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Dedicated to the memory of
Abraham Robinson

§ 1. Introduction*

This paper concerns questions on the interface between topology and model theory. We deal here with a notion of equivalence of topological spaces (elementary equivalence) which comes from model theory and which is properly weaker than the notion of homeomorphism. To say that X and Y are elementarily equivalent spaces is to say that X and Y have the same "first order properties". In order to make this idea precise it is necessary to consider what is meant by a "first order property" of topological spaces. This requires explanation because a topological space is not an algebraic structure of the type dealt with in model theory. In fact our approach is to replace each topological space by closely related algebraic structures to which the point of view of model theory does apply.

In dealing with groups, for example, the idea of a first-order property is well understood. Such a property is one which can be expressed in a formal language for groups which allows quantifiers ranging over elements of a group, but not over subsets or other higher type objects, and in which meaningful expressions are of finite length. For example, to say that a group is Abelian is first-order ($\forall a \forall b$: $ab = ba$) but, at least superficially, to say that a group is simple is not first-order, since it seems to require quantification over subsets ("there does not exist a proper normal subgroup.") It is first-order to say that every element of a group has order $\leqslant 2$ ($\forall a$: $a^2 = 1$) but it is not first-order to say that a group is a torsion group, since this requires a sentence of infinite length, as one might suspect.

It is true that properties which superficially seem not to be first-order often turn out to be so. Indeed, this fact is one reason why the model-theoretic point of view is useful and powerful in many mathematical contexts. Consider, for example, the ring of integers and the formal language of ring theory. The property that an integer is positive appears

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not to be first-order until one realizes that an integer is positive if and only if it is a sum of four squares; the latter property is evidently first-order.

Similar considerations apply to other algebraic structures such as lattices, but as mentioned above, topological spaces are not suited to such a treatment. We get around this by associating with each topological space X the lattice $\mathscr{L}(X)$ of its closed subsets. For us, then, a property of topological spaces is *first-order* if it corresponds to a first-order property of lattices via this association. For example, connectedness is a first-order property of topological spaces since X is connected if and only if $\mathscr{L}(X)$ has no nontrivial complemented elements, and this latter property is a first-order property of lattices. Most separation properties are first-order as is the property of having inductive dimension n, just to mention a few. In fact, so many topological properties turn out to be first-order that we have only a few examples of non-homeomorphic spaces with exactly the same first-order properties. On the other hand, there are obviously only countably many first-order properties of topological spaces.

This view of first-order properties leads us to define two topological spaces X and Y to be *elementarily equivalent* (e.e.) if $\mathscr{L}(X)$ is elementarily equivalent to $\mathscr{L}(Y)$ as lattices, that is, if $\mathscr{L}(X)$ and $\mathscr{L}(Y)$ have the same first-order properties. It should be observed that two T_1 spaces X and Y are homeomorphic iff $\mathscr{L}(X)$ and $\mathscr{L}(Y)$ are isomorphic lattices. For this and other reasons we restrict ourselves to T_1 spaces in this paper. As mentioned above, many important properties of topological spaces are preserved under elementary equivalence. Although compactness is *not* preserved under e.e., there are infinite spaces (such as the closed unit interval) which are e.e. only to compact spaces. Furthermore, if only metric spaces are considered, then a "completely arcwise connected" compact space is e.e. only to compact spaces (see § 4).

This is an example of the fact that it is often profitable to restrict attention to a topologically interesting class $\mathscr{C}$ of spaces (e.g. metric spaces, completely regular spaces, compact spaces, etc.) and to study the relation of elementary equivalence between pairs of spaces from $\mathscr{C}$. We are able to give a clear technical meaning for the notion of "characterizing" a space X within a class $\mathscr{C}$: we say $X \in \mathscr{C}$ is *categorical* within $\mathscr{C}$ if for each $Y \in \mathscr{C}$, Y e.e. X implies that Y is homeomorphic to X. We show, for example, that the closed unit interval and the closed unit disc are categorical within the class of all metric spaces. The proofs of these facts depend on the classical characterizations of the interval and the disc within the class of compact metric spaces together with a close analysis of spaces which are e.e. to the interval.

The major difficulty of our subject is that the class of lattices isomorphic to $\mathscr{L}(X)$ for some T_1 space X is not closed under elementary equivalence (see § 8). Thus certain important theorems of first-order logic fail badly when lattices only of the form $\mathscr{L}(X)$ are considered. For example, the Skolem–Löwenheim theorem fails to be true in this context. In § 9 we give a space X such that every space which is e.e. to X has cardinality at least $\aleph_\omega$. We also show that the extent to which the Skolem–Löwenheim theorem holds for topological spaces is governed in part by large cardinal axioms of set theory. Also we show that the compactness theorem of first-order logic fails for topological spaces (see § 8).

In another direction we consider certain decision problems which are connected with first-order properties of topological spaces. Grzegorczyk [8] showed that the first-order theory of $\mathscr{L}(E^2)$ is undecidable, where E^2 is the Euclidean plane. That is, there is no effective procedure for deciding which sentences of the first-order language for lattices are true in $\mathscr{L}(E^2)$. We strengthen this by showing that second-order number theory is effectively interpretable in the first-order theory of $\mathscr{L}(E^2)$. Also let Top denote the set of sentences, of the first-order language for lattices, which are true in $\mathscr{L}(X)$ for every T_1 topological space X. We show that Top is at least as complicated as second-order number theory. In particular, Top is not decidable (see § 7).

One may associate other algebraic structures with topological spaces and consider problems analogous to those discussed above. One possibility, which we do pursue here, is to associate with X the ring $C(X)$ of continuous real-valued functions on X or the lattice $Z(X)$ of zero-sets of functions in $C(X)$. Another possibility is the closure algebra of X; this was treated in [1], [16] and [17] but will not be considered here. In § 4 we examine $C(X)$ and $Z(X)$ in some detail, especially for totally disconnected, compact Hausdorff spaces. In this setting new phenomena arise and we are far from a complete understanding of the relationships among $C(X)$, $Z(X)$ and $\mathscr{L}(X)$. When X is a metric space, the first-order language associated with $C(X)$ is even more expressive for discussing properties of topological spaces than the language of $\mathscr{L}(X)$. The best comparison between $C(X)$ and $\mathscr{L}(X)$ known to us for general spaces is Theorem 6.1, which is expressed in terms of certain infinitary languages. This result (and Theorem 6.2) indicate that, in a certain sense, $\mathscr{L}(X)$ is generally more expressive than $C(X)$. Infinitary languages are also used as an important tool in the analysis of $C(X)$ given in § 5.

For $C(X)$ and compact spaces X we do have a downward Skolem–Löwenheim theorem (see § 9). This raises the possibility of constructing relatively small compact spaces with interesting properties (expressible as first-order properties of $C(X)$).

We conclude this paper with a short list of open problems. The interested reader will have no difficulty in adding to this list. It is our hope that topologists and logicians alike will share our fascination with revisiting topology from this new viewpoint.

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§ 2. Basic development

We use the standard terminology of logic (cf. e.g. [24]). In particular a first order language is determined by specifying its non-logical symbols, which are n-ary relation and function symbols (including constants). The formulas of the language are built up in the standard way from the nonlogical symbols, variables, the equals symbol $=$, parentheses, propositional connectives such as $\wedge$, $\vee$, $\neg$, $\rightarrow$, and the quantifiers $(\exists x)$, $(\forall x)$. A sentence is a formula without free variables. A structure for the language is a nonempty set (universe) together with a relation or operation on the universe corresponding to each nonlogical symbol of the language. $\langle A; R_1, R_2\rangle$ would be the notation for a structure $\mathscr{A}$ with universe A and relations R_1, R_2. If $\mathscr{A}$ is a structure, $a_1, \ldots, a_n$ are elements of its universe, and $\varphi(x_1, \ldots, x_n)$ is a formula having at most the variables $x_1, \ldots, x_n$ free, then $\mathscr{A} \models \varphi(a_1, \ldots, a_n)$ means that $\varphi(x_1, \ldots, x_n)$ is true in $\mathscr{A}$ when x_i is interpreted as a_i. In particular if φ is a sentence (has no free variables), $\mathscr{A} \models \varphi$ means that φ is true (or valid) in $\mathscr{A}$. If $\mathscr{A}$ is a structure with universe A and R is an n-ary relation on A, then R is said to be *definable* in $\mathscr{A}$ if there is a formula $\varphi(x_1, \ldots, x_n)$ such that $R(a_1, \ldots, a_n)$ iff $\mathscr{A} \models \varphi(a_1, \ldots, a_n)$ whenever $a_1, \ldots, a_n \in A$. If Σ is a set of sentences, a structure $\mathscr{A}$ is said to be a *model* of Σ if $\mathscr{A} \models \varphi$ for each $\varphi \in \Sigma$. $\mathscr{A}$ is a *model* of a sentence φ if $\mathscr{A} \models \varphi$. The *theory* of a structure $\mathscr{A}$ is the set of sentences φ such that $\mathscr{A} \models \varphi$. Two structures $\mathscr{A}$, $\mathscr{B}$ for the same language are said to be *elementarily equivalent* (denoted $\mathscr{A} \equiv \mathscr{B}$) if they have the same theory.

We shall in particular be concerned with a language $L_{\mathscr{L}}$ for lattice theory and a language $L_{\mathscr{R}}$ for ring theory. $L_{\mathscr{L}}$ has binary function symbols $\cup$, $\cap$, and $L_{\mathscr{R}}$ has binary function symbols $+$,

In § 1 it was proposed that $\mathscr{L}(X)$, the lattice of closed subset of the topological space X, could be used to replace X in logical investigations. As indicated there we henceforth restrict our attention to T_1-spaces, which will be called simply *spaces*. The letters X, Y, Z will always stand for spaces.

For any space X, $\mathscr{L}(X)$ is a structure for $L_{\mathscr{L}}$ in a natural way. We say X and Y are *elementarily equivalent* (*e.e.*) if $\mathscr{L}(X) \equiv \mathscr{L}(Y)$. If $\mathscr{P}$ is a property of spaces, $\mathscr{P}$ is *first-order* if there is a sentence φ such that for every space X, X has property $\mathscr{P}$ iff $\mathscr{L}(X) \models \varphi$. $\mathscr{P}$ is a first-order property of, e.g., metrizable spaces if there is a φ whose metrizable models coincide with the metrizable spaces having $\mathscr{P}$.

We desire to show that some standard topological notions are first-order. Since it would be cumbersome to write out the corresponding sentences, we introduce some abbreviations. If t is a term, let "$t = 0$" abbreviate $(\forall x)[t \cap x = t]$ and "$t = 1$" abbreviate $(\forall x)[t \cup x = t]$. (Here x is any variable not occuring in t. From now on, such restrictions on bound variables will be tacit.) In $\mathscr{L}(X)$, "$t = 0$" and "$t = 1$" assert that t represents $\emptyset$, X respectively. Now the property of connectivity is expressed by the formula

$$\neg(\exists u)(\exists v)[(\neg u = 0) \wedge (\neg v = 0) \wedge u \cup v = 1 \wedge u \cap v = 0].$$

Many topological properties of X refer to points of X, and it is important to be able to refer to points in our language. Since X is T_1, we can refer to points as atoms of the lattice $\mathscr{L}(X)$. Let $\mathrm{At}(x)$ be the formula

$$x \neq 0 \wedge (\forall y)[y \subseteq x \rightarrow y = 0 \vee y = x],$$

where $y \subseteq x$ is rendered in our language by $y \cap x = y$. Using At, it is easy to express the Hausdorff property by a first-order statement. However, some circumlocution is necessary because the Hausdorff property is stated in terms of *open sets*. We now set up a *generalized language* which has the same expressive power as our original language but which incorporates many topological notions directly into its symbols. The *generalized terms* are all variables, constants 0 and 1, and all expressions of the form $(t_1 \cup t_2)$, $(t_1 \cap t_2)$, t_1^c, $t_1 - t_2$, t_1^-, ∂t_1, t_1^0, and t_1', where t_1, t_2 are previously constructed generalized terms. (Here t_1^c is interpreted as the complement of t_1, $t_1 - t_2$ is the set-theoretical difference of t_1 and t_2, t_1^- is the closure of t_1, and ∂, ^{0}, $'$ refer to boundary, interior, and derived set, respectively. The variables still range only over closed sets, but generalized terms are interpreted as (finite) Boolean combinations of closed sets in the obvious way.) *Generalized atomic formulas* are expressions of the form $t_1 \subseteq t_2$, $t_1 = t_2$ where t_1, t_2 are generalized terms. Generalized formulas are built up from generalized atomic formulas in the same way that formulas were built up from atomic formulas. Generalized atomic formulas are translated into formulas of our original language using the following equivalences:

$$t_1 = t_2 \leftrightarrow t_1 \subseteq t_2 \wedge t_2 \subseteq t_1,$$
$$t_1 \subseteq t_2 \leftrightarrow (\forall x)\big(\mathrm{At}(x) \wedge x \subseteq t_1 \rightarrow \mathrm{At}(x) \wedge x \subseteq t_2\big),$$

$$\begin{aligned}
&\mathrm{At}(x)\wedge x\subseteq t_1\cup t_2 \leftrightarrow \big(\mathrm{At}(x)\wedge x\subseteq t_1\big)\vee\big(\mathrm{At}(x)\wedge x\subseteq t_2\big),\\
&\mathrm{At}(x)\wedge x\subseteq t_1\cap t_2 \leftrightarrow \big(\mathrm{At}(x)\wedge x\subseteq t_1\big)\wedge\big(\mathrm{At}(x)\wedge x\subseteq t_2\big),\\
&\quad\mathrm{At}(x)\wedge x\subseteq t_1^c \leftrightarrow \mathrm{At}(x)\wedge \neg x\subseteq t_1,\\
&\quad\mathrm{At}(x)\wedge x\subseteq t_1^- \leftrightarrow \mathrm{At}(x)\wedge (\forall y)[t_1\subseteq y\to x\subseteq y],\\
&\quad\mathrm{At}(x)\wedge x\subseteq t_1' \leftrightarrow \mathrm{At}(x)\wedge x\subseteq (t-x)^-
\end{aligned}$$

One simply uses the first two equivalences to replace generalized atomic formulas by those of the form $x\subseteq t$ in the presence of the conjunct $\mathrm{At}(x)$. The fact that such formulas can be expressed in the original language is then proved by induction on the length of t using the remaining equivalences and the observation that ∂t, t^0 can be replaced by $t^-\cap t^c$, t^{c-c} respectively. (Expressions of the form $\mathrm{At}(x)\wedge x\subseteq 0(1)$ are of course trivial.)

