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Sharp sufficient conditions for Hamiltonian cycles in tough graphs

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It is shown that if G is a 1-tough graph on $n \geqslant 3$ vertices such that $d(x)+d(y)+d(z) \geqslant n$ for every triple of mutually distinct and nonadjacent vertices x, y, z, and $\max\{d(u), d(v)\} \geqslant (n-5)/2$ for all vertices u, v at distance 2 in G then either G is Hamiltonian or else n is odd, $n \geqslant 15$, and G is a factor of a fixed maximally nonhamiltonian graph.

1. Introduction

Only simple graphs are considered. The letter G denotes an n-vertex graph with the vertex set V and the edge set E; $V = V(G)$ and $E = E(G)$. We sometimes write $|G|$ and $x \in G$ to abbreviate notation $|V(G)|$ and $x \in V(G)$, respectively. The number of components of G is denoted by $k(G)$. We call G a *tough graph* if G is 1-tough, i.e., if $k(G-W) \leqslant |W|$ for each subset W of V such that $k(G-W) \neq 1$. The smallest possible sum of degrees of m independent vertices in G is denoted by σ_m $(= \sigma_m(G))$, i.e.,

$$\sigma_m = \min_{I_m} \sum_{x \in I_m} d(x)$$

where I_m ranges over independent m-subsets of V. Some more special notation is introduced in the next section.

Some recent investigations into Hamiltonian tough graphs are inspired by the following result.

JUNG'S THEOREM [7]. *If G is a tough graph of order $n \geqslant 11$ with $\sigma_2 \geqslant n-4$ then G is Hamiltonian.* ∎

One can show that this result is sharp in a sense. To this end, let $*$ denote the nonassociative join on disjoint graphs and let $*\!\!\rightarrow$ stand for the injective join [10, 11]. Recall that, given mutually disjoint graphs K and G^i, $i = 1, \ldots, p$, with K complete and of order $\geqslant p$, and given an injection φ: $\{G^j | j = 1, \ldots, p\} \to V(K)$, the symbol $(\bigcup_{i=1}^{p} G^i) \ast\!\!\to K$ stands for the union $(\bigcup_i G^i) \cup K$ augmented by all G^i-$\varphi(G^i)$ edges, $i = 1, \ldots, p$. Hence

$$J := 3K_1 \ast\!\!\to K_3$$

is the *triangle* K_3 *with three disjoint short appendices*. Consider the following graphs each of which is easily seen to be tough and nonhamiltonian:

$$G_n^0 = K_1 \ast \Big(\big(\bigcup_{i=1}^{3} K_{m_i} \big) \ast\!\!\to K_3 \Big) \quad \text{with } 1 \leqslant m_1 \leqslant m_2 \leqslant m_3 \text{ and } n = 4 + \sum_i m_i \geqslant 7;$$

(1.1) $$G_n = ((n-7)/2) K_1 \ast K_{(n-5)/2} \ast J \quad \text{for odd } n \geqslant 7.$$

Next graphs are obtained by joining a new vertex to a maximal clique of G_{n-1}. Thus we get the following graphs for even n (three for each even $n \geqslant 10$):

(1.2) $$G_n = \big(K_2 \cup ((n-10)/2) K_1 \big) \ast K_{(n-6)/2} \ast J \quad \text{for even } n \geqslant 10,$$

(1.3) $$\left. \begin{array}{l} ((n-8)/2) K_1 \ast K_{(n-6)/2} \ast ((K_2 \cup 2K_1) \ast\!\!\to K_3) \\ ((n-8)/2) K_1 \ast K_{(n-6)/2} \ast (3K_1 \ast\!\!\to K_4) \end{array} \right\} \quad \text{for even } n \geqslant 8.$$

Notice that, for $n = 7$, G_7^0 $(= G_7)$ is the smallest nonhamiltonian tough graph of order $n \geqslant 3$, due to Chvátal. The graphs G_n^0, (1.2), and (1.3) appear in [10, 11], the graphs G_n^0 being generalizations of the graphs $K_1 \ast (3K_r \ast\!\!\to K_3)$ introduced in [1]; the graphs (1.1) appear in [3], their factors with $((n-5)/2) K_1$ in place of the complete part $K_{(n-5)/2}$ appear in [5]. Notice also that $\sigma_2(G_n^0) = n-2-m_3$ which is or can be $\geqslant n-4$ for $7 \leqslant n \leqslant 10$; moreover, $\sigma_2(G_n) = n-5$ for odd $n \geqslant 11$ and for $n = 12$. This shows that in Jung's Theorem both the bound on n and that on σ_2, as a linear function of n, cannot be relaxed. However, we did not find a corresponding example for any even $n \geqslant 14$. This has prompted the following substantial improvement of Jung's Theorem for larger n.

THEOREM 1. *If G is a tough graph on $n \geqslant 14$ vertices such that $\sigma_2 \geqslant n-5$ then either G is Hamiltonian or else n is odd, $n \geqslant 15$, and G is a factor of the graph G_n in* (1.1).

Hence and from Jung's Theorem we get the following sufficient condition for a tough graph G with $n \geqslant 11$ vertices to be Hamiltonian:

$$\sigma_2(G) \geqslant \begin{cases} n-4 & \text{for } n = 12 \text{ and for each odd } n \geqslant 11, \\ n-5 & \text{for each even } n \geqslant 14. \end{cases}$$

Examples presented above show sharpness of that result.

Before submitting the first version of this paper for publication, I received a preprint of [2], which prompted the following generalization of Theorem 1 above and of Theorem 4 in [2].

THEOREM 2. *Let G be a tough graph of order* $n \geqslant 3$ *such that*

(1.4) $$\sigma_3 \geqslant n, \quad \text{and}$$

(1.5) $\max\{d(u), d(w)\} \geqslant (n-5)/2$ *for all vertices u, w at the distance* $d(u, w)$ *equal to* 2 *in G.*

Then either G is Hamiltonian or else n is odd, $n \geqslant 15$, *and G, called an* ***exceptional graph****, is a factor of the graph* G_n *in* (1.1).

Notice that $\sigma_2 \geqslant n-5$ and $n \geqslant 14$ imply $\sigma_3 \geqslant n$ because the second smallest summand in σ_3 is then at least $\lceil (n-5)/2 \rceil \geqslant 5$. That is why our Theorem 2 generalizes Theorem 1.

