated with $\mathfrak{F}$ is determined by the $u, v \in C(W, A)$ such that $I\Gamma\{x | u(x) = v(x)\} \in \mathfrak{F}$. Now, this holds iff

$$I\Gamma \bigcup_{a \in A} u^{-1}(a) \cap v^{-1}(a) \in \mathfrak{F}$$

and hence also iff

$$\bigvee_{a \in A} E_a \in \mathfrak{F} \quad (E_a = I\Gamma u^{-1}(a) \cap I\Gamma v^{-1}(a))$$

where the join is in $\mathfrak{K}X$. On the other hand, the analogous condition for $u^{\#}$ and $v^{\#}$ is that (I and Γ now in the space $\Omega\mathfrak{K}X$)

$$I\Gamma \bigcup_{a \in A} \{\mathfrak{U} | I\Gamma u^{-1}(a) \cap I\Gamma v^{-1}(a) \in \mathfrak{U}\}$$

belong to the filter corresponding to $\mathfrak{F}$ via the isomorphism $\mathfrak{K}X \to \mathfrak{K}\Omega\mathfrak{K}X$. The latter is given by $V \rightsquigarrow \Omega_V = \{\mathfrak{U} | V \in \mathfrak{U} \in \Omega\mathfrak{K}X\}$, and thus the above holds iff

$$\bigvee_{a \in A} \Omega_{E_a} \in \{\Omega_V | V \in \mathfrak{F}\},$$

where the join is in $\mathfrak{K}\Omega\mathfrak{K}X$; clearly, this is equivalent to the previous condition for $u, v \in C(W, A)$, and this proves the assertion.

Remark 2. *For any infinite, powersimple, rigid algebra A and any compact metrizable space X without isolated points, $E(X, A) \notin \mathrm{SP}\{A\}$*, in fact, there is *no* homomorphism $\varphi: E(A, X) \to A$. For any such φ, $\mathrm{Ker}(\varphi)$ is determined by an ultrafilter $\mathfrak{U}$ in $\mathfrak{K}X$, and the map $\varphi_W: C(W, A) \to E(X, A) \to A$ ($W \subseteq X$ dense open) resulting from φ preserves the constants, by the hypotheses on A. Hence, if $\varphi_W(u) = a$ for $u \in C(W, A)$, then there exists a $V \in \mathfrak{U}$ such that $u | V \cap W$ is constant with value a. In particular, if W is the union of disjoint $W_1, W_2, \ldots$ in $\mathfrak{K}X$ and u is taken constant on each W_i, with different values for different i, then $V \cap W = V \cap W_i$ for some i, so that $\emptyset = U \cap W = U \cap W_i$ for all $U \subseteq V$ in $\mathfrak{U}$, and hence $W_i \in \mathfrak{U}$. It follows that $\mathfrak{U}$ is countably complete, a contradiction, since any countable basis $U_1 \supset U_2 \supset \ldots$ of open neighbourhoods of $x = \lim \mathfrak{U}$ determines the join $X = \bigvee ICU_i$ in $\mathfrak{K}X$ which belongs to $\mathfrak{U}$ whereas $ICU_i \notin \mathfrak{U}$ for all i. Algebras to which this applies are: the fields Q and R as rings with unit, Z as lattice-ordered ring with unit, and any infinite powersimple algebra with its elements added as nullary operations. The case of $\mathbf{Z}$, based on a somewhat different argument, was communicated to us by K. Keimel. It might be added that the above proof still works if the rigidity of A is relaxed to the condition that all monomorphisms $A \to A$ are onto since this suffices to show that $\varphi_W(u) = \varphi_W(v)$ for some constant v.

We close this discussion with the remark that no counterpart to Corollary 6 of Proposition 6 is to be expected in the present setting: Nothing meaningful can be proved concerning the reflection of equational compactness by $E(\Omega-, A)$ since any infinite $X \in$ CBooS is a non-trivial projective cover of some $Y \in$ BooS which is then not extremally disconnected but for which, of course, $E(Y, A) \simeq E(X, A)$ for all A. The reason for the existence of such Y is that the Boolean spaces which *do not* have a proper irreducible Boolean image are just the one-point compactifications of discrete spaces (Flachsmeyer [22]), and they are never extremally disconnected.

4. Exponent laws

For the powers of an algebra A by *sets*, one has the *exponent law* $(A^Y)^X \simeq A^{X \times Y}$, for any sets X and Y, the isomorphism given by the familiar adjointness map $u \rightsquigarrow \bar{u}$, where $\bar{u}(x, y) = u(x)(y)$ for all $x \in X$ and $y \in Y$, which results from the cartesian closedness of the category of sets. Here we are interested in analogous results concerning the iterates of the functors $C(X, -)$, $E(X, -)$, and variants of the latter.

For any topological spaces X and Y, the above correspondence $u \rightsquigarrow \bar{u}$ does indeed determine a homomorphism $C(X, C(Y, A)) \to C(X \times Y, A)$: given any $(x, y) \in X \times Y$, there exists a neighbourhood V of x such that $u|V$ is constant and then a neighbourhood W of y such that $u(x)$ is constant on W; $\bar{u}$ is then constant on the neighbourhood $V \times W$ of (x, y), and hence $u \in C(X \times Y, A)$. Moreover, $u \rightsquigarrow \bar{u}$ is obviously an embedding, and therefore the only question concerning the exponent law in this case is whether this map is onto, i.e. whether each $v \in C(X \times Y, A)$ is a $\bar{u}$ for some $u \in C(X, C(Y, A))$. Now, for any continuous map v: $X \times Y \to A$ the maps $v(x, -)$ are continuous for any $x \in X$, so that $v = \bar{u}$ iff $x \rightsquigarrow v(x, -)$ is a continuous map from X to $C(Y, A)$, the latter taken in the discrete topology. If Y is compact, then the discrete topology on $C(Y, A)$ coincides with the compact-open topology, and for the latter, the map $x \rightsquigarrow v(x, -)$ is always continuous (Bourbaki [10], p. 302). This proves

PROPOSITION 9. *If Y is compact, then the adjointness map $C(X, C(Y, A)) \to$ $\to C(X \times Y, A)$ is an isomorphism for any algebra A.*

Remark 1. The converse of this proposition does *not* hold, i.e. the exponent law does not characterize Y as compact: for rather trivial reasons, the law also holds for any connected Y, with arbitrary X; analogously, it holds for any Y and connected X. On the other hand, if Y is infinite discrete, then one can find a Boolean X such that the exponent law fails for $C(X \times Y, \mathbf{2})$: take X with a point which has a neighbourhood basis of the same cardinality as Y. It should be noted, that any composite $C(X, -)\circ \circ C(Y, -)$ can be expressed as a *limit power* with the set $|X| \times |Y|$ as exponent, by Węglorz [55]; however, this is quite different from the exponent law for $C(X \times Y, -)$.

Remark 2. The basic argument in the proof of Proposition 9 was pointed out to us by Brian Davey. The result for X and Y Boolean also appears in Burris [12]; however, there the following fact form Quackenbush [42] is used which we obtain here as a consequence of the exponent law:

COROLLARY. *For any Boolean algebras B and C, $B[C]^* \cong B \amalg C$.*

Proof. This follows immediately from the proposition since $C(Y, \mathbf{2}) \cong \Omega Y$ for any space Y.

We note that Quackenbush actually proved an analogous result for bounded distributive lattices, of which the above is a special case; a topological statement of that result appears in Davey [18].

The case of the functor $E(X, -)$ we want to consider in more general form. Recall from the preceding section that any filter $\mathfrak{F}$ in the Boolean algebra $\mathfrak{R}X$ determines a congruence $\theta_{\mathfrak{F}}$ on $D(X, A)$; in the following, $E_{\mathfrak{F}}(X, A)$ will be the *Boolean filter power* $D(X, A)/\theta_{\mathfrak{F}}$ of the algebra A. For an ultrafilter $\mathfrak{A}$ in $\mathfrak{R}X$, $E_{\mathfrak{A}}(X, A)$ is also referred to as a *Boolean ultrapower* of A.

It is clear that $E_{\mathfrak{F}}(X, A)$ can also be described à la Foster as the quotient of $A[\mathfrak{R}X]$ modulo the congruence consisting of all (u, v) such that $\bigvee u(a) \wedge v(a)$ $(a \in A)$ belongs to $\mathfrak{F}$. In particular, this shows that our Boolean ultrapowers are in fact the same as those of Mansfield [38]. Note, on the other hand, that $E_{\mathfrak{F}}(X, A) = E(X, A)$ if $\mathfrak{F}$ is the smallest filter, consisting only of X. The basic isomorphism $E(X, A) \simeq E(\Omega\mathfrak{R}X, A)$ (see the first section) induces analogous isomorphisms for Boolean filter powers as follows:

LEMMA 2. *For any space X and any algebra A, if $\mathfrak{F}$ is a filter in $\mathfrak{R}X$, and Σ the filter of regular open neighbourhoods in $\Omega\mathfrak{R}X$ of the closed set $\{\mathfrak{A} \mid \mathfrak{F} \subseteq \mathfrak{A} \in \Omega\mathfrak{R}X\}$, then $E_{\mathfrak{F}}(X, A) \cong E_{\Sigma}(\Omega\mathfrak{R}X, A)$.*

Proof. Let $u \rightsquigarrow u'$ be the map $D(X, A) \to D(\Omega\mathfrak{R}X, A)$ considered in the first section, i.e. $u'(\mathfrak{A}) = a$ iff $u^{-1}\{a\} \cap Du \in \mathfrak{A}$. Then for any $u, v \in D(X, A)$, if $\mathrm{Eq}(u, v) \supseteq Du \cap Dv \cap W$ for some $W \in \mathfrak{F}$ it follows that $\mathrm{Eq}(u', v') \supseteq Du' \cap \cap Dv' \cap W^*$, where $W^* = \{\mathfrak{A} \mid W \in \mathfrak{A} \in \Omega\mathfrak{R}X\}$, which belongs to Σ. Conversely, if $(u', v') \in \theta_{\Sigma}$, then there is a $W \in \mathfrak{F}$ such that $\mathrm{Eq}(u'v') \supseteq Du' \cap Dv' \cap W^*$, and this implies u and v coincide on $Du \cap Dv \cap W$. This shows the map $u \rightsquigarrow u'$ makes $\theta_{\mathfrak{F}}$ on $D(X, A)$ and θ_{Σ} on $D(\Omega\mathfrak{R}X, A)$ correspond to each other, and the assertion is an immediate consequence of this.

The following simple observation will be useful in our study of Boolean ultrapowers.