Now the property of being Hausdorff is expressed by the (generalized) formula

$$(\forall x)(\forall y)\big(\mathrm{At}(x)\wedge\mathrm{At}(y)\wedge x\cap y=0 \\ \to(\exists u)(\exists v)[x\subseteq u^c\wedge y\subseteq v^c\wedge u^c\cap v^c=0]\big).$$

The properties of regularity (T_3) and normality (T_4) are shown to be first-order by slight variations of the above formula.

To get further examples it is helpful to be able to make statements about the relative topologies of subspaces represented by generalized terms. Thus if φ is a formula (having no variables in common with a generalized term t) we define φ^t to be the generalized formula obtained from φ by replacing all atomic formulas $t_1=t_2$ in φ by $t\cap t_1=t\cap t_2$. Then it can be seen that if φ is a sentence and if elements of $\mathscr{L}(X)$ are assigned to the variables of t so that t represents a set $Y\subseteq X$, then φ^t holds in $\mathscr{L}(X)$ iff φ holds in $\mathscr{L}(Y)$.

It is not clear that the property of complete normality [10, p. 42] is first order since its definition refers to arbitrary subsets of X. However, Arthur Charlesworth has pointed out to us that a space is completely normal iff all its open subspaces are normal. Thus complete normality is defined by the formula $(\forall x)\varphi_N^{x^c}$, where φ_N is a sentence expressing normality.

With the aid of the generalized language and relativized formulas φ^t it is routine to show (by induction on n) that for each n there is a formula φ_n which defines the property of being of inductive dimension n. Thus if $n\neq m$, Euclidean n-space is not e.e. to Euclidean m-space. In § 4 we shall show in fact that any metric space e.e. to the unit interval or unit disc is homeomorphic to the unit interval, unit disc respectively. However, we shall turn now to illustrating some of the limitations of our language.

§ 3. Some elementarily equivalent spaces

Since there are at most $2^{\aleph_0}$ spaces up to elementary equivalence, it follows from cardinality arguments that e.e. spaces need not be homeomorphic. In this section we find some specific examples of this phenomenon.

It is easy to see (from the formula $(\forall x)(\exists y)[y = x^c]$) that the property of discreteness is first order and thus that an infinite discrete space can be e.e. only to infinite discrete spaces. Conversely, the following result of Skolem shows that there are no elementary differences among the infinite discrete spaces.

THEOREM 3.1 (Skolem, cf. [5] or [26]). *For any sentence φ there is a number n (which can be effectively found) such that if X and Y are any two finite or infinite discrete spaces of cardinality $\geqslant n$, then $\mathscr{L}(X) \models \varphi$ iff $\mathscr{L}(Y) \models \varphi$.*

The theorem is proved by elimination of quantifiers.

COROLLARY 3.2 (Skolem). *Any two infinite discrete spaces are e.e., and the first-order theory of any infinite discrete space is decidable.*

If X is a locally compact space, let X^* be the one-point compactification of X.

THEOREM 3.3. *If X and Y are infinite discrete spaces, then X^* and Y^* are e.e. and their common theory is decidable.*

Proof. The idea of the proof is that a statement about $\mathscr{L}(X^*)$ (X discrete) can be reduced to a statement about $\mathscr{L}(X)$, if $\mathscr{L}(X)$ is provided with an additional predicate which picks out the finite ($=$ compact) subsets of X.

For any nonempty set I, let $\mathscr{A}_I$ be the structure $\langle P(I); \cup, \cap, c, \mathrm{Fin}\rangle$ whose universe $P(I)$ is the power set of I, c denotes the unary operation of complementation (relative to I) and Fin is the unary predicate which holds of the finite subsets of I. As pointed out by Feferman and Vaught [5, pp. 85 and 86], it can be shown by elimination of quantifiers that if I and I' are infinite sets then $\mathscr{A}_I$ and $\mathscr{A}_{I'}$ are elementarily equivalent (and their common theory is decidable). Given a sentence φ of lattice theory we shall construct a sentence φ^* in the language of the structures $\mathscr{A}_I$ such that if $I = X^*$ for any discrete space X, then $\mathscr{L}(I) \models \varphi \Leftrightarrow \mathscr{A}_I \models \varphi^*$. Given φ, let φ^* be the sentence $(\exists x)[\mathrm{At}(x)$ and $\varphi(\mathrm{Rel}\,x)]$ where $\varphi(\mathrm{Rel}\,x)$ is obtained from φ by relativizing its quantifiers to range over the subsets of I which are closed when I is the one-point compactification of a discrete space with $x = \{\infty\}$. Since these sets are the finite subsets of I and those which contain x, $(\exists y)[\ldots]$ in φ is to be replaced by $(\exists y)\big[(y \cap x \neq 0 \vee \mathrm{Fin}(y)\big)\wedge$

$\wedge\ldots]$ in constructing $\varphi(\mathrm{Rel}\,x)$. In verifying that $\mathscr{L}(I) \models \varphi \Leftrightarrow \mathscr{A}_I \models \varphi^*$, one uses the fact that any element of I could be chosen to play the role of ∞ in making I a one-point compactification of a discrete space and the topology, up to homeomorphism, is independent of the choice.

The last result shows in particular that a countable compact space can be elementarily equivalent to an uncountable compact space. The next two proposition will be useful in proving that a countable compact space can be elementarily equivalent to a countable non-compact space.

If $\mathscr{A}$, $\mathscr{B}$ are structures for the same language, their direct product $\mathscr{A}\times\mathscr{B}$ is defined as in [5].

The following is a very special case (which can be proved by a routine induction) of the main result of [5].

PROPOSITION 3.4. *For any sentence φ there are sentences $\varphi_1, \ldots, \varphi_n$, $\psi_1, \ldots, \psi_n$ (in the same language) such that for any structures $\mathscr{A}$, $\mathscr{B}$ for the language of φ, $\mathscr{A}\times\mathscr{B} \models \varphi$ iff for some i, $1 \leqslant i \leqslant n$, $\mathscr{A} \models \varphi_i$ and $\mathscr{B} \models \psi_i$.*

DEFINITION. If X, Y are disjoint topological spaces, $X\dot{\cup} Y$ (the separated union of X and Y) is defined to be $X\cup Y$ with the topology whose closed sets are those of the form $F_1 \cup F_2$ where F_1 is closed in X and F_2 is closed in Y.

PROPOSITION 3.5. *If X, Y are disjoint spaces, $\mathscr{L}(X\dot{\cup} Y) \simeq \mathscr{L}(X)\times \times\mathscr{L}(Y)$ (where $\simeq$ denotes isomorphism).*

Proof. If $F_1 \in \mathscr{L}(X)$, $F_2 \in \mathscr{L}(Y)$ this isomorphism simply corresponds $F_1\cup F_2$ in $\mathscr{L}(X\dot{\cup} Y)$ with (F_1, F_2) in $\mathscr{L}(X)\times\mathscr{L}(Y)$.

The last two results immediately imply the following, which will be useful for future reference.

PROPOSITION 3.6. *If X_1, X_2 are e.e. spaces and Y is disjoint from each of them, then $X_1\dot{\cup} Y$, $X_2\dot{\cup} Y$ are e.e.*

THEOREM 3.7. *Let X be the negative integers with the discrete topology and Y the positive integers with the discrete topology. Then Y^* and $X\dot{\cup} Y^*$ are e.e.*

Proof. Let φ be a sentence of lattice theory and let $\varphi_1, \ldots, \varphi_n$, $\psi_1, \ldots, \psi_n$ be as in Proposition 3.4. Now $\mathscr{L}(X\dot{\cup} Y^*) \models \varphi$ iff $\mathscr{L}(X)\times \times\mathscr{L}(Y^*) \models \varphi$ iff for some i, $1 \leqslant i \leqslant n$, $\mathscr{L}(X) \models \varphi_i$ and $\mathscr{L}(Y^*) \models \psi_i$. Let F be a finite discrete space (disjoint from Y^*) sufficiently large in cardinality that $\mathscr{L}(F) \models \varphi_i$ iff $\mathscr{L}(X) \models \varphi_i$ for $1 \leqslant i \leqslant n$. ($F$ exists by Theorem 3.1.) Thus $\mathscr{L}(X\dot{\cup} Y^*) \models \varphi$ iff for some i, $\mathscr{L}(F) \models \varphi_i$ and $\mathscr{L}(Y) \models \psi_i$. Applying Proposition 3.4 again, $\mathscr{L}(X\dot{\cup} Y^*) \models \varphi$ iff $\mathscr{L}(F\dot{\cup} Y^*) \models \varphi$. But $F\dot{\cup} Y^*$ is homeomorphic to Y^*, so $\mathscr{L}(X\dot{\cup} Y^*) \models \varphi$ iff $\mathscr{L}(Y^*) \models \varphi$. Since φ was arbitrary, $X\cup Y^*$ is e.e. to Y^*.

COROLLARY 3.8. *There exist countable e.e. spaces X_0 and Y_0 such that X_0 is compact and Y_0 is not.*

Corollary 3.8 follows at once from Theorem 3.7 by choosing $X_0 = Y^*$, $Y_0 = X \dot{\cup} Y^*$.

In view of Corollary 3.2 and Theorem 3.3 one might conjecture that if X_0, Y_0 are arbitrary e.e. spaces, then X_0^*, Y_0^* are also e.e. However, the spaces used to establish Corollary 3.8 provide a counterexample to this conjecture. However, this counterexample has the defect that compactification is not a very natural operation to apply to spaces which are already compact. Consider instead the spaces $X_0 \dot{\cup} I$, $Y_0 \dot{\cup} I$ where X_0, Y_0 are the spaces used to establish Corollary 3.8 and I is the open interval $(-\frac{1}{2}, \frac{1}{2})$ in the relative topology of the real line. Then $X_0 \dot{\cup} I$, $Y_0 \dot{\cup} I$ are e.e. by Proposition 3.6 and Corollary 3.8 and obviously neither is compact. Also $(X_0 \dot{\cup} I)^*$ is homeomorphic to the disjoint union of the unit circle and the one-point compactification of the integers $\geqslant 2$, while $(Y_0 \dot{\cup} I)^*$ is homeomorphic to the circle together with a discrete sequence which converges to one of its points and another discrete sequence which converges to a point at ∞. Thus, $(Y_0 \dot{\cup} I)^*$ contains a point which is both a limit of isolated points and of nonisolated points, whereas $(X_0 \dot{\cup} I)^*$ does not. It is easy to express this difference in the generalized language of § 2, so we have established the following corollary.

Corollary 3.9. *There exist e.e. noncompact spaces X_1, Y_1 such that X_1^*, Y_1^* are not e.e.*

The following corollary is an extension of Corollary 3.2.

Corollary 3.10. *If X, Y are infinite discrete spaces and Z is any space, then $X \times Z$, $Y \times Z$ are e.e.*

Proof. If X is a discrete space of cardinality $\varkappa$, then $X \times Z$ is simply the separated union of $\varkappa$ copies of Z (in the obvious sense.) Hence, by the same argument as used to prove Proposition 3.5, $\mathscr{L}(X \times Z)$ is the product of $\varkappa$ copies of $\mathscr{L}(Z)$. Thus, by [5, Theorem 6], the theory of the power $\mathscr{L}(X \times Z)$ is independent of $\varkappa$, for $\varkappa$ infinite, and the corollary is proved.

A further example of spaces which are e.e. but not homeomorphic will be given at the end of § 7.

§ 4. Elementary characterizations of some familiar spaces

It is rather easy to show from results in the literature that the unit interval, circle, and disc can be characterized up to homeomorphism among compact metric spaces by first order statements about $\mathscr{L}(X)$. In this section we cite these results from the literature and also point out how the restriction to compact spaces can be removed. The results of this section will be useful in the remaining sections. We call a space X a *model* of a sentence φ if $\mathscr{L}(X) \models \varphi$.

In Hocking and Young [10, Th. 2.27] it is shown that if a compact metric space X has property $(*)$ of being connected and having exactly two non-cut points, then X is homeomorphic to the closed unit interval I. Since $(*)$ is clearly a first-order property, the property of being homeomorphic to I is a first-order property of compact metric spaces.