Remark. Each exceptional graph G includes J as an induced subgraph.

COROLLARY ([2, Theorem 4]). *A tough n-vertex graph G is Hamiltonian if* $n \geqslant 3$, $\sigma_3 \geqslant n$, *and*

(1.6) $$(u, w \in V,\ d(u, w) = 2) \Rightarrow \max\{d(u), d(w)\} \geqslant (n-4)/2. \ \blacksquare$$

The original proof of Theorem 1 and the proof of Theorem 2 given below involve arguments which have now become standard. Namely, for a nonhamiltonian graph G, its longest cycle C with a fixed orientation is considered. It is Nash-Williams who first introduced [8] such a method into Hamiltonian graph theory. Jung proceeds in [7] from that starting point too. Related papers [4, 2] show how using convenient notation as well as appropriate results and observations can lead to elegant proofs which are easy to follow. Other papers like [5, 3, 9] also influenced our reasoning. In the proof, we first note that C is a *dominating cycle* in G, i.e., $G-C$ comprises isolated vertices only, next that C can avoid a vertex of large degree and that C is of length $\geqslant n-2$. Finally, we show that exceptional graphs can arise only if C avoids exactly one vertex v whose degree $d(v)$ is large enough, namely $(n-5)/2 \leqslant d(v) \leqslant (n-3)/2$.

A number of standard observations used and stated in what follows are specifications of the following trivial one.

PROPOSITION. *Assume that the vertices of a longest cycle C of G form a proper subset of the vertices of the union of two disjoint subgraphs,* C' *and* C'', *each of which is a cycle or a path (possibly trivial) and* $C' \neq K_1$. *Let the phrase* ***a pair in a subgraph*** *mean an edge and end-vertices [a singleton] if the subgraph is a cycle and a path* $[K_1]$, *respectively. Then the following condition* $\mathrm{L0}^\alpha$ (= L0, L0′, L0″ *where the number of primes is that of paths among* C', C'') *holds:*

$\mathrm{L0}^\alpha$. *No pair in* C' *can be matched (improperly if* $C'' = K_1$) *onto one in* C''. ■

Our proof heavily depends on the following results.

LEMMA. *Let G be a tough graph of order $n \geqslant 3$ with $\sigma_3 \geqslant n$. Then* L1 *through* L3 *hold.*

L1. *If $n \leqslant 10$ then G is Hamiltonian.*

In fact, all maximally nonhamiltonian tough graphs of order $n \leqslant 10$, listed in [6], have $\sigma_3 < n$. ∎

The smallest known tough maximally nonhamiltonian graph of order n with $\sigma_3 \geqslant n$ is G_n in (1.1) for $n = 15$.

L2 ([3, Theorem 5]). *Each longest cycle of G is a dominating cycle.* ∎

L3 ([3, see proof of Theorem 9]). *If G is nonhamiltonian then G has a longest cycle C such that C avoids a vertex v of degree $d(v) \geqslant n/3$ in G.* ∎

Another important result taken from [3] (see L4 in the following section), under the hypotheses of L3, specifies two large independent subsets of $V(G)$, of cardinality $|G-C|+d(v)$ each.

2. Preliminaries

We assume that *paths* and *cycles* are simple graphs. In what follows, let C (or $\vec{C}$) be a cycle with a fixed orientation in a graph G. Then $\overleftarrow{C}$ denotes that cycle with the reverse orientation. If $u, w \in C$ then u^+ denotes the successor of u on C, u^- the predecessor, next uCw is the string of vertices on $\vec{C}$ from u to w inclusive, where $uCu := u$, and $[uCw]$ stands for the corresponding section (path) of C; however, $[vuCuv] = [vu]$ and $[uCu^+u] = [uu^+]$. Then the inverse of that string is clearly denoted by $w\overleftarrow{C}u$. However, clearly, $[uCw] = [w\overleftarrow{C}u]$. Similarly, if $W \subset V(C)$ then $W^+ = \{w^+ | w \in W\}$, $W^- = \{w^- | w \in W\}$, $u^{++} = (u^+)^+$.

L4 ([3, Theorem 5 and Lemma 8]). *Let G be a tough nonhamiltonian graph of order n with $\sigma_3 \geqslant n$, let C be a longest cycle in G, let $v \in G-C$, and let $X = N(v)$. Then $X \subset V(C)$ and $V(G-C) \cup X^+$ [also $V(G-C) \cup X^-$] is an independent set in G.* ∎

In what follows, *G is a nonhamiltonian graph, C is a longest cycle of G, $v \in G-C$, and $X = N(v) \cap V(C)$.* Moreover, let $x_0, x_1, \ldots, x_{|X|-1}$ be all vertices in X taken in their cyclic order along $\vec{C}$. Throughout, indices of vertices read modulo $|X|$. Because C is a longest cycle therefore $X \cap X^+ = \emptyset$. Hence both neighbours of any x_i on C are outside X. Following [5, 2] we use distinct letters to differentiate between those neighbours. Namely, let $x_i^- = z_{i-1}$ and $x_i^+ = y_i$ for each i, and let $X^- = Z$ and $X^+ = Y$. Let $T = X^+ \cap X^-$ and let t, possibly with a subscript, stand for an element of T.

The following observations will be helpful.

L5 (by L0′ or L4). *Y [also Z] is an independent set of vertices.* ■

Hence no t-$(Y \cup Z)$ edge is in E.

