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MATHEMATICAL PROBLEMS IN THE KINETIC THEORY OF GASES

CARLO CERCIGNANI

Istituto di Matematica, Politecnico di Milano, Milano, Italy

Chapter I

BASIC PRINCIPLES OF THE KINETIC THEORY OF GASES

1. Introduction

According to the molecular theory of matter, a macroscopic volume of gas (say, 1 cm^3) is a system of a very large number (say, 10^{20}) of molecules moving in a rather irregular way. In principle, we may assume, ignoring quantum effects, that the molecules are particles (mass points or other systems with a small number of degrees of freedom) obeying the laws of classical mechanics. We may also assume that the laws of interaction between the molecules are perfectly known so that, in principle, the evolution of the system is computable, provided suitable initial date are given. If the molecules are, e.g., mass points the equations of motion are:

$$\dot{\boldsymbol{\xi}}_i = \mathbf{X}_i, \qquad \dot{\mathbf{x}}_i = \boldsymbol{\xi}_i \tag{1.1a}$$

or

$$\ddot{\mathbf{x}}_i = \mathbf{X}_i \tag{1.1b}$$

where $\mathbf{x}_i$ is the position vector of the ith particle ($i = 1, \ldots, N$) and $\boldsymbol{\xi}_i$ its velocity vector; both $\mathbf{x}_i$ and $\boldsymbol{\xi}_i$ are functions of the time variable t and the dots denote, as usual, differentiation with respect to t. Here $\mathbf{X}_i$ is the force acting upon the ith particle divided by the mass of the particle. Such a force will in general be the sum of the resultant of external forces (e.g., gravity or, if the observer is not inertial, apparent forces such as centrifugal or Coriolis forces) and the forces describing the action of the other particles of the system on the ith particle.

2. Phase space and Liouville theorem

In order to discuss the behavior of a system of N mass points satisfying Eqs (1.1a), it is highly convenient to introduce the so-called *phase space*, i.e. a $6N$-dimensional space where the Cartesian coordinates are the $3N$ components of the N position vectors $\mathbf{x}_i$ and the $3N$ components of the N velocities $\boldsymbol{\xi}_i$.

In this space, the state of a system at a given time t, if known with absolute accuracy, is represented by a point whose coordinates are the $6N$ values of the components of the position vectors and velocities of the N particles. (Frequently, the momenta of the particles are used in place of their velocities, but the difference will not matter for our purposes.) Let us introduce the $6N$-dimensional vector $\mathbf{z}$ which gives the position of the representative point in phase space; clearly, the components of $\mathbf{z}$ are orderly given by the $3N$ components of the N three-dimensional vectors $\mathbf{x}_i$ and the $3N$ components of the N three-dimensional vectors $\boldsymbol{\xi}_i$. The evolution equation for $\mathbf{z}$ is from Eqs (1a)

$$\dot{\mathbf{z}} = \frac{d\mathbf{z}}{dt} = \mathbf{Z}, \tag{2.1}$$

where $\mathbf{Z}$ is a $6N$-dimensional vector, whose components are orderly given by the $3N$ components of the N three-dimensional vectors $\boldsymbol{\xi}_i$ and the $3N$ components of the N three-dimensional vectors $\mathbf{X}_i$. Given the initial state, i.e. a point $\mathbf{z}_0$ in phase space, Eq. (2.1) determines $\mathbf{z}$ at subsequent times (provided the conditions for existence and uniqueness of the solution are satisfied).

If the initial data are not known with absolute accuracy, we must introduce a probability density $P_0(\mathbf{z})$ which gives us the distribution of probability for the initial data and we can try to set up the problem of computing the probability density at subsequent times, $P(\mathbf{z}, t)$. In order to achieve this, we must find an evolution equation for $P(\mathbf{z}, t)$; this can easily be done, as we shall see, provided the forces are known, i.e., if the only uncertainty is on the initial data.

An intuitive way of deriving the equation satisfied by $P(\mathbf{z}, t)$ is the following. We replace each mass point by a continuous distribution with density proportional to the probability density; in such a way, the system of mass points is replaced by a sort of fluid with density proportional to P and velocity $\dot{\mathbf{z}} = \mathbf{Z}$. Hence conservation of mass will give:

$$\frac{\partial P}{\partial t} + \operatorname{div}(P\mathbf{Z}) = 0, \tag{2.2}$$

where, as usual, for any vector $\mathbf{u}$ of the phase space, we write

$$\operatorname{div} \mathbf{u} = \sum_{i=1}^{6N} \frac{\partial u_i}{\partial z_i} \equiv \frac{\partial}{\partial \mathbf{z}} \cdot \mathbf{u}. \tag{2.3}$$

Equation (2.2) is the Liouville equation; note that the components of $\mathbf{z}$ are independent variables. But:

$$\operatorname{div}(P\mathbf{Z}) = \mathbf{Z}\cdot\operatorname{grad} P + P\operatorname{div}\mathbf{Z}, \tag{2.4}$$

where, as usual, grad $P \equiv \partial P/\partial\mathbf{z}$ is the vector of components $\partial P/\partial z_i$. Hence P satisfies the equation

$$\frac{\partial P}{\partial t} + \mathbf{Z}\cdot\operatorname{grad} P + P\operatorname{div}\mathbf{Z} = 0. \tag{2.5}$$

Usually, div $\mathbf{Z} = 0$. In fact, since $\mathbf{x}_i$ and $\boldsymbol{\xi}_i$ are independent variables:

$$\operatorname{div}\mathbf{Z} = \sum_{i=1}^{N}\left(\frac{\partial}{\partial\mathbf{x}_i}\cdot\boldsymbol{\xi}_i + \frac{\partial}{\partial\boldsymbol{\xi}_i}\cdot\mathbf{X}_i\right) = \sum_{i=1}^{N}\frac{\partial}{\partial\boldsymbol{\xi}_i}\cdot\mathbf{X}_i. \tag{2.6}$$

If the force per unit mass is velocity-independent, then also $(\partial/\partial\boldsymbol{\xi}_i)\cdot\mathbf{X}_i = 0$, and div $\mathbf{Z} = 0$ as announced. Note, however, that for some velocity dependent forces $(\partial/\partial\boldsymbol{\xi}_i)\cdot\mathbf{X}_i = 0$, the most notable case being that of the Lorentz force acting on a charged particle in a magnetic field. We shall always consider forces such that div $\mathbf{Z} = 0$ (typically, velocity-independent forces). Hence we write the Liouville equation in the following form:

$$\frac{\partial P}{\partial t} + \mathbf{Z}\cdot\frac{\partial P}{\partial\mathbf{z}} = 0. \tag{2.7}$$

Equation (2.7) can be of course rewritten in terms of the variables $\mathbf{x}_i$, $\boldsymbol{\xi}_i$:

$$\frac{\partial P}{\partial t} + \sum_{i=1}^{N}\boldsymbol{\xi}_i\cdot\frac{\partial P}{\partial\mathbf{x}_i} + \sum_{i=1}^{N}\mathbf{X}_i\cdot\frac{\partial P}{\partial\boldsymbol{\xi}_i} = 0, \tag{2.8}$$

where $\partial P/\partial\mathbf{x}_i$ is the gradient in the three-dimensional space of the positions of the ith particle, $\partial P/\partial\boldsymbol{\xi}_i$ the gradient in the three-dimensional space of velocities of the ith particle.

3. Hard spheres and rigid walls. Mean free path

In the previous sections, we considered the case of mass points which continuosly interact with each other according to the equations of motion (1.1). It is frequently convenient to consider limiting cases in which the system has only discrete interactions of finite impulse (hard collisions); in such a case the forces are not describable by means of ordinary functions and the Liouville equation must be handled in a different way. The limiting case of hard collision is useful, because it gives a more intuitive idea of the evolution of the system and is a good approximation to the strong repulsive forces which actual molecules mutually exert when they are close to each other. These considerations lead to the concept of a gas of hard spheres; i.e., a system of many "billiard balls" which do not interact at distance and collide according to the laws of elastic impact. The diameter σ of the spheres is equivalent to the range of the force through which actual molecules interact; as a matter of fact a gas of rigid spheres can also be pictured as a system of mass points which do not interact when their mutual distance is larger than σ but interact with a formally infinite repulsive central force when this distance becomes exactly equal to σ so that a closer approach is impossible.

Another example of instantaneous interaction is considered when a molecule is assumed to be elastically reflected by a solid wall. This model is more unrealistic than the hard spheres model, because a solid wall has macroscopic dimensions and certainly shows a very detailed structure at a microscopic level. This structure will prevent an elastic collision on the regular geometric surface representing the wall in a macroscopic description.

In spite of this, it is useful to consider the case of perfectly elastic reflections on a rigid wall for illustrative purposes.

In $\mathbf{n}$ is the normal at the wall (assumed at rest), the effect of a collision will be to change the sign of the normal component while leaving the tangential component unmodified. Thus, if $\boldsymbol{\xi}'$ denotes the velocity of a molecule before the collision and $\boldsymbol{\xi}$ the velocity after the impact, $\boldsymbol{\xi}'$ and $\boldsymbol{\xi}$ will be related as follows

$$\boldsymbol{\xi} = \boldsymbol{\xi}' - 2\mathbf{n}(\mathbf{n}\cdot\boldsymbol{\xi}'). \tag{3.1}$$

Equation (3.1) simply expresses the fact that the molecules are specularly reflected by the wall. If the wall is not at rest, but moves with velocity $\mathbf{u}_0$ with respect to the reference frame chosen to describe the motion, then Eq. (3.1) will apply to the velocities relative to the wall, i.e., $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$ must be replaced by $\boldsymbol{\xi}-\mathbf{u}_0$ and $\boldsymbol{\xi}'-\mathbf{u}_0$. Hence Eq. (3.1) will be replaced by:

$$\boldsymbol{\xi} = \boldsymbol{\xi}' - 2\mathbf{n}[\mathbf{n}\cdot(\boldsymbol{\xi}-\mathbf{u}_0)]. \tag{3.2}$$

In the case of a collision between two identical rigid spheres, the equations which relate the velocities after impact, $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ to those before impact, $\boldsymbol{\xi}_1'$ and $\boldsymbol{\xi}_2'$, are:

$$\begin{aligned} \boldsymbol{\xi}_1 &= \boldsymbol{\xi}_1' - \mathbf{n}[\mathbf{n}\cdot(\boldsymbol{\xi}_1'-\boldsymbol{\xi}_2')], \\ \boldsymbol{\xi}_2 &= \boldsymbol{\xi}_2' - \mathbf{n}[\mathbf{n}\cdot(\boldsymbol{\xi}_2'-\boldsymbol{\xi}_1')], \end{aligned} \tag{3.3}$$

where $\mathbf{n}$ is the unit vector directed as the line joining the centers of the two spheres (orientation does not matter). Equations (3.3) can be derived by a suitable use of the laws of conservation of momentum and energy:

$$\begin{aligned} \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2 &= \boldsymbol{\xi}_1' + \boldsymbol{\xi}_2', \\ \xi_1^2 + \xi_2^2 &= \xi_1'^2 + \xi_2'^2. \end{aligned} \tag{3.4}$$