THEOREM 4.1. *The property of being homeomorphic to I is a first order property of metric spaces, i.e. there is a sentence φ_I whose metrizable models are exactly the spaces homeomorphic to I.*

Proof. If we can give a sentence ψ such that I is a model of ψ and all models of ψ are compact, the theorem will follow from the remarks before it by taking φ_I to be $\varphi \wedge \psi$, where φ is a sentence whose models are the spaces having property $(*)$.

The construction of ψ is based on the fact that the usual ordering of I is topologically definable, once the left-hand endpoint is chosen. More precisely, for any space X and points $x, y, u \in X$, let $x <_u y$ mean that every closed connected set which contains u and y also contains x. (Thus if $X = I$ and u is either endpoint, $<_u$ is the standard ordering of I or the dual ordering.) Let ψ be a first order sentence whose models are exactly the connected spaces X such that for some $u \in X$ the relation $<^u$ has the following properties:

1. $<_u$ is a linear ordering of X with endpoints.

2. The order topology of $<_u$ is the same as the given topology of X.

To complete the proof it obviously suffices to show that if X is a connected ordered topological space with endpoints, then X is compact. This is a matter of elementary analysis. First, such an X has the l.u.b. property because if $S \subseteq X$ failed to have a l.u.b. then the set of upper bounds to S would be both open and closed. Therefore X is compact as can be seen by considering, for a given open covering, l.u.b. $\{x: [u, x]$ is contained in a finite subcover$\}$, where u is the left-hand endpoint of X.

Let φ_I be the sentence constructed above to characterize I. We shall refer to models of φ_I as *elementary arcs*. By the above argument, elementary arcs are compact.

One can also show that the property of being homeomorphic to the circle (i.e. being a simple closed curve) is a first-order property of metric spaces. One simply combines the first-order characterization of simple closed curves among compact metric spaces [10, Th. 2.28] with the observation that any simple closed curve is the union of two elementary arcs.

THEOREM 4.2. *The property of being homeomorphic to the closed unit disc D in the plane is a first-order property of metric spaces.*

Proof. In [29, p. 92] there is a characterization of D due to Zippin which easily yields a first-order characterization of D among *compact* metric spaces. We first review this characterization and then show how to eliminate the restriction to compact spaces. An arc S is said to *span* a set J if $S \cap J$ consists exactly of the endpoints of S. Then Zippin's result implies that a compact metric space C is a disc with boundary J iff

(i) C is connected and locally connected,
(ii) J is a simple closed curve contained in C,
(iii) C contains an arc that spans J,
(iv) every arc of C that spans J separates C, and
(v) no closed proper subset of an arc spanning J separates C.

Clauses (i)–(v) are clearly first-order for metric spaces in view of the fact that the notions of arc and simple closed curve have already been shown to be first-order among metric spaces. Thus there is a sentence φ whose compact metric models are exactly the spaces homeomorphic to D.

To complete the proof it suffices to show there is a sentence ψ such that D is a model of ψ and all metric models of D are compact. (The characterization of D among metric spaces is then $\varphi \wedge \psi$.) Let ψ be a sentence which asserts of $\mathscr{L}(X)$ that X is Hausdorff and every closed discrete subset of X is contained in an elementary arc which is contained in X. (There is no difficulty in quantifying over elementary arcs since such sets are closed in X.) Let us call a space X *completely arcwise connected* if every finite subset of X is contained in an arc which is contained in X. Then the metric models of ψ are exactly the compact, metric, completely arcwise-connected spaces, and in particular D is a model of ψ. (If X is a model of ψ, every closed discrete subset of X is contained in a compact subset of X and is therefore finite.)

From the existence of ψ we have the following corollary.

COROLLARY 4.3. *If X is a compact, completely arcwise connected space, then every metric space to which X is e.e. is compact.*

COROLLARY 4.4. *The property of being simply connected is a first-order property of subspaces of the plane.*

Proof. A subspace of the plane is simply connected iff every simple closed curve in it bounds a disc in it. This is first-order by our previous results. This characterization does not work in higher dimensions because of the existence of knots. We do not know whether simple connectivity is first-order nor whether the restriction to metric spaces can be eliminated from 4.1–4.3.

Finally we remark that being homeomorphic to the 2-sphere S^2 is a first order property of metric spaces. This may be deduced from Zippin's characterization of S^2 [27, Theorem 4.2] using the same argument that was used for Theorem 4.2.

§ 5. First order properties of $C(X)$

Recall that $C(X)$ is the ring of continuous, real-valued functions on X. It is natural in studying $C(X)$ to restrict attention to completely regular spaces, since the topology on such a space is determined in a natural way by the elements of $C(X)$.

Recall that X is completely regular if for each closed set $C \subseteq X$ and each point $x \in X$, if $x \notin C$ then there is a function f in $C(X)$ such that $f(x) = 0$ and $f(y) = 1$ for all $y \in C$. Since a T_1 completely regular space is necessarily Hausdorff, we will consider only Hausdorff spaces in this section.

Locally compact Hausdorff spaces and metrizable spaces are completely regular. If X and Y are realcompact Hausdorff spaces (in particular if they are σ-compact or separable metric), then $C(X)$ is isomorphic to $C(Y)$ iff X is homeomorphic to Y. See Gillman and Jerison [7] for a thorough treatment of completely regular and realcompact spaces.

Closely related to $C(X)$ is the lattice $Z(X)$ of zero-sets of functions in $C(X)$, considered as a sublattice of $\mathscr{L}(X)$. For each $f \in C(X)$ the zero-set of f will be denoted by $Z(f)$, i.e.

$$Z(f) = \{x \in X: f(x) = 0\}.$$

If X is a metric space, then $Z(X)$ is equal to $\mathscr{L}(X)$; in general this equality does not hold.

In [13], A. MacIntyre showed that for compact metric spaces, first-order statements about $Z(X)$ are translatable into first-order statements about $C(X)$. This is true because the relation $Z(f) \subseteq Z(g)$ is definable by a first-order formula of ring theory, as we now show.

THEOREM 5.1. (a) *There is a formula* $\mathrm{SB}(f, g)$ *in* $L_{\mathscr{R}}$ *such that for each completely regular space* X *and each* $f, g \in C(X)$

$$Z(f) \subseteq Z(g) \Leftrightarrow C(X) \models \mathrm{SB}(f, g).$$

(b) *For each sentence* φ *in the language of lattice theory there is a sentence* ψ *of ring theory such that for every completely regular space* X,

$$Z(X) \models \varphi \Leftrightarrow C(X) \models \psi.$$

Proof. Our proof is only a slight variant of MacIntyre's argument. Let $U(f)$ be the ring-theoretic formula $(\exists g)(f \cdot g = 1)$, let $\mathrm{CZ}(f, g)$ be the formula $\neg U(f^2 + g^2)$ and let $\mathrm{SB}(f, g)$ be the formula

$$(\forall h)\big(\mathrm{CZ}(f, h) \to \mathrm{CZ}(g, h)\big).$$

(Obviously the use of 1 in $U(f)$ can be eliminated in favor of and $+$.) In the ring $C(X)$, where X is completely regular, $U(f)$ says that f does not vanish and $\mathrm{CZ}(f, g)$ says that f, g have a common zero. Also, $Z(f)$

$\subseteq Z(g)$ clearly implies that $\mathrm{SB}(f, g)$ holds. If $Z(f) \nsubseteq Z(g)$, then there exists $x \in X$ with $f(x) = 0$ but $g(x) \neq 0$. Since X is completely regular there is a function $h \in C(X)$ such that $h(x) = 0$ and $h(y) = 1$ for $y \in Z(g)$. But then h witnesses the falsity of $\mathrm{SB}(f, g)$. This proves (a); (b) is an immediate consequence of (a).

Among metric spaces then, the expressive power of $C(X)$ is at least as strong as that of $Z(X) = \mathscr{L}(X)$. Thus the properties shown to be first-order in § 2 continue to be first-order for metric spaces if $C(X)$ or $Z(X)$ is used in place of $\mathscr{L}(X)$.

Now we consider the results of § 3. If X is a discrete space, every function on X is continuous and so $C(X)$ is just the direct product of $\varkappa$ copies of the ring $\boldsymbol{R}$ of real numbers, where $\varkappa$ is the cardinality of X. But by [5, Th. 6.5] the theory of the product of $\varkappa$ copies of a structure $\mathscr{A}$ is independent of $\varkappa$, for $\varkappa$ infinite, and is decidable if $\mathscr{A}$ has decidable theory. Thus, if X and Y are infinite discrete spaces, $C(X) \equiv C(Y)$ and their common theory is decidable (since the ring $\boldsymbol{R}$ has decidable theory by [28]).

However, the following result of J. Isbell shows that Theorem 3.3 fails for $C(X)$ and for $Z(X)$.

THEOREM 5.2 (Isbell). *If X, Y are discrete spaces, X is countable, and Y is uncountable, then $Z(X^*) \not\equiv Z(Y^*)$ and $C(X^*) \not\equiv C(Y^*)$.*

Proof. An atom in $Z(X^*)$ or $Z(Y^*)$ is just a singleton set $\{x\}$ such that x is the unique zero of some continuous function. The point x is isolated exactly when $\{x\}$ is complemented in the lattice of zero sets. Now in $Z(X^*)$ there is an atom which is not complemented, namely $\{x_0\}$ where x_0 is the point at infinity. However, in $Z(Y^*)$ every atom is complemented, since any continuous function on Y^* which is 0 at the point at infinity must be 0 elsewhere too. This shows $Z(X^*) \not\equiv Z(Y^*)$; $C(X^*) \not\equiv C(Y^*)$ follows by Theorem 5.1. (However, it will follow from Theorem 5.11 that $C(X^*) \equiv C(Y^*)$ whenever X, Y are both uncountable discrete spaces.)

The remaining results from § 3 go over to $Z(X)$ and $C(X)$ without difficulty.

THEOREM 5.3. *If X, Y are the discrete negative, positive integers respectively, then $C(Y^*) \equiv C(X \dot{\cup} Y^*)$.*

The proof is analogous to that for Theorem 3.7 except that the appeal to Theorem 3.1 is replaced by an application of Theorem 6.6 of [5]. Of course the analogue to Corollary 3.8 follows immediately. The analogue to Corollary 3.9 can be proved as before except that one uses Theorem 5.1 to show that $C(X_1^*) \not\equiv C(Y_1^*)$.

As remarked above, the results in § 4 go over to $C(X)$ trivially if we restrict attention to metric spaces. These characterization results can be improved if we consider $C(X)$ as an algebra over $\boldsymbol{R}$. Let $C(X)_{\boldsymbol{R}}$ be a twosorted structure with domains $C(X)$ and $\boldsymbol{R}$. The operations in $C(X)_{\boldsymbol{R}}$ are the ring operations on $C(X)$, the field operations on $\boldsymbol{R}$ and the scalar multiplication operation from $\boldsymbol{R}\times C(X)$ to $C(X)$. (Note that this is essentially equivalent to adding the set of constant functions to $C(X)$ as a distinguished unary relation.)

THEOREM 5.4. *There is a sentence φ_I in the language of $C(I)_{\boldsymbol{R}}$ such that for any completely regular Hausdorff space X,*

$$C(X)_{\boldsymbol{R}} \models \varphi_I \Leftrightarrow X \text{ is homeomorphic to } \mathrm{I}.$$

Proof. Let X be a completely regular Hausdorff space. Because $C(X)$ separates points in X, any atom in the lattice $Z(X)$ must be a singleton $\{x\}$ for some point $x\in X$. For convenience let us to refer to these as the *zero-points* of X; denote the set of zero-points in X by $\mathrm{ZP}(X)$.

Given Z in $Z(X)$, we say that Z is *Z-connected* if there do not exist $Z_1, Z_2\in Z(X)$ satisfying $Z_1\cap Z\neq\emptyset$, $Z_2\cap Z\neq\emptyset$, $Z_1\cup Z_2\supseteq Z$ and $Z_1\cap Z_2\cap Z=\emptyset$. Note that this is simply a translation into $Z(X)$ of the usual connectedness condition in $\mathscr{L}(X)$.

If x, y, u are in $\mathrm{ZP}(X)$, we define $x<'_u y$ to mean: every Z-connected zero-set which contains u and y must also contain x. Now consider the following condition on X: there exist $u\in\mathrm{ZP}(X)$ and $f\in C(X)$ such that

(1) $<'_u$ is a dense linear ordering with end points on $\mathrm{ZP}(X)$.

(2) For each $x, y\in\mathrm{ZP}(X)$ with $x<'_u y$ there is a zero-set Z such that for all $z\in\mathrm{ZP}(X)$

$$z\notin Z \Leftrightarrow x<'_u z<'_u y.$$

(3) For each zero-set Z and each $x\in\mathrm{ZP}(X)$, if $x\notin Z$ then there exist $y, z\in\mathrm{ZP}(X)$ such that $y<'_u x<'_u z$ and for all $w\in\mathrm{ZP}(X)$, $y<'_u w<'_u z$ implies $w\notin Z$.

(4) For all $x, y\in\mathrm{ZP}(X)$,

$$x<'_u y\Leftrightarrow f(x)<f(y).$$

(5) For all zero-sets Z_1, Z_2, if $Z_1\nsubseteq Z_2$, then there exists $x\in\mathrm{ZP}(X)$ such that $x\in Z_1$ and $x\notin Z_2$.

(6) X is Z-connected.