L6 (by L0′). *Given an edge ww^+ on C, if the path $P := [x_i^+ C x_j^-]$, where $i \neq j$ (and possibly $P = K_1$), avoids both vertices w and w^+, then G has no two other edges which could match ww^+ (improperly if $x_i^+ = x_j^-$) onto $\{x_i^+, x_j^-\}$.* ■

L7 (by L0″). *Let $i \neq j$ and let $\{u_i, u_j\}$ be $\{x_i^+, x_j^+\}$ or $\{x_i^-, x_j^-\}$ where $u_i = x_i^{\pm}$. Let $w_i w_j \in E(C - \{u_i, u_j\})$ where notation is chosen so that the pairs u_i, w_j and u_j, w_i intertwine on C. Then at most one of mutually crossing chords $u_i w_j$, $u_j w_i$ is in E.*

To use L0″ in the proof, find a u_i-w_i path and a disjoint u_j-w_j path whose lengths sum up to $|C|-1$. ■

L8. *If $t \in T$ and $u \in N(t) \cap V(C)$ then $X^+ \cup \{u^+\}$ [also $X^- \cup \{u^-\}$] is an independent set of vertices.*

L8 follows from L5, L6, and L7. ■

L9 (by L0). *If $y_h z_f \in E$ then no edge of the subgraph $[y_h C z_f y_h]$ can be matched onto any edge of the subgraph $[v x_{f+1} C x_h v]$.* ■

For $\alpha = 4, 5, 7, 8, 15$, $L\alpha^+$ and $L\alpha^-$ will denote the part of the observation $L\alpha$ on X^+ and X^-, respectively (see the next section for L15).

Let components of $C - X$ (represented sometimes by strings of the form $y_i C z_i$) be called *segments* of C. The number of vertices in a segment S, denoted by $|S|$, is called the *order* of S; and S will be named an *$|S|$-segment*. A segment S is called *trivial* if S is a 1-segment; otherwise S is called *nontrivial*. Thus $|T|$ is the number of trivial segments of C. In what follows

$$\nu = |X| - |T|, \tag{2.1}$$

whence ν is the *number of nontrivial segments of C*, each of which will be denoted by S, with a subscript if $\nu > 1$; S itself will stand for the *union of all nontrivial segments of C*, $|S|$ being the number of vertices in that union.

Now, assume additionally that in what follows *G is tough, nonhamiltonian, and $\sigma_3 \geqslant n \geqslant 3$*. Thus, by L2, C is a dominating cycle of G.

L10. *$|C| = 2|X| + r$ for some $r \geqslant 2$ and $1 \leqslant \nu \leqslant r$. Additionally, if $r \leqslant 3$ then $\nu > 1$ and, moreover, $|C| = n - 1$ for $r = 2$.*

Proof. If $|C| \leqslant 2|X| + 1$ then all segments of C are trivial, except possibly one which can be a 2-segment. Then, by L4, $k(G - X) > |X|$, contrary to the toughness of G. Hence, $|C| = 2|X| + r$ for some $r \geqslant 2$; $1 \leqslant \nu \leqslant r$ is obvious. Assume $r \leqslant 3$ and suppose G is still a counterexample. Suppose $\nu = 1$. Then, for some $y \in Y$, we have (i) $r = 2$ and $S = yy^+z$; or (ii) $r = 3$ and $S = yy^+z^-z$.

If no T-S edge is in E then, by L4, $k(G-X) > |X|$, a contradiction with the toughness of G. On the other hand, if $u \in S$ and $ut' \in E$ for some $t' \in T$ then, by L5, u is an inner vertex of S, whence, by L6, u is the only vertex in S adjacent to T; next, by L8, $S-u$ consists of two parts nonadjacent in G. Then, by L4, $k(G-(X \cup \{u\})) > |X|+1$, again a contradiction. Thus $\nu > 1$. It remains to consider the case $|C| < n-1$ and $r = 2 = \nu$. Then S is the union of two 2-segments, whence, by L4, $k(G-X) > |X|$, which is impossible. ∎

3. Proof of Theorem 2

In what follows we assume that G *satisfies the hypotheses of Theorem* 2 *and is nonhamiltonian*, whence, by L1, $n \geq 11$. Let C be a longest cycle of G, with a fixed orientation, such that the maximum degree in G of vertices in $G-C$ is as large as possible among all longest cycles in G. Let v be a vertex of $G-C$ of the largest degree in G. By L2, C is a dominating cycle. Hence $X = N(v)$ and, by L3, $|X| \geq n/3$, whence, by (2.1),

L11. $|X| = (n-5)/2 \Rightarrow |T| \geq 5-\nu$; $|X| = (n-4)/2 \Rightarrow |T| \geq 4-\nu$. ∎

L12. $|X| \geq (n-5)/2$.

Proof. Suppose $|X| < (n-5)/2$. Then, by (1.5), $d(u) \geq (n-5)/2$ for each $u \in Y \cup Z$. If y is a 1-segment of C then replacing $y^- y y^+$ in C by $y^- v y^+$ gives a longest cycle of G which avoids the vertex y with $d(y) > d(v)$, a contradiction with the choice of C. Hence C contains no 1-segment and therefore $d(v) = |X| \leq |C|/3 < n/3$, a contradiction with L3. ∎

Hence, because G is tough and $V(G-C) \cup Y$ is independent by L4$^+$, therefore $|G-C|+|Y| \leq n/2$, whence $|C| \geq n/2+|X| \geq n-5/2$. Moreover, by L10,

$$2|X|+r = |C| \leq n-1 \leq 2|X|+4 \quad \text{where } r \geq 2, \tag{3.1}$$

whence, by (2.1), $1 \leq \nu \leq r \leq 4$ and $\nu \leq |X|$.

Case 1: $\nu = 1$. Then, by L10 and (3.1), we have $r = 4$, $|C| = n-1$, and $|X| = (n-5)/2$, whence n is odd and, by L11, $|T| \geq 4$. We may assume $S = y_0 y_0^+ y_0^{++} z_0^- z_0$. Suppose that two vertices, say u and w, in S are adjacent to T. Then, by L6 and L5, $\{u, w\} = \{y_0^+, z_0^-\}$. By L5 and L8, however,

$$k(G-(X \cup \{y_0^+, z_0^-\})) > |X|+2, \tag{3.2}$$

a contradiction with the toughness of G. Similarly, if G has no S-T edge then $k(G-X) > |X|$, again a contradiction. Thus exactly one of inner vertices in S has a neighbour in T. Suppose $ty_0^+ \in E$. Then, by L8, $y_0 y_0^{++}, y_0 z_0 \notin E$. Now $y_0 z_0^- \in E$ because otherwise $k(G-(X \cup \{y_0^+\})) > |X|+1$, a contradiction. However, $y_0^{++} z_0 \notin E$ because otherwise $[v x_0 C^{\leftarrow} t y_0^+ y_0 z_0^- y_0^{++} z_0 C t^- v]$ is a Hamiltonian cycle of G. Hence (3.2). Therefore, no edge $ty_0^+ \in E$ and, equivalently, no $tz_0^- \in E$.

