

MATROIDAL DECOMPOSITION OF A GRAPH

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Decomposition of graphs connected with problems of combinatorial optimization is studied. All the graphs considered are finite, nonoriented, without loops and multiple edges.

1. Preliminary definitions and facts

Let V be a finite nonempty set, and let I be a nonempty collection of its subsets satisfying the following

AXIOM 1. $(X \in I, Y \subset X) \Rightarrow (Y \in I)$.

Then the pair (V, I) is called an *independence system*. The set V is called the *support* of the system (V, I) , elements of I are called *independent sets*. By abuse of language, in this situation we also call the set I an independence system.

We note two known interpretations of independence systems. The first one is connected with monotonic Boolean functions [2]. Let $V = \{1, \dots, n\}$. For an arbitrary subset $X \subseteq V$ we define its characteristic vector $x = (x_1, \dots, x_n)$ putting

$$x_i = \begin{cases} 1 & \text{if } i \in X, \\ 0 & \text{otherwise.} \end{cases}$$

Evidently the mapping $X \rightarrow x$ is a bijection between 2^V and the n -dimensional cube B^n , $B = \{0, 1\}$. Now let $f: B^n \rightarrow B$ be an arbitrary monotonic Boolean function, $f \neq 1$. A subset $X \subseteq V$ is called *independent* if $f(x) = 0$. If I is the set of all independent subsets of V then (V, I) is an independence system. One says that this independence system is *determined by the function* f .

Conversely, let (V, I) be an arbitrary independence system and $V = \{1, \dots, n\}$. We define the Boolean function $f: B^n \rightarrow B$ putting $f(x) = 0$ iff

$X \in I$. Here X is the subset whose characteristic vector is x . Evidently, f is the monotonic Boolean function determining the independence system (V, I) .

Systems of linear inequalities with real coefficients provide another interpretation of independence systems [3, 8]. Consider the system

$$(1) \quad \sum_{j=1}^n a_{ij}x_j \leq b_i, \quad i = 1, \dots, n, \quad a_{ij}, b_i \geq 0.$$

A subset $X \subseteq V = \{1, \dots, n\}$ will be called *independent* if its characteristic vector x satisfies (1). If I is the set of all independent subsets X , then the pair (V, I) is an independence system. One says that this independence system is *determined by* (1). Every independence system is known to be determined by a system of linear inequalities.

A significant part of combinatorial optimization problems can be formulated as follows:

$$\text{P: find } \max(c_1x_1 + \dots + c_nx_n) \\ \text{provided that } f(x_1, \dots, x_n) = 0, \quad x_i \in \{0, 1\}.$$

Here (c_1, \dots, c_n) is a nonnegative real vector, f is a monotonic Boolean function.

A particular case of P is the well-known packing problem

$$\text{PP: find } \max(c_1x_1 + \dots + c_nx_n) \\ \text{provided that } \sum_{j=1}^n a_{ij}x_j \leq 1, \quad a_{ij} \geq 0, \quad x_j \in \{0, 1\}.$$

PP is known to be NP-hard even with the complementary conditions $a_{ij} \in \{0, 1\}$, $c_j = 1$ (the independence number of graph problem) [9].

An independence system I is called a *matroid* if it satisfies the following

$$\text{AXIOM 2. } (X, Y \in I, |X| < |Y|) \Rightarrow (\exists y \in Y \setminus X, X \cup \{y\} \in I).$$

Partition matroids form a narrow class of matroids. Let $(V_i: i = 1, \dots, p)$ be a partition of a finite set V . We call a subset $X \subseteq V$ *independent* if $|X \cap V_i| \leq 1$ for $i = 1, \dots, p$. Denoting by I the set of all independent subsets of V , we obtain the *partition matroid* (V, I) .

Each independence system is known to be the intersection of matroids. Namely, there exist matroids (V, I_p) , $p = 1, \dots, m$, with the same support V such that

$$(2) \quad I = \bigcap_{p=1}^m I_p.$$

We shall call the representation (2) a *matroidal decomposition* of (V, I) and the minimal number m of components in a matroidal decomposition (2) the *matroidal number* $m(I)$.