We want to show, now, that the Liouville theorem (conservation of the volume in phase space) remains valid for the instantaneous interactions considered in this section. We observe that both Eq. (3.1) and Eqs (3.3) are reversible: i.e., one can solve for the primed variables to find

$$\boldsymbol{\xi}' = \boldsymbol{\xi} - 2\mathbf{n}(\mathbf{n}\cdot\boldsymbol{\xi}) \tag{3.5}$$

and

$$\begin{aligned} \boldsymbol{\xi}_1' &= \boldsymbol{\xi}_1 - \mathbf{n}[\mathbf{n}\cdot(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)], \\ \boldsymbol{\xi}_2' &= \boldsymbol{\xi}_2 - \mathbf{n}[\mathbf{n}\cdot(\boldsymbol{\xi}_2 - \boldsymbol{\xi}_1)], \end{aligned} \tag{3.6}$$

respectively, i.e., the same equations as (3.1) and (3.3), with the primed variables interchanged with the unprimed ones. To derive Eq. (3.5), it is sufficient to compute $\mathbf{n}\cdot\boldsymbol{\xi}'$ first (and this is easily done by scalarly multiplying Eq. (3.1) by $\mathbf{n}$) and replace the obtained value $\mathbf{n}\cdot\boldsymbol{\xi}' = -\mathbf{n}\cdot\boldsymbol{\xi}$ into Eq. (3.1). Analogously, to obtain Eqs. (3.6), we first compute the value of $\mathbf{n}\cdot(\boldsymbol{\xi}_1' - \boldsymbol{\xi}_2')$ in terms of the unprimed variables; to this end, we deduce by subtraction the relation between relative velocities

$$\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 = \boldsymbol{\xi}_1' - \boldsymbol{\xi}_2' - 2\mathbf{n}[\mathbf{n}\cdot(\boldsymbol{\xi}_1' - \boldsymbol{\xi}_2')]. \tag{3.7}$$

Scalar multiplications by $\mathbf{n}$ yields $\mathbf{n}\cdot(\boldsymbol{\xi}_1' - \boldsymbol{\xi}_2') = -\mathbf{n}\cdot(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)$ and insertion of this result into Eqs (3.3) produces the desired relations, Eqs (3.6).

We observe now that in both cases (interaction with the wall and impact between two rigid spheres), the components of the velocities involved undergo a linear transformation described by some matrix $\mathbf{A}$ (3×3 or 6×6), whose elements depend on $\mathbf{n}$. In both cases comparison of the direct and inverse transformations (Eqs (3.1) and (3.5); Eqs (3.3) and (3.6)) shows that the inverse matrix $\mathbf{A}^{-1}$ equals $\mathbf{A}$, i.e., $\mathbf{A}^2$ is the identity matrix. Hence the square of the determinant of $\mathbf{A}$ (which is nothing else than the Jacobian T_1 of the linear transformation) is unity, i.e. $T_1 = \pm 1$.

In connection with hard spheres, it is convenient to introduce the notion of free path. It is the distance travelled by a sphere S_1 between two subsequent collisions. This distance, of course, depends upon the number n of spheres per unit volume, the velocity of the chosen sphere S_1 and the velocity of the sphere S_2 with which S_1 will have the next impact. Accordingly, only the notion of mean free path will be meaningful.

A simple estimate of the value of the mean free path l of a hard sphere is obtained by assuming the other spheres at rest and surrounding each of them by a sphere of radius equal to the diameter σ of the particles, while the travelling sphere S_1 is represented by a point. Then is S_1 travels a distance l in average, between two impacts, this means that there is only one molecule, namely S_1, in a cylinder of base $\pi\sigma^2$ and height l or:

$$n\pi\sigma^2 l \simeq 1. \tag{3.8}$$

Hence

$$l \simeq \frac{1}{n\pi\sigma^2}. \tag{3.9}$$

4. Equations for the many particle distribution functions for a gas of rigid spheres

In this section we derive the equations satisfied by the s-particle distribution functions ($s = 1, \ldots, N$) for a gas of N rigid spheres contained within a region R of volume V. We have for the probability density $P_N = P_N(\mathbf{x}_i, \xi_i, t)$:

$$P_N = 0 \quad (|\mathbf{x}_i - \mathbf{x}_j| < \sigma,\ i \neq j) \tag{4.1}$$

since the spheres cannot penetrate into each other; accordingly, P_N, in general, will be discontinuous at the points of phase space where $|\mathbf{x}_i - \mathbf{x}_j| = \sigma$, the limit from one side being zero and from the other side ($|\mathbf{x}_i - \mathbf{x}_j| > \sigma$), generally speaking, different from zero. Accordingly, when we consider P_N for some $|\mathbf{x}_j - \mathbf{x}_i| = \sigma$, we shall always understand the limiting value from "outside", i.e., from the region $|\mathbf{x}_j - \mathbf{x}_i| > \sigma$. In the latter region the state of molecules corresponds to inertial motion; hence the Liouville equation, Eq. (2.8) reduces to:

$$\frac{\partial P_N}{\partial t} + \sum_{i=1}^{N} \xi_i \cdot \frac{\partial P_N}{\partial \mathbf{x}_i} = 0 \quad (|\mathbf{x}_i - \mathbf{x}_j| > \sigma,\ i \neq j) \tag{4.2}$$

where it is appropriate to assume that P_N is symmetric with respect to an exchange of the N molecules.

Let us integrate Eq. (4.2) over its domain of validity, with respect to the coordinates and velocities of $N-s$ particles; without loss of generality, we shall integrate with respect to the particles numbered from $s+1$ to N. If we introduce the s-particle distribution function $P_N^{(s)}$ by integration of P_N with respect to the coordinates and velocities of the last s particles, Eq. (4.2) gives:

$$\frac{\partial P_N^{(s)}}{\partial t} + \sum_{i=1}^{s} \int \xi_i \cdot \frac{\partial P_N}{\partial \mathbf{x}_i} \prod_{l=s+1}^{N} d\mathbf{x}_l \, d\xi_l + \sum_{j=s+1}^{N} \int \xi_j \cdot \frac{\partial P_N}{\partial \mathbf{x}_j} \prod_{l=s+1}^{N} d\mathbf{x}_l \, d\xi_l = 0, \tag{4.3}$$

where integration extends to the whole space with respect to ξ_l ($l = s+{}$ $+1, \ldots, N$) and to the volume V deprived of the sets $|\mathbf{x}_i - \mathbf{x}_l| < \sigma$ ($i = 1, \ldots, N$, $i \neq l$) with respect to $\mathbf{x}_l$ ($l = s+1, \ldots, N$). Terms with $1 \leqslant i \leqslant s$ have been separated from those with $s+1 \leqslant i \leqslant N$ for later convenience.

A typical term in the first sum in Eq.. (4.3) contains the integral of a derivative with respect to a variable, $\mathbf{x}_i$, over which one does not integrate; it is not possible, however, to exchange the orders of integration and differentiation because the domain has boundaries ($|\mathbf{x}_i - \mathbf{x}_l| = \sigma$) depending upon $\mathbf{x}_i$. To obtain the correct result, a boundary term has to be added:

$$(4.4)\quad \int \xi_i \cdot \frac{\partial P_N}{\partial \mathbf{x}_i} \prod_{l=s+1}^{N} d\mathbf{x}_l\, d\xi_l = \xi_i \cdot \frac{\partial}{\partial \mathbf{x}_i} \int P_N \prod_{l=s+1}^{N} d\mathbf{x}_l\, d\xi_l - \sum_{j=s+1}^{N} \int P_N\, \xi_i \cdot \mathbf{n}_{ij}\, d\sigma_{ij}\, d\xi_j \prod_{\substack{l=s+1\\ l\neq j}}^{N} d\mathbf{x}_l\, d\xi_l = \xi_i \cdot \frac{\partial P_N^{(s)}}{\partial \mathbf{x}_i} - \sum_{j=1}^{N} \int P_N^{(s+1)}\, \xi_j \cdot \mathbf{n}_{ij}\, d\sigma_{ij}\, d\xi_j$$

where $\mathbf{n}_{ij}$ is the outer normal to the sphere $|\mathbf{x}_i - \mathbf{x}_j| = \sigma$ (with center at $\mathbf{x}_j$), $d\sigma_{ij}$ the surface element on the same sphere and $P_N^{(s+1)}$ the $(s+1)$-particle distribution function with arguments $(\mathbf{x}_k, \xi_k)$ $(k = 1, 2, \ldots, s, j)$.

A typical term in the second sum in Eq. (4.3) can be immediately integrated by means of the Gauss lemma, since it involves the integration of a derivative taken with respect to one of the variables of integration. We find:

$$(4.5)\quad \int \xi_j \cdot \frac{\partial P_N}{\partial \mathbf{x}_j} \prod_{l=s+1}^{N} d\mathbf{x}_l\, d\xi_l = \sum_{i=1}^{s} \int P_N^{(s+1)}\, \xi_j \cdot \mathbf{n}_{ij}\, d\sigma_{ij}\, d\xi_j + \sum_{\substack{k=s+1\\ i\neq j}}^{N} \int P_N^{(s+2)}\, \xi_j \cdot \mathbf{n}_{kj}\, d\sigma_{ij}\, d\xi_j\, d\mathbf{x}_k\, d\xi_k + \int P_N^{(s+1)}\, \xi_j \cdot \mathbf{n}_j\, dA_j\, d\xi_j,$$

where dA_j is the surface element of the boundary of the volume V in the three-dimensional subspace described by $\mathbf{x}_j$, and $\mathbf{n}_j$ the unit vector normal to such surface element and pointing into the gas. The last term in Eq. (4.5) is the contribution from the solid boundary of R; if the molecules are specularly reflected there, then the term under consideration is obviously zero because $\xi_j \cdot \mathbf{n}_{ij}$ changes its sign under specular reflection. The boundary term, however, is zero under more general assumptions; it is sufficient to assume that the effect of an interaction of a rigid sphere with the wall is independent of the evolution of the state of the other spheres and no particles are captured by the solid walls. We shall discuss this point at the end of this section and neglect the boundary term for the moment.