Thus $t_0y_0^{++} \in E$ for some $t_0 \in T$. Then, by L8, $y_0z_0^-, y_0^+z_0 \notin E$. Hence, if none of the edges $y_0z_0, y_0^+z_0^-$ is in E then $k(G-(X \cup \{y_0^{++}\})) > |X|+1$, a contradiction; however, those edges are not both in E because otherwise G would clearly have a cycle longer than C. Suppose $y_0z_0 \in E$ and $y_0^+z_0^- \notin E$. By L9, $y_0^+t_0^-, y_0^+t_0^+ \notin E$. Also $y_0^+x_0, y_0^+x_1 \notin E$ because otherwise G has a cycle longer than C. Hence, since $n \geqslant 11$, $t_0^+ \neq x_0$ or $t_0^- \neq x_1$. Therefore at least three vertices in X are nonadjacent to y_0^+, whence $d(y_0^+) \leqslant |X|-1 < (n-5)/2$. By symmetry, also $d(z_0^-) < (n-5)/2$, contrary to (1.5).

Thus $y_0z_0 \notin E$ and $y_0^+z_0^- \in E$. Now, if $t \neq t_0$ then $ty_0^{++} \notin E$ (whence $N(t) \subseteq X$), because otherwise G is clearly Hamiltonian. Similarly, $y_0y_0^{++}, z_0y_0^{++} \notin E$. Hence, J is induced by $\{t_0, y_0, z_0, y_0^+, y_0^{++}, z_0^-\}$ and $G \subseteq G_n$ in (1.1). ∎

Definition. Given $\alpha, \beta \in Z_{|X|}$, let $T'_{\alpha\beta} = \{t \in T\colon t \in [y_\alpha C z_\beta]\}$. Given T', let $T'' = T - T'$ and, moreover, let $t' \in T'$ and $t'' \in T''$.

Case 2: $v = 2 = r$. We may assume that $S = S_0 \cup S_i$ where $S_0 = y_0z_0$ and $S_i = y_iz_i$ for some $i > 0$. By L10, n is odd and $|C| = n-1$, whence, by (3.1), $|X| = (n-3)/2$. Since $n \geqslant 11$, $|T| = |X|-2 \geqslant 2$. Moreover, $N(t) \subseteq X$ by L5. If $y_0z_i, z_0y_i \notin E$ then, by L5, $k(G-X) > |X|$, a contradiction with the toughness of G. On the other hand, by L6, only one of those two edges can be in E.

Assume $z_0y_i \in E$ and let $T' = T'_{i0}$ (the remaining case $y_0z_i \in E$ is clearly equivalent). Hence $y_0z_i \notin E$. Moreover, by L6 and L7$^+$, $y_0x_i, y_0x_1 \notin E$, whence $N(y_0) \subseteq \{z_0\} \cup X - \{x_1, x_i\}$. Similarly, $N\{z_i\} \subseteq \{y_i\} \cup X - \{x_1, x_i\}$.

Suppose $i > 1$. Then $d(y_0)$, $d(z_i)$ are both $\leqslant (n-5)/2$. Suppose $i+1 < |X|$. Then $T' \neq \emptyset$ and, by L6 (or L7), no t'-$\{x_1, x_i\}$ edge is in E, whence $d(x_0^-)$, $d(x_{i+1}^+)$ are both $\leqslant |X|-2 < (n-5)/2$. Hence, by (1.5), $d(y_0) = d(z_i) = (n-5)/2$; and, moreover, $x_0^- = x_{i+1}^+$. Now, if $n = 11$ then $\sigma_3 \leqslant d(x_0^-)+d(y_0)+d(v) < n$, contrary to (1.4). If $n > 11$ then $i > 2$, whence $y_0x_2, z_ix_2 \in E$ and therefore, by L9, G has no $\{y_1, y_2\}$-$\{x_0, x_{i+1}\}$ edge where $y_1, y_2 \in T$. Hence, $d(y_1), d(y_2) < (n-5)/2$, contrary to (1.5). Therefore $T' = \emptyset$. Hence, by (1.5), y_0x_2 or z_ix_2 is in E. Without loss of generality, assume $y_0x_2 \in E$. Then $y_1, y_2 \in T$ and, by L9, we have $y_1x_0, y_2x_0 \notin E$, whence $d(y_1), d(y_2) \leqslant (n-5)/2$. Now, by (1.5), y_1x_i or $y_2x_1 \in E$. In either case, however, G has a Hamiltonian cycle: through $y_iz_0y_0x_2$ and either $x_iy_1x_1vx_0$ or $x_1y_2Cx_ivx_0$, a contradiction.

Thus $i = 1$. By L6 (or L7), no $tx_1 \in E$, whence $N(t) \subseteq X - \{x_1\}$ and equality holds for all but possibly one $t \in T$ (by (1.5)). Now, $G \subseteq G_n$ in (1.1) and J is induced by $\{v, y_0, z_0, x_1, y_1, z_1\}$. ∎

L13. *If $v = 2 < r \leqslant 4$ then $r = 4$ and nontrivial segments of C have orders 4 and 2.*

Proof. Suppose the contrary. Then we may assume that $S = S_0 \cup S_i$ where $S_0 = y_0y_0^+z_0$ and, for some $i > 0$, either $S_i = y_iz_i$ if $r = 3$, or $S_i = y_iy_i^+z_i$ if

$r = 4$. Moreover, by (3.1) and L11, we have either $3 \leqslant r \leqslant 4$, $|X| = (n-5)/2$ and $|T| \geqslant 3$, or $r = 3$, $|X| = (n-4)/2$ and $|T| \geqslant 2$; furthermore, $|C| = n-1$ if $r = 4$.