The number $m(I)$ is an important parameter. If $m(I) = 1$, then I is a matroid; in this situation the problem P is quickly solved by the greedy algorithm. For $m(I) = 2$, P is effectively solved by the method of alternating sequences [9]. For $m(I) \geq 3$ even the independence number of a graph problem is NP-hard [9]), but it is always useful to know the value of $m(I)$. The point is that the relative error of the gradient algorithm of the coordinate lifting for the problem P on the intersection of m matroids does not exceed $(m+1)^{-1}$ (if a is the obtained value of the maximized function and a_0 is the maximal value, then $a_0 \leq a(m+1)$ [4]).

But finding $m(I)$ and, moreover, the construction of an optimal matroidal decomposition for an arbitrary independence system I seem to be difficult.

The representation of a Boolean function as the disjunction of functions of a special form corresponds to a matroidal decomposition of an independence system: if (2) is a matroidal decomposition, f_p , $p = 1, \dots, m$, and f are the monotonic Boolean functions determining I_p and I , respectively, then

$$(3) \quad f = f_1 \vee \dots \vee f_m.$$

We shall call (3) a *matroidal decomposition* of f .

2. Graphical independence systems

Let G be an arbitrary graph and let VG be the set of its vertices. A subset $X \subseteq VG$ is called *independent* if it includes no edges of G . We denote by IG the set whose elements are all independent subsets $X \subseteq VG$ (the empty set \emptyset is not excluded). Evidently, IG is an independence system. We call it the independence system of G , or *graph independence system*. The maximal size $\alpha_0(G)$ of elements from IG is called the *independence number* of G .

The number of nonzero coordinates of a Boolean function f is called the *norm* of f . A Boolean function f is called *graphical* if the norm of its every lowest unit is equal to 2 or if $f \equiv 0$. An independence system is known to be graphical iff it is determined by a graphical Boolean function. Also, an independence system is graphical iff it can be determined by a system of linear inequalities (1) with $a_{ij} \in \{0, 1\}$, $b_i = 1$ [3].

A graph will be called an *M-graph* if all its connected components are complete graphs. The following statement is used below.

STATEMENT 1 [2]. *A graphical independence system IG is a matroid iff G is an M-graph.*

Let K be the set of all *M-graphs*. Since a 1-edge graph is an *M-graph*, every graph G can be represented as the union

$$(4) \quad G = G_1 \cup \dots \cup G_m, \quad G_i \in K,$$

of M -graphs with the same vertex set VG ; we call such a representation a *matroidal decomposition* of G .

STATEMENT 2. *An independence system I is graphical iff it is the intersection of partition matroids.*

Proof. Let (4) be a matroidal decomposition of a graph G , let G_k be one of the components in (4) and let

$$(5) \quad VG_k = V_1 \cup \dots \cup V_p$$

be the partition into connected parts. Since all the induced subgraphs $G(V_j)$, $j = 1, \dots, p$, are complete, IG_k coincides with the matroid of the partition (5). Now (4) implies that IG is the intersection of the matroids of the partitions of VG into the connected parts of G_k , $k = 1, \dots, m$.

Coversely, let $I = \bigcap_{k=1}^m I_k$ be the intersection of partition matroids I_k , and let $V = V_1 \cup \dots \cup V_p$ be the partition corresponding to I_k . We denote the complete graph with the vertex set V_j by $K(V_j)$ and set

$$G_k = K(V_1) \cup \dots \cup K(V_p), \quad G = G_1 \cup \dots \cup G_m.$$

Evidently $I_k = IG_k$, $I = IG = \bigcap_{k=1}^m IG_k$.

The minimal number m of components in matroidal decompositions of a graph G is called the *equivalent covering number* of G and is denoted by $eq G$ [5]. For example, $eq G = 2$ for the graph G in Fig. 1.

In what follows EG denotes the set of edges of a graph G .

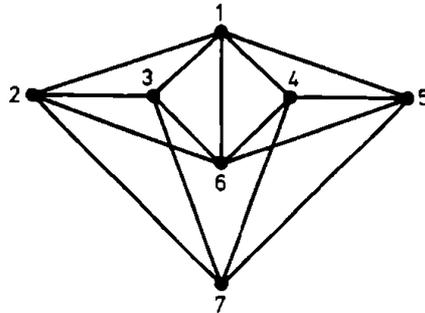


Fig. 1

THEOREM 1. *For every graph G*

$$(6) \quad m(IG) = eq G.$$

Proof. If (4) is a matroidal decomposition of G , then $IG = \bigcap_{k=1}^m IG_k$ is a matroidal decomposition of IG , so $m(IG) \leq eq G$.