Inserting Eqs (4.4) and (4.5) into Eq. (4.3), we find:

$$(4.6)\quad \frac{\partial P_N^{(s)}}{\partial t} + \sum_{i=1}^{s} \xi_i \cdot \frac{\partial P_N^{(s)}}{\partial \mathbf{x}_i} - \sum_{i=1}^{s} \sum_{j=s+1}^{N} \int P_N^{(s+1)}\, \mathbf{V}_{ij} \cdot \mathbf{n}_{ij}\, d\sigma_{ij}\, d\xi_j - \frac{1}{2} \sum_{\substack{k,j=s+1\\ i\neq j}}^{N} \int P_N^{(s+2)}\, \mathbf{V}_{kj} \cdot \mathbf{n}_{kj}\, d\sigma_{kj}\, d\xi_j\, d\mathbf{x}_k\, d\xi_k,$$

where $\mathbf{V}_{ij} = \boldsymbol{\xi}_i - \boldsymbol{\xi}_j$ is the relative velocity of the ith particle with respcet to the jth and we took into account that $\boldsymbol{\xi}_i \cdot \mathbf{n}_{ij}$ can be replaced by $(1/2)\mathbf{V}_{ij} \cdot \mathbf{n}_{ij}$ in the second sum because of the antisymmetry of $\mathbf{n}_{ij}$ with respect to its own indexes. $\mathbf{x}_j$ and $\boldsymbol{\xi}_j$ are integration variables in Eq. (2.6); hence the sums over j are made of identical terms, as well as the sum over k in the second integral. We shall write $\mathbf{x}_*$, $\boldsymbol{\xi}_*$ in place of $\mathbf{x}_j$, $\boldsymbol{\xi}_j$ in order to emphasize that the index j is dummy, and $\mathbf{x}_0$, $\boldsymbol{\xi}_0$ in place of $\mathbf{x}_k$, $\boldsymbol{\xi}_k$ while we shall simply write $\mathbf{V}_i$, $\mathbf{n}_i$, $d\sigma_i$ for $\mathbf{V}_{ij}$, $\mathbf{n}_{ij}$, $d\sigma_{ij}$, $\mathbf{V}_0$, $\mathbf{n}_0$, $d\sigma_0$ for $\mathbf{V}_{kj}$, $\mathbf{n}_{kj}$, $d\sigma_{kj}$. Accordingly we obtain:

$$\frac{\partial P_N^{(s)}}{\partial t} + \sum_{i=1}^{s} \boldsymbol{\xi}_i \cdot \frac{\partial P_N^{(s)}}{\partial \mathbf{x}_i} - (N-s) \sum_{i=1}^{s} \int P_N^{(s+1)} \mathbf{V}_i \cdot \mathbf{n}_i \, d\sigma_i \, d\boldsymbol{\xi}_* - \frac{(N-s)(N-s-1)}{2} \int P_N^{(s+2)} \mathbf{V}_0 \cdot \mathbf{n}_0 \, d\sigma_0 \, d\boldsymbol{\xi}_* \, d\boldsymbol{\xi}_0 \, d\mathbf{x}_0, \tag{4.7}$$

where the arguments of $P_N^{(s+1)}$ are $(\mathbf{x}_1, \boldsymbol{\xi}_1, \ldots, \mathbf{x}_s, \boldsymbol{\xi}_s, \mathbf{x}_*, \boldsymbol{\xi}_*, t)$ and those of $P_N^{(s+2)}(\mathbf{x}_1, \boldsymbol{\xi}_1, \ldots, \mathbf{x}_s, \boldsymbol{\xi}_s, \mathbf{x}_*, \boldsymbol{\xi}_*, \mathbf{x}_0, \boldsymbol{\xi}_0, t)$.

We observe now that multiple collisions (i.e., simultaneous contacts of more than two spheres) contribute nothing to the above integrals (at least, if $P_N^{(s+1)}$, $P_N^{(s+2)}$ are ordinary integrable functions). In fact, the integrals with respect to $d\sigma_i$, $d\sigma_0$ are extended over the surface

$$|\mathbf{x}_* - \mathbf{x}_i| = \sigma \qquad (i = 0, 1, \ldots, s;\ s \leqslant N) \tag{4.8}$$

with center at $\mathbf{x}_i$ but not the whole surface, because we must cut out those parts which are inside the other similar surfaces

$$|\mathbf{x}_* - \mathbf{x}_j| = \sigma \qquad (j = 0, 1, \ldots, N;\ j \neq i). \tag{4.9}$$

(This is due to the fact that $P_N = 0$ inside such surfaces.) Multiple collisions correspond to the boundary of the integration region, hence to a one-dimensional subset; accordingly, their contribution to the integrals is zero, unless singularities occur, which we have excluded by using a smoothed out P_N.

It is important, now, to separate each of the integrals appearing in Eq. (4.7) into the corresponding integrals extended to the subsets $\mathbf{V}_i \cdot \mathbf{n}_i < 0$ and $\mathbf{V}_i \cdot \mathbf{n}_i > 0$ $(i = 0, 1, \ldots, s)$, respectively. The first set corresponds to molecules entering a collision, while the second set corresponds to molecules which have just collided (remember that $P_N^{(s+1)}$ is the limiting value from outside the sphere $|\mathbf{x}_* - \mathbf{x}_i| = \sigma$ with center at $\mathbf{x}_i$ so that the value of $P_N^{(s+1)}$ for $|\mathbf{x}_i - \mathbf{x}_*| = \sigma$ and $\mathbf{V}_i \cdot \mathbf{n}_i > 0$ corresponds to the limit from the state just after collision, since $\mathbf{n}_i = (\mathbf{x}_i - \mathbf{x}_*)/\sigma$). Accordingly, Eq. (2.7) becomes

$$\frac{\partial P_N^{(s)}}{\partial t} + \sum_{i=1}^{s} \boldsymbol{\xi}_i \cdot \frac{\partial P_N^{(s)}}{\partial \mathbf{x}_i} - (N-s) \sum_{i=1}^{s} \left[\int^{(+)} P_N^{(s+1)} |\mathbf{V}_i \cdot \mathbf{n}_i| \, d\sigma_i \, d\boldsymbol{\xi}_* - \int^{(-)} P_N^{(s+1)} |\mathbf{V}_i \cdot \mathbf{n}_i| \, d\sigma_i \, d\boldsymbol{\xi}_* \right] - \tag{4.10}$$

$$-\frac{(N-s)(N-s-1)}{2}\left[\int^{(+)} P_N^{(s+2)}|\mathbf{V}_0\cdot\mathbf{n}_0|\,d\sigma_0\,d\xi_*\,d\xi_0\,d\mathbf{x}_0 - \right.$$
$$\left. - \int^{(-)} P_N^{(s+2)}|\mathbf{V}_0\cdot\mathbf{n}_0|\,d\sigma_0\,d\xi_*\,d\xi_0\,d\mathbf{x}_0\right]$$
$$(s = 1, \ldots, N)$$

where the $(+)$ and $(-)$ superscripts in the integrals correspond to $\mathbf{V}_i\cdot\mathbf{n}_i \gtrless 0$ $(i = 0, 1, \ldots, s)$, respectively).

The equations used so far are incomplete, because we have not used the laws of elastic impact, Eqs (3.3) or (3.6).

According to these laws, the velocities after collision, ξ_i and ξ_*, are related to the ones before collision, ξ_i' and ξ_*', by

$$\text{(4.11)}\qquad \begin{aligned}\xi_i &= \xi_i' - \mathbf{n}_i(\mathbf{n}_i\cdot\mathbf{V}_i'),\\ \xi_* &= \xi_*' + \mathbf{n}_i(\mathbf{n}_i\cdot\mathbf{V}_i'),\end{aligned}\qquad (\mathbf{V}_i' = \xi_i' - \xi_*')$$

or

$$\text{(4.12)}\qquad \begin{aligned}\xi_i' &= \xi_i - \mathbf{n}_i(\mathbf{n}_i\cdot\mathbf{V}_i),\\ \xi_*' &= \xi_* + \mathbf{n}_i(\mathbf{n}_i\cdot\mathbf{V}_i),\end{aligned}\qquad (\mathbf{V}_i = \xi_i - \xi_*),$$

where $\mathbf{n}_i = (\mathbf{x}_i - \mathbf{x}_*)/\sigma$, the unit vector directed as the line joining the centers of the two spheres, coincides with the outer normal to the sphere $|\mathbf{x}_i - \mathbf{x}_*| = \sigma$ with center at $\mathbf{x}_*$ and variable $\mathbf{x}_i$ and variable x. $\mathbf{V}_i'$ and $\mathbf{V}_i$ are, of course, the relative velocities before and after collision, related to each other by Eq. (4.4.12):

$$\text{(4.13)}\qquad \mathbf{V}_i = \mathbf{V}_i' - 2\mathbf{n}_i(\mathbf{n}_i\cdot\mathbf{V}_i').$$

Accordingly, any molecule entering a collision with velocity ξ_i' at $\mathbf{x}_i$ is at the same time (or a vanishingly short time later) in an after-collision state with velocity ξ_i related to ξ_i' and $\mathbf{n}_i = (\mathbf{x}_i - \mathbf{x}_*)/\sigma$ by Eq. (4.11); accordingly:

$$\text{(4.14)}\qquad P_N^{(s+1)}(\mathbf{x}_1, \xi_1, \ldots, \mathbf{x}_i, \xi_i, \ldots, \mathbf{x}_s, \xi_s, \mathbf{x}_*, \xi_*, t)$$
$$= P_N^{(s+1)}(\mathbf{x}_1, \xi_1, \ldots, \mathbf{x}_i, \xi_i - \mathbf{n}_i(\mathbf{n}_i\cdot\mathbf{V}_i), \ldots, \mathbf{x}_s, \xi_s, \mathbf{x}_*, \xi_* + \mathbf{n}_i(\mathbf{n}_i\cdot\mathbf{V}_i)),$$
$$(i = 0, 1, \ldots, s;\ 1 \leqslant s \leqslant N-1)$$

We first examine the term involving $P_N^{(s+2)}$ in Eq. (4.11) and claim that it is zero, i.e.,

$$\text{(4.15)}\qquad \int P_N^{(s+2)}|\mathbf{V}_0\cdot\mathbf{n}_0|\,d\sigma_0\,d\xi_*\,d\xi_0\,d\mathbf{x}_0 = \int^{(-)} P_N^{(s+2)}|\mathbf{V}_0\cdot\mathbf{n}_0|\,d\sigma_0\,d\xi_*\,d\xi_0\,d\mathbf{x}_0.$$

In fact, changing the variables from ξ_0 and ξ_* to ξ_0' and ξ_*' given by Eqs (4.12) with $(i = 0)$ and taking Eq. (4.14) (with $i = 0$ and s replaced by $s+1$) into account, the *left*-hand side of Eq. (4.15) becomes:

$$\text{(4.16)}\qquad \text{l.h.s.} = \int^{(-)} P_N^{(s+2)'}|\mathbf{V}_0'\cdot\mathbf{n}_0|\,d\sigma_0\,d\xi_*'\,d\xi_0'\,d\mathbf{x}_0,$$

where the arguments of $P_N^{(s+2)'}$ are the same as in $P_N^{(s+2)}$ with $\boldsymbol{\xi}_*$ and $\boldsymbol{\xi}_0$ replaced by $\boldsymbol{\xi}'_*$ and $\boldsymbol{\xi}'_0$, and we took into account that the absolute value of the Jacobian determinant of the transformation from $\boldsymbol{\xi}_*$, $\boldsymbol{\xi}_0$ to $\boldsymbol{\xi}'_*$, $\boldsymbol{\xi}'_0$ is 1 (see Section 3). The integral extends to the hemisphere $\mathbf{V}'_0 \cdot \mathbf{n}_0 < 0$, because Eq. (4.13) (with $i = 0$) implies

$$\mathbf{V}_0 \cdot \mathbf{n}_0 = -\mathbf{V}'_0 \cdot \mathbf{n}_0. \tag{4.17}$$

We can now drop the primes in Eq. (4.16) since $\boldsymbol{\xi}'_0$ and $\boldsymbol{\xi}'_*$ are integration variables; we find

$$\text{l.h.s.} = \int^{(-)} P_N^{(s+2)} |\mathbf{V}_0 \cdot \mathbf{n}_0|\, d\sigma_0\, d\mathbf{x}_0\, d\boldsymbol{\xi}_0\, d\boldsymbol{\xi}_* = \text{r.h.s.} \tag{4.18}$$

and Eq. (4.15) is proved.