Suppose $y_0 z_i \in E$ and let $T' = T'_{0i}$. Then, by L7, $t'' y_0^+$, $t'' z_i^- \notin E$, whence, by L4, $N(t'') \subseteq X$. By L6 (or L7), G has no t'-$\{y_0^+, x_0, x_{i+1}, z_i^-\}$ edge, whence, by L4, $d(t') < |X|$ even if $x_{i+1} = x_0$. Therefore, by (1.5), $i < 3$, i.e., $|T'| \leqslant 1$, whence $x_{i+1} \neq x_0$. Similarly, no y_i-$\{x_0, y_0^+, x_{i+1}\}$ edge is in E. Moreover, if $r = 3$ then $y_i z_0 \notin E$ by L6, whence, by L4, $d(y_i) < |X|$. Consequently, (1.5) implies $T' = \emptyset$ if $r = 3$. Hence, if $r = 3$ then $y_{i+1}, y_{i+2} \in T''$. The same is true if $r = 4$ because then $|T| \geqslant 3$. Therefore, by (1.5), $tx_1 \in E$ for some $t = y_{i+1}, y_{i+2}$. Hence, by L7$^+$, $y_i z_0 \notin E$ also if $r = 4$. Thus, in each case, $N(y_i) \subseteq \{z_i, y_i^+\} \cup X - \{x_0, x_{i+1}\}$. Similarly, $N(z_0) \subseteq \{y_0, y_0^+\} \cup X - \{x_0, x_{i+1}\}$. Hence, by (1.5), there is $u \in \{z_0, y_i\}$ such that $d(u) = |X|$ (so $u = z_0$ if $r = 3$). Therefore $ux_{i+2} \in E$. Hence, by (1.5), an edge of C incident to x_{i+2} can be matched onto $z_0 x_1$ if $u = z_0$ or onto $x_i y_i$ if $u = y_i$, a contradiction with L9. Thus $y_0 z_i \notin E$ and similarly $z_0 y_i \notin E$.

Now, using L4, we shall conclude that G is not tough or has a cycle longer than C. Namely, if both S_0 and S_i are induced paths in G then, for $X' = X \cup \{y_0^+\}$ if $r = 3$ and $X' = X \cup \{y_0^+, y_i^+\}$ if $r = 4$, $k(G - X') > |X'|$. Otherwise, we (change notation if necessary so as to) assume that S_0 induces a triangle. For $r = 4$, assume additionally that $y_i z_i \notin E$ or $y_0^+ y_i^+ \notin E$. Notice that, by L7, the center of that segment which induces a triangle (hence y_0^+) is adjacent to no end-vertex of another segment. Hence, for $X'' = X$ if $y_i z_i \in E$ or $X'' = X \cup \{y_i^+\}$ if $r = 4$ and $y_i z_i \notin E$, we clearly get $k(G - X'') > |X''|$ wherein we may use the fact that $|C| = n-1$ if $r = 4$. It remains to consider the case that $r = 4$, both S_0 and S_i induce triangles and $y_0^+ y_i^+ \in E$. Then, however, G is clearly Hamiltonian, a contradiction. ∎

Condition (1.4) implies the following.

L14. *If* $\{u_1, u_2, u_3\}$ *is an independent 3-subset of* $V(G)$ *then* $d(u_1) \geqslant n - d(u_2) - d(u_3)$. ∎

Case 3: $v = 2 < r$. Owing to (3.1) and L13, $r = 4$, $|C| = n-1$, $|X| = (n-5)/2$ and, moreover, we may assume that $S = S_0 \cup S_i$ where $S_0 = y_0 y_0^+ z_0^- z_0$ and $S_i = y_i z_i$ for some $i > 0$. Hence, by L11, $|T| \geqslant 3$.

Subcase 3.1: *An end-vertex of* S_0 *is adjacent to one of* S_i. Without loss of generality, due to L5, assume $z_0 y_i \in E$ and let $T' = T'_{i0}$. By L8$^+$ and L5, $N(t) \subseteq \{y_0^+\} \cup X$ for each $t \in T$, and, by L6, $t' x_1$, $t' x_i \notin E$, whence $d(t') < |X|$ if $i > 1$. By L6 and L7, no z_i-$\{y_0, z_0^-, x_1, x_i\}$ edge as well as no y_0-$\{x_1, x_i\}$ edge is in E, whence, by L5, the sets of possible neighbours of z_i and y_0 outside $X - \{x_1, x_i\}$ are $\{y_0^+, y_i\}$ and $\{y_0^+, z_0^-, z_0\}$, respectively. However, at most one of the edges $z_i y_0^+$ and $y_0 z_0^-$ is in E because otherwise G has a Hamiltonian cycle: $[v x_1 C y_i z_0 z_0^- y_0 y_0^+ z_i C x_0 v]$, contrary to the assumption. Therefore, because by L14, $d(y_0) \geqslant n - d(v) - d(z_i)$, if $i > 1$ then the vertex y_0 has three or more neighbours in $X - \{x_1, x_i\}$.

Suppose $i > 1$. Then, by (1.5), because $|X| = (n-5)/2$, $|T'| \leqslant 1$, whence $|T''| \geqslant 2$. Hence $i > 2$ and y_0 is adjacent to some x_j with $1 < j < i$. Then z_{j-1}, $y_j \in T''$, whence, by (1.5), $z_{j-1}x_j$ or x_jy_j can be matched in G onto one of the edges $y_0y_0^+$ and y_0x_0, which contradicts L9.

Thus $i = 1$, whence $T = T'$ and $N(t) \subseteq \{y_0^+\} \cup X - \{x_1\}$. Therefore (1.5) implies that, for some $t_0 \in T$, $t_0y_0^+ \in E$. Then, by L8, $y_0z_0^-$, y_0z_0, $z_0^-y_1 \notin E$. Moreover, $G - z_0^-$ can easily be seen to have a Hamiltonian cycle through z_0x_1, whence, by L0′, $z_0^-x_1 \notin E$. Consequently, the set $\{v, z_0^-, z_0, x_1, y_1, z_1\}$ induces J in G and $G \subseteq G_n$ in (1.1).

Subcase 3.2: *No end-vertex of* S_0 *is adjacent in* G *to an end-vertex of* S_i. Suppose that y_0^+ and z_0^- are not both covered by S_i-S_0 edges. Then either (α) G has no S_i-S_0 edge or (β) for $\{w, w_0\} := \{y_0^+, z_0^-\}$, assume without loss of generality that $y_iw \in E$ for a fixed w and that each S_i-S_0 edge of G covers w. In case (α), G has a T-S edge because otherwise, by L5, $k(G-X) > |X|$, a contradiction with the toughness of G.