Conversely, let $IG = \bigcap_{k=1}^m I_k$ be an arbitrary matroidal decomposition of the independence system IG , and let

$$f = f_1 \vee \dots \vee f_m$$

be the corresponding decomposition of the graphical monotonic Boolean function determining IG . For $k = 1, \dots, m$ let g_k be the graphical monotonic Boolean function whose lowest units coincide with those of f_k with norm 2. If f_k has no such lowest units, we put $g_k \equiv 0$. Evidently,

$$(7) \quad f = g_1 \vee \dots \vee g_m,$$

because the norm of every lowest unit of f is equal to 2. Now let G_k be the graph corresponding to g_k , i.e. the graph whose independence system is determined by g_k : $VG_k = VG = \{1, \dots, n\}$, $uv \in EG_k$ iff $g_k(x) = 1$ for the Boolean vector $x = (x_1, \dots, x_n)$ with norm 2, $x_u = x_v = 1$. (7) implies

$$(8) \quad G = G_1 \cup \dots \cup G_m.$$

Each of the components G_k in (8) is an M -graph. Indeed, let $uv, vw \in EG_k$. Then the sets $\{u, v\}$ and $\{v, w\}$ are cycles of the matroid IG_k . Consequently, the set $\{u, w\}$ is dependent. But all one-element subsets are independent with respect to IG and so to IG_k . So the vector

$$x = (x_1, \dots, x_n), \quad x_u = x_w = 1, \quad x_v = 0, \quad v \neq u, w,$$

is a lowest unit of f_k and so of g_k . But then $uw \in EG_k$. We have proved that G_k is an M -graph and (8) is a matroidal decomposition. So, $\text{eq } G \leq m(I)$, which implies (6).

Computing $\text{eq } G$ in the class of all graphs seems to be difficult.

Let $N(v)$ be the neighbourhood of a vertex v in a graph G , i.e. the set of vertices adjacent to v . The vertices u and v are called *paired* if $\{u\} \cup N(u) = \{v\} \cup N(v)$. A graph is called *supercompact* if it has no paired vertices. The *compressing operation* $G \rightarrow C(G)$ is the process of deletion of one of paired vertices at a time until the supercompact graph $C(G)$ is obtained.

STATEMENT 3 [2]. *If v is a paired vertex of a graph G , then $\text{eq } G = \text{eq}(G - v)$.*

COROLLARY 1. *For every graph G , $\text{eq } G = \text{eq } C(G)$.*

Evidently, $\text{eq } G = \max \text{eq } G_i$ over all the connected components G_i of G .

Thus the study of matroidal decompositions of graphs is reduced to the same problem for supercompact connected graphs.

3. Some estimates of $\text{eq } G$

In what follows C_n is the n -vertex chordless cycle, \bar{G} is the graph complementary to G , $\Delta(G)$ and $\delta(G)$ are the maximal and the minimal degree of vertices of G , respectively.

A set of pairwise adjacent vertices of a graph is called a *clique*. The *clique graph* $Q(C)$ is the graph whose vertices bijectively correspond to maximal

cliques of G (with respect to inclusion), and two vertices are adjacent iff the intersection of the corresponding cliques is nonempty.

A system $(G_i; i = 1, \dots, k)$ of complete subgraphs of G such that $EG = EG_1 \cup \dots \cup EG_k$ is called a *clique covering* of G . The minimal number k of components in a clique covering of G is called the *clique covering number* and denoted by $cc(G)$.

$\chi(G)$ and $\chi'(G)$ denote the chromatic number and the chromatic class (the edge chromatic number) of a graph G , respectively, and $\chi_Q(G) = \chi(Q(G))$.

The following estimates of $\text{eq } G$ are known:

- 1) [6]. $\text{eq } \bar{C}_n \geq t$ for $n \geq 3t!$.
- 2) [1]. $\log_2 n - 1 \leq \text{eq } \bar{C}_n \leq \log_2 n + 3$ for $n \geq 3$.
- 3) [1]. If $|VG| = n$, $\Delta(G) \leq d$, $\delta(G) \geq 1$, then

$$\log_2 n - \log_2 d \leq \text{eq } \bar{G} \leq 2e^2(d+1)^2 \log_2 n.$$

Here e is the base of natural logarithms.

The estimates given below have a rather theoretical significance because they are expressed in terms of hardly computable parameters.