A similar argument cannot be used in connection with the integrals involving $P_N^{(s+1)}$. We can use, however, Eq. (4.14) for expressing both integrals in terms of the distribution function holding before the collision, or, alternatively, in terms of the one prevailing after the collision. In fact, Eq. (4.14) is perfectly reversible and can be used for expressing either the distribution function of molecules which have just collided in terms of the one of molecules entering a collision or the latter in terms of the former. It is obvious that the first way is to be used if we want to predict the future from the past and not *vice versa*; it is equally clear, however, that, at this point, we commit ourselves to a definite time arrow, i.e., we introduce a difference between past and future, as we shall see in more detail later.

According to this discussion, we use Eq. (2.14) to obtain:

$$\int^{(+)} P_N^{(s+1)} |\mathbf{V}_i \cdot \mathbf{n}_i|\, d\sigma_i\, d\boldsymbol{\xi}_* = \int^{(+)} P_N^{(s+1)'} |\mathbf{V}_i \cdot \mathbf{n}_i|\, d\sigma_i\, d\boldsymbol{\xi}_* \tag{4.19}$$

where $P_N^{(s+1)'}$ means the value taken by $P_N^{(s+1)}$ when the arguments $\boldsymbol{\xi}_i$ and $\boldsymbol{\xi}_*$ are replaced by $\boldsymbol{\xi}'_i$ and $\boldsymbol{\xi}'_*$ given by Eqs (4.12). If we insert Eqs (4.19) and (4.15) into Eq. (4.10), we obtain

$$\begin{aligned} \frac{\partial P_N^{(s)}}{\partial t} + \sum_{i=1}^{s} \boldsymbol{\xi}_i \cdot \frac{\partial P_N^{(s)}}{\partial \mathbf{x}_i} = (N-s)\sigma^2 \sum_{i=1}^{s} \Bigg[\int^{(+)} P_N^{(s+1)'} |\mathbf{V}_i \cdot \mathbf{n}_i|\, d\mathbf{n}_i\, d\boldsymbol{\xi}_* - \\ - \int^{(-)} P_N^{(s+2)} |\mathbf{V}_i \cdot \mathbf{n}_i|\, d\mathbf{n}_i\, d\boldsymbol{\xi}_* \Bigg] \quad (s = 1, \ldots, N) \end{aligned} \tag{4.20}$$

where we replaced $d\sigma_i$ by its expression $\sigma^2 d\mathbf{n}_i$ in terms of the radius σ of the sphere given by Eq. (2.8) and the elements of solid angle $d\mathbf{n}_i$. We may even abolish the index i in $\mathbf{n}_i$ provided the argument $\mathbf{x}_*$ in the ith integral is replaced by

$$\mathbf{x}_* = \mathbf{x}_i - \mathbf{n}\sigma. \tag{4.21}$$

Finally, we may transform the two integrals extended to $\mathbf{V}_i \cdot \mathbf{n} < 0$ and

$\mathbf{V}_i \cdot \mathbf{n} > 0$ into a single integral by changing, e.g., $\mathbf{n}$ into $-\mathbf{n}$ in the second integral. Thus we have

$$\frac{\partial P_N^{(s)}}{\partial t} + \sum_{i=1}^{s} \boldsymbol{\xi}_i \cdot \frac{\partial P_N^{(s)}}{\partial \mathbf{x}_i} = (N-s)\sigma^2 \sum_{i=1}^{s} \int [P_N^{(s+1)\prime} - P_N^{(s+1)}] \, |\mathbf{V}_i \cdot \mathbf{n}| \, d\mathbf{n} \, d\boldsymbol{\xi}_* \qquad (s = 1, \ldots, N) \tag{4.22}$$

where integration is extended to the hemisphere $\mathbf{V}_i \cdot \mathbf{n} > 0$ and the arguments in $P_N^{(s+1)\prime}$ are the same as those in $P_N^{(s+1)}$ except for $\boldsymbol{\xi}_i$, $\boldsymbol{\xi}_*$ which are replaced by $\boldsymbol{\xi}_i'$, $\boldsymbol{\xi}_*'$ given by Eqs (4.12) and $\mathbf{x}_i - \mathbf{n}\sigma$ which is replaced by $\mathbf{x}_i + \mathbf{n}\sigma$.

We stress the fact that Eq. (4.22) was derived under the only assumptions of symmetrical dependence of $P_N^{(s)}$ upon the molecules and sufficient regularity of $P_N^{(s)}$ (the latter is required in order to neglect the contribution of a *line* to a surface integral, i.e., to neglect the effect of triple collisions). In addition we neglected the surface integral in Eq. (4.5).

To justify the fact that we disregarded the boundary integral, we must discuss the boundary conditions satisfied by P_N when $\mathbf{x}_i$ belongs to the boundary ∂R of the region R where the gas is enclosed. This is a topic which will be considered in detail in paper [1]. Here we assume that a molecule hitting the solid boundary ∂R at some point $\mathbf{x}$ with some velocity $\boldsymbol{\xi}'$ reemerges at practically the same point with some other velocity $\boldsymbol{\xi}$, the duration of the molecule-wall interaction being negligible.

The nature of the interaction determines a probability density $R(\boldsymbol{\xi}' \to \boldsymbol{\xi}; \mathbf{x}, t)$ of a transition from a velocity $\boldsymbol{\xi}'$ to a velocity $\boldsymbol{\xi}$ at point $\mathbf{x}$ and time t; we shall assume that this probability is independent of the state of the other molecules and no particles are captured by the solid walls.

The probability that the jth molecule emerges from the surface element dA_j about $\mathbf{x}_j$ during the time interval dt with velocity between $\boldsymbol{\xi}_j$ and $\boldsymbol{\xi}_j + d\boldsymbol{\xi}_j$ when the lth molecule $(l \neq j)$ is in the volume element $d\mathbf{x}_l$ with velocity between $\boldsymbol{\xi}_l$ and $\boldsymbol{\xi}_l + d\boldsymbol{\xi}_l$ is

$$d^* \mathscr{P} = P_N |\boldsymbol{\xi}_j \cdot \mathbf{n}_j| \, dt \, dA_j \, d\boldsymbol{\xi}_j \prod_{\substack{l=1 \\ l \neq j}}^{N} d\mathbf{x}_l \, d\boldsymbol{\xi}_l \tag{4.23}$$

where $\mathbf{n}_i$ is the normal unit vector pointing into the gas at $\mathbf{x}_j$; in fact, this is the probability to find the molecule in the cylinder filled by points leaving dA_j during the time interval dt with velocity between $\boldsymbol{\xi}_j$ and $\boldsymbol{\xi}_j + d\boldsymbol{\xi}_j$ when the lth molecule $(l \neq j)$ is in the volume element $d\mathbf{x}_l$ with velocity between $\boldsymbol{\xi}_l$ and $\boldsymbol{\xi}_l + d\boldsymbol{\xi}_l$ and the two probabilities are obviously the same thing.

Analogously, the probability that the same molecule impinges upon the same surface element with velocity between $\boldsymbol{\xi}_j'$ and $\boldsymbol{\xi}_j' + d\boldsymbol{\xi}_j'$ during dt when the lth molecule is in $d\mathbf{x}_l$ with velocity between $\boldsymbol{\xi}_l$ and $\boldsymbol{\xi}_l + d\boldsymbol{\xi}_l$ $(l \neq j)$ is

$$d^* \mathscr{P}' = P_N' |\boldsymbol{\xi}_j' \cdot \mathbf{n}_j| \, dt \, dA_j \, d\boldsymbol{\xi}_j' \prod_{\substack{l=1 \\ l \neq j}}^{N} d\boldsymbol{\xi}_l \, d\mathbf{x}_l \tag{4.24}$$

where the arguments of P'_N are the same as those of P_N except for the fact that ξ'_j replaces ξ_j. If we multiply $d^*\mathscr{P}'$ by the probability of a scattering from the wall from velocity ξ'_j to a velocity between ξ_j and $\xi_j+d\xi_j$ i.e. $R(\xi'_j \rightarrow \xi_j;\ \mathbf{x}_j,\ t)d\xi_j$ and "sum", i.e. integrate, over all the possible values of ξ'_j, we must obtain $d^*\mathscr{P}$:

$$d^*\mathscr{P} = d\xi_j \int_{\xi'_j\cdot\mathbf{n}_j<0} R(\xi'_j\rightarrow\xi_j;\ \mathbf{x}_j,\ t)\,d^*\mathscr{P}' \qquad (\xi_j\cdot\mathbf{n}_j>0) \tag{4.25}$$

or, using Eqs (4.23) and (4.24) and canceling the common factor $d\xi_j(\Pi d\xi_l\,d\mathbf{x}_l)\,dA_j\,dt$:

$$\begin{aligned} &|\xi_j\cdot\mathbf{n}_j|\,P_N(\mathbf{x}_1,\xi_1,\dots,\mathbf{x}_j,\xi_j,\dots,\mathbf{x}_N,\xi_N,t) \\ &= \int_{\xi'_j\cdot\mathbf{n}_j<0} R(\xi'_j\rightarrow\xi_j;\ \mathbf{x}_j,\ t)\,P_N(\mathbf{x}_1,\xi_1,\dots,\mathbf{x}_j,\xi_j,\dots,\mathbf{x}_N,\xi_N,t)\,|\xi'_j\cdot\mathbf{n}_j|\,d\xi'_j \\ &\qquad\qquad (\mathbf{x}_j\in\partial R;\ \xi_j\cdot\mathbf{n}_j>0) \end{aligned} \tag{4.26}$$

This is the boundary condition satisfied by P_N at the solid boundaries, under the assumptions of instantaneous interaction. In particular, if the wall specularly reflects the molecules, we have

$$R(\xi'\rightarrow\xi;\ \mathbf{x},\ t) = \delta(\xi'-\xi+2\mathbf{n}(\mathbf{n}\cdot\xi)). \tag{4.27}$$

In general, R must satisfy some restrictions, which is discussed in paper [1]. The only restriction to be presently considered is the one related to the assumption that no molecules are captured by the walls; this means that any molecule impinging upon the wall eventually re-emerges with some velocity ξ and consequently the "sum" of the elementary probabilities $R(\xi'\rightarrow\xi)d\xi$ over all possible values of ξ must be unity:

$$\int_{\xi\cdot\mathbf{n}>0} R(\xi'\rightarrow\xi;\ \mathbf{x},\ t)\,d\xi = 1 \qquad (\mathbf{x}\in\partial R,\ \xi'\cdot\mathbf{n}<0). \tag{4.28}$$

Hence Eq. (4.26) gives, after integration over ξ_j and using Eq. (4.28):

(4.29)

$$\int_{\xi_j\cdot\mathbf{n}_j>0} |\xi_j\cdot\mathbf{n}_j|\,P_N\,d\xi_j = \int_{\xi'_j\cdot\mathbf{n}_j<0} P'_N\,|\xi'_j\cdot\mathbf{n}_j|\,d\xi'_j = \int_{\xi_j\cdot\mathbf{n}_j<0} P_n\,|\xi_j\cdot\mathbf{n}_j|\,d\xi_j \qquad (\mathbf{x}_j\in\partial R),$$

where the third expression comes from replacing ξ'_j by ξ_j. Equation (4.29) can be rewritten as follows:

$$\int \xi_j\cdot\mathbf{n}_j\,P_N\,d\xi_j = 0 \qquad (\mathbf{x}_j\in\partial R), \tag{4.30}$$

where ξ_j unrestrictedly varies throughout both the half-spaces $\xi_j\cdot\mathbf{n}_j \gtrless 0$.