LEMMA 3. *The natural embedding $\mathbf{2} \to E_{\mathfrak{F}}(X, \mathbf{2})$ is an isomorphism iff $\mathfrak{F}$ is an ultrafilter on $\mathfrak{R}X$.*

Proof. If $\mathfrak{F}$ is an ultrafilter, then for each $f \in D(X, 2)$ since the sets $I\Gamma(f^{-1}\{0\} \cap Df)$ and $I\Gamma(f^{-1}\{1\} \cap Df)$ have dense union, one of them belongs to $\mathfrak{F}$. But $I\Gamma(f^{-1}\{a\} \cap Df) \in \mathfrak{F}$ implies that f is congruent, modulo $\theta_{\mathfrak{F}}$, to the constant function with value a; consequently the map $\mathbf{2} \to E_{\mathfrak{F}}(X, \mathbf{2})$ is onto, and hence is an isomorphism.

Conversely, if this map is an isomorphism, then for $U \in \mathfrak{R}X$, the map $f\colon X \to \mathbf{2}$ which is constant on U, with value 1, and constant on CU with value 0, is congruent, modulo $\theta_{\mathfrak{F}}$, to a constant function g. If g has walue 0, then for some $V \in \mathfrak{F}, f|V \cap (U \cup ICU)$ is constant with value 0, and so $V \subseteq U$ which implies $U \in \mathfrak{F}$; if g has value 1, then $CU \in \mathfrak{F}$ and this shows that $\mathfrak{F}$ is an ultrafilter.

If $\mathfrak{F}$ and $\mathfrak{G}$ are any filters in $\mathfrak{R}X$ and $\mathfrak{R}Y$, respectively, for any spaces X and Y, we let $\mathfrak{F} * \mathfrak{G}$ be the filter consisting of all $S \in \mathfrak{R}(X \times Y)$ such that $I\Gamma I\bar{S} \in \mathfrak{F}$, where $\bar{S}$ is the set of all $x \in X$ for which $I\Gamma\{y|(x, y) \in S\} \in \mathfrak{G}$. That this is indeed a filter follows from the fact that (i) $\overline{S \cap T} = \bar{S} \cap \bar{T}$ because $\mathfrak{G}$ is filter, and (ii) the topological operator $I\Gamma I$ preserves finite intersections. Note that the sets $U \times V$, $U \in \mathfrak{F}$ and $V \in \mathfrak{G}$, belong to $\mathfrak{F} * \mathfrak{G}$; also, if $\mathfrak{F} = \{X\}$ and $\mathfrak{G} = \{Y\}$, then $\mathfrak{F} * \mathfrak{G} = \{X \times Y\}$, since $I\bar{S}$ cannot be dense in X for any $S \in \mathfrak{R}(X \times Y)$ different from $X \times Y$.

Now, for any algebra A, and any spaces X and Y, the adjointness map $u \rightsquigarrow \bar{u}$ maps $D(X, D(Y, A))$ into $D(X \times Y, A)$: let $U_\alpha \subseteq X$ be open such that u is constant on each U_α and $\bigcup U_\alpha$ is dense in X; now, for $x \in U_\alpha$, $u(x)$ is continuous on some dense open $V_\alpha \subseteq Y$, and hence $\bar{u}$ is continuous on $\bigcup U_\alpha \times V_\alpha$, which is dense open in $X \times Y$.

Proposition 10. *For any spaces X and Y, any algebra A, and any filters $\mathfrak{F}$ in $\mathfrak{R}X$ and $\mathfrak{G}$ in $\mathfrak{R}Y$, the adjointness map induces an embedding $E_{\mathfrak{F}}(X, E_{\mathfrak{G}}(Y, A)) \to E_{\mathfrak{F}*\mathfrak{G}}(X \times Y, A)$.*

Proof. Consider the diagram

$$\begin{tikzcd}
D(X, D(Y, A)) \arrow[r, "\varphi"] \arrow[d, "\bar{\mu}"] & D(X \times Y, A) \arrow[dd, "\varrho"] \\
D(X, E_{\mathfrak{G}}(Y, A)) \arrow[d, "\nu"] & \\
E_{\mathfrak{F}}(X, E_{\mathfrak{G}}(Y, A)) \arrow[r, dashed] & E_{\mathfrak{F}*\mathfrak{G}}(X \times Y, A)
\end{tikzcd}$$

where φ is the adjointness map, ν and ϱ are the quotient maps, and $\bar{\mu}$ is induced by the quotient map $\mu\colon D(Y, A) \to E_{\mathfrak{G}}(Y, A)$. The claim, then, is that $\operatorname{Ker}(\nu\bar{\mu}) = \operatorname{Ker}(\varrho\varphi)$ since this will imply that $\varrho\varphi$ factors through the onto homomorphism $\nu\bar{\mu}$ such that the factoring map is an embedding.

If $(u, v) \in \operatorname{Ker}(\nu\bar{\mu})$, then $\nu(\mu u) = \nu(\mu v)$, and therefore $\operatorname{Eq}(\mu u, \mu v) \supseteq U \cap V$, where U is dense open and contained in $D(\mu u) \cap D(\mu v)$, and $V \in \mathfrak{F}$. Let $U = \bigcup U_\alpha$ be an open partition of U such that μu and μv are constant on each U_α. For $x \in U_\alpha \cap V$, one then has $\mu(u(x)) = \mu(v(x))$ and therefore there exists an $S_\alpha \in \mathfrak{G}$ and a dense open $W_\alpha \subseteq Du(x) \cap Dv(x)$ such that $u(x)|S_\alpha \cap W_\alpha = v(x)|S_\alpha \cap W_\alpha$. Now, let $T = \bigcup (V \cap U_\alpha) \times S_\alpha$ and $S = I\Gamma T$. We claim that $S \in \mathfrak{F} * \mathfrak{G}$: if $x \in U_\alpha \cap V$, then $S_\alpha \subseteq \{y | (x, y) \in S\}$, and since $S_\alpha \in \mathfrak{G}$ this shows that $U_\alpha \cap V \subseteq \bar{S}$ for all α; hence $U \cap V \subseteq \bar{S}$, and therefore $V \subseteq I\Gamma(U \cap V) \subseteq I\Gamma I\bar{S}$, so that $I\Gamma I\bar{S} \in \mathfrak{F}$, which shows $S \in \mathfrak{F} * \mathfrak{G}$. Finally, consider any $(x, y) \in T \cap W$, where $W = \bigcup U_\alpha \times W_\alpha$, which is easily seen to be dense open and contained in $D\bar{u} \cap D\bar{v}$. Then $x \in U_\alpha$ for a unique α, hence $(x, y) \in (V \cap U_\alpha) \times (S_\alpha \cap W_\alpha)$, and this implies $\bar{u}(x, y) = \bar{v}(x, y)$ by the choice of S_α and W_α. It follows that $\mathrm{Ker}(\nu\mu) \subseteq \mathrm{Ker}(\varrho\varphi)$. For the reverse inclusion, consider now any $(u, v) \in \mathrm{Ker}(\varrho\varphi)$. Let $U = \bigcup U_\alpha$ be a dense open set with an open decomposition such that u and v are constant on each U_α, and $W_\alpha \subseteq Y$ dense open and contained in $Du(x) \cap Dv(x)$ for $x \in U_\alpha$, so that, as before, $\bar{u}$ and $\bar{v}$ are continuous on the dense open set $W = \bigcup U_\alpha \times W_\alpha$. Now $(u, v) \in \mathrm{Ker}(\varrho\varphi)$ implies that $\bar{u}|S \cap W = \bar{v}|S \cap W$ for some $S \in \mathfrak{F} * \mathfrak{G}$, and if $x \in \bar{S} \cap U_\alpha$ and $y \in V \cap W_\alpha$ for $V = \{y | (x, y) \in S\}$, then $\bar{u}(x, y) = \bar{v}(x, y)$, so that $u(x)|V \cap W_\alpha = v(x)|V \cap W_\alpha$ for $x \in \bar{S} \cap U_\alpha$. By continuity it then follows that $u(x)|W_\alpha \cap I\Gamma V = v(x)|W_\alpha \cap I\Gamma V$, and since $x \in \bar{S}$, and therefore $I\Gamma V \in \mathfrak{G}$, this implies that $\mu(u(x)) = \mu(v(x))$. In all, this shows that $\mu u|\bar{S} \cap U = \mu v|\bar{S} \cap U$ which implies, again by continuity, that $\mu u|U \cap I\Gamma I\bar{S} = \mu v|U \cap I\Gamma I\bar{S}$, where $I\Gamma I\bar{S} \in \mathfrak{F}$ by definition of $\mathfrak{F} * \mathfrak{G}$. It now follows that $\nu(\mu u) = \nu(\mu v)$, and hence $(u, v) \in \mathrm{Ker}(\nu\mu)$.

In particular, for the filters $\mathfrak{F} = \{X\}$ and $\mathfrak{G} = \{Y\}$, one has

COROLLARY 1. *For any spaces X and Y, and any algebra A, the adjointness map induces an embedding $E(X, E(Y, A)) \to E(X \times Y, A)$.*

If φ in the above diagram is onto modulo $\sim$ then $\varrho\varphi$ is onto for any $\mathfrak{F}$ and $\mathfrak{G}$, which yields:

COROLLARY 2. *If the embedding $E(X, E(Y, A)) \to E(X \times Y, A)$ is an isomorphism, then for all filters $\mathfrak{F}$ in $\mathfrak{K}X$, $\mathfrak{G}$ in $\mathfrak{K}Y$, the embedding $E_{\mathfrak{F}}(X, E_{\mathfrak{G}}(Y, A)) \to E_{\mathfrak{F}*\mathfrak{G}}(X \times Y, A)$ is an isomorphism.*

Of particular interest is the case when both $\mathfrak{F}$ and $\mathfrak{G}$ are ultrafilters; the next corollary, together with the one above, provides an exponent law for Boolean ultrapowers.

COROLLARY 3. *If the embedding $E(X, E(Y, 2)) \to E(X \times Y, 2)$ is an isomorphism, then $\mathfrak{F} * \mathfrak{G}$ is an ultrafilter whenever $\mathfrak{F}$ and $\mathfrak{G}$ are ultrafilters in $\mathfrak{K}X$ and $\mathfrak{K}Y$ respectively.*

Proof. If $\mathfrak{F}$ and $\mathfrak{G}$ are ultrafilters, then by Lemma 3, the embedding $2 \to E_{\mathfrak{G}}(Y, 2) \to E_{\mathfrak{F}}(X, E_{\mathfrak{G}}(Y, 2))$ is an isomorphism and hence by Corollary 2, the embedding $2 \to E_{\mathfrak{F}*\mathfrak{G}}(X \times Y, 2)$ is also an isomorphism, and this, by Lemma 3, implies that $\mathfrak{F} * \mathfrak{G}$ is an ultrafilter.