Suppose that X satisfies this condition. Conditions (2) and (3) imply that the relative topology on $\mathrm{ZP}(X)$ is the same as the $<'_u$ order topology. Suppose that a, b, are the $<'_u$-end points and $a<'_u b$. Then (4) implies that f maps $\mathrm{ZP}(X)$ into the interval $[f(a), f(b)]$. Since X is Z-connected, it must be that the image of X under f contains this whole interval. Moreover, (5) implies that $f(X)$ equals $f(\mathrm{ZP}(X))$, so that f maps $\mathrm{ZP}(X)$ onto $[f(a), f(b)]$. But then (4) implies that f is a homeomorphism on $\mathrm{ZP}(X)$, which must therefore be a compact (and hence closed) subset of X. Finally, (5) implies that $\mathrm{ZP}(X)$ is dense in X and therefore is equal to X. Thus X is homeomorphic to a closed interval, via f.

It is obvious that I satisfies the condition above. Thus it suffices to find a sentence φ_I such that X satisfies the condition iff $C(X)_{\boldsymbol{R}} \models \varphi_I$. With the possible exception of property (4), it is easy to translate the parts of this condition using Theorem 5.1. Moreover, (4) is easily dealt with once it is observed that $f(x) < f(y)$ is equivalent to

$$(\exists \alpha)(\exists \beta)\big(x \in Z(f-\alpha\cdot 1) \wedge y \in Z(f-\beta\cdot 1) \wedge \alpha < \beta\big)$$

which is readily put into the language of $C(X)_{\boldsymbol{R}}$. (α and β are variables ranging over the $\boldsymbol{R}$-sort in $C(X)_{\boldsymbol{R}}$.)

For the remainder of this section we will consider $C(X)$ and $Z(X)$ for compact, zero-dimensional Hausdorff spaces — the so-called *Boolean spaces* which arise in Stone's representation theory for Boolean algebras. (See [9].) Given a Boolean space X, let $\mathscr{B}(X)$ be the Boolean algebra of clopen subsets of X (that is, the closed-and-open subsets of X). Evidently $\mathscr{B}(X)$ is contained in $Z(X)$ as the sub-lattice of complemented elements. Therefore every first-order property of $\mathscr{B}(X)$ can be translated into a first-order property of $Z(X)$ (and then, using Theorem 5.1, into a first-order property of $C(X)$).

THEOREM 5.5. *There is a sentence β in the language of lattice theory such that, for any compact Hausdorff space X,*

$$X \text{ is a Boolean space} \Leftrightarrow Z(X) \models \beta.$$

Proof. This follows from the fact that a compact Hausdorff space is a Boolean space iff each disjoint pair of zero-sets can be separated by a clopen set. Obviously this is true of Boolean spaces, since each disjoint pair of closed sets can be separated by a clopen set. If X is compact Hausdorff but not Boolean, then there must be a closed set C and a point $x \notin C$ such that x cannot be separated from C by a clopen set. But x and C can be separated by a disjoint pair of zero-sets since X is completely regular.

THEOREM 5.6. *There is a sentence γ in the language of lattice theory such that, for any Boolean space X,*

$$\mathscr{B}(X) \text{ is a countably complete Boolean algebra} \Leftrightarrow Z(X) \models \gamma.$$

Proof. Let X be a Boolean space. Consider a d creasing chain $C_1 \supseteq C_2 \supseteq \ldots$ in $\mathscr{B}(X)$. Define $f\colon \boldsymbol{X} \to \boldsymbol{R}$ so that $f(X) = 1$ for $x \notin C_1$, $f(x) = 1/n$ for $x \in C_{n+1} \sim C_n$ $(n = 1, 2, \ldots)$ and $f(x) = 0$ for $x \in \bigcap C_n$. Evidently f is continuous; and therefore $Z = \bigcap C_n$ is a zero-set. Moreover, $\inf_n C_n$ exists in $\mathscr{B}(X)$ iff there is a largest clopen set contained in Z.

Now suppose $Z = Z(f)$ is any zero-set in X. For each $n = 1, 2, \ldots$ let C_n be a clopen set which separates $Z_n = \{x| \ |f(x)| \geqslant 1/2n\}$ from $Z'_n = \{x| \ |f(x)| \leqslant 1/(2n+1)\}$. That is, for $x \in X$

$$|f(x)| \leqslant \frac{1}{2n+1} \Rightarrow x \in C_n \Rightarrow |f(x)| < \frac{1}{2n}.$$

This is possible since Z'_n, Z_n are disjoint zero-sets for each n. But then $Z = \bigcap C_n$; therefore every zero-set is the intersection of a decreasing chain of clopen sets.

It follows from this argument that $\mathscr{B}(X)$ is countably complete iff each zero-set in X contains a largest clopen subset. From this the sentence γ is easily obtained.

Theorem 5.6 is an example of a general and interesting phenomenon: in the finitary first-order language of $Z(X)$ or $C(X)$ one can express properties of $\mathscr{B}(X)$ which require infinitary formulas to be expressed in the anguage of Boolean algebras.

THEOREM 5.7. *There is a formula $F(x)$ in the language of lattice theory such that for each Boolean space X and eaoh clopen set $C \subseteq X$,*

$$C \text{ is finite} \Leftrightarrow Z(X) \models F(C).$$

Proof. Suppose X is a Boolean space and C is a clopen subset of X. We will show that C is a finite set iff every zero-set contained in C is clopen. From this the formula $F(x)$ can easily be constructed.

Obviously every subset of a finite, clopen set C is clopen, since C must simply be a finite set of isolated points. Conversely, suppose C is clopen but is not finite. Then we can find a strictly decreasing chain $C_1 \supset C_2 \supset$ of clopen subsets of C. The set $Z = \bigcap C_n$ is a zero-set (as shown in the proof of 5.6) which cannot be open (since the chain $(C_1 \sim Z) \supset (C_2 \sim Z) \supset$ has empty intersection).

THEOREM 5.8. *There is a sentence δ in the language of lattice theory such that for each Boolean space X the set of isolated points in X is countable $\Leftrightarrow Z(X) \models \delta$.*

Proof. Let X be a Boolean space. Suppose first that the set A of all isolated points in X is countable. Then we easily see that $Z = X \sim A$ is a zero-set. If C is a clopen set disjoint from Z, then C is finite (since C is a set of isolated points of X and X is compact).

Suppose, conversely, Z is a zero-set which contains no isolated point of X and every clopen set disjoint from Z is finite. As in the proof of Theorem 5.6, we may write Z as an intersection of countably many clopen sets $\{C_n\colon n = 1, 2, \ldots\}$. But then each set $X \sim C_n$ is a finite set of isolated points of X and every isolated points in X is contained in $X \sim Z = \bigcup (X \sim C_n)$. Therefore the number of isolated points in X is countable.

Finally, note that the isolated points of X correspond exactly to the atoms of the Boolean algebra $\mathscr{B}(X)$. Thus it suffices to take δ to be the sentence

$$(\exists z)\big[(\forall y)\big(y \text{ an atom in } \mathscr{B}(X) \to y \nsubseteq z\big) \wedge \\ \wedge (\forall y)\big(y \text{ in } \mathscr{B}(X) \wedge y \cap z = \emptyset \to F(y)\big)\big],$$

where F is the formula in Theorem 5.7. ∎

THEOREM 5.9. *Let X_0 be a countably infinite discrete space and let X be either the one-point compactification of X_0 or the Stone–Čech compactification of X_0. There is a sentence σ_X in the language of lattice theory such that for any compact Hausdorff space Y,*

$$Z(Y) \models \sigma_X \Leftrightarrow Y \text{ is homeomorphic to } X.$$

Proof. First let σ_1 be the conjunction of the sentence β from Theorem 5.5, a sentence asserting that the set of isolated points is dense in X and the sentence $\neg F(X) \wedge \delta$ obtained from Theorems 5.7 and 5.8, which asserts that the set of isolated points of X is countably infinite. If Y is compact and $Z(Y) \models \sigma_1$, then Y is a Boolean space and $\mathscr{B}(Y)$ is an atomic Boolean algebra with a countably infinite number of atoms.

When X is the one-point compactification of X_0, take σ_X to be the conjunction of σ_1 and the sentence

$$(\forall y)\big(y \text{ clopen} \to \big(F(y) \vee F(X \sim y)\big)\big).$$

If $Z(Y)$ is a model of this σ_X, then by Theorem 5.7 every set in $\mathscr{B}(Y)$ is finite or co-finite. Thus, Y and the one-point compactification of X_0 are Boolean spaces which represent the same Boolean algebra. Hence they are homeomorphic.

When X is the Stone–Čech compactification of X_0, take σ_X to be the conjunction of σ_1 and the sentence γ from Theorem 5.6. If Y is compact and Hausdorff and $Z(Y)$ is a model of this σ_X, then $\mathscr{B}(Y)$ is a countably complete, atomic Boolean algebra with a countably infinite number of atoms. Thus Y and X are Boolean spaces which both represent the power set of a countable set; hence, Y and X are homeomorphic. We remind the reader that, by Theorem 5.1, the $Z(X)$ characterizations of properties and spaces, which are given in Theorems 5.5 through 5.9, can be automatically translated into $C(X)$ characterizations.

Next we present what seems to be a useful tool for proving the elementary equivalence of $C(X)$ rings, at least in some interesting special cases. The statement of this result involves the concept of elementary equivalence relative to the infinitary languages $L_{\infty,\varkappa}$. Given a first-order language L, the infinitary formulas in $L_{\infty,\varkappa}$ are formed by repeatedly using conjunctions and disjunctions of arbitrary sets of formulas and by using quantifiers over sequences of variables of length less than $\varkappa$.

If $\mathscr{A}$, $\mathscr{B}$ are structures for L, then we write $\mathscr{A} \equiv_{\infty,\varkappa} \mathscr{B}$ iff $\mathscr{A}$, $\mathscr{B}$ satisfy exactly the same sentences of $L_{\infty,\varkappa}$. Infinitary languages of this type are treated in [11]. We will make use of the back-and-forth criterion for $\equiv_{\infty,\varkappa}$ [2], [3] and of a preservation theorem for $\equiv_{\infty,\varkappa}$ due to Feferman [6]. The reader may consult these references for a more precise description of $L_{\infty,\varkappa}$ and a treatment of the facts about $\equiv_{\infty,\varkappa}$ which we will use. Note that $L_{\infty,\varkappa}$ contains L, so that $\equiv_{\infty,\varkappa}$ is a stronger equivalence than $\equiv$.

THEOREM 5.10. *If X, Y are Boolean spaces, then the following conditions are equivalent, for each uncountable cardinal $\varkappa$:*

(1) $\mathscr{B}(X) \equiv_{\infty,\varkappa} \mathscr{B}(Y)$,

(2) $Z(X) \equiv_{\infty,\varkappa} Z(Y)$,

(3) $C(X) \equiv_{\infty,\varkappa} C(Y)$.

Proof. The implications (3) ⇒ (2) and (2) ⇒ (1) are immediate, using Theorem 5.1 and the analogous interpretation of $\mathscr{B}(X)$ in $Z(X)$.

To prove (1) ⇒ (3) we apply a slight modification of a theorem due to Feferman [6]. Consider the functor F which assings to each Boolean algebra $\mathscr{B}$ the ring $C(X_{\mathscr{B}})$, where $X_{\mathscr{B}}$ is the Stone representation space of $\mathscr{B}$. Recall that the elements of $X_{\mathscr{B}}$ may be viewed as the Boolean homomorphisms from $\mathscr{B}$ into the two-element Boolean algebra. Given a Boolean homomorphism $\varphi\colon \mathscr{B}_1 \to \mathscr{B}_2$, the ring homomorphism $F(\varphi)\colon F(\mathscr{B}_1) \to F(\mathscr{B}_2)$ is defined as follows: given $f_1 \in C(X_{\mathscr{B}_1})$, the image f_2 of f_1 under $F(\varphi)$ satisfies

$$f_2(x) = f_1(x \circ \varphi) \qquad \text{for each } x \in X_{\mathscr{B}_2}.$$

We will show that F is essentially an ω_1-local functor, in the language of [6]. First, if $\mathscr{B}_1 \subseteq \mathscr{B}_2$ are Boolean algebras and $i\colon \mathscr{B}_1 \to \mathscr{B}_2$ is the inclusion mapping, then it is an easy exercise to show that $F(i)$ is an embedding of $F(\mathscr{B}_1)$ into $F(\mathscr{B}_2)$. (In [6] $F(i)$ is assumed to be actually an inclusion mapping, but this is inessential.) Second, we will show below that if $\mathscr{B}$ is a Boolean algebra and S is a countable subset of $F(\mathscr{B})$, then there is a countable subalgebra $\mathscr{B}'$ of $\mathscr{B}$, with inclusion map $i\colon \mathscr{B}' \to \mathscr{B}$, such that S is contained in the image under $F(i)$ of $F(\mathscr{B}')$. From these facts it follows, essentially as in [6, Theorem 6] that F preserves $\equiv_{\infty,\varkappa}$ for each $\varkappa \geqslant \omega_1$, which is just what the implication (1) ⇒ (3) asserts.