Finally, by integrating Eq. (4.30) with respect to dA_j and the coordinates and velocities of $N-s-1$ molecules (other than the jth one), we obtain

$$\int \boldsymbol{\xi}_j \cdot \mathbf{n}_j P_N^{(s+1)} d\boldsymbol{\xi}_j \, dA_j = 0 \tag{4.31}$$

which proves that the last term in Eq. (4.5) is indeed zero, under our assumptions.

5. The Boltzmann equation for rigid spheres

The previous section was devoted to deriving the equations satisfied by $P_N^{(s)}(s = 1, \ldots, N)$, Eq. (4.22). In particular, for $s = 1$, we have:

$$\frac{\partial P_N^{(1)}}{\partial t} + \boldsymbol{\xi}_1 \cdot \frac{\partial P_N^{(1)}}{\partial \mathbf{x}_1} = (N-1)\sigma^2 \int [P_N^{(2)\prime} - P_N^{(2)}] |\mathbf{V}_1 \cdot \mathbf{n}| \, d\mathbf{n} \, d\boldsymbol{\xi}_*. \tag{5.1}$$

This equation shows that the time evolution of the one-particle distribution function, $P_N^{(1)}$, depends upon the two-particle distribution function, $P_N^{(2)}$. In order to have a closed form equation for $P_N^{(1)}$, it is necessary to express $P_N^{(2)}$ in terms of $P_N^{(1)}$; a simple intuitive way of doing this is to assume the absence of correlation, i.e., to write:

$$P_N^{(2)}(\mathbf{x}_1, \boldsymbol{\xi}_1, \mathbf{x}_*, \boldsymbol{\xi}_*, t) = P_N^{(1)}(\mathbf{x}_1, \boldsymbol{\xi}_1, t) P_N^{(1)}(\mathbf{x}_*, \boldsymbol{\xi}_*, t). \tag{5.2}$$

This relation was obtained in the case of thermal equilibrium for $N \to \infty$. If we accept it even in the case of non-equilibrium and insert Eq. (5.2) into Eq. (5.1), and equation involving $P_N^{(1)}$ alone is found. This is essentially the "*stosszahlansatz*" used by Boltzmann (1872) to derive the equation for $P_N^{(1)}$, which is accordingly called the *Boltzmann equation.*

We have no right, however, to postulate Eq. (5.2) because $P_N^{(2)}$ is determined by another equation (Eq. (4.2) with $s = 2$) involving $P_N^{(3)}$, and the latter in turn, by another equation, Eq. (4.2) with $s = 3$, involving $P_N^{(4)}$, etc. The least requirement is, therefore, to show that Eq. (5.2) is not in contrast with the equation regulating the time evolution of $P_N^{(s)}(s \geqslant 2)$. Now, we cannot prove this statement, at least if we take it literally.

In fact: Equation (5.2) means that the states of the two molecules considered are statistically uncorrelated. Now, this makes sense for any two randomly chosen molecules of the gas since they do not interact when they are far apart, and therefore behave independently. In particular, this seems true for two molecules which are going to collide, because they are just two random molecules whose paths happen to cross; but the same statistical independence is far from being true for two molecules which have just collided. We note, however, that Eq. (5.1) involves $P_N^{(2)}$ for molecules that are entering a collision, because we used Eq. (4.14) to eliminate the values of $P_N^{(s+1)}$ corresponding to after-collision states. This remark is important, but problems still arise because Eq. (4.2) for $P_N^{(s)}$ is valid provided $\mathbf{x}_i$ $(i = 1, \ldots, s)$

is outside the sets $|\mathbf{x}_i - \mathbf{x}_j| \leqslant \sigma$ ($j = 1, \ldots, s; j \neq i$); the volume of these sets grows linearly with s, being proportional to $s\sigma^3$. These sets are, however, negligible in the limit $\sigma \to 0$, $N \to \infty$ for a fixed s (or even if we let s grow with N ($s \leqslant N$), provided $N\sigma^3 \to 0$ as is the case for a perfect gas). We conclude that Boltzmann's *ansatz*, Eq. (5.2), is not true in a literal sense, but could become true for $N \to \infty$, $\sigma \to 0$ provided we specify that Eq. (3.2) is valid almost everywhere, i.e., ceases to be valid in exceptional sets of zero measure (among which the set of after-collision states).

Accordingly, we must prove that Eq. (5.2) (for $N \to \infty$, $\sigma \to 0$) is not in contrast with the equations governing the time evolution of $P_N^{(s)}$ ($s \geqslant 2$). We shall prove more, i.e., that the factorization property:

$$P^{(s)} = \prod_{i=1}^{s} P^{(1)}(\mathbf{x}_i, \xi_i, t) \tag{5.3}$$

(where

$$P^{(s)} = \lim_{N \to \infty} P_N^{(s)}) \tag{5.4}$$

is not in contrast with Eqs (4.2) provided $\sigma \to 0$ in such a way that $N\sigma^2$ is bounded (hence $N\sigma^3 \to 0$). In order to prove this, we shall assume that the limit shown in Eq. (5.4) exists for any finite s and the resulting function $P^{(s)}$ is sufficiently smooth.

Then, is we fix s and let $N \to \infty$, $\sigma \to 0$ in Eq. (4.2), in such a way that $N\sigma^2$ is bounded, we obtain:

$$\frac{\partial P^{(s)}}{\partial t} + \sum_{i=1}^{s} \xi_i \cdot \frac{\partial P^{(s)}}{\partial \mathbf{x}_i} = (N\sigma^2) \sum_{i=1}^{s} \int [P^{(s+1)\prime} - P^{(s+1)}] |\mathbf{V}_i \cdot \mathbf{n}| \, d\mathbf{n} \, d\xi_* \tag{5.5}$$

$$(s = 1, 2, 3, \ldots)$$

where the arguments in $P^{(s+1)\prime}$ and $P^{(s+1)}$ are the same as above, except for the fact that $\mathbf{x}_* = \mathbf{x}'_* = \mathbf{x}_i$, in agreement with Eq. (4.21) for $\sigma \to 0$. Eqs (3.5) give a complete description of the time evolution of a Boltzmann gas, provided the initial value problems is well set for this system of equations.

A particular solution of Eqs (5.5) can be found in the form given by Eq. (5.3) provided the one-particle distribution function satisfies

$$\frac{\partial P}{\partial t} + \xi \cdot \frac{\partial P}{\partial \mathbf{x}} = (N\sigma^2) \int (P' P'_* - P P_*) |\mathbf{V} \cdot \mathbf{n}| \, d\mathbf{n} \, d\xi_* \tag{5.6}$$

where we wrote ξ and $\mathbf{x}$ in place of ξ_1 and $\mathbf{x}_1$, P in place of $P^{(1)}$ while P_*, P', P'_* denote that the argument ξ appearing in P is to be replaced by ξ_*, ξ', ξ'_*, respectively. The above statement is straightforwardly verified by substituting Eq. (5.3) into Eq. (5.5), provided Leibniz's rule for differentiating a product is used when evaluating the time derivative of $P^{(s)}$.

Hence, if the system of Eqs (5.5) admits a unique solution for a given initial datum, we conclude that the solution corresponding to a datum satisfying the "chaos assumption":

$$P^{(s)} = \prod_{i=1}^{s} P^{(1)}(\mathbf{x}_i, \xi_i, 0) \qquad (t = 0) \tag{5.7}$$

will remain factored for all subsequent times and the one-particle distribution function $P = P^{(1)}$ will satisfy the Boltzmann equation. Therefore the factorization assumption, Eq. (5.3), is not inconsistent with the dynamics of rigid spheres in the limit $N \to \infty$, $\sigma \to 0$ ($N\sigma^2$ bounded) and leads to the Boltzmann equation.

The integral in the right-hand side of Eq. (5.6), which is called the *collision term*, is extended to all the values of ξ_* and the hemisphere $|\mathbf{n}| = 1$, $\mathbf{V} \cdot \mathbf{n} > 0$. We observe that it could be equivalently extended to the whole unit sphere and divide the result by 2, because changing $\mathbf{n}$ into $-\mathbf{n}$ does not alter the integrand.

Frequently, when dealing with the Boltzmann equation, one introduces a different unknown f which is related to P by:

$$f = NmP = MP, \tag{5.8}$$

where N is the number of molecules, m the mass of a molecule and M the total mass. The meaning of f is an (expected) mass density in the phase space of a single particle, i.e., the (expected) "mass per unit volume" in the six-dimensional space described by $(\mathbf{x}, \xi)$. We note that because of the normalization condition

$$\int P d\mathbf{x} d\xi = 1 \tag{5.9}$$

we have

$$\int f d\mathbf{x} d\xi = M. \tag{5.10}$$

It is clear that in terms of f we have:

$$\frac{\partial f}{\partial t} + \xi \cdot \frac{\partial f}{\partial \mathbf{x}} = \frac{\sigma^2}{m} \int (f' f'_* - f f_*) |\mathbf{V} \cdot \mathbf{n}| \, d\mathbf{n} d\xi_*, \tag{5.11}$$

where $f_* = f(\xi_*)$, $f'_* = f(\xi'_*)$, $f' = f(\xi')$. This is the form of the Boltzmann equation for a gas of rigid spheres which will be used in the following.

The above considerations could be repeated if an external force per unit mass, $\mathbf{X}$, acts on the molecules, the only influence of this force being that one should add a term $\mathbf{X} \cdot \partial f / \partial \xi$ to the left-hand side of Eq. (5.11). Since we shall usually consider cases where the external action on the gas is exerted through solid boundaries (surface forces), we shall not usually write the above mentioned term describing body forces; it should be kept in mind, however, that such simplification implies neglecting, *inter alia*, gravity.