We are interested in conditions which ensure that the map $E(X, E(Y, A)) \to E(X \times Y, A)$ is an isomorphism; incidentally, here is a situation in which the fact that something holds for $A = 2$ need *not* imply the same for arbitrary A. There is one simple case in which this exponent law holds: if X is discrete then the dense open subsets of $X \times Y$ are exactly the sets $\{x\} \times V_x$, where V_x is dense open in Y, and this implies the desired result for all A. Below, we give a characterization of the ontoness of $E(X, E(Y, A)) \to E(X \times Y, A)$ for arbitrary X and *discrete* Y. For this, recall that a complete Boolean algebra is called *(m, n)-distributive*, where m and n are any cardinal numbers, iff, for any family $(a_{ij})_{i<m, j<n}$ in the algebra, one has the distribution law

$$\bigwedge_{i<m} \bigvee_{j<n} a_{ij} = \bigvee_{\varphi \in n^m} \bigwedge_{i<m} a_{i\varphi(i)}.$$

Note that a complete Boolean algebra is (m, n)-distributive iff $\bigvee_{\varphi \in n^m} \bigwedge_{i<m} a_{i\varphi(i)} = 1$ whenever $(a_{ij})_{i<m, j<n}$ is a family in the algebra with $a_{ij} \wedge a_{ik} = 0$ for $j \neq k$ and $\bigvee_{j<n} a_{ij} = 1$ for all $i < m$ (Sikorski [46], 19.2 and the remark following 20.3). A Boolean algebra is called *(m, ∞)-distributive* iff it is (m, n)-distributive for all n.

PROPOSITION 11. *For any space X, any discrete Y and any non-trivial algebra A, the following are equivalent:*

(1) *The embedding $E(X, E(Y, A)) \to E(X \times Y, A)$ is an isomorphism.*

(2) *$\mathfrak{R}X$ is $(|Y|, |A|)$-distributive.*

(3) *The embedding $E(X, E(Y, B)) \to E(X \times Y, B)$ is an isomorphism for all algebras B with $|B| < |A|^{|Y|}$.*

(4) *$\mathfrak{R}X$ is $(|Y|, |A|^{|Y|})$-distributive.*

Proof. (1)$\Rightarrow$(2): Suppose (U_{ya}) $(y \in Y, a \in A)$ are regular open subsets of X such that $U_{ya} \cap U_{yb} = \emptyset$ for $a \neq b$ and $\bigcup U_{ya} (a \in A)$ is dense for each $y \in Y$. Let u_y: $X \to A$ be functions such that $u_y | U_{ya}$ is constant, with value a. Then u: $X \times Y \to A$ given by $u(x, y) = u_y(x)$ is continuous on $\bigcup V_y \times \{y\}$, where $V_y = \bigcup U_{ya}$ is open dense in X, and thus $u \in D(X \times Y, A)$. By (1), there exists $v \in D(X, D(Y, A))$ with $\bar{v} \sim u$. Then for each $\varphi \in A^Y = D(X, A)$, since $p_y v$ is continuous on Dv, p_y: $D(Y, A) \to A$ the yth projection, it follows that

$$\begin{aligned} v^{-1}(\varphi) \cap Dv &= \bigcap_{y \in Y} (p_y v)^{-1}(\varphi(y)) \cap Dv \\ &\subseteq \bigcap_{y \in Y} I\Gamma\big((p_y v)^{-1}(\varphi(y)) \cap Dv \cap V_y\big) \\ &= \bigcap_{y \in Y} I\Gamma\big(u_y^{-1}(\varphi(y)) \cap Dv \cap V_y\big) \\ &= \bigcap_{y \in Y} U_{y\varphi(y)} \end{aligned}$$

and consequently $v^{-1}(\varphi)\cap Dv \subseteq I\bigcap_{y\in Y} U_{y\varphi(y)}$. Thus it follows that $\bigvee_{\varphi\in A^Y}\bigwedge_{y\in Y} U_{y\varphi(y)} = I\Gamma \bigcup_{\varphi\in A^Y} I\bigcap_{y\in Y} U_{y\varphi(y)} \supseteq I\Gamma \bigcup_{\varphi\in A^Y} v^{-1}(\varphi)\cap Dv = I\Gamma Dv = x$, and by the above remark, this is sufficient to prove (2).

(2) $\Rightarrow$ (1): Assume (2), and suppose $u\in D(X\times Y, A)$. Then $Du = \bigcup_{y\in Y} V_y\times\{y\}$ for $V_y\subseteq X$ open dense, and if $U_{ya} = \{x\in V_y | u(x, y) = a\}$, then $\bigcup_y U_{ya}\times\{y\} = u^{-1}\{\dot{a}\}\cap Df$ is open, and consequently, U_{ya} is open for each y and a. Since $\bigcup_{y\in Y} I\Gamma U_{ya}\supseteq V_y$, which is open dense, it follows from hypothesis (2) that $I\Gamma \bigcup_{\varphi\in A^Y} I\bigcap_{y\in Y} I\Gamma U_{y\varphi(y)} = X$. Now the sets $W_\varphi = I\bigcap_{y\in Y} I\Gamma U_{y\varphi(y)}$, for $\varphi\in A^Y$, are pairwise disjoint regular open sets with dense open union; and so any $v\colon X\to D(Y, A)$ with $v|W_\varphi$ constant, with value φ, belongs to $D(X, D(Y, A))$, and then $\bar{v}\sim u$.

(1) $\Rightarrow$ (4): Assuming (1), we have the following natural isomorphisms:

$$\begin{aligned}
E(X, E(Y, E(Y, A))) &\cong E(X, E(Y\times Y, A)) && \text{because } Y \text{ is discrete,}\\
&\cong E(X\times(Y\times Y), A) && \text{since } |Y| = |Y\times Y|,\\
&\cong E(Y, E(X\times Y, A)) && \text{by discreteness of } Y,\\
&\cong E(Y, E(X, E(Y, A))) && \text{by (1),}\\
&\cong E(X\times Y, E(Y, A)) && \text{by discreteness of } Y.
\end{aligned}$$

But now the proof of (1) $\Rightarrow$ (2) above implies that $\mathfrak{R}X$ is $(|Y|, |A|^{|Y|})$-distributive.

(4) $\Rightarrow$ (3): Same argument as (2) $\Rightarrow$ (1) above.

(3) $\Rightarrow$ (1): Trivial.

Corollary 1. *For any space X, the embedding $E(X, E(Y, 2))\to E(X\times Y, 2)$ is an isomorphism for all discrete Y iff $\mathfrak{R}X$ is completely distributive.*

Corollary 2. *$\mathfrak{R}X$ is $(|Y|, \infty)$-distributive iff the embedding $E(X, E(Y, A))\to E(X\times Y, A)$, for discrete Y, is an isomorphism for all A.*

Although there is no general exponent law for Boolean ultrapowers, we have the following special result of this sort as a consequence of Corollary 2, and Corollaries 2 and 3 of Proposition 10.

Corollary 3. *If $\mathfrak{R}X$ is $(|Y|, \infty)$-distributive and Y is discrete, then for all ultrafilters $\mathfrak{A}$ in $\mathfrak{R}X$ and $\mathfrak{B}$ in $\mathfrak{R}Y$, $\mathfrak{A}*\mathfrak{B}$ is an ultrafilter in $\mathfrak{R}(X\times Y)$ and $E_{\mathfrak{A}}(X, E_{\mathfrak{B}}(Y, A))\cong E_{\mathfrak{A}*\mathfrak{B}}(X\times Y, A)$ for all algebras A.*

Remark. That the embedding $E(X, E(Y, A))\to E(X\times Y, A)$ may fail to be an isomorphism for $\mathfrak{R}X(|Y|, \infty)$-distributive but Y non-discrete is shown by the following example: If $X\in\mathrm{CBooS}$, then for $Y = \beta Z$ (Z discrete) the embedding $E(X, E(Y, 2))\to E(X\times Y, 2))$ being an isomorphism would imply that $E(\beta(X\times Z), 2)\cong E(X\times\beta Z, 2)$ since $E(X, E(Z, 2))\simeq E(X\times Z, 2)\simeq$

$E(\beta(X\times Z),2)$, the latter because $\Omega\mathfrak{R}(X\times Z)=\Omega(\mathfrak{R}X)^Z=\coprod_Z X=\beta(X\times Z)$, and hence $\beta(X\times Z)\simeq X\times\beta Z$ which does not hold for infinite Z (Glicksberg [25]).

5. Finite A

In this section we investigate the special properties of the functor $C(-,A)$ for a *finite algebra* A. The most notable and immediate consequence of the finiteness of A is that the adjoint $H(-,A)$ of $C(-,A)$ maps $\mathrm{SP}\{A\}$ into BooS since $H(B,A)$ is a closed subspace of the Boolean space $A^{|B|}$, and hence $C(-,A)$ determines a (contravariant) adjointness between BooS and $\mathrm{SP}\{A\}$. (See also Davey [19].) It might be noted that this fact characterizes the finite A: for infinite A, the functor $C(-,A)$ restricted to BooS *cannot* have an adjoint on the right as one immediately sees by considering infinite coproducts in BooS. Throughout the remainder of this section, the algebra A will always be finite.

An obvious consequence of the above observation is that the functor $C(\Omega-,A)=C(-,A)\Omega$: Boo $\to\mathrm{SP}\{A\}$ has a left adjoint, namely $\mathfrak{L}H(-,A)$; this also occurs in Day [20], although a somewhat different description of the adjoint is given there. Now, right adjoint functors preserve equational compactness (Banaschewski [3]), and thus $C(\Omega-,A)$ has this property, a fact first proved by Burris [12] who further notes that this no longer holds, either for $C(\Omega-,A)$ or for $E(\Omega-,A)$, if A is infinite.

Since the equationally compact Boolean algebras are exactly the complete ones, it also follows that $C(X,A)$ is equationally compact for $X\in$ CBooS. Alternatively, this, and hence the preservation of equational compactness by $C(\Omega-,A)$, follows from the fact that $C(X,A)$ *is a retract of a power of A for any* $X\in$ CBooS: If D is the discrete space with the same points as X, βD its Stone–Čech compactification, f: $\beta D\to X$ the continuous map determined by the identity map $D\to X$, and g: $X\to\beta D$ a continuous right inverse of f by the projectivity of X in BooS, then the sequence of homomorphisms

$$C(X,A)\to A^D\to C(\beta D,A)\to C(X,A),\qquad u\rightsquigarrow u\rightsquigarrow\beta u=uf\rightsquigarrow(\beta u)g=u,$$

provides a retraction of A^D to $C(X,A)$ (see also Davey [19]). In particular, this shows $C(X,A)$ is injective in $\mathrm{SP}\{A\}$, or some equational class containing A, whenever A is, and, in any case, $C(X,A)$ is equationally compact.

In the following, we give a sufficient condition for injectivity in $\mathrm{SP}\{A\}$ under suitable additional assumptions on A.