THEOREM 2. *For every graph G*

$$(9) \quad \text{eq } G \leq cc(G),$$

$$(10) \quad \text{eq } G \leq \chi'(G),$$

$$(11) \quad \text{eq } G \leq \chi_Q(G).$$

All these estimates can be reached and are independent. Each of the differences $cc(G) - \text{eq } G$, $\chi'(G) - \text{eq } G$, $\chi_Q(G) - \text{eq } G$ can be arbitrarily large. If G does not contain triangles, then

$$(12) \quad \text{eq } G = \chi'(G) = \chi_Q(G).$$

For a bipartite graph G

$$(13) \quad \text{eq } G = \Delta(G).$$

Proof. (9) is evident.

We fix a right colouring of edges of G and consider the partition

$$(14) \quad EG = E_1 \cup \dots \cup E_k$$

into coloured classes. Evidently, the graph G_i with $VG_i = VG$, $EG_i = E_i$ is an M -graph and

$$(15) \quad G = G_1 \cup \dots \cup G_k$$

is a matroidal decomposition. Thus the partition into coloured classes (14) determines a matroidal decomposition (15), which proves (10).

The subgraph of G induced by the union of cliques contained in the same

coloured class for a right vertex colouring of the clique graph $Q(G)$ is an M -graph. Consequently, if $VQ(G) = V_1 \cup \dots \cup V_k$ is the partition of the vertex set of $Q(G)$ into coloured classes and G_i is obtained from $G(V_i)$ by addition of all the vertices from $VG \setminus V_i$ as isolated ones, then $G = G_1 \cup \dots \cup G_k$ is a matroidal decomposition. So (11) is proved.

If G has no triangles, then the complete subgraphs are edges or vertices, so (12) is evident.

It is known that $\chi'(G) = \Delta(G)$ for a bipartite graph G [11], which implies (13).

The remaining statements of the theorem are proved by the following examples. Let G be the graph pictured in Fig. 2. It is the "path" consisting of

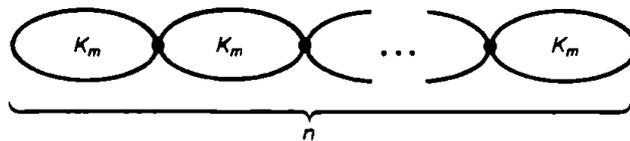


Fig. 2

$n > 1$ copies of the complete graph K_m in which the neighbouring graphs are "sticked together" in one vertex; m is even. For this graph

$$\text{eq } G = 2, \quad \text{cc}(G) = n, \quad \chi'(G) = 2(m-1), \quad \chi_Q(G) = 2.$$

One more example. Let H be obtained from the star $K_{2,m}$ by adding a new vertex which is adjacent with all vertices of $K_{2,m}$, $m > 3$. For this graph

$$\text{eq } H \leq \chi'(H) \leq m+3, \quad \chi_Q(H) = 2m.$$

Finally, for the graph F in Fig. 3

$$\text{cc}(F) = 3, \quad \chi_Q(F) = 4.$$

The theorem is proved.

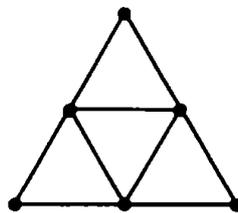


Fig. 3

4. The graphs with $\text{eq } G = 2$

The characterization of graphs with $\text{eq } G = 2$ in terms of forbidden subgraphs is obtained in [10]. Let $K_{1,3}$ be the 4-vertex star, let W_4 be the 5-vertex wheel and $W_4 - e$ the graph in Fig. 4.

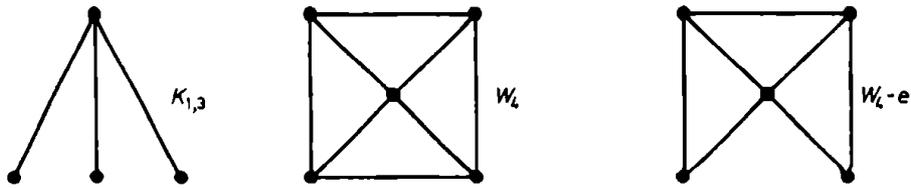


Fig. 4

THEOREM 3 [10]. *For a connected graph $G \neq K_n$ the following statements are equivalent:*

- 1) $eq G = 2$;
- 2) G contains none of the graphs $K_{1,3}$, W_4 , $W_4 - e$ and C_{2n+1} , $n \geq 2$, as an induced subgraph.