6. Generalizations

In the previous section, following a paper of the author it was shown that, under certain assumptions, the Boltzmann equation follows from the Liouville equation for a gas of identical rigid spheres in the Boltzmann limit, defined by $N \to \infty$, $\sigma \to 0$, $N\sigma^2$ finite. Three possible generalizations suggest themselves: 1) molecules interacting with at-distance force, 2) systems composed of several species of molecules, i.e., a mixture of gases, 3) polyatomic gases, 4) dense gases ($N \to \infty$, $\sigma \to 0$, $N\sigma^3$ finite).

At first sight, the case of molecules interacting with at-distance force seems to yield equations completely different from the Boltzmann equation. In fact, the Liouville equation, Eq. (2.8), can be written as follows:

$$\frac{\partial P_N}{\partial t} + \sum_{i=1}^{N} \boldsymbol{\xi}_i \cdot \frac{\partial P_N}{\partial \mathbf{x}_i} + \sum_{i,j=1}^{N} \mathbf{X}_{ij} \cdot \frac{\partial P_N}{\partial \boldsymbol{\xi}_i} = 0. \tag{6.1}$$

Here we assume that the force per unit mass acting on the ith molecule, $\mathbf{X}_i$, is the resultant of $N-1$ two-body forces $\mathbf{X}_{ij}$ ($\mathbf{X}_{ii} = 0$) due to the interaction with remaining molecules and such that $\mathbf{X}_{ij} = \mathbf{X}(\mathbf{x}_i, \mathbf{x}_j)$ depends on the coordinates $\mathbf{x}_i$ and $\mathbf{x}_j$ alone.

If we integrate Eq. (6.1) with respect to the coordinates and velocities of $N-s$ molecules and use Eq. (1.3) defining the s-particle distribution function $P_N^{(s)}$, we obtain

$$\frac{\partial P_N^{(s)}}{\partial t} + \sum_{i=1}^{s} \boldsymbol{\xi}_i \cdot \frac{\partial P_N^{(s)}}{\partial \mathbf{x}_i} + \sum_{i=1}^{s} \sum_{j=1}^{s} \mathbf{X}_{ij} \cdot \frac{\partial P_N^{(s)}}{\partial \boldsymbol{\xi}_j} + \\ + (N-s) \sum_{i=1}^{s} \frac{\partial}{\partial \boldsymbol{\xi}_i} \cdot \int P_N^{(s+1)} \mathbf{X}_i \, d\mathbf{x}_* \, d\boldsymbol{\xi}_* = 0, \tag{6.2}$$

where $\mathbf{X}_i = \mathbf{X}(\mathbf{x}_i, \mathbf{x}_*)$. In order to obtain Eq. (4.2), it is sufficient to observe that the terms of the last sum in Eq. (6.2) with $i \geqslant s+1$ integrate to zero, because they can be transformed into a surface integral at infinity in the velocity space of the ith molecule (we assume that $P \to 0$ when $\boldsymbol{\xi}_i \to \infty$ while the terms with $j \geqslant s+1$ give identical contributions; finally, terms involving space derivatives with respect to $\mathbf{x}_i$ ($i \geqslant s+1$) are transformed into surface integrals extended to the physical boundary of the system, which are assumed to be zero by the same kind of arguments used in Section 2.

Equations (6.2) constitute the so-called *BBGKY-hyerarchy* (from the names of Bogoliubov, Born and Green, Kirkwood, Yvon). It is not obvious how to handle these equations in the Boltzmann limit. There is another limit, however, in which Eq. (6.2) lends itself to deriving a simple result. If each of the forces $\mathbf{X}_{ij}$ is uniformly small, of order ε, in such a way that we may let N

$\to \infty$ and $\varepsilon \to 0$ and keep $N\varepsilon$ finite (i.e., the order of magnitude of the total force is finite), then we obtain from Eq. (6.2):

$$\frac{\partial P^{(s)}}{\partial t} + \sum_{i=1}^{s} \boldsymbol{\xi}_i \cdot \frac{\partial P^{(s)}}{\partial \mathbf{x}_i} + N \sum_{i=1}^{s} \frac{\partial}{\partial \boldsymbol{\xi}_i} \cdot \int P^{(s+1)} \mathbf{X}_i \, d\mathbf{x}_* \, d\boldsymbol{\xi}_* = 0, \tag{6.3}$$

where $P^{(s)} = \lim\limits_{N \to \infty} P_N^{(s)}$, as above. This system of equations, however, possesses a particular solution having the factorization property expressed by Eq. (6.3), as is verified by direct substitution; the one-particle probability density $P = P^{(1)}$ satisfies:

$$\frac{\partial P}{\partial t} + \boldsymbol{\xi} \cdot \frac{\partial P}{\partial \mathbf{x}} + \mathbf{X} \cdot \frac{\partial P}{\partial \boldsymbol{\xi}} = 0. \tag{6.4}$$

Here

$$\begin{aligned} \mathbf{X}(\mathbf{x}) &= N \int P(\mathbf{x}_*, \boldsymbol{\xi}_*, t) \mathbf{X}(\mathbf{x}, \mathbf{x}_*) \, d\mathbf{x}_* \, d\boldsymbol{\xi}_* \\ &= \int n(\mathbf{x}_*, t) \mathbf{X}(\mathbf{x}, \mathbf{x}_*) \, d\mathbf{x}_* \end{aligned} \tag{6.5}$$

where

$$n(\mathbf{x}_*, t) = N \int P(\mathbf{x}_*, \boldsymbol{\xi}_*, t) \, d\boldsymbol{\xi}_* \tag{6.6}$$

is the number density in physical space, i.e., the number of molecules per unit volume, in a neighborhood of $\mathbf{x}_*$ at time t.

Equation (6.4) is a remarkable equation, called the *Vlasov equation*. It is completely different from the Boltzmann equation and is unseful to describe the short time behavior of a system of weakly interacting mass points; this is the case of a rarefied gas whose particles interact with relatively weak, long range forces, such as the electrons of an ionized gas (Coulomb force) ot the stars of a stellar system (gravitational force). In a ordinary gas, however, the intermolecular force is extremely weak when the molecules are close to each other; hence the model of hard collisions, though extremely crude, is closer to a significant description of the state of affairs than the model of a continuously distributed, weak force.

In the kinetic theory of gases, it is customary to consider some molecular models which take the molecular interaction into account in a more or less accurate fashion. One of these is the hard sphere model which was discussed in detail before; the other models are based on mass points interacting with central, hence conservative, forces and differ from each other only for the expression of the potential U of these forces. The simplest assumption is $U(\varrho) = \varkappa \varrho^{1-n}$ where ϱ is the distance between two interacting molecules and the force $\mathbf{X} = -\operatorname{grad} U$ is assumed to be repulsive ($\varkappa > 0$). A great use, especially in the computation of transport coefficients, has been made of the model of Lennard-Jones, which includes both a repulsive and an attractive part

$$U = \frac{\varkappa}{\varrho^{n-1}} - \frac{\varkappa'}{\varrho^{n'-1}} \qquad (n > n') \tag{6.7}$$

with the typical choice $n = 13$, $n' = 7$. Other models replace the first of these terms by an exponential in r or by a rigid sphere potential, i.e., 0 for $\varrho > \sigma$, ∞ for $\varrho < \sigma$. The force corresponding to a potential of the Lennard–Jones type, Eq. (6.7), is well approximated by a power law potential for short distances ($r \leqslant (\varkappa/\varkappa')^{n-n'}$) and may be replaced by a cutoff force with potential:

$$U(\varrho) = \begin{cases} \varkappa\varrho^{1-n}, & \varrho \leqslant \sigma, \\ \varkappa\sigma^{1-n}, & \varrho \geqslant \sigma. \end{cases} \tag{6.8}$$

If we adopt such a cutoff potential, it is possible to derive the Boltzmann equation under the assumption $\sigma \to 0$, $N \to \infty$, $N\sigma^2$ finite, provided $U(\sigma)$ is small of the order of the molecular mass $m \simeq \sigma^2 Ml/V$ (V volume, l mean free path). The latter circumstance is reasonably well verified for monatomic gases, since $U(\sigma)/(mR)$ (where R is the gas constant) is of the order of magnitude of a typical temperature (ranging from 10 K to 230 K).

In order to prove what we asserted, we introduce the truncated distribution functions:

$$^{\sigma}P_N^{(s)} = \int_{D_\sigma} P_N \prod_{l=s+1}^{N} d\mathbf{x}_l \, d\boldsymbol{\xi}_l, \tag{6.9}$$

where the domain of integration excludes those regions, where P_N would be zero by definition if the molecules were rigid spheres of radius $\sigma/2$.

We can then repeat the derivation given in Sections 2 and 3 except for two facts:

(a) Multiple collisions are not a set of zero measure in the set of all collisions, because now collisions are replaced by finite duration interactions. If we let $N\sigma^3 \to 0$, however, as is correct for a Boltzmann gas, the measure tends to zero, because the probability of a triple collision is small with $N\sigma^3/V$ (V volume). Hence, for a Boltzmann gas, it is safe to neglect multiple collisions, and treat each collision as a two-body problem, even if we are not dealing with a gas of rigid spheres.

(b) A molecules leaves the protection sphere ($\varrho = \sigma$) of another molecule at a point different from the one at which the molecule entered the same sphere. The law of scattering can be written in the form given by Eq. (4.14) provided $\mathbf{n}_i$ is directed as the apse line of the orbit of the „bullet" molecule with respect to the "target" molecule (see; the apse line is the line through the target molecule and the point of closes t approach) and $\mathbf{x}_i$, $\mathbf{x}_*$, t in the right-hand side are replaced by some $\mathbf{x}_i'$, $\mathbf{x}_*'$ t' differing from $\mathbf{x}_i$, $\mathbf{x}_*$, t by terms small with σ. The latter correction disappears when $\sigma \to 0$. There is an additional point, however, which will be presently considered. Let $\mathbf{n}^{(in)}$ and $\mathbf{n}^{(out)}$ be the normal unit vectors at the points where the "bullet" molecules reaches and leaves the protection sphere; then

$$\sigma^2 |\mathbf{V}' \cdot \mathbf{n}^{(in)}| \, d\mathbf{n}^{(in)} = \sigma^2 |\mathbf{V} \cdot \mathbf{n}^{(out)}| \, d\mathbf{n}^{(out)}, \tag{6.10}$$

because the trajectories are fully simmetrical with respect to the apse line and $\mathbf{V}' = \mathbf{V}$. But, in general,

$$\frac{|\mathbf{V}' \cdot \mathbf{n}^{(in)}|\, d\mathbf{n}^{(in)}}{|\mathbf{V} \cdot \mathbf{n}|\, d\mathbf{n}} = \frac{|\mathbf{V} \cdot \mathbf{n}^{(out)}|\, d\mathbf{n}^{(out)}}{|\mathbf{V} \cdot \mathbf{n}|\, d\mathbf{n}} \tag{6.11}$$

will not be unity. Accordingly, if we want to use $\mathbf{n}$ throughout, we must compute this ratio. We note that an elementary geometrical argument gives

$$\sigma^2 |\mathbf{V} \cdot \mathbf{n}^{(out)}|\, d\mathbf{n}^{(out)} = Vr\, dr\, d\varepsilon, \tag{6.12}$$

where V is the relative speed and r, ε the polar coordinates in a plane orthogonal to $\mathbf{V}$ so that $r\,dr\,d\varepsilon$ is the surface element into which the surface element $\sigma^2 d\mathbf{n}^{(out)}$ of the protection sphere is projected. When $\sigma\mathbf{n}$ described the protection sphere, the point (r, ε) describes the corresponding disk twice, but only once is the image of a point in the "plus" hemisphere $(\mathbf{V} \cdot \mathbf{n} > 0)$ r is nothing else than the impact parameter, i.e., the distance of closest approach of the two particles, had they continued their motion without interacting. The problem is to compute r as a function of V and θ where θ is the angle between $\mathbf{n}$ and $\mathbf{V}$; in fact, $d\mathbf{n} = \sin\theta\, d\theta\, d\varepsilon$, and consequently:

$$\begin{aligned} \sigma^2 |\mathbf{V} \cdot \mathbf{n}^{(out)}|\, d\mathbf{n}^{(out)} &= Vr\frac{\partial r}{\partial \theta}\, d\theta\, d\varepsilon = B(\theta, V)\, d\theta\, d\varepsilon \\ &= Vs(\theta, V)\, d\mathbf{n}, \end{aligned} \tag{6.13}$$

where

$$B(\theta, V) = Vr\frac{\partial r}{\partial \theta}, \tag{6.14}$$

$$s(\theta, V) = \frac{1}{\sin\theta}\, r\frac{\partial r}{\partial \theta}. \tag{6.15}$$