PROPOSITION 12. *If A is finitely powersimple, then any injective in* $\mathrm{SP}\{A\}$ *is isomorphic to some $C(X,A)$ with $X\in$ CBooS.*

Proof. Since the evaluation map $B \to C(H(B, A), A)$ is clearly an embedding for all $B \in \mathrm{SP}\{A\}$, any injective B is a retract of $C(H(B, A), A)$ and hence isomorphic to $C(X, A)$ for some closed subspace X of $H(B, A)$, by Corollary 3 of Proposition 6 and the remarks preceding the corollary of Proposition 5. Now, if $Y \to X$ is the projective cover of X in BooS, then the corresponding embedding $C(X, A) \to C(Y, A)$ has a left inverse which clearly preserves the constants; hence Corollary 5 of Proposition 6 implies that X is a retract of Y and therefore extremally disconnected.

Remark 1. In particular, the above proposition applies to quasi-primal A; moreover, in that case the information actually concerns $\mathrm{HSP}\{A\}$ since this is then equal to $\mathrm{SP}\{A\}$ by Pixley [40].

Remark 2. If A is rigid, i.e. the identity map $A \to A$ is its only endomorphism, in addition to being powersimple, and $\mathrm{SP}\{A\}$ has any non-trivial injectives at all, then A is injective and consequently, as noted earlier, so are all $C(X, A)$ for $X \in$ CBooS. The point is that rigidity makes A an absolute retract in $\mathrm{SP}\{A\}$, and for any non-trivial injective B of $\mathrm{SP}\{A\}$ one has an embedding $A \to B$ by the proposition.

It should be added that, in general, algebras of the form $C(X, A)$, where $X \in$ CBoos and A is finitely powersimple, need not be injective in $\mathrm{SP}\{A\}$; in fact, any simple non-distributive lattice A is not injective in $\mathrm{SP}\{A\}$ since any such A contains a 3-chain, and the latter retracts to a 2-chain.

Concerning equational compactness, we have the following characterization:

PROPOSITION 13. *The equationally compact algebras in the equational class $\mathscr{A}$ generated by a quasi-primal algebra A are, up to isomorphism, precisely the finite products of algebras $C(X, B)$, where $X \in$ CBooS and B is a subalgebra of A.*

Proof. Evidently, by the above remarks, every such product is equationally compact. On the other hand, since each algebra in $\mathscr{A}$ has a pure embedding into a product $A_1^{I_1} \times \ldots \times A_n^{I_n}$, where the A_i are subalgebras of A (Taylor [50]), any equationally compact $B \in \mathscr{A}$ is a retract of such a product. Now $\mathscr{A}$ has factorizable congruences, and hence the kernel of this retraction is a product of congruences θ_i on the factors $A_i^{I_i}$, so that $B \cong \Pi A_i^{I_i}/\theta_i$; further, since each A_i is quasi-primal, one has $A_i^{I_i}/\theta_i \cong C(X_i, A_i)$ for some Boolean space X_i, by the fact that $A_i^{I_i} \cong C(\beta I_i, A_i)$, the third corollary of Proposition 6 and the remark preceding the corollary of Proposition 5. It follows that each equationally compact algebra in $\mathscr{A}$ is a product $C(X_1, A_1) \times \ldots \times C(X_n, A_n)$, where the A_i are subalgebras of A and the $X_i \in$ BooS; it remains to show that the X_i are extremally disconnected.

Now, for such a product, let $p_i\colon \tilde{X}_i \to X_i$ be the projective cover of X_i

in BooS. Then $C(p_i, A_i)$: $C(X_i, A_i) \to C(\tilde{X}_i, A_i)$ is pure for each i; products of pure embeddings being again pure, it follows that $\Pi C(p_i, A_i)$: $\Pi C(X_i, A_i) \to \Pi C(\tilde{X}_i, A_i)$ is pure, and so there is a retraction r: $\Pi C(\tilde{X}_i, A_i) \to \Pi C(X_i, A_i)$. Next, since $\mathscr{A}$ has factorizable congruences and each of the A_i is quasi-primal, there exist closed $Y_i \subseteq \hat{X}_i$ such that $r(u_1, \ldots, u_n) = r(v_1, \ldots, v_n)$ iff $u_i|Y_i = v_i|Y_i$ for all $i \leqslant n$. Finally, an argument analogous to that in the proof of Corollary 6 of Proposition 6 establishes that $Y_i = X_i$ for all i and so r is one-one, yielding the desired result.

COROLLARY. *If A is quasi-primal without subalgebras, then the equationally compact algebras in* HSP$\{A\}$ *are precisely the $C(X, A)$ for $X \in$* CBooS.

Remark. The above proposition generalizes in the following way: The proof of Theorem 1.6 of Taylor [50] can easily be modified to show, for quasi-primal algebras $A_1, \ldots, A_n$ whose quaternary discriminators are given by one and the same derived operation, that every algebra in HSP $\{A_1, \ldots, A_n\}$ has a pure embedding into a product of subalgebras of the A_i. The above argument then shows that the equationally compact algebras in HSP $\{A_1, \ldots, A_n\}$ are precisely the finite products of algebras $C(X, B)$, where $X \in$ CBooS and B is a subalgebra of one of the A_i. This result was announced in Bulman-Fleming and Werner [11].

We close this section with a brief discussion of the well-known connection between Boolean powers and primal algebras due to Hu [31], [32]. An algebra A is, by definition, *primal* iff it is finite, non-trivial, and every function $A^n \to A$ is a derived operation, i.e. equals p_A for some $p \in F_A(x_1, \ldots, x_n)$, the free algebra on $x_1, \ldots, x_n$ in the equational class generated by A, where $p_A(a_1, \ldots, a_n)$ is the image of p under the homomorphism $F_A(x_1, \ldots, x_n) \to A$ mapping x_i to a_i.

If A is primal, then it is quasi-primal and thus HSP$\{A\}$ = SP$\{A\}$. Also, A is rigid and finitely powersimple and hence by Corollary 5 of Proposition 6, $C(-, A)$: BooS $\to$ HSP$\{A\}$ is a dual equivalence.

For the converse, suppose the functor $C(-, A)$: BooS $\to$ HSP$\{A\}$ is a dual equivalence. Then A is non-trivial, and finite since any element of any A^I has only finitely many different images with respect to the homomorphism $A^I \to A$, this being so for the $C(X, A)$, $X \in$ BooS. Now, let u: $A^n \to A$ be any function. Since the free algebras in HSP$\{A\}$ are isomorphic to algebras $C(X, A)$, there exists $X \in$ BooS and $u_1, \ldots, u_n \in C(X, A)$ such that $C(X, A)$ is the free algebra on $\{u_1, \ldots, u_n\}$ in HSP$\{A\}$. Define p: $X \to A$ by $p(x) = u(u_1(x), \ldots, u_n(x))$ so that $p \in C(X, A)$, as a composite of continuous maps. But then, for $a_1, \ldots, a_n \in A$, the homomorphism h: $C(X, A) \to A$ mapping u_i to a_i must, in view of the fullness of $C(-, A)$, be evaluation at some $x \in X$, and consequently $u(a_1, \ldots, a_n) = u(u_1(x), \ldots, u_n(x)) = p(x) = h(p) = p_A(a_1, \ldots, a_n)$. Thus every function $A^n \to A$ is a derived operation, and so A is primal. Note that the entire argument still holds for SP $\{A\}$ in place of HSP $\{A\}$.

Finally, if T: BooS $\to \mathscr{A}$, is *any* dual equivalence, for any equational class $\mathscr{A}$, then T is naturally equivalent to $C(-, T1)$ by Proposition 2, and $\mathscr{A} = \mathrm{SP}\{T\mathbf{1}\}$ since any $X \in$ BooS is a quotient of a copower of $\mathbf{1}$. Hence, $T\mathbf{1}$ is primal, by the preceeding discussion.

6. Boolean ultrapowers

This section deals with the first order properties of Boolean ultrapowers, and presents a proof of Mansfield's characterization of elementary equivalence in terms of Boolean ultrapowers [38]. First, we provide a criterion for satisfaction in Boolean ultrapowers, where X is an arbitrary topological space. The special case for $X \in$ CBooS and atomic formulae was given in Daigneault [17].

Throughout this section we write "$[u]$" for the image of u under the quotient map $D(X, A) \to E_{\mathfrak{A}}(X, A)$.

Proposition 14. *For any algebra A, any formula φ in the language of A with n free variables, any ultrafilter $\mathfrak{A}$ in $\mathfrak{R}X$ and $u_1, \ldots, u_n \in D(X, A)$,*

$$E_{\mathfrak{A}}(X, A) \models \varphi([u_1], \ldots, [u_n]) \quad \textit{iff} \quad I\Gamma\{x \in U \mid A \models \varphi(u_1(x), \ldots, u_n(x))\} \in \mathfrak{A}$$

for some dense open $U \subseteq X$ on which all the u_i are continuous.

Proof. Note that if the latter condition holds for some dense open U on which all the u_i are continuous, then it holds for *all* such dense open U.

The proof is by induction on the complexity of the formula φ.

For an atomic formula, i.e. equation (p, q), if $p([u_1], \ldots, [u_n]) = q([u_1], \ldots, [u_n])$, then $p(u_1(x), \ldots, u_n(x)) = q(u_1(x), \ldots, u_n(x))$ for all $x \in U \cap V$ for some $V \in \mathfrak{A}$, where $U = Du_1 \cap \ldots \cap Du_n$, and so $I\Gamma\{x \in U \mid p(u_1(x), \ldots, u_n(x)) = q(u_1(x), \ldots, u_n(x))\} \in \mathfrak{A}$. Conversely, if the latter holds for appropriate U, then $W = \{x \in U \mid p(u_1(x), \ldots, u_n(x)) = q(u_1(x), \ldots, u_n(x))\}$ and $W' = U - W$ are disjoint open sets by the continuity of the u_i on U, and so $W' \cap I\Gamma W = \emptyset$, which implies that $p(u_1(x), \ldots, u_n(x)) = q(u_1(x), \ldots, u_n(x))$ for all $x \in U \cap I\Gamma W$ from which it follows that $p([u_1], \ldots, [u_n]) = q([u_1], \ldots, [u_n])$.

The induction step for negation (indicated by "$\neg$") employs the fact that for any dense open U on which $u_1', \ldots, u_n$ are continuous, $I\Gamma\{x \in U \mid A \models \varphi(u_1(x), \ldots, u_n(x))\}$ and $I\Gamma\{x \in U \mid A \models \varphi(u_1(x), \ldots, u_n(x))\}$ are disjoint regular open sets whose join in $\mathfrak{R}X$ is X, and hence exactly one of them belongs to $\mathfrak{A}$.

The induction step for disjunction is a straightforward consequence of the fact that, for any open sets U and V, $I\Gamma(U \cup V) = I\Gamma(I\Gamma U \cup I\Gamma V)$ and the latter is the join in $\mathfrak{R}X$ of $I\Gamma U$ and $I\Gamma V$.