Let $\mathscr{B}$ and $S \subseteq F(\mathscr{B})$ be given, with S countable. Since $X_{\mathscr{B}}$ is a Boolean space, for each $f \in S$ and each pair $r < s$ of rational numbers, there is a clopen subset $B_f(r, s)$ of $X_{\mathscr{B}}$ such that

$$f(x) \leqslant r \Rightarrow x \in B_f(r, s) \Rightarrow f(x) < s \qquad \text{for all } x \in X_{\mathscr{B}}.$$

Moreover, by the Stone duality theory, there is an element $C_f(r, s)$ of $\mathscr{B}$ such that for $x \in X_{\mathscr{B}}$

$$x \in B_f(r, s) \Leftrightarrow x\big(C_f(r, s)\big) = \mathbf{1}.$$

(Again we consider the elements of $X_{\mathscr{B}}$ as 2-valued homomorphisms on $\mathscr{B}$.)

Let $\mathscr{B}'$ be the subalgebra of $\mathscr{B}$ which is generated by the sets $C_f(r, s)$. Given $f \in S$, we must show that f is in the image of $F(\mathscr{B}')$ under $F(i)$. This is easily seen to be the same as saying that if $x, y \in X_{\mathscr{B}}$ are homomorphisms on $\mathscr{B}$ which agree on $\mathscr{B}'$, then $f(x) = f(y)$. If this is not so, we may assume that there are rational numbers $r < s$ which satisfy

$$f(x) < r < s < f(y).$$

Therefore $x \in B_f(r, s)$ and $y \notin B_f(r, s)$. This implies $x\big(C_f(r, s)\big) = \mathbf{1} \neq \mathbf{0} = y\big(C_f(r, s)\big)$, so that x, y do not agree on $\mathscr{B}'$. This shows that F is an ω_1-local functor and completes the proof of the theorem.

In applying Theorem 5.10 we will use the well known back-and-forth criterion for $\equiv_{\infty,\varkappa}$. (See [2], [3] or [6] for a discussion of this criterion.) Given structures $\mathscr{A}$, $\mathscr{B}$ for L, this criterion states that $\mathscr{A} \equiv_{\infty,\varkappa} \mathscr{B}$ iff there is a family J of mappings which satisfies these conditions:

(A) each $\alpha \in J$ is an isomorphism from some substructure of $\mathscr{A}$ onto a substructure of $\mathscr{B}$,

(B) for each $\alpha \in J$, each subset S of $|\mathscr{A}|$ and each subset T of $|\mathscr{B}|$ with card $S < \varkappa$ and card $T < \varkappa$, there exists an extension β of α in J whose domain contains S and whose range contains T.

We will refer to a family of mappings which satisfies (A) and (B) as a *$\varkappa$-partial isomorphism* from $\mathscr{A}$ onto $\mathscr{B}$.

THEOREM 5.11. *If X, Y are uncountable discrete spaces, then $C(X^*) \equiv_{\infty,\omega_1} C(Y^*)$.*

Proof. The Boolean algebra $\mathscr{B}(X^*)$ may be identified with the algebra of finite and co-finite subsets of X. Let J be the set of all isomorphisms α from a countable subalgebra of $\mathscr{B}(X^*)$ into $\mathscr{B}(Y^*)$ such that for each set A in the domain of α

(i) A is finite $\Leftrightarrow \alpha(A)$ is finite,

(ii) A finite $\Rightarrow \operatorname{card} A = \operatorname{card} \alpha(A)$.

It is easily seen that J is an ω_1-partial isomorphism from $\mathscr{B}(X^*)$ to $\mathscr{B}(Y^*)$, and thus $\mathscr{B}(X^*) \equiv_{\infty,\omega_1} \mathscr{B}(Y^*)$. The desired result follows using Theorem 5.10.

Our next result concerns the Stone–Čech compactification of a discrete space X; as usual we denote this space by βX.

THEOREM 5.12. *If X, Y are discrete spaces of cardinality $> 2^{\aleph_0}$, then $C(\beta X) \equiv_{\infty,\omega_1} C(\beta Y)$.*

Proof. The Boolean algebra $\mathscr{B}(\beta X)$ may be identified with $P(X)$, the algebra of all subsets of X. By Theorem 5.10, we need to show that if X, Y are sets of cardinality $> 2^{\aleph_0}$, then $P(X) \equiv_{\infty,\omega_1} P(Y)$. We do this by constructing an ω_1-partial isomorphism from $P(X)$ to $P(Y)$.

Let P be any partition of X, Q any partition of Y, and $f\colon P \to Q$ a bijection. We say the triple (P, Q, f) is *admissible* if P, Q have cardinality $\leqslant 2^{\aleph_0}$ and for each set A in P

(i) $\operatorname{card} A \leqslant 2^{\aleph_0} \Leftrightarrow \operatorname{card}(fA) \leqslant 2^{\aleph_0}$,

(ii) if $\operatorname{card} A \leqslant 2^{\aleph_0}$, then $\operatorname{card} A = \operatorname{card}(fA)$.

Given an admissible triple (P, Q, f) we define a mapping a as follows: the domain of a is the algebra of subsets of X which are of the form $\bigcup P'$ for some $P' \subseteq P$; for such sets a is defined by

$$a(\bigcup P') = \bigcup \{fA \mid A \in P'\}.$$

Evidently a is an isomorphism from a subalgebra of $P(X)$ into $P(Y)$. Let J be the set of all mappings a obtained as above from admissible triples (P, Q, f).

To show that J is an ω_1-partial isomorphism it suffices to prove the following fact: Let (P_1, Q_1, f_1) be an admissible triple and P_2 a partition of X which refines P_1 and has cardinality $\leqslant 2^{\aleph_0}$. Then there is a refinement Q_2 of Q_1 and a bijection $f_2\colon P_2 \to Q_2$ such that (P_2, Q_2, f_2) is an admissible triple and for each $A_1 \in P_1$ and $A_2 \in P_2$,

$$A_2 \subseteq A_1 \Rightarrow f_2(A_2) \subseteq f_1(A_1).$$

This fact is proved by an easy cardinality argument which we leave to the reader.

Note that Theorems 5.9 and 5.12 do not apply to $C(\beta(X))$, where X is a discrete space with $\aleph_0 < \operatorname{card} X \leqslant 2^{\aleph_0}$. It can be shown that if X is such a space and Y is a discrete space, then $\operatorname{card} Y \neq \operatorname{card} X$ implies $C(\beta X) \not\equiv_{\infty,\omega_1} C(\beta Y)$. But if Y is uncountable and X is as above, we do not know whether $C(\beta X) \equiv C(\beta Y)$ holds or not.

For our last example in this section we consider the product spaces $\{0, 1\}^{\varkappa}$, where $\{0, 1\}$ is given the discrete topology and $\varkappa$ is an infinite cardinal. As is well known, the clopen subsets of $\{0, 1\}^{\varkappa}$ depend on only a finite number of coordinates. That is, if $B \subseteq \{0, 1\}^{\varkappa}$ is a clopen set, then there is a finite set $F \subseteq \varkappa$ such that for all $x, y \in \{0, 1\}^{\varkappa}$ $x \in B$ and x, y agree on F imply $y \in B$. It follows that if $Z \subseteq \{0, 1\}^{\varkappa}$ is a zero set, then Z depends only on a countable number of coordinates. If $\varkappa$ is uncountable, then this implies that $Z(\{0, 1\}^{\varkappa})$ has no atoms. On the other hand, $\{0, 1\}^{\omega}$ is metrizable, so that $Z(\{0, 1\}^{\omega})$ is atomic. This shows that for $\varkappa$ uncountable $Z(\{0, 1\}^{\omega}) \not\equiv Z(\{0, 1\}^{\varkappa})$ and $C(\{0, 1\}^{\omega}) \not\equiv C(\{0, 1\}^{\varkappa})$.

THEOREM 5.13. *If $\varkappa_1$, $\varkappa_2$ are uncountable cardinals, then* $C(\{0, 1\}^{\varkappa_1}) \equiv_{\infty,\omega_1} C(\{0, 1\}^{\varkappa_2})$.

Proof. For $i = 1, 2$ let $\mathscr{B}_i$ be the Boolean algebra of all clopen subsets of $\{0, 1\}^{\varkappa_i}$. Given $A \subseteq \varkappa_i$, let $\mathscr{B}_i^A$ be the subalgebra of all $B \in \mathscr{B}_i$ which depend only on coordinates in A. That is, $B \in \mathscr{B}_i^A$ iff $B \in \mathscr{B}_i$ and for each $x, y \in \{0, 1\}^{\varkappa_i}$

$x \in B$ and x, y agree on A imply $y \in B$.

If $A_1 \subseteq \varkappa_1$, $A_2 \subseteq \varkappa_2$ and $f\colon A_1 \to A_2$ is a bijection, then there is a canonical isomorphism $\hat{f}\colon \mathscr{B}_1^{A_1} \to \mathscr{B}_2^{A_2}$. Namely, if $C \in \mathscr{B}_1^{A_1}$ we define

$$\hat{f}(C) = \{y \in \{0,1\}^{\varkappa_2} \mid \text{there exists } x \in C \text{ such that } x(a) = y\big(f(a)\big) \text{ for all } a \in A_1\}.$$

Let J be the set of all such isomorphisms $\hat{f}$ obtained as above from *countable* sets A_1, A_2 and bijections $f\colon A_1 \to A_2$.

If $\hat{f} \in J$ is given as above and $S \subseteq \mathscr{B}_1$, $T \subseteq \mathscr{B}_2$ are countable sets, then we may find countable sets $A_1' \subseteq K_1$, $A_2' \subseteq K_2$ and a bijection $f'\colon A_1' \to A_2'$ such that $A_1 \subseteq A_1'$, $A_2 \subseteq A_2'$, f' extends f, $S \subseteq \mathscr{B}_1^{A_1'}$ and $T \subseteq \mathscr{B}_2^{A_2'}$. Then $\hat{f}'$ is in J; moreover, it is an extension of $\hat{f}$ whose domain contains S and whose range contains T. This shows that J is an ω_1-partial isomorphism from $\mathscr{B}(\{0,1\}^{\varkappa_1})$ to $\mathscr{B}(\{0,1\}^{\varkappa_2})$ and so, by Theorem 5.10, the proof is complete.

§ 6. $\mathscr{L}(X)$ and $C(X)$ compared

If X, Y are infinite discrete spaces, X countable and Y uncountable, then $\mathscr{L}(X^*) \equiv \mathscr{L}(Y^*)$ (by Theorem 3.3) while $C(X^*) \not\equiv C(Y^*)$ (by Theorem 5.2). Our next result shows that if a strong enough equivalence assumption is made about $\mathscr{L}(X)$, $\mathscr{L}(Y)$ (where X, Y are arbitrary spaces), then $C(X)$, $C(Y)$ must be elementarily equivalent. However, as we show later by example, no similar result is possible in the opposite direction.

THEOREM 6.1. *If X, Y are T_1 spaces and $\varkappa$ is an uncountable cardinal, then*

$$\mathscr{L}(X) \equiv_{\infty,\varkappa} \mathscr{L}(Y) \quad \textit{implies} \quad C(X) \equiv_{\infty,\varkappa} C(Y).$$

Proof. First we establish some notation. Suppose Z is any T_1 space and $\mathscr{L}$ is a sublattice of $\mathscr{L}(Z)$. By $C(Z;\ \mathscr{L})$ we mean the set of all functions f in $C(Z)$ such that $\{z \in Z\colon f(z) \leqslant r\}$ and $\{z \in Z\colon f(z) \geqslant r\}$ are in $\mathscr{L}$ for every rational number r.

Now assume that X, Y are T_1 spaces and $\mathscr{L}(X) \equiv_{\infty,\varkappa} \mathscr{L}(Y)$, with $\varkappa \geqslant \omega_1$. By the back-and-forth criterion for $\equiv_{\infty,\varkappa}$ there exists a $\varkappa$-partial isomorphism J from $\mathscr{L}(X)$ to $\mathscr{L}(Y)$. It is easy to see that if a is in J, then $\big(\mathscr{L}(X), A\big)_{A \in \operatorname{dom} a} \equiv \big(\mathscr{L}(Y), a(A)\big)_{A \in \operatorname{dom} a}$. In particular, if $a \in J$, then a and a^{-1} map atoms to atoms.

Given $a \in J$ we define a relation R_a (contained in $X \times Y$) by:

$$R_a = \{(x, y) \in X \times Y\colon \text{for all } A \in \operatorname{dom} a,\ x \in A \leftrightarrow y \in a(A)\}.$$

Note that if β extends α then $xR_\beta y \Rightarrow xR_\alpha y$. Also, if $\alpha(\{x\}) = \{y\}$, then $xR_\alpha y$.

If $x \in X$, then there must exist $y \in Y$ satisfying $xR_\alpha y$. To see this, extend α to $\beta \in J$ with $\{x\} \in \operatorname{dom}\beta$. Then choose $y \in Y$ so that $\beta(\{x\}) = \{y\}$. It follows that $xR_\alpha y$. Similarly, for each $y \in Y$ there exists $x \in X$ such that $xR_\alpha y$.

Next suppose $\mathscr{L} = \operatorname{dom}\alpha$ and $f \in C(X; \mathscr{L})$. Then if $xR_\alpha y$ and $x'R_\alpha y$ it must follow that $f(x) = f(x')$. For if not there must be a rational number r with (say) $f(x) < r < f(x')$. But let $A = \{x \in X: f(x) \leqslant r\}$, which is in $\mathscr{L}$. We must have

$$x \in A \Leftrightarrow y \in \alpha(A) \Leftrightarrow x' \in A$$

which is a contradiction. Similarly, if $\mathscr{L}' = \operatorname{range}\alpha$, $g \in C(Y; \mathscr{L}')$, $xR_\alpha y$ and $xR_\alpha y'$, then $g(y) = g(y')$.