$s(\theta, V)$ is called *differential scattering cross-section*, since is has the dimension of an area; for rigid spheres $r = \sigma \sin\theta$, $B(\theta, V) = V\sigma^2 \sin\theta \cos\theta$, $s(\theta, V) = \sigma^2 \cos\theta$. Details of the computation of $B(\theta, V)$ or, equivalently, $s(\theta, V)$ for a given potential $U(\varrho)$ will be given in the next section.

If we take into account these remarks, the Boltzmann equation follows in the limit $N \to \infty$, $\sigma \to 0$, $N\sigma^2$ finite, exactly as for the case of rigid spheres, provided $|\mathbf{V} \cdot \mathbf{n}^{(out)}|\, d\mathbf{n}^{(out)} = B(\theta, V)\, d\theta\, d\varepsilon$ replaces $|\mathbf{V} \cdot \mathbf{n}|\, d\mathbf{n}$. Accordingly, the Boltzmann equation for mass points interacting with a central force can be written as follows:

$$\frac{\partial f}{\partial t} + \boldsymbol{\xi} \cdot \frac{\partial f}{\partial \mathbf{x}} = \iiint (f'f'_* - ff_*)\, B(\theta, V)\, d\theta\, d\varepsilon\, d\boldsymbol{\xi}_*. \tag{6.16}$$

7. Details of the collision term

In order to specify completely the right-hand side of the Boltzmann equation:

$$\frac{\partial f}{\partial t}+\boldsymbol{\xi}\cdot\frac{\partial f}{\partial \mathbf{x}}=\frac{1}{m}\int(f'f'_*-ff_*)\,B(\theta,\,V)\,d\theta\,d\varepsilon\,d\boldsymbol{\xi}_* \tag{7.1}$$

we have to find the expression of $B(\theta, V)$ defined by Eq. (6.14). To this end it is necessary to study the two-body problem for a given potential $U(\varrho)$. Let m, m_* be the masses of the two molecules; then it is well known that the relative motion takes place as if one of the molecules (the "target" molecule) were at rest and the other (the "bullet" molecule) had a mass equal to the reduced mass

$$\mu=\frac{mm_*}{m+m_*} \tag{7.2}$$

(in particular, if $m=m_*$, $\mu=m/2$). If ϱ, φ are the radial and angular coordinates in the plane of motion, then conservation of energy and angular momentum (with respect to a pole located at the position of the target molecule) give:

$$\begin{gathered}\tfrac{1}{2}\mu(\dot{\varrho}^2+\varrho^2\dot{\varphi}^2)+U(\varrho)=\tfrac{1}{2}\mu V^2+U(\sigma) \qquad (\varrho\leqslant\sigma),\\ \varrho^2\dot{\varphi}=rV,\end{gathered} \tag{7.3}$$

where r is the impact parameter and V the relative speed; the right-hand sides of these equations are evaluated when the bullet molecule is outside the sphere of interaction and the kinetic energy is constant and equal to $\mu V^2/2$, potential energy is also constant and equal to $U(\sigma)$ and angular momentum equals the product of linear momentum and impact parameter. We could omit $U(\sigma)$ in Eq. (7.3) by stipulating $U(\sigma)=0$ as is always possible; it is more instructive, however, to retain the constant explicitly. Also, we shall restrict our considerations to repulsive potentials, which is the important case for close interactions between molecules in a gas, as we saw before.

Now we can easily integrate the above equations (one can eliminate time derivatives by using φ as independent variable); the orbit is, as we anticipated in Section 6, symmetric with respect to the apse line. The angle θ can easily be evaluated since it is the angle between $\mathbf{V}$ and the apse line (directed along $\mathbf{n}$) and the solution of Eqs (7.3) gives:

$$\theta=\left(\frac{\mu}{2}\right)^{1/2}Vr\int_{\varrho_0}^{\sigma}\varrho^{-2}\left[\frac{\mu}{2}V^2\left(1-\frac{r^2}{\varrho^2}\right)-U(\varrho)+U(\sigma)\right]^{-1/2}d\varrho+\sin^{-1}\left(\frac{r}{\sigma}\right), \tag{7.4}$$

where ϱ_0 is the distance of closest approach which satisfies:

$$\frac{\mu}{2}V^2\left(1-\frac{r^2}{\varrho_0^2}\right)=U(\varrho_0)-U(\sigma). \tag{7.5}$$

We note that $\varrho_0 \leqslant \sigma$ (otherwise no deflection arises, since the molecules do not enter into an interaction); it is also clear that $r \leqslant \sigma$ as follows from $\varrho_0 \leqslant \sigma$ and the assumption of repulsive potential [which implies $U(\varrho_0)-U(\sigma)\geqslant 0$].

What one should now do is to invert Eq. (5.4) to give $r=r(\theta)$ (the assumption of repulsive potential guarantees that $\theta=\theta(r)$ is a monotonic function) and insert it into Eq. (6.14) to obtain $B(\theta, V)$.

It is seen that all the complicated details of the two-body interactions are summarized by the quantity $B(\theta, V)$ giving the (unnormalized) probability density of a relative deflection equal to $\pi-2\theta$ for a pair of molecules having relative speed V. $B(\theta, V)$ cannot be expressed in terms of elementary functions even for such simple potentials as inverse power potentials ($U=\varkappa\varrho^{1-n}$; $n\neq 2, 3$); the cases of inverse-square and inverse-cube force laws are amenable to an analytic treatment, but describe too soft an interaction at small distances to be realistic for a neutral gas. In spite of these negative remarks, it is worth-while to consider the case of power law potentials in more detail. Equation (7.4) becomes

$$\theta=\left(\frac{\mu}{2}\right)^{1/2}Vr\int_{\varrho_0}^{\sigma}\varrho^{-2}\left[\frac{\mu}{2}V^2\left(1-\frac{r^2}{\varrho^2}\right)-\frac{\varkappa}{\varrho^{n-1}}+\frac{\varkappa}{\sigma^{n-1}}\right]^{-1/2}d\varrho+\sin^{-1}\left(\frac{r}{\sigma}\right), \tag{7.6}$$

where ϱ_0 satisfies

$$\frac{\mu}{2}V^2\left(1-\frac{r^2}{\varrho_0^2}\right)-\frac{\varkappa}{\varrho_0^{n-1}}+\frac{\varkappa}{\sigma^{n-1}}=0. \tag{7.7}$$

If we now put

$$\begin{aligned} b&=r\left(\frac{\mu}{2\varkappa}V^2+\frac{\varkappa}{\sigma^{n-1}}\right)^{1/(n-1)},\\ x&=\frac{r}{\varrho}\left(1+\frac{2\varkappa}{\mu V^2\sigma^{n-1}}\right)^{1/2},\\ \lambda&=\frac{r}{\sigma}\left(1+\frac{2\varkappa}{\mu V^2\sigma^{n-1}}\right)^{1/2}. \end{aligned} \tag{7.8}$$

Equation (5.13) becomes

$$\theta=\int_{\lambda}^{x_0}\frac{dx}{\sqrt{1-x^2-(x/b)^{n-1}}}+\sin^{-1}\left(\frac{r}{\sigma}\right), \tag{7.9}$$

where x_0 satisfies

$$1-x_0^2-(x_0/b)^{n-1}=0. \tag{7.10}$$

It is clear that the computation of $\theta=\theta(r, V)$ is a rather complicated task. An essential simplification occurs in the limiting case $\sigma \to \infty$, i.e., when we analyse a many-body interaction as a sequence of grazing binary collisions. Since all the standard work on the computation of viscosity and heat conduction coefficients is based on this assumption, we give the relevant formulas:

$$\theta=\int_0^{x_0} \frac{dx}{\sqrt{1-x^2-(x/b)^{n-1}}}, \tag{7.11}$$

$$b=r\left(\frac{\mu}{2\varkappa}\right)^{1/(n-1)} V^{2/(n-1)}, \tag{7.12}$$

where x_0 satisfies Eq. (7.10). Equations (7.11) and (7.12) give $\theta=\theta(b)$ and, inverting, $b=b(\theta)$. Hence Eq. (7.12) gives

$$r=\left(\frac{2\varkappa}{\mu}\right)^{1/(n-1)} V^{-2/(n-1)} b(\theta). \tag{7.13}$$

Equation (7.13) shows that the dependence of r upon V and θ factorizes and consequently, Eq. (6.14) gives:

$$B(\theta, V)=V^{(n-5)/(n-1)}\left(\frac{2\varkappa}{\mu}\right)^{2/(n-1)} \frac{db}{d\theta}=V^{\gamma} \beta(\theta), \tag{7.14}$$

where $\gamma=(n-5)/(n-1)$ and

$$\beta(\theta)=\left(\frac{2\varkappa}{\mu}\right)^{2/(n-1)} b \frac{db}{d\theta}.$$

The relevant simplification for inverse-power laws without cutoff distance is therefore that $B(\theta, V)$ becomes the product of a function of θ alone times a fractional power of V. A particularly noticeable simplification arises when $n=5$, because then V disappears. This simplification was discovered by Maxwell and the fictious molecules interacting in this way are usually called *Maxwell molecules*. Although actual molecules are not Maxwell molecules, yet the concept is useful, because the assumption of an inverse fifth-power law frequently simplifies the calculations in a considerable fashion and gives satisfactory answers or, at least, first- approximations to satisfactory answers.