Re existential quantification: If

$$E_{\mathfrak{A}}(X, A) \models (\exists z)\,\varphi(z, [u_1], \ldots, [u_n]),$$

then $\varphi([u], [u_1], \ldots, [u_n])$ holds in $E_{\mathfrak{A}}(X, A)$ for some $u \in D(X, A)$ so that the induction hypothesis implies $I\Gamma\{x \in U | A \models \varphi(u(x), u_1(x), \ldots, u_n(x))\} \in \mathfrak{A}$ and consequently $I\Gamma\{x \in U | A \models (\exists z)\,\varphi(z, u_1(x), \ldots, u_n(x))\} \in \mathfrak{A}$, where U is any dense open set on which u and all the u_i are continuous. Conversely, if for a dense open U on which all u_i are continuous,

$$V = I\Gamma\{x \in U | A \models (\exists z)\,\varphi(z, u_1(x), \ldots, u_n(x))\} \in \mathfrak{A},$$

then for any partition of U into open blocks U_λ $(\lambda \in \Lambda)$ on which each of the u_i is constant, let $u\colon X \to A$ be any function which is constant on each U_λ such that if $x \in U_\lambda$ and $A \models (\exists z)\,\varphi(z, u_1(x), \ldots, u_n(x))$, then $\varphi(u(x), u_1(x), \ldots, u_n(x))$ holds in A. Such a u is continuous on U, and $I\Gamma\{x \in U | A \models \varphi(u(x), u_1(x), \ldots, u_n(x))\} = V \in \mathfrak{A}$ and hence by induction $E_{\mathfrak{A}}(X, A) \models \varphi([u], [u_1], \ldots, [u_n])$ so that $(\exists z)\,\varphi(z, [u_1], \ldots, [u_n])$ holds in $E_{\mathfrak{A}}(X, A)$, completing the proof.

We now have the following immediate consequence of the proposition (see also Daigneault [17]).

COROLLARY. *For any ultrafilter $\mathfrak{A}$ in $\mathfrak{R}X$ and any algebra A the embedding $A \to E_{\mathfrak{A}}(X, A)$ is elementary.*

For the next proposition, recall that an ultrafilter in a Boolean algebra is called *countably incomplete* whenever it fails to be closed under countable meets, and that an algebra A is *$\aleph_1$-saturated* iff any countable set of formulae with constants from A in one free variable is simultaneously satisfiable in A whenever this is the case for each of its finite subsets.

PROPOSITION 15 (Mansfield [38]). *Boolean ultrapowers with respect to countably incomplete ultrafilters are $\aleph_1$-saturated.*

Proof. If $\mathfrak{A}$ is countably incomplete, then there exist $U_n \in \mathfrak{A}$ $(n \in \omega)$ with $U_n \supseteq U_{n+1}$ for all n and $I \bigcap U_n = \emptyset$.

Suppose φ_n $(n \in \omega)$ are formulae with one free variable, possibly with constants from $E_{\mathfrak{A}}(X, A)$, such that $\{(\exists z)\,\varphi_n(z) | n \in \omega\}$ is finitely satisfiable in $E_{\mathfrak{A}}(X, A)$. We may assume without loss of generality that $\varphi_{n+1} \to \varphi_n$ is a tautology for all n (otherwise replace φ_n by $\varphi_n \wedge \varphi_{n-1} \wedge \ldots \wedge \varphi_0$). Thus there exist $u_n \in D(X, A)$ such that $E_{\mathfrak{A}}(X, A) \models \varphi_n([u_n])$, i.e., $I\Gamma\{x \in Du_n | A \models \varphi_n(u_n(x))\} \in \mathfrak{A}$.

Let $V_n = \{x \in U_n \cap Du_0 \cap \ldots \cap Du_n | A \models \varphi_k(u_k(x))$ for all $k \leqslant n\}$. Then for all n, V_n is open, $I\Gamma V_n \in \mathfrak{A}$ (this follows from the fact that $I\Gamma$ distributes over finite intersections of open sets) and $V_{n+1} \subseteq V_n$. Also $I \bigcap I\Gamma V_n \subseteq I \bigcap U_n = \emptyset$. Now let $W_n = V_n \cap ICV_{n+1}$; then the W_n are pairwise disjoint open sets. For any $u\colon X \to A$ with $u|W_n = u_n|W_n$ and $u|C\bigcup W_n$ constant, $Du \supseteq \bigcup W_n \cup IC \cup W_n$ which is open dense and hence $u \in D(X, A)$. Also,

$$\begin{aligned}\{x \in Du \mid A \models \varphi_n(u(x))\} &\supseteq \bigcup_{k \geqslant n} \{x \in W_k \mid A \models \varphi_n(u(x))\} \\ &= \bigcup_{k \geqslant n} \{x \in W_k \mid A \models \varphi_n(u_k(x))\} \\ &\supseteq \bigcup_{k \geqslant n} \{x \in W_k \mid A \models \varphi_k(u_k(x))\} \\ &= \bigcup_{k \geqslant n} W_k.\end{aligned}$$

Now an easy induction shows that for all $k \geqslant n$, $W_n \cup \ldots \cup W_k = V_n \cap ICV_k \cap U_k$ for some open dense U_k, and thus $\Gamma(W_n \cup \ldots \cup W_k) = \Gamma(V_n \cap ICV_k)$. Consequently $\Gamma \bigcup_{k \geqslant n} W_k \supseteq \bigcup_{k \geqslant n} \Gamma(V_n \cap ICV_k) \supseteq V_n \cap \bigcup_{k \geqslant n} ICV_k$.

Now

$$\Gamma \bigcup_{k \geqslant n} ICV_k = \Gamma \bigcup_{k \geqslant n} \Gamma ICV_k = \Gamma \bigcup_{k \geqslant n} CI\Gamma V_k = \Gamma C \bigcap_{k \geqslant n} I\Gamma V_k = CI \bigcap_{k \geqslant n} I\Gamma V_k = X,$$

and so $\Gamma \bigcup_{k \geqslant n} W_k \supseteq \Gamma\big(V_n \cap \bigcup_{k \geqslant n} ICV_k\big) = \Gamma V_n$; and the reverse inclusion is trivial. Finally, this implies $I\Gamma \bigcup_{k \geqslant n} W_k = I\Gamma V_n$ and so $I\Gamma\{x \in Du \mid A \models \varphi_n(u(x))\} \in \mathfrak{A}$ which means $E_{\mathfrak{A}}(X, A) \models \varphi_n([u])$. This holds for all $n \in \omega$, and so we have the desired result.

An analogue of the following lemma is implicitly used in Mansfield [38], and a generalization and partial converse can be found in Bankston [8]. We include a proof for the sake of completeness.

Proposition 16. *If a topological space X has a basis of open sets which is closed under countable intersections, such that the intersection of any countable decreasing sequence of non-empty basic open sets is non-empty, then $\mathfrak{R}X$ is $(\aleph_0, \infty)$-distributive.*

Proof. Note that the hypotheses imply that the intersection of any countable collection of open sets is open. Further, if U_n $(n \in \omega)$ are open dense subsets of X and V is a non-empty basic open set then a simple induction produces a decreasing sequence of non-empty basic open V_n with $V_n \subseteq V \cap U_1 \cap \ldots \cap U_n$ and hence $\emptyset \neq \bigcap V_n \subseteq V \cap \bigcap U_n$ and so $\bigcap U_n$ is dense. Thus the union of countably many closed nowhere dense sets is nowhere dense.

But then for all open $U_n \subseteq X$ $(n \in \omega)$

$$\begin{aligned}\bigcap I\Gamma U_n \cap C\Gamma \bigcap U_n &= \bigcap I\Gamma U_n \cap IC \bigcap U_n \\ &= I\big(\bigcap I\Gamma U_n \cap \bigcup CU_n\big) \subseteq I\big(\bigcup \Gamma U_n \cap CU_n\big) = \emptyset\end{aligned}$$

which implies $\bigcap I\Gamma U_n \subseteq \Gamma \bigcap U_n$ and so $\bigcap I\Gamma U_n \subseteq I\Gamma \bigcap U_n$; the reverse inclusion being trivial it follows that $I\Gamma$ distributes over countable intersections of open sets.

Consequently, if $U_{ni} \in X$ $(n \in \omega, i \in I)$, then

$$\begin{aligned}\bigwedge_n \bigvee_i U_{ni} &= I \bigcap_n I\Gamma \bigcup_i U_{ni} = I\Gamma \bigcap_n \bigcup_i U_{ni} \\ &= I\Gamma \bigcup_{\varphi \in I^\omega} \bigcap_n U_{n\varphi(n)} = I\Gamma \bigcup_\varphi I \bigcap_n U_{n\varphi(n)} \\ &= \bigvee_\varphi \bigwedge_n U_{n\varphi(n)}.\end{aligned}$$

The following result and the basic ideas used in its proofs, are contained in Mansfield [38].

PROPOSITION 17. *For $\aleph_1$-saturated algebras A and B, A is elementarily equivalent with B iff there exists a topological space X with $\mathfrak{R}X$ $(\aleph_0, \infty)$-distributive such that $E_{\mathfrak{A}}(X, A) \simeq E_{\mathfrak{A}}(X, B)$ for all filters $\mathfrak{A}$ in $\mathfrak{R}X$.*

Proof. $(\Rightarrow)$: Let X be the topological space whose underlying set consists of all partial elementary maps from A to B, i.e. the graphs of all functions $f\colon S \to B$, where $S \subseteq A$ and A, with the elements of S as additional nullary operations, is elementarily equivalent with B with the elements $f(s)$ $(s \in S)$ as corresponding nullary operations, such that X has as basis for open sets, the sets

$$[f] = \{t \in X \mid f \subseteq t\},$$

where f is any at most countable partial function from A to B. Note that $[f] \neq \emptyset$ iff f is a partial elementary function.

Then for countable $f_n \in X$ $(n \in \omega)$ either $\bigcup f_n$ is a partial elementary map, and so $\bigcap [f_n] = [\bigcup f_n] \neq \emptyset$ or $\bigcup_{n \leq m} f_n$ is not elementary for some $m \in \omega$, and then $\bigcap_{n \leq m} [f_n] = \emptyset$. It then follows from the above proposition that $\mathfrak{R}X$ is $(\aleph_0, \infty)$-distributive.

For $(a, b) \in A \times B$, $[(a, b)] = [\{(a, b)\}]$ is open, and consequently $U_a = \bigcup_b [(a, b)] = \{t \in X \mid a \in \mathrm{dom}(t)\}$ is open in X. Since $A \equiv B$, it follows from the $\aleph_1$-saturicity of A and B that for each at most countable $t \in X$ there exists $f \supseteq t$ with $f \in X$ and $a \in \mathrm{dom}(f)$, so that $[t] \cap U_a \neq \emptyset$, which shows that U_a is dense for each $a \in A$.