Now for each $\alpha \in J$ with $\mathscr{L} = \operatorname{dom}\alpha$ and $\mathscr{L}' = \operatorname{range}\alpha$ we construct a function $\hat{\alpha}$ from $C(X; \mathscr{L})$ into $C(Y; \mathscr{L}')$. Given f in $C(X; \mathscr{L})$ we define $\hat{\alpha}(f)$ on Y by

$$[\hat{\alpha}(f)](y) = s \Leftrightarrow \text{for some } x \in X \ xR_\alpha y \text{ and } f(x) = s.$$

Our discussion above shows that $\hat{\alpha}(f)$ is properly defined on all of Y. For each rational number r, let $A = \{x \in X: f(x) \leqslant r\}$ and $B = \{x \in X: f(x) \geqslant r\}$; both sets are in $\mathscr{L}$. Given $y \in Y$ choose $x \in X$ such that $xR_\alpha y$. Then

$$[\hat{\alpha}(f)](y) \leqslant r \Leftrightarrow f(x) \leqslant r \Leftrightarrow x \in A \Leftrightarrow y \in \alpha(A).$$

Similarly $[\hat{\alpha}(f)](y) \geqslant r \Leftrightarrow y \in \alpha(B)$. This shows that $\hat{\alpha}(f)$ is in $C(Y; \mathscr{L}')$. A symmetric argument shows that $\hat{\alpha}$ is a bijection of $C(X; \mathscr{L})$ onto $C(Y; \mathscr{L}')$. Moreover, it is clear from the definition that if f, g, $f+g$ and $f \cdot g$ are all in $C(X; \mathscr{L})$, then

$$\hat{\alpha}(f+g) = \hat{\alpha}(f) + \hat{\alpha}(g),$$
$$\hat{\alpha}(f \cdot g) = \hat{\alpha}(f) \cdot \hat{\alpha}(g).$$

We now define $\hat{J}$ to be the set of all mappings which are the restriction of some $\hat{\alpha}$ to a subring of $C(X)$, letting α range over J. Each element of $\hat{J}$ is then an isomorphism from a subring of $C(X)$ onto a subring of $C(Y)$. The proof of the theorem will be complete once it is shown that $\hat{J}$ is a $\varkappa$-partial isomorphism from $C(X)$ to $C(Y)$. To prove this, let $\alpha \in J$ with $\mathscr{L}_1 = \operatorname{dom}\alpha$ and $\mathscr{L}_1' = \operatorname{range}\alpha$. If $S \subseteq C(X)$ and $T \subseteq C(Y)$ are subrings having cardinality $< \varkappa$, then since $\varkappa$ is uncountable there exist sublattices $\mathscr{L}_2 \subseteq \mathscr{L}(X)$ and $\mathscr{L}_2' \subseteq \mathscr{L}(Y)$ such that $\mathscr{L}_1 \subseteq \mathscr{L}_2$, $\mathscr{L}_1' \subseteq \mathscr{L}_2'$, $\mathscr{L}_2$ and $\mathscr{L}_2'$ have cardinality $< \varkappa$ and $S \subseteq C(X; \mathscr{L}_2)$, $T \subseteq C(Y; \mathscr{L}_2')$. Since J is a $\varkappa$-partial isomorphism, there is an extension β of α with $\beta \in J$, $\mathscr{L}_2 \subseteq \operatorname{dom}\beta$ and $\mathscr{L}_2' \subseteq \operatorname{range}\beta$. Then $\hat{\beta}$ is an extension of $\hat{\alpha}$ whose domain contains S and whose range contains T. This shows that $\hat{J}$ is a $\varkappa$-partial isomorphism, therefore $C(X) \equiv_{\infty,\varkappa} C(Y)$.

We remark that the maps $\hat{a}$ constructed in the proof above also satisfy

$$\hat{a}(sf) = s\hat{a}(f)$$

whenever s is a real scalar and f, $sf \in C(X, \operatorname{dom} a)$. Therefore the conclusion of Theorem 6.1 can be streng hened to read $C(X)_{\boldsymbol{R}} \equiv_{\infty,\varkappa} C(Y)_{\boldsymbol{R}}$.

Note also that Theorem 6.1 does not hold for the equivalence $\equiv_{\infty,\omega}$. Indeed, if X, Y are infinite discrete spaces, then it can be shown that $\mathscr{L}(X^*) \equiv_{\infty,\omega} \mathscr{L}(Y^*)$. However, if X is countable and Y is uncountable, then $C(X^*) \not\equiv C(Y^*)$ by Theorem 5.2.

The next result shows that the converse of Theorem 6.1 does not hold, even in a weak form.

THEOREM 6.2. *For each $\varkappa$ there exist Boolean spaces X, Y such that $C(X) \equiv_{\infty,\varkappa} C(Y)$ but $\mathscr{L}(X) \not\equiv \mathscr{L}(Y)$.*

Proof. Let us suppose that there exist Boolean algebras $\mathscr{A}$ and $\mathscr{B}$ such that $\mathscr{A} \equiv_{\infty,\varkappa} \mathscr{B}$, $\mathscr{A}$ is complete and $\mathscr{B}$ is not complete. Take X, Y to be the Stone representation spaces of $\mathscr{A}$, $\mathscr{B}$ (respectively). If $\varkappa \geqslant \omega_1$, then by Theorem 5.10 $C(X) \equiv_{\infty,\varkappa} C(Y)$. Now a Boolean space is extremally disconnected iff its algebra of clopen sets is complete [9]. (A Boolean space is extremally disconnected if the closure of every open set is open.) Therefore, X is extremally disconnected while Y is not. Evidently this implies $\mathscr{L}(X) \not\equiv \mathscr{L}(Y)$.

Thus Theorem 6.2 is an immediate consequence of the following:

LEMMA 6.3. *For each $\varkappa$ there exist Boolean algebras $\mathscr{A}$, $\mathscr{B}$ such tha $\mathscr{A} \equiv_{\infty,\varkappa} \mathscr{B}$, but $\mathscr{A}$ is complete while $\mathscr{B}$ is not.*

Proof. We just give a sketch of a proof. For convenience we may suppose $\varkappa = \tau^+$ where $\tau \geqslant \omega$. Let S be a set of cardinality $(2^\tau)^{++}$. Take $\mathscr{A}$ to be the algebra of all subsets of S; take $\mathscr{B}$ to be the algebra of all subsets A of S such that $\operatorname{card} A \leqslant (2^\tau)^+$ or $\operatorname{card}(S \sim A) \leqslant (2^\tau)^+$. Then $\mathscr{A}$ is complete while $\mathscr{B}$ is not. Note however that $\mathscr{B}$ is closed under unions and intersections of families with cardinality $\leqslant 2^\tau$.

The proof that $\mathscr{A} \equiv_{\infty,\varkappa} \mathscr{B}$ is similar to the argument given in the proof of Theorem 5.12. Here an admissibie triple (P, Q, f) is a pair of partitions P, Q of S and a bijection $f\colon P \to Q$ such that:

(i) P, Q have cardinality $\leqslant 2^\tau$ and every set in Q is an element of $\mathscr{B}$;

(ii) for each $A \in P$, $\operatorname{card} A \leqslant 2^\tau \Leftrightarrow \operatorname{card}(fA) \leqslant 2^\tau$;

(iii) for each $A \in P$, if $\operatorname{card} A \leqslant 2^\tau$, then $\operatorname{card} A = \operatorname{card}(fA)$.

Given an admissible triple (P, Q, f), let $\mathscr{A}_P$ be the algebra of all sets of the form $\bigcup P'$, where $P' \subseteq P$; similarly define $\mathscr{B}_Q$. Note that $\mathscr{A}_P \subseteq \mathscr{A}$ and $\mathscr{B}_Q \subseteq \mathscr{B}$. The bijection f gives rise to an isomorphism $\hat{f}$ of $\mathscr{A}_P$ onto $\mathscr{B}_Q$ defined by

$$f(\bigcup P') = \bigcup \{fA \mid A \in P'\} \quad \text{for} \quad P' \subseteq P.$$

Now let J be the set of all maps $\hat{f}$ obtained as above from admissible triples. To show that J is a $\varkappa$-partial isomorphism it suffices to check that the following extension property is true for admissible triples (P, Q, f): If $S \subseteq \mathscr{A}$ and $T \subseteq \mathscr{B}$ have cardinality $\leqslant \tau$, then there is an admisisble triple (P', Q', f') such that P' refines P, Q' refines Q, $S \subseteq \mathscr{A}_{P'}$, $T \subseteq \mathscr{B}_{Q'}$ and for each $A \in P$ and $A' \in P'$

$$A' \subseteq A \Leftrightarrow f'(A') \subseteq f(A).$$

(Then $\hat{f}'$ is an extension of $\hat{f}$ whose domain contains S and whose range contains T.) The details of the construction of (P', Q', f') will be omitted.

§ 7. Some results on undecidability

Let E^n denote Euclidean n-space. It was shown by Grzegorczyk [8, § 3] that if $n \geqslant 2$, then $\mathrm{Th}(\mathscr{L}(E^n))$ (the set of sentences true in $\mathscr{L}(E^n)$) is undecidable. On the other hand, Rabin [19, Th. 2.9] showed that $\mathrm{Th}(\mathscr{L}(E^1))$ is decidable. In this section we sharpen Grzegorczyk's result by showing that for $n \geqslant 2$, $\mathrm{Th}(\mathscr{L}(E^n))$ is "exactly as undecidable" as second order arithmetic. We also show that the set of "topologically valid" sentences (i.e. sentences valid in $\mathscr{L}(X)$ for every space X) is "at least as undecidable" as 2nd order arithmetic.

Our reductions between decision problems are just m-reductions in the sense of recursion theory. If L_1, L_2 are languages and S_i is a set of sentences of L_i, we write $S_1 \leqslant_m S_2$ if there is an effective procedure which associates to each sentence φ of L_1 a sentence φ^* of L_2 such that $\varphi \in S_1$ iff $\varphi^* \in S_2$. We write $S_1 \equiv_m S_2$ if $S_1 \leqslant_m S_2$ and $S_2 \leqslant_m S_1$. Clearly $\leqslant_m$ is a transitive relation. (If S_1, S_2 are closed under conjunction and $S_1 \equiv_m S_2$ it is easy to see that the corresponding sets of Gödel numbers are *recursively isomorphic* [21, Ch. 7].)

If $\mathscr{A}$ is a structure, the second order language for $\mathscr{A}$ has the same nonlogical symbols as the first order language for $\mathscr{A}$ but has in addition for each n an infinite list of variables $\{X_i^n\}$ for n-ary relations. For such a variable X_i^n, and (first order) terms $t_1, \ldots, t_n$, the expression $X_i^n(t_1, \ldots \ldots, t_n)$ is admitted as an atomic formula and quantifiers $(\exists X_i^n)$, $(\forall X_i^n)$ may be used in building up formulas. Otherwise, formulas are constructed as in the first order case, and truth is defined in the natural way. If this is done only for $n = 1$, one obtains the *monadic* (*second-order*) *language* of $\mathscr{A}$. We write $\mathrm{Th}_2(\mathscr{A})[\mathrm{Th}_M(\mathscr{A})]$ for the set of [monadic] second order sentences true in $\mathscr{A}$. (Recall that $\mathrm{Th}(\mathscr{A})$ is just the set of first-order sentences true in $\mathscr{A}$.) In particular, we write $\mathscr{N}$ for the structure $\langle \omega, +, \cdot \rangle$, so $\mathrm{Th}_2(\mathscr{N})$ is often called *true second-order arithmetic.*

THEOREM 7.1. *If* $n \geqslant 2$, $\mathrm{Th}(\mathscr{L}(E^n)) \equiv_m \mathrm{Th}_2(\mathscr{N})$.

Proof. It is routine to show that $\mathrm{Th}(\mathscr{L}(E^n)) \leqslant_m \mathrm{Th}_2(\mathscr{N})$ since basic open sets in E^n (e.g. balls with rational radii and centers having all coordinates rational) can be effectively encoded by natural numbers, and hence open sets in E^n can be encoded by sets (= unary relations) of natural numbers in such a way that the inclusion relation between open sets corresponds to a definable relation between their codes.

To show that $\mathrm{Th}_2(\mathscr{N}) \leqslant_m \mathrm{Th}(\mathscr{L}(E^n))$ we combine the methods of Grzegorczyk [8] with the use of definable orderings $<_u$ of § 4. We could have followed Grzegorczyk's methods more closely to obtain the theorem at hand, but it is not clear they would yield Corollary 7.3.

LEMMA 7.2. *If X is a Hausdorff space and the unit disc D is embedded in X,* $\mathrm{Th}_2(\mathscr{N}) \leqslant_m \mathrm{Th}(\mathscr{L}(X))$.

Proof. The argument can be simplified by using Raphael Robinson's observation [20, p. 239] that multiplication (of natural numbers) can be defined in terms of addition in the language of monadic logic. Of course once addition and multiplication are defined, a pairing function and its inverses can be defined. Thus

$$\mathrm{Th}_2(\mathscr{N}) \leqslant_m \mathrm{Th}_M(\mathscr{N}) \leqslant_m \mathrm{Th}_M(\mathscr{N}_0), \quad \text{where} \quad \mathscr{N}_0 = \langle \omega, T \rangle$$

and T is the ternary relation on ω corresponding to addition, i.e. $T = \{(a, b, c) \colon c = a+b\}$. (Thus the language for $\mathscr{N}_0$ has no function symbols, which yields a slight technical simplification.) Thus it will suffice to show that $\mathrm{Th}_M(\mathscr{N}_0) \leqslant_m \mathrm{Th}(\mathscr{L}(X))$ when X satisfies the hypotheses of the lemma. This will be done by showing that the unit disc D (and therefore X) has closed subsets which encode $\mathscr{N}_0$ in a definable way which we now describe.