COROLLARY 2. *For a connected supercompact graph G , $eq G = 2$ iff G has no induced subgraphs $K_{1,3}$, $K_4 - e$ and C_{2n+1} , $n \geq 2$.*

COROLLARY 3. *A supercompact graph G with $eq G = 2$ is a line graph.*

To prove this it is sufficient to compare Corollary 2 with the list of forbidden induced subgraphs for line graphs [7].

STATEMENT 4. *A supercompact graph G with $eq G = 2$ is the line graph $L(H)$ of some bipartite graph H .*

Proof. By Corollary 3, $G \cong L(H)$ for some graph H . We shall prove that H is bipartite. By Corollary 2, G has no induced chordless cycle C_{2n+1} with $n \geq 2$. Consequently, H has no such cycle either, because $L(C_k) \cong C_k$.

Now suppose that there are vertices u, v, w in H on which K_3 is induced. We denote by \overline{xy} the vertex of G corresponding to an edge xy of H . It is clear that G induces the triangle on the set $\{\overline{uv}, \overline{vw}, \overline{uw}\}$. Since G is supercompact, for the edge uv, vw there exists a vertex z that $\overline{zuv} \in EG$ and $\overline{zvw} \in EG$. Then, evidently, $z = \overline{ux}$. The edges ux and uw are adjacent in H . So $\overline{zuw} \in EG$. We have obtained the induced subgraph $K_4 - e$ in G , which contradicts Corollary 2. Thus H has no triangles and is bipartite.

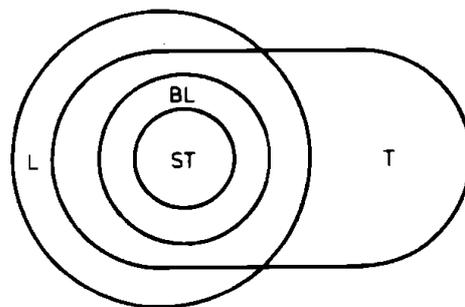


Fig. 5

The relations between the class of graphs with $\text{eq } G = 2$ and line graphs $L(H)$ are illustrated in Fig. 5. Here L, BL, T and ST are the classes of all line graphs, line graphs $L(H)$ with bipartite H , all graphs with $\text{eq } G = 2$ and supercompact graphs with $\text{eq } G = 2$, respectively. The graphs in Figs. 6, 7 both have $\text{eq } G = 2$, but the first one is the line graph of a nonbipartite graph and the second one is not a line graph.

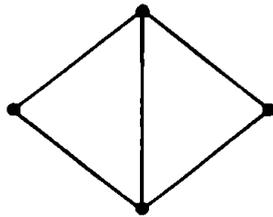


Fig. 6

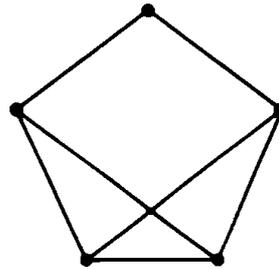


Fig. 7

We note in connection with the equation considered above that $\chi(H) = m$ implies $\text{eq}(L(H)) \leq m$. The corresponding matroidal decomposition of $L(H)$ is constructed in the evident way.

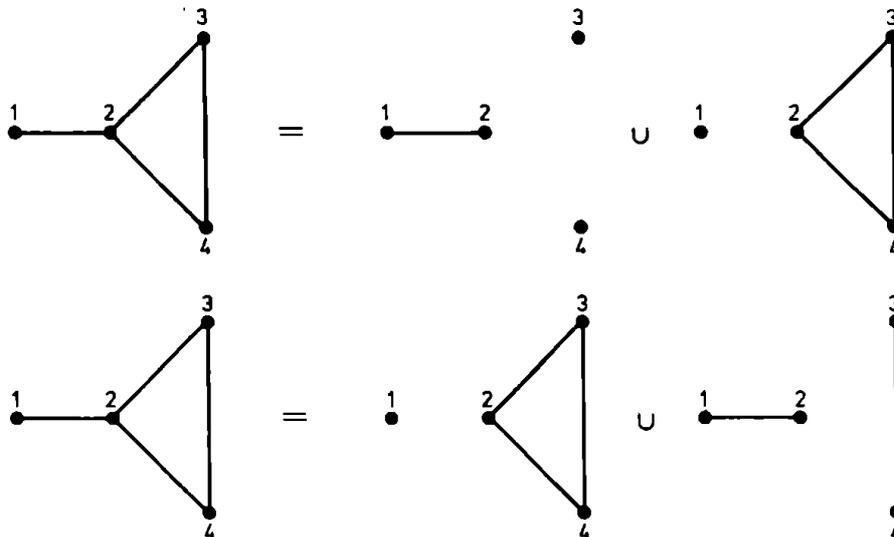


Fig. 8

A matroidal decomposition of a graph G with $\text{eq } G = 2$ is not unique, an example is in Fig. 8.