Suppose there is $u \in S$ such that $t_0u \in E$ for some $t_0 \in T$. Then, by L5, $u \in \{w, w_0\}$. Next, by L6, $N(t) \subseteq \{u\} \cup X$ for each t. Moreover, by L8, we have y_0u^+, $z_0u^- \notin E$. Hence, (α) does not hold and, moreover, $u = w_0$ ($\neq w$ and $y_iw \in E$) because otherwise $k(G-(X \cup \{u\})) > |X|+1$ both in case (β) for $u = w$ and in case (α). By L7$^+$, $y_0w^+ \notin E$, whence $N(y_0) \subseteq \{y_0^+\} \cup X$. Furthermore, $w = y_0^+$ because otherwise $t_0y_0^+$, $y_iz_0^- \in E$, which contradicts L8$^+$. Thus $t_0w^+ \in E$, whence, due to L6 (or L7$^+$), $t_0 \notin [y_iCx_0]$. Hence, by L6, $z_iw \notin E$, whence $N(z_i) \subseteq \{y_i\} \cup X$. Define the cycles $\tilde{C}' = [y_iCy_0^+y_i]$ and $\tilde{C}'' = [vt_0^-C^{\leftarrow}z_0^-t_0Cx_iv]$. Now, by L0, if $j = i+1, \ldots, |X|$, then no neighbour of x_j on $\tilde{C}'$ (hence neither z_i nor y_0) is adjacent to x_i or t_0^- ($\in X$), the neighbours of v of $\tilde{C}''$. Therefore, because $x_0^- \in T$ or $x_0^- = z_i$, we have $d(y_0)$, $d(x_0^-) < |X|$, a contradiction with (1.5) because $|X| = (n-5)/2$.

Thus u does not exist, whence $N(t) \subseteq X$. Moreover, (β) holds. Hence, by L7$^+$, $y_0w^+ \notin E$. However, $w^-z_0 \in E$ because otherwise $k(G-(X \cup \{w\})) > |X|+1$, a contradiction. Hence, by L7$^-$, $z_iw \notin E$, whence $N(z_i) \subseteq \{y_i\} \cup X$. Let $C' = [wy_iCw^-z_0C^{\leftarrow}w]$ and either $C'' = [vx_1Cx_iv]$, a cycle, if $i > 1$, or else $C'' = [vx_1]$, a path, if $i = 1$. Now, $i \geqslant |X|-2$ because otherwise the vertices y_{i+1}, $y_{i+2} \in T$ but, by L0 (or L0′ if $i = 1$) none of them is adjacent to x_1, the neighbour of v on C'', whence $d(y_h) < |X|$ for $h = i+1, i+2$, which contradicts (1.5). Hence, $i > 2$ and either x_0^- is the only element of T on C' or $x_0^- = z_i$. By L14, because the set $\{x_0^-, y_1, v\}$ is independent, $d(x_0^-) \geqslant 5$. Hence x_0^- has three or more neighbours in $X \cap V(C'')$, whence $x_0^-x_j \in E$ for some j, $1 < j < i$. Therefore, by L0, both neighbours of x_j on C'', which are both in T, are nonadjacent to x_0, again a contradiction with (1.5).

Thus S_i-S_0 edges cover both y_0^+ and z_0^- and, by L6, those edges cover either y_i or z_i (but not both). Assume without loss of generality that they cover

y_i. Then, by L7$^+$, no y_0-$\{z_0^-, z_0\}$ edge is in E. Also $z_0 y_0^+ \notin E$ because otherwise we can arrive at a contradiction by considering C' and C'' defined as above with $w = z_0^-$. Thus the set $\{y_0, y_0^+, z_0^-, z_0, y_i, z_i\}$ induces J in G and $G \subseteq G_n$ in (1.1). ■

L15. *No two vertices in X^+ [in X^-] have degrees smaller than $(n-5)/2$.*

Proof. Let $U = Y$ or $U = Z$. Suppose that two vertices u_1, u_2 in U have degrees $d(u_1)$, $d(u_2) < (n-5)/2$. By L4, u_1 and u_2 have neighbours in $C-U$ only and, by (1.5), no neighbour in common, whence $d(u_1)+d(u_2) \leqslant |C|-d(v)$. Moreover, the set $\{u_1, u_2, v\}$ is independent, whence $\sigma_3 \leqslant |C| < n$, contrary to (1.4). ■

L16. *If $3 \leqslant v \leqslant 4$ then $v = 3$ and $r = 4$.*

Proof. Suppose the contrary. Then, by (3.1), $3 \leqslant v = r \leqslant 4$ and, for $v = 3$, either $|C| = n-2$ and $|X| = (n-5)/2$ or $|C| = n-1$ and $|X| = (n-4)/2$, else $v = 4$, $|C| = n-1$ and $|X| = (n-5)/2$. Moreover, we may assume that $S = S_0 \cup S_i \cup S_j \cup S_g$ where $0 < i < j \leqslant g < |X|$, $S_l = y_l z_l$ for $l = 0, i, j, g$, and $g = j$ iff $v = 3$. By L11, $|T| \geqslant 1$ and, by L4, $N(t) \subseteq X$.

If no edge connects distinct segments then G is not tough, because then, by L4, $k(G-X) > |X|$. On the other hand, due to symmetry, it is enough to show that contradiction follows in each of the following three cases:

(α) $z_0 y_i \in E$;
(β) $z_0 y_j \in E$;
(γ) $v = 4$ and $E \cap E' \neq \emptyset$ where $E' = \{z_0 y_g, z_i y_0, z_j y_i, z_g y_j\}$.