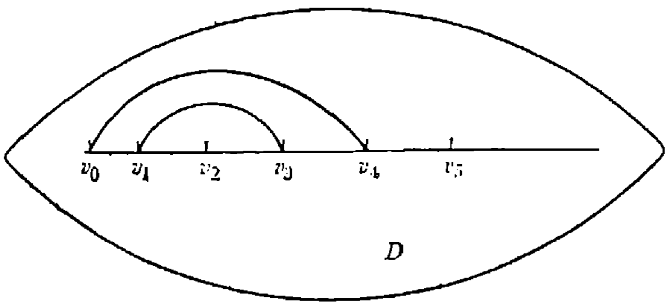
A triple (u, U, V) is called *ω-like* if U is a closed subset of X, V is a discrete subset of U, $u \in U$, and $<$ orders V with order type ω, where $<$ is the restriction to V of the relation $<_u$ on U defined in the proof of Theorem 4.1 (replacing X there by U). We would like to assert that the set of ω-like triples is definable in $\mathscr{L}(X)$, but this does not quite make sense because V need not be closed. However, any discrete set is the difference of two closed sets (i.e. its closure and its derived set.) We call a quadruple of closed sets (F, U, H_1, H_2) an *ω-code* if $F = \{u\}$ and $(u, U, H_1 - H_2)$ is ω-like. Clearly each ω-like triple comes from an ω-code, and we claim also that the set of ω-codes is definable in $\mathscr{L}(X)$. This is because a set has order type ω in an ordering iff it is well-ordered, has no greatest element, and each element except the least is a successor. It is possible to say that a discrete set is well-ordered (by $<_u$) since all of its subsets are discrete and thus differences of closed sets. If F_1, F_2 are closed subsets of X, let $F_1 \approx F_2$ mean that there is a closed set $G \subseteq X$ such that each component of G contains exactly one element of F_1 and exactly one element of F_2 (cf. [8]). Clearly if F_1, F_2 are finite and $F_1 \approx F_2$, then F_1 and F_2 have the same cardinality (although the converse need not hold). Also $\approx$ is a definable relation in $\mathscr{L}(X)$.

Let (u, U, V) be ω-like. If $x, y \in V$, let

$$[x, y) = \{z \in V\colon x \leqslant_u z <_u y\}.$$

Let v_0 be the $<_u$-least element of V. If $x, y, z \in V$, define $T_0(x, y, z)$ to hold iff $[v_0, x) \approx [y, z)$ and $z \geqslant_u y$. Let v_i be the ith element of V in general. If $T_0(v_j, v_k, v_l)$ holds, then $l = j+k$ since $[v_0, v_j)$ is of cardinality j and $[v_k, v_l)$ is of cardinality $l-k$. Again the converse need not be true, so we simply define (u, U, V) to be *ω-T like* if it is ω-like and for each $x, y \in V$ there exists $z \in V$ such that $T_0(x, y, z)$. Clearly if (u, U, V) is ω-T like then the order isomorphism of ω and V sends T to T_0. Being ω-T like is definable in $\mathscr{L}(X)$ in the sense that the corresponding set of ω-codes (called *ω-T codes*) is definable.

We digress to note that the unit disc D has an ω-T like triple (u, U, V) as can be seen by letting U be the closed unit interval and V a sequence of points converging to one endpoint of U (with u as the other endpoint). To see, for instance, that $T_0(v_2, v_3, v_5)$ holds, consider the picture where



the G corresponding to $[v_0, v_2)$ and $[v_3, v_5)$ is the union of the two curved lines. Since D is embedded in X it is easy to see that X has an ω-T like triple since D is compact and X is Hausdorff, so sets closed in the relative topology of D are closed in X.

For any sentence φ in the monadic second-order language corresponding to $\mathscr{N}_0$, one may obtain in a natural inductive way a formula $\tilde{\varphi}(F, U, H_1, H_2)$ such that whenever (F, U, H_1, H_2) is an ω-T code, $\mathscr{N}_0 \models \varphi$ iff $\mathscr{L}(X) \models \tilde{\varphi}(F, U, H_1, H_2)$. Quantification over (subsets of)

ω in φ corresponds to quantification over (subsets of) $V(=G-H)$ in $\tilde{\varphi}$, and the predicate symbol for T corresponds to a formula defining T_0. Given φ as above, let φ^* be

$$(\forall F, U, H_1, H_2)[\psi(F, U, H_1, H_2)\to\tilde{\varphi}(F, U, H_1, H_2)]$$

where ψ is a formula defining the class of ω-T codes. If $\mathcal{N}_0 \models \varphi$, then $\mathcal{L}(X) \models \varphi^*$ for every space X. If $\mathcal{N}_0 \models \neg\varphi$, then $\mathcal{L}(X) \models \neg\varphi^*$ for every X containing an ω-T code and in particular for every X in which D is embedded. From this and the fact that φ^* depends only on φ and not on X we immediately obtain the following corollary.

COROLLARY 7.3. *If* Top *is the set of sentences of* $L_{\mathcal{L}}$ *such that* $\mathcal{L}(X) \models \varphi$ *for every space* X, $\mathrm{Th}_2(\mathcal{N}) \leqslant_m \mathrm{Top}$.

We now give an example of compact metric spaces which are e.e. but not homeomorphic. This example is given here rather than in § 3 because the corresponding proof uses results about monadic second-order logic.

THEOREM 7.4. *There are two countable compact spaces, each embeddable in the real line (and hence metric) which are e.e. but not homeomorphic.*

Proof. Let A be a set which is linearly ordered by an ordering $<_A$, and also let A have the order topology of $<_A$. Then the family of closed subsets of A is definable (within the power set of A) by a formula of the monadic second-order language whose only nonlogical symbols is $<$ (representing $<_A$). (In fact, this formula is first-order except for a free variable ranging over the power set of A). Since set-theoretical union and intersection are obviously also definable, any first order statement about $\mathcal{L}(A)$ can be translated into a monadic second-order statement about $\langle A, <_A\rangle$. Since the translation is independent of A, it follows that if $\mathrm{Th}_M(\langle A, <_A\rangle) = \mathrm{Th}_M(\langle B, <_B\rangle)$, then A and B are e.e. equivalent as spaces in their order topology. In [4, p. 93], there is given a necessary and sufficient condition for two countable ordinals to have the same monadic theory as ordered sets. This condition shows in particular that $\omega^\omega+1$ and $\omega^\omega\cdot 2+1$ have the same monadic theory and thus are e.e. as ordered spaces. It is easy to see that these ordered spaces are homeomorphic to compact subspaces of the real line. However, they are not homeomorphic to each other because the ωth Cantor–Bendixson derivative of $\omega^\omega+1$ consists of a single point while the ωth Cantor–Bendixson derivative of $\omega^\omega\cdot 2+1$ consists of two points (cf. [15, p. 21]).

Since the translation used in the preceding proof is effective, and $\mathrm{Th}_M(\langle a; <\rangle)$ is decidable for any ordinal $a \leqslant \omega_1$ [4, pp. 96, 124], it follows that $\mathrm{Th}(\mathcal{L}(a))$ is decidable for any $a \leqslant \omega_1$, where a has the order topology.

§ 8. The class of topology lattices

Let $\mathscr{L}_1$ be the class of all topology lattices, i.e. lattices isomorphic to $\mathscr{L}(X)$ for some space X. A basic difficulty with our topic is that there is no set of axioms whose models are the topology lattices. In this section we illustrate this failure by considering various classes of lattices whose first-order properties resemble those of the topology lattices.

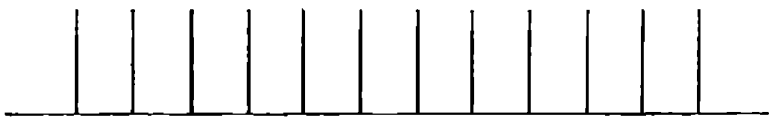
Let $\mathscr{L}_2$ be the class of lattices elementarily equivalent to some topology lattice. Let $\mathscr{L}_3$ be the class of all models of the theory of $\mathscr{L}_1$, i.e. lattices which satisfy all statements true in all topology lattices. Let $\mathscr{L}_4$ be the class of all distributive atomic lattices L with least and greatest element such that any subset of L which is first order definable with parameters from L has a greatest lower bound in L. (An example of the topological interpretation of a typical axiom for $\mathscr{L}_4$ would be: for any closed set S_1, the intersection of all closed connected sets containing S_1 is itself closed.) Let $\mathscr{L}_5$ be the set of atomic distributive lattices with least and greatest elements. We will show that

$$\mathscr{L}_1 \subsetneq \mathscr{L}_2 \subsetneq \mathscr{L}_3 \subsetneq \mathscr{L}_4 \subsetneq \mathscr{L}_5.$$

Each inclusion above is obvious, so it remains to check that the inclusions are all proper. That $\mathscr{L}_1 \neq \mathscr{L}_2$ in an easy consequence of the Skolem–Löwenheim theorem. Let X be an infinite Hausdorff space, and let L be a countable lattice elementarily equivalent to $\mathscr{L}(X)$. Obviously $L \in \mathscr{L}_2$. If $L \in \mathscr{L}_1$, $L \simeq \mathscr{L}(Y)$ say, then Y is an infinite Hausdorff space and so Y has uncountably many closed sets, as a simple argument shows (If the set I of isolated points is infinite, each of the uncountably many subsets of I is closed. Otherwise one may successively choose infinitely many pairwise disjoint open sets using the Hausdorff property. Any subcollection of those open sets has an open union.) The following result will show that $\mathscr{L}_2 \neq \mathscr{L}_3$.

THEOREM 8.1. *$\mathscr{L}_1$ is not compact, i.e. there exists a set Σ of sentences of lattice theory such that every finite subset of Σ has a model in $\mathscr{L}_1$ but Σ has no model in $\mathscr{L}_1$.*

Proof. We got the idea for this proof after seeing A. Adler's proof of the corresponding result for closure algebras (private correspondence). Let $C(x)$ be a first-order formula asserting of $\mathscr{L}(X)$ that x is a singleton and $X-x$ has exactly three components. Let φ_0 be a sentence asserting that there is a closed set which is an elementary arc (cf. § 4) containing all sets (i.e. points) satisfying $C(x)$. Let φ_n $(n \geqslant 1)$ assert that there are at least n distinct points satisfying $C(x)$. Let φ_ω assert that the points satisfying $C(x)$ have no limit point. Let $\Sigma = \{\varphi_i\colon 0 \leqslant i \leqslant \omega\}$. Every finite subset of Σ has a model in $\mathscr{L}_1$, i.e. $\mathscr{L}(X)$, for X a "comb" of the form



However, if $\mathscr{L}(X)$ satisfied all of Σ, then it would contain an elementary arc having an infinite subset with no limit point, which is impossible since elementary arcs are compact.

COROLLARY 8.2. $\mathscr{L}_2 \subsetneq \mathscr{L}_3$.

Proof. Let Σ be as in Theorem 8.1 and let Σ' be the theory of $\mathscr{L}_1$. Then $\Sigma \cup \Sigma'$ has a model L by the compactness theorem of logic. Clearly $L \in \mathscr{L}_3 - \mathscr{L}_2$. (In fact Theorem 8.1 and Corollary 8.2 are equivalent since any class K of structures is compact iff each model of the theory of K is elementarily equivalent to some structure in K (cf. [14, p. 315]).

The definition of $\mathscr{L}_4$ makes it clear that $\mathscr{L}_4$ is the class of models of some recursively enumerable set Σ of axioms. The logical closure of Σ is the theory of $\mathscr{L}_4$ by the completeness theorem and is obviously recursively enumerable. On the other hand, $\mathscr{L}_1$, $\mathscr{L}_2$, and $\mathscr{L}_3$ all have the same theory, and this theory is not definable in second order arithmetic by Corollary 7.3 (and so is certainly not recursively enumerable). Thus $\mathscr{L}_3 \neq \mathscr{L}_4$. (However, we do not have a specific example of a lattice in $\mathscr{L}_4 - \mathscr{L}_3$.) To show that $\mathscr{L}_4 \neq \mathscr{L}_5$, let L be the lattice of countable and cofinite subsets of the real line (with set-theoretic union and intersection as the lattice operations). Clearly $L \in \mathscr{L}_5$. However $L \notin \mathscr{L}_4$ because there s an $x \in L$ (namely any denumerably infinite set of reals) such that $\{y\colon x \cup y = 1\}$ has no g.l.b. (Intuitively L fails to satisfy the statement "every open set has a closure" and this statement is an axiom for $\mathscr{L}_4$.)