THEOREM 4. *Every connected supercompact graph G with $\text{eq } G = 2$ has a unique matroidal decomposition*

$$(16) \quad G = G_1 \cup G_2.$$

Moreover,

$$(17) \quad EG_1 \cap EG_2 = \emptyset.$$

Proof. Let (16) be an arbitrary matroidal decomposition, $e = uv \in EG_1 \cap EG_2$. G does not permit pressings, so one of the ends of e , e.g. u , is adjacent to some $w \notin N(v) \cup \{v\}$. We have $uw \notin EG_1$ and $uw \notin EG_2$, otherwise G_1 or G_2 has an induced subgraph $K_{1,2}$ on the vertices u, v, w , which is impossible in an M -graph. Thus $uw \notin EG_1 \cup EG_2$, which contradicts (16). (17) is proved.

Now we shall prove the uniqueness of (16). We choose an arbitrary edge $e = uv \in EG_1$. Let K be some maximal complete subgraph of G containing e . We shall prove that $EK \subseteq EG_1$. Suppose otherwise: there exists $e' = xy \in EK \cap EG_2$. If e and e' are adjacent (e.g. $u = x$), then vy ($v \neq y$) belongs neither to EG_1 nor to EG_2 since otherwise either $e \in EG_2$ or $e' \in EG_1$, which contradicts (17). We thus obtain a contradiction with (16): $vy \notin EG_1 \cup EG_2$. If e and e' are not adjacent, then ux is adjacent to e . As we have proved above, $ux \in EG_1$. The edge e' is adjacent to ux and so $e' \in EG_1$. We have a contradiction. Thus $EK \subseteq EG_1$.

Evidently, all the edges having exactly one end in VK belong to EG_2 . We take an arbitrary such edge f and some maximal complete subgraph K' containing f . As above, $EK' \subseteq EG_2$.

Due to the connectedness of the graph G , its edges are uniquely distributed between EG_1 and EG_2 . Then theorem is proved.

Theorem 4 implies the correctness of the following algorithm for recognizing graphs with $\text{eq } G = 2$ and constructing the corresponding decomposition.

INPUT: a connected supercompact graph $G \neq K_1$.

STEP 0. Put $n = 1$ and label an arbitrary edge by 1.

STEP 1. Choose an arbitrary labeled edge e .

STEP 2. Choose an arbitrary maximal complete subgraph K containing e .

STEP 3. If there is an edge in EK with label different from n , then OUTPUT 1. Otherwise label all the edges of K by n .

STEP 4. Put $n = n + 1 \pmod{2}$.

STEP 5. If there is an edge with label different from n among the edges having exactly one end in K , then OUTPUT 1. Otherwise label all these edges by n .

STEP 6. Delete all the edges of K from the graph.

STEP 7. If the empty graph is obtained, go to STEP 8. Otherwise go to STEP 1.

STEP 8. If the edges labeled by 1 and edges labeled by 0 form M -graphs, then OUTPUT 2. Otherwise OUTPUT 1.

OUTPUT 1. $\text{eq } G \geq 3$.

OUTPUT 2. eq $G = 2$ and the edges labeled by 0 and 1 induce M -graphs G_1 and G_2 , respectively, and $G = G_1 \cup G_2$.

Note. The dual problem is considered in [2]: an independence system I is represented as a union of matroids. This is better suited for combinatorial optimization problems, because a decomposition of an optimization problem into several such problems on matroids is obtained. In particular, this can be used for graphical independence systems. But we need to decompose graphical independence systems into an intersection of matroids, and not a union, if we want to preserve the graph-theoretic picture. The point is that an arbitrary graphical system is not necessarily represented as a union of graphical independence systems which are matroids, i.e. of partition matroids. The standard transition to the dual Boolean function does not help because the function fails to be graphical and matroidal here. The results of [2] and ours complete each other.

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