Define $\varphi\colon D(X, A) \to D(X, B)$ as follows: for $u \in D(X, A)$, let $\varphi(u)$ be any function $X \to B$ with $\varphi(u) | Du \cap u^{-1}\{a\} \cap U_a = \mathrm{ev}(a)$, the evaluation map at a (i.e. $\varphi(u)(t) = t(a)$ for $a \in Du \cap u^{-1}\{a\} \cap U_a$) for each $a \in A$. It is clear that the sets $Du \cap u^{-1}\{a\} \cap U_a$ are pairwise disjoint and open (for $a \in A$); moreover, for $b \in B$

$$\varphi(u)^{-1}\{b\} \cap Du \cap u^{-1}\{a\} \cap U_a = [(a, b)] \cap Du \cap u^{-1}\{a\} \cap U_a$$

which is open, and consequently $\varphi(u)$ is continuous on $\bigcup_a Du \cap u^{-1}\{a\} \cap U_a$, which in view of the fact that Du and all the U_a are open dense, is itself open dense.

ψ: $D(X,B) \to D(X,A)$ is defined analogously, that is, for $v \in D(X,B)$, $\psi(v)(t) = t^{-1}(b)$ for each $t \in Dv \cap v^{-1}\{b\} \cap V_b$, where $V_b = \bigcup_{a\in A} [(a,b)]$.

Since for each $t \in X$, t^{-1} is a partial elementary map from B to A, it follows that $\varphi\psi(u) \sim u$ and $\psi\varphi(u) \sim u$ for all u in $D(X,B)$ or $D(X,A)$ respectively.

Moreover, for all $u, v \in D(X,A)$, there exists an open dense $U \subseteq X$ with $U \cap \mathrm{Eq}(u,v) \subseteq \mathrm{Eq}(\varphi(u), \varphi(v))$ (where $\mathrm{Eq}(u,v) = \{x \in X \mid u(x) = v(x)\}$); this can be seem as follows: Let $U = \bigcup_a Du \cap Dv \cap u^{-1}\{a\} \cap U_a$. Then U is open dense, and for $t \in U$, $u(t) = v(t) = a$, say, implies $\varphi(u)(t) = t(a) = \varphi(v)(t)$, which proves the point.

Now suppose that λ is an n-ary operation of the type of A, and that $u_1, \dots, u_n \in D(X,A)$. Then there exist open W_α $(\alpha \in I)$ with $\bigcup W_\alpha$ dense, and each u_i constant on W_α, say with value α_i. Let $\alpha_{n+1} = \lambda(\alpha_1, \dots, \alpha_n)$. Then for $t \in W_\alpha \cap U_{\alpha_1} \cap \dots \cap U_{\alpha_n} \cap U_{\alpha_{n+1}}$

$$\begin{aligned}\lambda(\varphi(u_1), \dots, \varphi(u_n))(t) &= \lambda(\varphi(u_1)(t), \dots, \varphi(u_n)(t))\\ &= \lambda(t(\alpha_1), \dots, t(\alpha_n)) = t(\lambda(\alpha_1, \dots, \alpha_n))\\ &= t(\alpha_{n+1}) = \varphi(\lambda(u_1, \dots, u_n))(t).\end{aligned}$$

Since each U_{α_i} is dense, it follows that $\bigcup_{\alpha\in I} W_\alpha \cap U_{\alpha_1} \cap \dots \cap U_{\alpha_{n+1}}$ is dense, and this establishes the fact that, up to $\sim$, φ is a homomorphism. Analogously, ψ is, up to $\sim$, a homomorphism.

But now these facts, together with the facts that $\varphi\psi$ and $\psi\varphi$ are, up to $\sim$, identity maps, and that for u, $v \in D(X,A)$ there exists an open dense U with $\mathrm{Eq}(u,v) \cap U \subseteq \mathrm{Eq}(\varphi(u), \varphi(v))$, and the analogous fact for ψ, imply that for any filter $\mathfrak{A}$ in $\mathfrak{R}X$, φ and ψ factor through $E_{\mathfrak{A}}(X,A)$ and $E_{\mathfrak{A}}(X,B)$ to produce homomorphisms which are inverse to one another, and this implies $E_{\mathfrak{A}}(X,A) \cong E_{\mathfrak{A}}(X,B)$.

The reverse implication is trivial.

COROLLARY (Mansfield [38]). *Two algebras are elementarily equivalent iff they have isomorphic Boolean ultrapowers.*

Proof. If $A \equiv B$, then for any non-principal ultrafilter $\mathfrak{A}$ on ω, $\mathfrak{A}$ is countably incomplete, and $E_{\mathfrak{A}}(\omega, A)$ and $E_{\mathfrak{A}}(\omega, B)$ are $\aleph_1$-saturated besides being elementarily equivalent. The result then follows from the proposition and the restricted exponent law for Boolean ultrapowers expressed in Corollary 3 of Proposition 11.

We now consider certain properties of Boolean ultrapowers with respect to countably complete ultrafilters; the appropriate setting for this deals with the language $L_{\omega_1\omega}$ which allows countable disjunctions (and conjunctions) of formulae in addition to the usual first order rules for formation of formulae, and its extension $L_{\omega_1\omega_1}$, which in addition allows simultaneous quantification over countably many variables.

PROPOSITION 18. *For any algebra A, any formula φ in the $L_{\omega_1\omega}$ language of A with n free variables, any countably complete ultrafilter $\mathfrak{A}$ in $\mathfrak{R}X$ and any $u_1, \ldots, u_n \in D(X, A)$,*

$$E_{\mathfrak{A}}(X, A) \models \varphi([u_1], \ldots, [u_n]) \quad \textit{iff} \quad I\Gamma\{x \in U | A \models \varphi(u_1(x), \ldots, u_n(x))\} \in \mathfrak{A}$$

for any open U on which all the u_i are continuous.

Proof. The proof is the same as that of Proposition 14, except for the additional induction step for countable disjunction which relies on the fact that for open U_n $(n \in \omega)$, $I\Gamma \bigcup U_n = I\Gamma \bigcup I\Gamma U_n$, where the latter is the join in $\mathfrak{R}X$ of the sets $I\Gamma U_n$, and the fact that, for a countably complete ultrafilter $\mathfrak{A}$, $U_n \in \mathfrak{A}$ for some n whenever $\bigvee U_n \in \mathfrak{A}$.

COROLLARY. *For $\mathfrak{A}$ a countably complete ultrafilter in $\mathfrak{R}X$ the embedding $A \to E_{\mathfrak{A}}(X, A)$ is $L_{\omega_1\omega}$-elementary.*

If $\mathfrak{R}X$ is $(\aleph_0, \infty)$-distributive, then the intersection of any countably many dense open sets has dense interior, and so for any $u_n \in D(X, A)$ $(n \in \omega)$ there is a dense open $U \subseteq X$ on which all the u_n are continuous. But then for such X the argument in the proof of Proposition 14 can be extended to $L_{\omega_1\omega_1}$ formulae, to show the following

PROPOSITION 19. *For any algebra A, any countably complete ultrafilter $\mathfrak{A}$ in $\mathfrak{R}X$, any formula φ in the $L_{\omega_1\omega_1}$-language of A, any $u_n \in D(X, A)$ $(n \in \omega)$, if $\mathfrak{R}X$ is $(\aleph_0, \infty)$-distributive, then*

$$E_{\mathfrak{A}}(X, A) \models \varphi([u_n]_{n\in\omega}) \quad \textit{iff} \quad I\Gamma\{x \in U | A \models \varphi(((u_n(x))_{n\in\omega})\} \in \mathfrak{A}$$

for some dense open U on which all the u_n are continuous.

COROLLARY. *For $\mathfrak{A}$ countably complete and $\mathfrak{R}X$ $(\aleph_0, \infty)$-distributive, the embedding $A \to E_{\mathfrak{A}}(X, A)$ is $L_{\omega_1\omega_1}$-elementary for any algebra A.*

The above corollary generalizes a result on ultrapowers, see Chang–Keisler [15], p. 183.

Next we direct our attention to certain rather special $L_{\omega_1\omega}$-formulae, namely those of the form

$$(*) \qquad (\exists x_1) \ldots (\exists x_m)(\forall y_1) \ldots (\forall y_k) \bigvee_n \varphi_n(x_1, \ldots, x_m, y_1, \ldots, y_k),$$

where the φ_n are first order formulae in $m+k$ free variables.

Note that if an $\aleph_1$-saturated algebra A satisfies $(*)$, then it satisfies

$$(\exists x_1) \ldots (\exists x_m)(\forall y_1) \ldots (\forall y_k) \bigvee_{n \leqslant r} \varphi_n(x_1, \ldots, x_m, y_1, \ldots, y_k)$$

for some finite r. This can be seen as follows: suppose $a_1, \ldots, a_m \in A$ and for each r, the formula $\neg \bigvee_{n \leqslant r} \varphi_n(a_1, \ldots, a_m, y_1, \ldots, y_k)$ is satisfiable in A. Now, A being $\aleph_1$-saturated implies that every countable set of formulae with finitely many constants from A and k free variables is simultaneously satisfiable in A whenever every finite subset has this property (see Chang–Keisler [15], p. 98), and hence A satisfies

$$(\exists y_1) \ldots (\exists y_k) \neg \bigvee_n \varphi_n(a_1, \ldots, a_n, y_1, \ldots, y_n)$$

contradicting (*).

Consequently if P is a property of algebras definable by a sentence of the form (*), then any $\aleph_1$-saturated algebra with property P satisfies a P-sentence (see Burris and Jeffers [13]), i.e. a sentence, each of whose models has P.

Combining this with the previous results we have:

PROPOSITION 20. *For a countably complete ultrafilter* $\mathfrak{A}$ *in* $\mathfrak{K}X$, *and a* (*)*-property* P, *an algebra* A *has* P *iff* $E_{\mathfrak{A}}(X, A)$ *has* P; *moreover,* $E_{\mathfrak{A}}(X, A)$ *has* P *for* $\mathfrak{A}$ *countably incomplete iff* $E_{\mathfrak{A}}(X, A)$, *and hence also* A, *satisfies a* P-*sentence.*

Now, it follows from Mal'cev's characterization of principal congruences (see Taylor [49]) that for any at most countable type τ of finitary algebras, if $\psi_n(x, y, u, v)$ $(n \in \omega)$ is an enumeration of all formulae of type τ, in four free variables, of the form

$$(\exists x_1) \ldots (\exists x_k) \big(u = p_1(x, y, x_1, \ldots, x_k) \wedge v = p_m(y, x, x_1, \ldots, x_k) \wedge \\ \wedge \bigwedge_{i \leqslant m} p_i(y, x, x_1, \ldots, x_k) = p_{i+1}(x, y, x_1, \ldots, x_k)\big),$$

the p_i $(i \leqslant m)$ being derived operations, $k, m \in \omega$, then for any algebra A of type τ and $a, b, c, d \in A$, $(a, b) \in \theta(c, d)$ (the principal congruence generated by (c, d)) iff $A \models \bigvee_n \psi_n(a, b, c, d)$. Consequently an algebra A is simple iff for all $a, b, c, d \in A$, either $(a, b) \in \theta(c, d)$ or $c = d$, i.e. iff

$$A \models \bigvee \psi_n^*(a, b, c, d),$$

where $\psi_n^*(a, b, c, d) = \psi_n(a, b, c, d) \vee (c = d)$.