§ 9. Some bounds on the Löwenheim number for topology lattices

Because of the Skolem–Löwenheim theorem it is natural to inquire whether every space is e.e. to a space that is "small" in some sense, e.g. satisfies some sort of countability or separability axiom. We have already seen in § 8 that each infinite Hausdorff space is e.e. only to spaces having at least $2^{\aleph_0}$ closed sets. Also from § 4 it follows that any space e.e. to the closed unit interval is a compact Hausdorff space without isolated points and thus has at least the cardinality c of the continuum. Isbell has pointed out a stronger result relative to compact spaces. Consider βN, the Stone-Čech compactification of the discrete integers. If Y e.e. βN and Y is compact, then by a remark of Isbell (private correspondence) Y has cardinality at least 2^c since Y is an infinite extremally disconnected space and thus contains a copy of βN. The following result gives further negative information in this vein.

THEOREM 9.1. *There is a space X such that each space Y e.e. to X satisfies the following for each n:*

(i) *there is a family of $\aleph_n$ pairwise disjoint open sets in Y;*

(ii) *there is a point p_n in Y such that no basis of neighborhoods of p_n has cardinality $\leqslant \aleph_n$;*

(iii) *each nonempty open set in Y has cardinality $\geqslant c$.*

Proof. Let X be the "$\aleph_\omega$ long line" i.e. the space obtained by placing a copy of the open unit interval between a and $a+1$ for each ordinal $a<\aleph_\omega$ and using the order topology. Let Y be elementarily equivalent to X.

If a space has exactly one non-cut point u and is linearly ordered by the relation $<_u$ defined in § 4, let us call $<_u$ its *canonical ordering.* Clearly X has the order topology of its canonical ordering and so the same can be said of Y.

Requirement (iii) holds of Y by essentially the same argument used to show that any space e.e. to the unit interval has cardinality $\geqslant c$.

The proof of (i) and (ii) hinges on the fact that for each n the collection of discrete subsets of Y which are well-ordered (in the canonical ordering) and of order type $\aleph_n$ is definable in $\mathscr{L}(Y)$. (As in § 7, a family $\mathscr{F}$ of discrete sets is definable if the set of pairs of closed sets F_1, F_2 such that $F_1-F_2\in\mathscr{F}$ is definable.) As in § 7 we observe that every subset of a discrete set is discrete and hence a difference of closed sets so that in effect it is possible to express quantification over arbitrary subsets of a discrete set in terms of (first-order) quantification over $\mathscr{L}(Y)$. Thus if φ is any monadic second order sentence of the language of ordering, the set of discrete subsets of Y which satisfy φ (w.r.t. the canonical ordering) is definable in $\mathscr{L}(Y)$. Since by [4, p. 19] there is for each n a sentence of the monadic theory of ordering whose models are exactly those sets of order type $\aleph_n$, it follows that for each n the set of discrete subsets of Y of order type $\aleph_n$ is definable in Y. (Observe that the definition is independent of Y, provided only that Y has exactly one non-cut point.)

For any n, X has a discrete well-ordered subset of order type $\aleph_n$, i.e. the set of midpoints of the intervals $(a, a+1)$ for $a<\aleph_n$. Thus Y also has a discrete well-ordered subset W_n of order type $\aleph_n$. Each element z of W_n has an immediate successor z' in W_n, and the open interval (z, z') (w.r.t. the canonical ordering) is open in Y. The family of all these open intervals (z, z') for $z\in W_n$ is the required family of $\aleph_n$ pairwise disjoint open sets in Y. To prove (ii), let $\psi_n(p)$ be a first-order formula which asserts that p is a point and that there is a discrete well-ordered set of order type $\aleph_n$ which has p as a limit point and lies entirely to the left of p in the canonical order. Let θ_n be the sentence $(\exists p)[\psi_{n+1}(p) \wedge \neg\psi_n(p)]$. Then θ_n holds in $\mathscr{L}(X)$ as can be seen by taking $p = \aleph_{n+1}$. ($\psi_{n+1}(p)$ holds because of the set of midpoints used to prove (i) and $\psi_n(p)$ fails because $\aleph_n$ is a regular cardinal.) Therefore θ_n holds in $\mathscr{L}(Y)$. Let p_n be a point which makes θ_n hold in $\mathscr{L}(Y)$, and let V_n be a discrete well-ordered subset of Y which has p_n as a limit point, lies to the left of p_n, and has order type $\aleph_{n+1}$. Assume there is a basis Ω_n of neighborhoods of p_n such that the cardinality of Ω_n is $\leqslant \aleph_n$. Each neighborhood $G \in \Omega_n$ intersects V_n, and we may choose a point q_G in $V_n \cap G$. The set S of all points $q_G (G \in \Omega_n)$ is contained in V_n and is therefore discrete and well-ordered. But S has cardinality $\leqslant \aleph_n$ since Ω_n does, p_n is a limit point of S, and S lies to the left of p_n. This contradicts the assumption that $\psi_n(p_n)$ fails, so no such basis of neighborhoods Ω_n can exist.

COROLLARY 9.2. *There is a space X such that for any space Y e.e. to X, Y has cardinality at least* $\max\{\aleph_\omega, c\}$.

Shelah [23] has observed that if Gödel's axiom of constructibility (V = L) holds, then for each n there is a sentence of the monadic theory of orderings whose models are exactly those orderings whose order type is the nth weakly compact cardinal. (A cardinal $\varkappa$ is *weakly compact* if every partition of the two-element subsets of $\varkappa$ into two classes has a homogeneous set of power $\varkappa$. Weak compactness is a "large cardinal" property strictly intermediate between strong inaccessibility and measurability.) Hence if there are infinitely many weakly compact cardinals and V = L holds, then Theorem 9.1 and Corollary 9.2 hold with $\aleph_n$ replaced by the nth weakly compact cardinal $\varkappa_n$ and $\aleph_\omega$ replaced by $\sup_n \varkappa_n$.

By considering elementary equivalence with respect to both $\mathscr{L}(X)$ and $C(X)$, restricting to compact spaces, and assuming the existence of a measurable cardinal, Isbell has pointed out that one may obtain a much higher bound on possible Skolem–Löwenheim results than any of the preceding. A space X is called a *P-space* if every prime ideal in $C(X)$ is maximal. By [7, Ex. 4J], X is a P-space iff $C(X)$ is a regular ring, i.e. $C(X) \models (\forall f)(\exists g)[f^2 g = f]$. By [7, Ex. 12H], every extremally disconnected non-discrete P-space is of measurable cardinal and (if there exist measurable cardinals) there exist compact extremally disconnected non-discrete P-spaces. If X is such a space and $\mathscr{L}(Y) \equiv \mathscr{L}(X)$, $C(Y) \equiv C(X)$, and Y is compact, then Y is also such a space and hence is of measurable cardinal. (In [7], only completely regular spaces are considered, but compactness implies complete regularity.)

We have no positive results of the Skolem–Löwenheim sort for $\mathscr{L}(X)$. However, it is obviously true that there is a cardinal $\varkappa$ such that every space is e.e. to a space of cardinality $\leqslant \varkappa$, since there are only c spaces up to elementary equivalence. Let $\varkappa_0$ by the least such $\varkappa$. The results of this section show that $\varkappa_0$ must be $\geqslant \aleph_\omega$ and c. The remarks after Corollary 9.2 and [25] show that if it is consistent with the usual axioms of set theory (ZFC) to assume the existence of infinitely many weakly compact cardinals, then it is consistent with ZFC to assume that $\varkappa_0$ exceeds infinitely many weakly compact cardinals. Thus it seems unlikely that the value of $\varkappa_0$ can be found in ZFC, although it might be characterizable in a suitable extension of ZFC.

For $C(X)$ the situation is somewhat different, at least for compact spaces.

THEOREM 9.3. *For each compact Hausdorff space X there is a compact Hausdorff space Y such that $C(Y) \equiv_{\omega_1,\omega_1} C(X)$,* $\operatorname{card} Y \leqslant 2^c$ *and* $\operatorname{card} C(Y) \leqslant c$.

Proof. Using the Downward Löwenheim–Skolem Theorem for L_{ω_1,ω_1} we may find a subring $\mathscr{A}$ of $C(X)$ which has cardinality $\leqslant c$, contains the constant functions and satisfies $\mathscr{A} <_{\omega_1,\omega_1} C(X)$. For $f \in C(X)$, $\|f\| \leqslant 1$ is equivalent to

$$(\exists g)(\exists h)(1-f = \dot{h}^2 \text{ and } f+1 = g^2).$$

It follows immediately that $|\mathscr{A}|$ is a closed subset of $C(X)$, in the supremum norm topology.

Define an equivalence relation E on X by

$$xEy \Leftrightarrow f(x) = f(y) \quad \text{for all } f \in \mathscr{A}.$$

Let Y be the quotient space X/E; since E is closed in $X \times X$, Y is a compact Hausdorff space. Evidently there is a norm-preserving, ring isomorphism of $\mathscr{A}$ onto a subring $\mathscr{A}'$ of $C(Y)$. Then $\mathscr{A}'$ is a closed subring of $C(Y)$ which contains the constant functions and separates points in Y. By the Stone–Weierstrass Theorem [7] this means that $\mathscr{A}' = C(Y)$. Thus Y is compact Hausdorff, $C(Y) \equiv_{\omega_1,\omega_1} C(X)$ and $\operatorname{card} C(Y) \leqslant c$. Since $C(Y)$ separates points in Y, it must follow that $\operatorname{card} Y \leqslant 2^c$, which completes the proof.

We remark that the cardinality bounds in Theorem 9.3 are best possible, even if $\equiv_{\omega_1,\omega_1}$ is replaced by $\equiv$. For if X is βX_0, where X_0 is a countable discrete space, then $\operatorname{card} X = 2^c$ and $\operatorname{card} C(X) = c$. Moreover, if Y is compact and $C(Y) \equiv C(X)$, then Y is homeomorphic to X by Theorem 5.9.

§ 10. Open questions

A few questions which are related to the results in this paper are listed below:

Q 1. Call a space X *categorical* if every space Y satisfying $\mathscr{L}(Y)$

$\equiv \mathscr{L}(X)$ is actually homeomorphic to X. Is there any infinite categorical space? Is the unit interval I categorical?

Remark. In § 4 it was shown that if $\mathscr{L}(X) \equiv \mathscr{L}(I)$, then the topology on X is the order topology given by a complete, dense linear order $<$ with end points. By using the ideas in § 7 it is easily shown that X does not have a discrete subset with order type ω_1 under $<$ or under $>$. Therefore, X must be σ-compact and first countable. (This fact was also observed by A. Swett [27].) Similar reasoning shows that no discrete subset of X has the order type of a complete, dense linear order. However we are not able to show that every discrete subset of X must be countable. In the presence of Souslin's Hypothesis, this would suffice to prove that I is categorical. A Swett has also announced [27] that if $\mathscr{L}(X) \equiv \mathscr{L}(I)$ and $\mathscr{L}(X^n) \equiv \mathscr{L}(I^n)$ for some $n \geqslant 2$ then X is homeomorphic to I. Swett has also claimed (in a personal communication) that there exist spaces X, Y such that X e.e. Y but X^2 is not e.e. to Y^2. (This answers a question raised in an earlier version of this paper.)

Q 2. Does $\mathscr{L}(X)$ have a decidable theory for every subspace X of the unit interval?

Q 3. Are any two 0-dimensional separable metric spaces without isolated points e.e.? In particular, are the rationals, the irrationals and the Cantor set e.e. as topological spaces?

Q 4. Which spaces are e.e. to spaces of arbitrarily large cardinality?

Q 5. Are any two infinite dimensional Banach spaces e.e. as topological spaces?

Q 6. Given two (connected, σ-compact) manifolds X, Y, does X e.e. Y imply that X is homeomorphic to Y?

Q 7. If X, Y are Boolean spaces and $\mathscr{B}(X)$, $\mathscr{B}(Y)$ are both countable does $C(X) \equiv C(Y)$ imply that X, Y are homeomorphic?

Q 8. If X, Y are Boolean spaces, does $C(X) \equiv C(Y)$ imply $C(X) \equiv_{\omega_1,\omega_1} C(Y)$, or even $C(X) \equiv_{\infty,\omega_1} C(Y)$?

Q 9. Given a space X and a closed subset C of X, the *type* of C is the set of formulas $\varphi(x)$ (in the language of lattice theory) such that $\varphi(C)$ is true in $\mathscr{L}(X)$. Is it true that if p, q are non-isolated points of βN, the Stone–Čech compactification of the positive integers, then $\{p\}$, $\{q\}$ have the same type in $\mathscr{L}(\beta N)$? (This is a first-order analogue of the question whether $\beta N \sim N$ is homogeneous. The non-homogeneity of $\beta N \sim N$ was proved by W. Rudin [22] (assuming the continuum hypothesis) and Z. Frolík [30].)

Added note (September, 1975). A. K. Swett (*The first order topology of the real line*, preprint) has used a result of Shelah [23] to show that there is a space e.e. to the unit interval but not homeomorphic to it, thus answering negatively the last part of Q 1. Swett has also obtained a negative answer to Q 2 by considering Cantor–Bendixson derivatives. In addition Swett has obtained many further results in the area of this paper, including a characterization (due jointly to W. Fleissner) of Souslin lines as exactly those spaces which are $\mathscr{L}_{\infty,\omega}$-equivalent to the real line.

G. Cherlin (*Undecidable rings of continuous functions*, preprint) has a general theorem which implies that the theory of $C(R)$ (where R is the real line) is undecidable and in fact is recursively isomorphic to true second-order arithmetic. Along the way he shows that the set of constant functions is first-order definable in $C(R)$. From this and the analogue of Theorem 5.4 for R it can be shown that if X is a completely regular Hausdorff space such that $C(X) \equiv C(R)$, then X is homeomorphic to R.

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