Suppose (α) holds. Let $T' = T'_{i0}$. Then, by L6, no y_0-$\{x_i, z_i, z_j, z_g\}$ edge is in E and, by L7$^+$, $y_0 x_1 \notin E$, whence, by L4, $N(y_0) \subseteq \{z_0\} \cup X - \{x_1, x_i\}$. Similarly, $N(z_i) \subseteq \{y_i\} \cup X - \{x_1, x_i\}$ and also $t'x_1$, $t'x_i \notin E$. Hence $d(t') < |X|$, whence, by L15, $|T'| \leqslant 1$. Suppose $i > 1$. Then $d(y_0)$, $d(z_i) < |X|$, whence, by L15, $T' = \emptyset$. Moreover, by (1.5), y_0 and z_i have no common neighbour. Hence $\sigma_3 \leqslant d(y_0)+d(z_i)+d(v) \leqslant 2|X| < n$, contrary to (1.4). Thus $i = 1$. Then $T = T'$, $|T| = 1$ and $|X| = v+|T|$. If $t \in T$ then $\sigma_3 \leqslant d(t)+d(y_0)+d(v) \leqslant 3v+2 < n$, contrary to (1.4).

Thus $z_0 y_i \notin E$ and similarly $z_i y_j$, $z_j y_g$ for $g > j$, and $z_g y_0 \notin E$.

Suppose (β) holds, i.e., $z_0 y_j \in E$. By L6 and L7, $N(y_0) \subseteq \{z_0, z_i\} \cup X - \{x_1, x_j\}$ and $N(z_j) \subseteq \{y_j, y_i\} \cup X - \{x_1, x_j\}$. Similarly, if $t' \in T'_{j0}$ then $t'x_1$, $t'x_j \notin E$, whence $d(t') \leqslant |X|-2$. Therefore, by L15, $|T'_{j0}| \leqslant 1$. Hence, because $|T| \geqslant 1$, if $y_0 z_i$, $z_j y_i \in E$ then one of the sets T'_{j0}, T'_{0i} and T'_{ij} is a singleton and the remaining two are empty, and $d(t) \leqslant |X|-2$. Then $|X| = v+1$ and therefore $\sigma_3 \leqslant d(y_0)+d(z_j)+d(t) \leqslant 3v+1 < n$, contrary to (1.4). Therefore $d(y_0)$ or $d(z_j)$ is smaller than $|X|$. Hence, by L15, $T'_{j0} = \emptyset$ and $N(t) = X$, whence $tx_0 \in E$ for each t. Moreover, the set $\{y_0, z_j, v\}$ is independent, whence, by L14, $d(y_0)$

$\geqslant v+1$. Therefore y_0 has a neighbour adjacent on C to some t, which contradicts L9 with $y_j z_0 \in E$.

Thus $z_0 y_j \notin E$ and similarly the edges $z_i y_0$ and $z_j y_i$ if $v = 3$ as well as $z_i y_g$, $z_j y_0$, and $z_g y_i$ if $v = 4$ do not belong to E. This is a contradiction if $v = 3$.

Thus (γ) holds. Now E' contains all Y-Z chords of C that are in E. Suppose that $z_0 y_g \in E$. Then, by L6 and L7, $N(y_0) \subseteq \{z_0, z_i\} \cup X - \{x_1, x_g\}$ and $N(z_g) \subseteq \{y_g, y_j\} \cup X - \{x_1, x_g\}$. By L7, if $t \in T'_{g0}$ then $N(t) \subseteq X - \{x_1, x_g\}$. Hence $E' \not\subseteq E$ because otherwise $d(t) < |X|$ for each t, whence, by L15, $|T| = 1$; moreover, $d(u) \geqslant |X|$ (with equality in fact) for each $u \in (Y \cup Z) - T$, and therefore $y_0 x_j$, $y_j x_0 \in E$, contrary to L9 with $y_g z_0 \in E$. Thus, without loss of generality, assume that $z_0 y_g \in E$ and $z_g y_j \notin E$. Then $d(z_g) < |X|$, whence, by (1.5), $T'_{g0} = \emptyset$ but $d(y_0) = |X|$. Hence $y_0 z_i \in E$. Moreover, by L15, $N(t) = X$ for each t and therefore $T'_{0i} = \emptyset$. Thus $y_{i+1} \in T$ or $y_{g-1} \in T$ because $T \neq \emptyset$; moreover, $t x_0 \in E$ for each t. Hence, by L9, $y_0 x_{i+1}$ or $y_0 x_{g-1}$ is not in E, which contradicts $d(y_0) = |X|$. ∎

Case 4: $v \geqslant 3$. Then, by L16 and (3.1), $v = 3$, $r = 4$, $|C| = n-1$, $|X| = (n-5)/2$, and we may assume that $S = S_0 \cup S_i \cup S_j$ where $0 < i < j < |X|$, $S_0 = y_0 y_0^+ z_0$, and $S_l = y_l z_l$ for $l = i, j$. By L11, $|T| \geqslant 2$; by L5, $N(t) \subseteq \{y_0^+\} \cup X$. Notice that $y_h z_g \in E$ for some $g, h \in \{0, i, j\}$ such that either $h \neq g$ or $h = g = 0$. In fact, otherwise by L5, $k(G-(X \cup \{y_0^+\})) > |X|+1$, contrary to the toughness of G.

Suppose $y_0 z_i \in E$ and let $T' = T'_{0i}$. Then, by L6, $y_i y_0^+$, $y_i x_0$, $y_i z_0 \notin E$ and, by L7$^+$, $y_i x_{i+1} \notin E$, whence, by L5, $N(y_i) \subseteq \{z_i, z_j\} \cup X - \{x_0, x_{i+1}\}$. Similarly, $N(z_0) \subseteq \{y_0^+, y_0, y_j\} \cup X - \{x_0, x_{i+1}\}$, $N(t') \subseteq X - \{x_0, x_{i+1}\}$ and $N(t'') \subseteq X$. Hence, by (1.5), $|T'| \leqslant 1$, whence $T'' \neq \emptyset$. Suppose $d(y_i) \geqslant |X|$. Then, by L15, $x_i t_0 \in E$ for some $t_0 \in T''$, whence $x_i y_i$ can be matched in G onto an edge of C incident to t_0, contrary to L9. Else $d(y_i) < |X|$. Then, by L15, $N(t) = X$, whence $T' = \emptyset$ (i.e., $i = 1$). Therefore, by (1.5), $d(z_0) \geqslant |X|$. Consequently, because $|T''| \geqslant 2$, $z_0 x_1$ can be matched onto an edge of C incident to some t'', again a contradiction with L9. Hence $y_0 z_i \notin E$ and similarly $y_i z_0 \notin E$.