Further, the congruence lattice of A is atomic (i.e. every non-zero element contains an atom) with at most m atoms iff A satisfies

$$(\exists a_1) \ldots (\exists a_m)(\exists b_1) \ldots (\exists b_m)(\forall a)(\forall b) \bigvee_{n,i} \psi_n^*(a_i, b_i, a, b).$$

(For $m = 1$ this means A is subdirectly irreducible.)

Consequently, as an application of the above proposition, we obtain the following result which is contained in Burris and Jeffers [13] for the special case of simplicity and subdirect irreducibility.

COROLLARY 1. *For $\mathfrak{U}$ a countably complete ultrafilter in $\mathfrak{K}X$, A is simple or the congruence lattice of A is atomic with exactly n atoms, iff $E_{\mathfrak{U}}(X, A)$ has the same property; moreover, for a countably incomplete ultrafilter $\mathfrak{U}$ in $\mathfrak{K}X$ if $E_{\mathfrak{U}}(X, A)$ is simple, or has an atomic congruence lattice with at most m atoms, then A satisfies a sentence, each of whose models has this property.*

Since each Boolean ultrapower of a powersimple algebra is simple, a further consequence of the proposition is the following:

COROLLARY 2. *Every powersimple algebra satisfies a simplicity sentence.*

Note that the converse of this does not hold in general, the two-element group providing a counterexample, but does hold in special cases:

COROLLARY 3. *In a congruence factorable equational class, every algebra satisfying a simplicity sentence is powersimple.*

Proof. If A satisfies a simplicity sentence, then every Boolean ultrapower of A by a countably incomplete ultrafilter is both $\aleph_1$-saturated and simple, and so it, and hence also A, satisfies $\bigvee_{n \leqslant r} \psi_n(x, y, u, v) \vee (x = y)$ for some finite r, where the ψ_n are the formulae discussed immediately after Proposition 20, and characterize principal congruences in algebras of the type of A.

To show that A is power simple, it is enough to show that every principal congruence on a power A^I is a filter congruence. For $u, v \in A^I$, let $S = \{i \in I | u(i) = v(i)\}$ and let $T = I - S$, then $A^I \simeq A^T \times A^S$ and the congruence on A^S generated by $(u|S, v|S)$ is just the diagonal. By congruence factorability, it is enough to show that the congruence on A^T generated by $(u|T, v|T)$ is a filter congruence. For this, consider any $t, w \in A^T$. For each $i \in T$, A satisfies $\psi_n(u(i), v(i), w(i))$ for some $n \leqslant r$, and thus there is a partition $S_1, \ldots, S_r$ of T such that A satisfies $\psi_n(u(i), v(i), t(i), w(i))$ for all $i \in S_n$. Consequently $(t|S_n, w|S_n)$ belongs to the congruence on A^{S_n} generated by $(u|S_n, v|S_n)$, and so by congruence factorability, (t, w) belongs to the congruence on A^T generated by $(u|T, v|T)$ which proves that this congruence is $A^T \times A^T$.

It might be pointed out that all that was needed for the above proof was that A satisfied a simplicity sentence, and every congruence on a product $A^I \times A^J$, any I and J, was a product of congruences from A^I and A^J.

As an application of Corollary 3, we see that the lattice $M_\varkappa$ consisting of $\varkappa$ pairwise incomparable elements together with a 0 and 1 are powersimple for $\varkappa \geqslant 3$ since they satisfy the simplicity sentence "L is a lattice with at least five elements in which all chains contain at most three elements".

A further application provides an alternative proof that the lattice ordered ring $\mathbf{Z}$ of integers is powersimple: every partially ordered ring with unit satisfying

$$(\forall x)\,(x \neq 0 \rightarrow x \geqslant 1 \ \vee\ -x \geqslant 1)$$

is simple.

One might point out that in general the fact that an algebra A has a $(*)$-property P does not imply that all its Boolean ultrapowers (or even ultrapowers) have P. Regarding simplicity, this is the case for any infinite simple equationally compact algebra since such an algebra is a retract of each of its ultrapowers; the existence of such groups, for instance, is mentioned in Taylor [48].

7. Elementary properties

In this section we apply the extension to the limit reduced powers, of the Feferman–Vaught techniques [21] for proving elementary equivalence of products in the setting of Boolean powers and bounded Boolean powers, both of these being special cases of limit reduced powers.

For a filter $\mathfrak{F}$ on $I \times I$ and an algebra A, the *limit* power $A^I|\mathfrak{F}$ is the subalgebra of A^I consisting of all functions $u\colon I \rightarrow A$ with $\operatorname{Ker} u \in \mathfrak{F}$. Note that these u are exactly the functions $I \rightarrow A$ which are uniformly continuous with respect to the uniformity on I generated by the equivalence relations belonging to $\mathfrak{F}$, A taken with its discrete uniformity. Since A is finitary, this implies that $A^I|\mathfrak{F}$ is a subalgebra of A^I. Also, $A^I|\mathfrak{F}$ is a direct limit of powers of A, namely $A^I|\mathfrak{F} = \varinjlim A^{I/E}$ ($E \in \mathfrak{F}$, equivalence relation).

For any space X, let $\mathfrak{A}_X$ be the filter on $X \times X$ generated by the equivalence relations on X with open blocks. Then, $\mathfrak{A}_X$ is a uniformity on the underlying set $|X|$ of X and the continuous functions $X \rightarrow A$ are precisely the functions uniformly continuous with respect to $\mathfrak{A}_X$. Hence $C(X, A) = A^{|X|}|\mathfrak{A}_X$.

Every limit power of A evidently contains the constant maps, and is closed under the pointwise quaternary discriminator (since $\operatorname{Ker} Q(t, u, v, w) \supseteq \operatorname{Ker} t \cap \operatorname{Ker} u \cap \operatorname{Ker} v \cap \operatorname{Ker} w$) and so, by Proposition 2, *if A is finite then the limit powers of A are precisely the bounded Boolean powers of A.*

Now if ϑ is a filter on I, and $\mathfrak{F}$ a filter on $I \times I$, then the *limit reduced power* $A^{(\vartheta)}|\mathfrak{F}$ is the image of $A^I|\mathfrak{F}$ under the quotient map $\nu\colon A^I \rightarrow A^{(\vartheta)}$, where $\nu(u) = \nu(v)$ iff u and v agree on a set in ϑ.

Our interest in this notion lies in the fact that $E(X, A)$ is a limit reduced power for any space X and any algebra A. To see this, let $\mathfrak{G}$ be the filter on $X \times X$ generated by the filter basis of all kernels of maps from X to a discrete space which are continuous on an open dense subset of X.

Then $D(X, A) \subseteq A^X|\mathfrak{G}$, and every $u \in A^X|\mathfrak{G}$ factors through a map from X to a discrete space which is continuous on an open dense subset, and hence is itself continuous on an open dense subset. This shows that $D(X, A) = A^X|\mathfrak{G}$. But then if ϑ is the filter on X generated by the open dense sets, we evidently have $E(X, A) = A^{(\vartheta)}|\mathfrak{G}$. Note that the choice of ϑ and $\mathfrak{G}$ does not depend on A! Analogously, if $\mathfrak{F}$ is any filter in $\mathfrak{R}X$, then $E_{\mathfrak{F}}(X, A) = A^{(\bar{\mathfrak{F}})}|\mathfrak{G}$ for the filter $\bar{\mathfrak{F}}$ generated by $\mathfrak{F}$ on X.

For each formula θ in the language L_τ of algebras of the (finitary) type τ, with n free variables, each algebra A of type τ, and $u_1, \ldots, u_n \in A^I|\mathfrak{F}$ ($\mathfrak{F}$ any filter on $I \times I$) we define a function $|\theta|(u_1, \ldots, u_n)\colon I \to \mathbf{2}$ by

$$|\theta|(u_1, \ldots, u_n)(\alpha) = 1 \quad \text{iff} \quad A \models \theta(u_1(\alpha), \ldots, u_n(\alpha)) \quad \text{for } \alpha \in I.$$

Since $\bigcap_{1 \leqslant i \leqslant n} \operatorname{Ker} u_i \subseteq \operatorname{Ker} |\theta|(u_1, \ldots, u_n)$, it follows that $|\theta|(u_1, \ldots, u_n) \in \mathbf{2}^I|\mathfrak{F}$.

The proof of the following proposition can be obtained by adapting the proof for the special case of reduced powers, as it appears in Chang–Keisler [15], Proposition 6.3.2. The proposition is also derived in Burris–Werner [14] as a corollary to a result attributed to Comer. Further, it appears implicitly in Volger [51].

PROPOSITION 21. *For every formula φ of L_τ with n free variables there is a "determining sequence" $(\sigma, \psi_1, \ldots, \psi_m)$, where σ is a Boolean algebra formula with m free variables, and $\psi_1, \ldots, \psi_m$ are formulae of L_τ with the same free variables as φ, such that for all τ-algebras A, all filters $\mathfrak{F}$ on $I \times I$, ϑ on I, and all $u_1, \ldots, u_n \in A^I|\mathfrak{F}$.*

$$A^{(\vartheta)}|\mathfrak{F} \models \varphi(\mu(u_1), \ldots, \mu(u_n))$$

iff

$$\mathbf{2}^{(\vartheta)}|\mathfrak{F} \models \sigma\big(\nu(|\psi_1|(u_1, \ldots, u_n)), \ldots, \nu(|\psi_m|(u_1, \ldots, u_n))\big),$$

where $\mu\colon A^I \to A^{(\vartheta)}$ and $\nu\colon \mathbf{2}^I \to \mathbf{2}^{(\vartheta)}$ are quotient maps.

It might be added that the method of assigning a determining sequence to each formula in L_τ is constructive, so that this proposition can be used to obtain a variety of decidability results from the corresponding result for Boolean algebras. Some investigations along these lines can be found in Burris [12], Comer [16], Werner [56], and Burris and Werner [14].

The next two corollaries are further instances of the principle that whatever holds for $\mathbf{2}$ holds for every algebra A.

COROLLARY 1. *For algebras A, C, of the same type, filters ϑ on I, $\mathfrak{E}$ on J, $\mathfrak{F}$ on $I \times I$, $\mathfrak{J}$ on $J \times J$, if $A \equiv C$ and $\mathbf{2}^{(\vartheta)}|\mathfrak{F} \equiv \mathbf{2}^{(\mathfrak{E})}|\mathfrak{G}$, then $A^{(\vartheta)}|\mathfrak{F} \equiv C^{(\mathfrak{E})}|\mathfrak{G}$.*

Part of the above corollary appears in Węglorz [55], and Waszkiewicz and Węglorz [52].

Given filter $\mathfrak{H}$ on I, $\mathfrak{E}$ on J, $\mathfrak{F}$ on $I\times I$, $\mathfrak{G}$ on $J\times J$, suppose $f\colon J\to I$ is a set map such that for all $S\in\mathfrak{H}$, $f^{-1}(S)\in\mathfrak{E}$ and for all $U\in\mathfrak{F}$, $f^{-2}(U)\in\mathfrak{G}$. Then for any algebra A there is a homomorphism $A^I\to A^J$ given by $u \rightsquigarrow uf$: if $\mathrm{Ker}\, u\in\mathfrak{F}$, then $\mathrm{Ker}\, uf\in\mathfrak{G}$, and if u and v coincide on a set $S\in\mathfrak{H}$, then uf and vf coincide on $f^{-1}(S)\in\mathfrak{E}$; consequently there is a homomorphism $f_A\colon A^{(\mathfrak{H})}|\mathfrak{F}\to A^{(\mathfrak{E})}|\mathfrak{G}$ such that for all $u\in A^I|\mathfrak{F}$, $f_A(\nu(u)) = \mu(uf)$, where $\nu\colon A^I\to A^{(\mathfrak{H})}$, $\mu\colon A^J\to A^{(\mathfrak{E})}$ are the quotient maps.

COROLLARY 2. *If $f\colon J\to I$ is compatible with the filters $\mathfrak{H}$, $\mathfrak{E}$, $\mathfrak{F}$ and $\mathfrak{I}$ in the above sense, and if $f_2\colon 2^{(\mathfrak{H})}|\mathfrak{F}\to 2^{(\mathfrak{E})}|\mathfrak{G}$ is an elementary embedding, then $f_A\colon A^{(\mathfrak{H})}|\mathfrak{F}\to A^{(\mathfrak{E})}|\mathfrak{G}$ is an elementary embedding for all algebras A.*

The correspondence $A \rightsquigarrow A^{(\mathfrak{H})}|\mathfrak{F}$, for any filters $\mathfrak{F}$ on $I\times I$ and $\mathfrak{H}$ on I, is actually functorial on the category of all algebras of any given type: If $f\colon A\to C$ is a homomorphism, then clearly $\mathrm{Ker}\, fu\in\mathfrak{F}$ whenever $\mathrm{Ker}\, u\in\mathfrak{F}$, for all $u\in A^I$, and so there is a homomorphism $A^I|\mathfrak{F}\to C^I|\mathfrak{F}$, given by $u\rightsquigarrow fu$. Moreover, if $u, v\in A^I|\mathfrak{F}$ agree on some subset of I, then so do fu and fv and hence there is a homomorphism $f^{(\mathfrak{H})}|\mathfrak{F}\colon A^{(\mathfrak{H})}|\mathfrak{F}\to C^{(\mathfrak{H})}|\mathfrak{F}$ such that $f^{(\mathfrak{H})}|\mathfrak{F}(\nu_A(u)) = \nu_C(fu)$, where $\nu_A\colon A^I\to A^{(\mathfrak{H})}$ and $\nu_C\colon C^I\to C^{(\mathfrak{H})}$ are the quotient maps.

COROLLARY 3. *If $f\colon A\to C$ is an elementary embedding, then for all filters $\mathfrak{F}$ on $I\times I$, $\mathfrak{H}$ on I, $f^{(\mathfrak{H})}|\mathfrak{F}\colon A^{(\mathfrak{H})}|\mathfrak{F}\to C^{(\mathfrak{H})}|\mathfrak{F}$ is an elementary embedding.*

We now have the following results (also see Ash [1], Burris [12], Comer [16], Volger [51]), by first using Corollaries 1 and 3, and then the fact that $C(\Omega B, 2)\cong B$ for any $B\in\mathrm{Boo}$ and $E(\Omega B, 2)\cong B$ for any $B\in\mathrm{CBoo}$, together with Corollary 2.

PROPOSITION 22. *The functors $C(X, -)$, $E(X, -)$, and $E_{\mathfrak{F}}(X, -)$ for any space X and any filter $\mathfrak{F}$ in $\mathfrak{K}X$, and, for any algebra A, the functors $C(\Omega-, A)$ on* Boo *and $E(\Omega-, A)$ on* CBoo *all preserve elementary equivalence and elementary embeddings.*

As a further application of Corollaries 1 and 2, we have

PROPOSITION 23. *For any space X and Y, $C(X, A)\equiv C(Y, A)$ for all A whenever this holds for 2, and likewise for $E(X, A)$. Similarly, for any continuous map $f\colon X\to Y$, $C(f, A)\colon C(Y, A)\to C(X, A)$ is elementary for all A whenever this holds for 2, and likewise for $E(f, A)$ provided inverse images, relative to f, of dense open sets are dense.*

As a consequence of Proposition 22, and the fact that $A \amalg B \simeq C(\Omega A, B)$ for any Boolean algebras A and B, one obtains that coproducts of Boolean algebras preserve elementary equivalence, a result which can also be found in Burris [12], Olin [41], and Węglorz [54]. This contrasts rather sharply with the fact that coproducts do not preserve elementary equivalence in *any* non-trivial equational class of lattices (Jónsson and Olin [34]).

Proposition 23, together with the first corollary of Proposition 21 and the exponent law for $C(-, A)$, also leads to the following result of Węglorz [55]: For any algebra A, and any filters $\mathfrak{H}$ on I, $\mathfrak{F}$ on $I \times I$, $\mathfrak{E}$ on J and $\mathfrak{J}$ on $J \times J$, $(A^{(\mathfrak{H})}|\mathfrak{F})^{(\mathfrak{E})}|\mathfrak{G} \equiv (A^{(\mathfrak{E})}|\mathfrak{G})^{(\mathfrak{H})}|\mathfrak{F}$. The proof of this is based on the equivalences

$$(A^{(\mathfrak{H})}|\mathfrak{F})^{(\mathfrak{E})}|\mathfrak{G} \equiv C(Y, A^{(\mathfrak{H})}|\mathfrak{F}) \equiv C(Y, C(X, A)),$$

where $Y = \Omega(2^{(\mathfrak{E})}|\mathfrak{G})$ and $X = \Omega(2^{(\mathfrak{H})}|\mathfrak{F})$, so that, for instance, $A^{(\mathfrak{H})}|\mathfrak{F} \equiv C(X, A)$ since $2^{(\mathfrak{H})}|\mathfrak{F} \equiv C(X, 2)$.

We conclude this section with a discussion as to when the natural map $C(X, A) \to E(X, A)$ is elementary, and when $C(X, A) \equiv E(X, A)$, for any Boolean space X. Since the natural map $C(X, 2) \to E(X, 2)$ corresponds to the natural embedding $\mathfrak{L}X \to \mathfrak{K}X$, which is the injective hull (alias MacNeille completion) of $\mathfrak{L}X$, it follows that $C(X, A) \to E(X, A)$ is an elementary embedding for *all* A iff the embedding of $\mathfrak{L}X$ into its injective hull is elementary, and $C(X, A) \equiv E(X, A)$ for all A iff $\mathfrak{L}X$ is elementarily equivalent with its injective hull. In actual fact, it will turn out that these two conditions on $\mathfrak{L}X$ are equivalent, as was first pointed out to us by Phil Olin.

Since injective Boolean algebras are complete, it is clear that if a Boolean algebra B is elementarily equivalent to its injective hull, then the set of all atoms in B has a least upper bound in B, say a, and a' is then atomless, so that every element of B is the join of an atomic and an atomless element. Such Boolean algebras are called *separable*. Conversely, if B is separable, then $1 = a \vee a'$, where a is atomic and a' is atomless and then a is the least upper bound of the set of all atoms in B. Now the theory of separable Boolean algebras with additional unary predicates $Q, P_1, P_2, \ldots$ such that $Q(x)$ iff x is atomic, and $P_n(x)$ iff x contains at least n atoms, admits elimination of quantifiers (Kreisel–Krivine [35], p. 65 ff.) and consequently, by Sacks [45], Theorem 13.1, an embedding $f\colon B \to C$, for separable Boolean algebras, is elementary iff for all $b \in B$, b is atomic iff $f(b)$ is atomic, and either $f(b)$ contains the same finite number of atoms in C as b does in B, or both contain infinitely many atoms. Since the embedding of a Boolean algebra B into its injective hull is join-dense, the atoms of B are precisely the atoms in the injective hull, and so this embedding has the above property. Consequently, the *embedding of any separable Boolean algebra into its hull is elementary*.

Since the atoms of B corrlvd to the principal ultrafilters, and so to the isolated points of ΩB, it is easy to see that the set of atoms in B has a least upper bound (i.e. B is separable) iff the closure in ΩB of the set of isolated points is open. This proves:

PROPOSITION 24. *For a Boolean space X, the following conditions are equivalent*:

1° *The natural map $C(X, A) \to E(X, A)$ is elementary for all algebras A.*

2° $C(X, A) \equiv E(X, A)$ *for all algebras* A.

3° $\mathfrak{L}X$ *is separable.*

4° *The closure of the set of isolated points of* X *is open.*

We note that these conditions hold, in particular, for extremally disconnected Boolean spaces X: $\mathfrak{L}X$ is complete and hence separable.

Concluding remarks

Although we have restricted our attention to Boolean powers of algebras, it is clear that the definitions of $C(X, A)$ and $E(X, A)$ are applicable to arbitrary finitary structures A having both, relations and operations. All the foregoing results hold in this wider setting, with the exception of the three corollaries of Proposition 20; for a considerable part of them this is immediate, and the only cases where additional considerations are needed at all are those involving congruences. In particular, it has to be specified what relations will be considered as congruences. Since the Boolean powers of any structure A always belong to Horn$\{A\}$, the natural choice for congruences in this setting are the kernels of homomorphisms to structures belonging to Horn$\{A\}$. The properties of Horn$\{A\}$ then ensure that arbitrary intersections and direct unions of such congruences are again such congruences. Moreover, it is obvious that the congruences θ considered in Section 3 are indeed of this type. The only difficulty with these congruences is that there is no description of principal congruences of the type available in the case of algebras, and it is for this reason that the proofs of the corollaries of Proposition 20 do not apply here.

A generalization of the results presented here, in a different direction deals with the following modification of the $E(X, A)$: for each infinite cardinal m, let $D_m(X, A) \subseteq D(X, A)$ consist of those $u \in D(X, A)$ for which Du is the union of fewer than m open-closed sets, and let $E_m(X, A)$ be the corresponding subalgebra of $E(X, A)$. It is easy to see that, for suitable Boolean spaces X, $E_m(X, A)$ has a description à la Foster, and then $E_m(X, A)$ is $A[\mathfrak{L}X]_m$ in the notation of Ash [1]. Since $C(X, A) \subseteq E_m(X, A) \subseteq E(X, A)$, all results on congruences which hold for both $C(X, A)$ and $E(X, A)$ also hold for $E_m(X, A)$, and since $E_m(X, A)$ is a limit reduced power of A, the same applies to the results on elementary properties, some of which are given in Ash [1].

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