Suppose $z_0 y_i \in E$ and let $T' = T'_{i0}$. By L6 and/or L7, $N(z_i) \subseteq \{y_i\} \cup X - \{x_1, x_i\}$, $N(t') \subseteq X - \{x_1, x_i\}$, and $N(t'') \subseteq X$. Hence $i > 1$ because otherwise $T'' = \emptyset$ and $|T'| \geqslant 2$, contrary to L15. Then $d(z_i) < |X|$, whence, by L15, $T' = \emptyset$ and therefore $i > 2$; moreover, $N(t) = X$. Furthermore, by L14, $d(z_i) \geqslant n - d(t) - d(v) \geqslant 5$. Consequently, $z_i x_h \in E$ for some h, $1 < h < i$. Then $y_h \in T$ and therefore $z_i x_{i+1}$ can be matched in G onto $x_h y_h$, contrary to L9. Hence $z_0 y_i \notin E$ and similarly $y_0 z_j \notin E$.

Suppose $y_i z_j \in E$ and let $T' = T'_{ij}$. By L6, $t' x_i$, $t' x_{j+1} \notin E$, whence $d(t') < |X|$. By (1.5), $d(z_i)$ or $d(y_j) \geqslant |X|$. Without loss of generality, suppose $d(y_j) \geqslant |X|$. Then, by the above, $N(y_j) = \{z_j, y_0^+\} \cup X - \{x_i, x_{j+1}\}$. Hence, by L9, no x_j-$\{y_0, t''\}$ edge is in E. Moreover, by L7$^+$, $y_0 z_0 \notin E$ because $y_j y_0^+ \in E$. Also, if $u = y_0$ or $u = t_0 \in [x_{j+1} C x_0]$ then $u x_i \notin E$ [if $t_1 \in [x_1 C x_i]$ then

$t_1 x_{j+1} \notin E$] because otherwise there is a Hamiltonian u^+-v path [Hamiltonian t_1^--v path] of $G-[y_i C z_j]$ whose end-vertices can be matched in G onto the edge $y_j x_j$ of the cycle $[z_j y_i C z_j]$, contrary to L0′. Thus $N(y_0) \subseteq \{y_0^+\} \cup X - \{x_j, x_i\}$, whence $d(y_0) < |X|$. Hence, by L15, $T' = \emptyset$, i.e., $j = i+1$. Therefore $d(t_0), d(t_1) < |X|$, whence, by L15, $T'' = \emptyset$, a contradiction with $T \neq \emptyset$. Hence $y_i z_j \notin E$.

Assume $z_i y_j \in E$ and let $T' = T'_{ji}$. Now, $t' x_{i+1}, t' x_j \notin E$. Moreover, $N(y_i) \subseteq \{y_0^+, z_i\} \cup X - \{x_{i+1}, x_j\}$, $N(y_0) \subseteq \{y_0^+, z_0\} \cup X - \{x_{i+1}, x_j\}$ and, by L7$^+$, at most one of the edges $y_i y_0^+$, $y_0 z_0$ is in E. Suppose $j > i+1$. Then $d(t') < |X|$ and, by L15, one of $d(y_i)$, $d(y_0)$ is $|X|$ and the other smaller. Hence $T' = \emptyset$, $j > i+2$ because $|T| \geqslant 2$, and $d(t) \geqslant |X|$. By L14, because $\{y_0, y_i, v\}$ is independent, $d(y_0) \geqslant 5$. Hence $y_0 x_h \in E$ for some h, $i+1 < h < j$. Therefore $y_h \in T$ and, by L9, we have $y_h y_0^+$, $y_h x_0 \notin E$, whence $d(y_h) < |X|$, contrary to L15$^+$. Thus $j = i+1$ if $z_i y_j \in E$.

Suppose $y_0 z_0 \in E$. Then $z_i y_j \notin E$ because otherwise $j = i+1$ and no t-$\{y_0^+, x_j\}$ edge is in E, whence $d(t) < |X|$, contrary to L15 and $|T| \geqslant 2$. Moreover, by L7, no y_0^+-$(Y \cup Z - \{y_0, z_0\})$ edge is in E. Therefore and by the above, $k(G-X) > |X|$, a contradiction with the toughness of G.

Hence finally, $y_j z_i$ is the only chord of C of the form $y_h z_g$ which is in E. Then $j = i+1$, $\{v, y_i, z_i, x_j, y_j, z_j\}$ induces J in G, and $G \subseteq G_n$ in (1.1). ∎

4. Concluding remarks

The graphs G_n and G_n^0 are used in [2] to show that the sufficient condition in Corollary (Section 1) is sharp for large n's. Namely, for $G = G_n^0$ with $n \geqslant 7$, the Fan-type condition (1.6) holds if $m_2 = m_3 = \lceil (n-6)/2 \rceil$ but $\sigma_3(G_n^0) = n-1$. On the other hand, for $G = G_n$ with $n = 15$ and $n \geqslant 17$, $\sigma_3(G) = n$ but (1.6) holds with the bound $(n-4)/2$ replaced with one by 1 smaller, i.e.,

$$(4.1) \qquad (u, w \in V,\ d(u, w) = 2) \Rightarrow \max\{d(u), d(w)\} \geqslant (n-6)/2.$$

CONJECTURE 1. Theorem 2 (Section 1) remains valid if (1.5) is replaced with (4.1), the phrase "n is odd, $n \geqslant 15$" with "$15 \leqslant n \neq 16$", and "the graph G_n in (1.1)" with "a graph in (1.1), (1.2), or (1.3)".

CONJECTURE 2. For some integer n_1 and for a tough 3-connected n-vertex graph G, if $n \geqslant n_1$ and (4.1) holds then either G is Hamiltonian or else G is a factor of a graph in (1.1), (1.2), or (1.3).

The Tietze graph (cubic, on 12 vertices) shows that $n_1 \geqslant 13$.

Conjecture 2, if true, generalizes Theorem 5 of [2] which gives the following sufficient condition for a tough n-vertex graph G to be Hamiltonian:

$$G \text{ is 3-connected, (1.6) holds and } n \geqslant n_0$$

where $n_0 \leqslant 35$. The graph G_{12} in (1.2) shows that $n_0 \geqslant 13$.

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