We note that $\beta(\theta)$ has the following behavior:

$$\begin{aligned} &\beta(\theta)=O(\theta) && (\theta \to 0), \\ &\beta(\theta)=O\left[(\pi/2-\theta)^{-(n+1)/(n-1)}\right] && (\theta \to \pi/2), \end{aligned} \tag{7.15}$$

where $O(x)$ denotes a quantity of the order of x. The first of these relations is easily obtained by noting that when $\theta \to 0$, $x_0 \to 0$ from Eq. (7.11), and hence $b \simeq x_0 \to 0$ from Eq. (7.10), while Eq. (7.11) becomes

$$\theta = \int_0^b \frac{dx}{\sqrt{1-(x/b)^{n-1}}} = b\int_0^1 \frac{du}{\sqrt{1-u^{n-1}}},$$

i.e., $b \simeq K\theta$ where $K \neq 0$, $b\,db \simeq \kappa^2 \theta\,d\theta$, $\beta(\theta) = O(\theta)$. When $\theta \to \pi/2$, $b \to \infty$ (as is seen by letting $b \to \infty$ in Eq. (7.11)). Hence $x_0^2 \simeq 1-b^{1-n}$, as follows from Eq. (7.10) and:

(7.16)

$$\begin{aligned}\theta &= x_0\int_0^1 \frac{dy}{\sqrt{1-x_0^2 y^2-(x_0/b)^{n-1}y^{n-1}}} \simeq x_0\int_0^1 \frac{dy}{\sqrt{1-y^2+(1/b^{n-1})(y^2-y^{n-1})}}\\ &\simeq x_0\int_0^1 \frac{dy}{\sqrt{1-y^2}}\left[1-\tfrac{1}{2}\frac{1}{b^{n-1}}\left(\frac{y^2-y^{n-1}}{1-y^2}\right)\right]\\ &\simeq \frac{\pi}{2}-\frac{\pi}{2}\tfrac{1}{2}\frac{1}{b^{n-1}}-\tfrac{1}{2}\frac{1}{b^{n-1}}\int_0^1 \frac{y^2(1-y^{n-3})}{(1-y^2)^{3/2}}\,dy\\ &= \frac{\pi}{2}+O\left(\frac{1}{b^{n-1}}\right) \quad (n>3).\end{aligned}$$

Hence

$$b = O\left[\left(\frac{\pi}{2}-\theta\right)^{-1/(n+1)}\right], \quad b\frac{db}{d\theta} = O\left[\left(\frac{\pi}{2}-\theta\right)^{-(n+1)/(n-1)}\right], \tag{7.17}$$

and Eq. (7.23) follows.

8. Elementary properties of the collision operator. Collision invariants

The right-hand side of Eq. (7.1) contains a quadratic expression $Q(f,f)$ defined by:

$$Q(f,f) = \frac{1}{m}\int (f'f'_* - ff_*)B(\theta, V)\,d\xi_1\,d\varepsilon\,d\theta. \tag{8.1}$$

The operator Q acts on the velocity-dependence of f; it describes the effect of interactions, and is accordingly called the *collision operator*. $Q(f,f)$, i.e., the integral in Eq. (8.1), is called the *collision integral* or, simply, the *collision term*. In this section we shall study some properties which make the manipulation of Q possible in many problems of basic character in spite of its complicated form. Actually, we shall study here a slightly more general expression, the bilinear quantity

$$Q(f,g)=\frac{1}{2m}\int(f'g'_*+f'_*g'-fg_*-f_*g)\,B(\theta,V)\,d\xi_*\,d\varepsilon\,d\theta. \tag{8.2}$$

It is clear that when $g=f$, Eq. (8.2) reduces to Eq. (8.1); in addition,

$$Q(f,g)=Q(g,f). \tag{8.3}$$

Our first aim is to study some manipulations of the eightfold integral

$$\int Q(f,g)\,\varphi(\xi)\,d\xi=\frac{1}{2m}\int(f'g'_*+f'_*g'-fg_*-f_*g)\,\varphi(\xi)\,B(\theta,V)\,d\xi\,d\xi_*\,d\theta\,d\varepsilon, \tag{8.4}$$

where the integrals with respect to ξ are extended to the whole velocity space and $\varphi(\xi)$ is any function of ξ such that the indicated integrals exist.

We now perform the interchange of variables $\xi\to\xi_*$, $\xi_*\to\xi$ (which implies also $\xi'\to\xi'_*$, $\xi'_*\to\xi'$ because of Eqs (4.11)). Then, since both $B(\theta,V)$ and the quantity within parentheses transform into themselves, and the Jacobian of the transformation is obviously unit, we have

$$\int Q(f,g)\,\varphi(\xi)\,d\xi=\frac{1}{2m}\int(f'g'_*+f'_*g'-fg_*-f_*g)\,\varphi(\xi_*)\,B(\theta,V)\,d\xi\,d\xi_*\,d\theta\,d\varepsilon. \tag{8.5}$$

This equation is identical to Eq. (8.4) except for having $\varphi(\xi_*)$ in place of $\varphi(\xi)$. Now we consider another transformation of variables in Eq. (8.4): $\xi\to\xi'$ and $\xi_*\to\xi'_*$ (here, as above, the unit vector $\mathbf{n}$ in Eq. (3.11) is considered as fixed). As we know (Sect. 3), the absolute value of the Jacobian of this transformation is unity, i.e., $d\xi\,d\xi_*=d\xi'\,d\xi'_*$ and Eq. (8.5) becomes

$$\int Q(f,g)\,\varphi(\xi)\,d\xi=\frac{1}{2m}\int(f'g'_*+f'_*g'-fg_*-f_*g)\,\varphi(\xi)\,B(\theta,V)\,d\xi'\,d\xi'_*\,d\theta\,d\varepsilon, \tag{8.6}$$

where now, since ξ' and ξ'_* are integration variables, we must express ξ and ξ_* by means of the relations inverting Eqs. (4.12), which are

$$\xi=\xi'-\mathbf{n}(\mathbf{n}\cdot\mathbf{V}'),\qquad \xi_*=\xi'_*+\mathbf{n}(\mathbf{n}\cdot\mathbf{V}'), \tag{8.7}$$

where $\mathbf{V}'=\xi'-\xi'_*$ is related to $\mathbf{V}=\xi-\xi_*$ by Eq. (3.7) and consequently

$$\mathbf{V}'\cdot\mathbf{n}=-\mathbf{V}\cdot\mathbf{n} \tag{8.8}$$

and the hemisphere $\mathbf{V}\cdot\mathbf{n}>0$ corresponds to $\mathbf{V}'\cdot\mathbf{n}<0$; we may change, however, $\mathbf{n}$ into $-\mathbf{n}$, without altering the expressions of ξ, ξ_* and integrate over the hemisphere $\mathbf{V}'\cdot\mathbf{n}>0$. We can also change the names of integration variables and call ξ, ξ_* what we called ξ', ξ'_* before. Then, because of Eqs (4.12) and (8.7), we can consistently call ξ' and ξ'_* what we called ξ and ξ_* before, and write Eq. (8.8) as follows:

(8.9)
$$\int Q(f, g)\varphi(\xi)\,d\xi = \frac{1}{2m}\int (fg_* + f_* g - f'g'_* - f'_* g')\varphi(\xi')B(\theta, V)\,d\xi\,d\xi_*\,d\theta\,d\varepsilon$$

where $B(\theta, V)$ is not affected by the change, since Eq. (4.12) implies $V' = V$. We can rewrite Eq. (8.9) as follows:

$$(8.10)\quad \int Q(f, g)\varphi(\xi)\,d\xi$$
$$= -\frac{1}{2m}\int (f'g'_* + f'_* g' - fg_* - f_* g)\varphi(\xi')B(\theta, V)\,d\xi\,d\xi_*\,d\theta\,d\varepsilon.$$

This equation is identical to Eq. (6.4) except for a minus sign and having $\varphi(\xi')$ in place of $\varphi(\xi)$.

Finally, let us interchange ξ and ξ_* in Eq. (8.10) as we did in Eq. (8.4) to obtain Eq. (8.5). The result is:

(8.11)
$$\int Q(f, g)\varphi(\xi)\,d\xi = -\frac{1}{2m}\int (f'g'_* + f'_* g' - fg_* - f_* g)\varphi(\xi'_*)B(\theta, V)\,d\xi\,d\xi_*\,d\theta\,d\varepsilon$$

which is identical to Eq. (8.4) except for a minus sign and for having $\varphi(\xi'_*)$ in place of $\varphi(\xi)$.

We have thus obtained four different expressions for the same quantity: Eqs (8.4), (8.5), (8.10), (8.11). We can now obtain more expressions by taking appropriate linear combinations of the four basic ones; we are particularly interested in the combination which is obtained by adding the above four expressions and dividing by four. The result is:

$$(8.12)\quad \int Q(f, g)\varphi(\xi)\,d\xi$$
$$= \frac{1}{8m}\int (f'g'_* + f'_* g' - fg_* - f_* g)(\varphi + \varphi_* - \varphi' - \varphi'_*)B(\theta, V)\,d\xi\,d\xi_*\,d\theta\,d\varepsilon.$$

This equation expresses a basic property of the collision term, which will be frequently used in the following. In the particular case of $g = f$ Eq. (8.12) reads:

(8.13)
$$\int Q(f, f)\varphi(\xi)\,d\xi = \frac{1}{4m}\int (f'f'_* - ff_*)(\varphi + \varphi_* - \varphi' - \varphi'_*)B(\theta, V)\,d\xi\,d\xi_*\,d\theta\,d\varepsilon.$$

We now observe that the integral appearing in Eq. (8.12) is zero, independent of the particular f and g, if

$$\varphi+\varphi_* = \varphi'+\varphi'_* \tag{8.14}$$

is valid almost everywhere in velocity space. Since the integral appearing in the left-hand side of Eq. (6.11) is the average change of the function $\varphi(\boldsymbol{\xi})$ in unit time by the effect of the collisions, the functions satisfying Eq. (8.14) are usually called "collision invariants". We now have the property that, if $\varphi(\boldsymbol{\xi})$ is assumed to be continuous, then Eq. (8.14) is satisfied if and only if

$$\varphi(\boldsymbol{\xi}) = a+\mathbf{b}\cdot\boldsymbol{\xi}+c\xi^2 \tag{8.15}$$

where a and c are constant scalars and $\mathbf{b}$ a constant vector. The functions $\psi_0 = 1$, $(\psi_1, \psi_2, \psi_3) = \boldsymbol{\xi}$, $\psi_4 = \xi^2$ are usually called the *elementary collision invariants*; thus a general collision invariant is a linear combination of the five ψ's.

9. Solution of the equation $Q(f,f) = 0$

In this section we investigate the existence of positive functions f which give a vanishing collision integral:

$$Q(f,f) = \int (f'f'_* - ff_*) B(\theta, V)\, d\boldsymbol{\xi}_*\, d\theta\, d\varepsilon = 0. \tag{9.1}$$

We want to show that such functions exist and are all given by

$$f(\boldsymbol{\xi}) = \exp(a+\mathbf{b}\cdot\boldsymbol{\xi}+c\xi^2), \tag{9.2}$$

where a, $\mathbf{b}$, c have the same meaning as in Eq. (6.15). In order to show that this statement is true, we prove a preliminary result which will also be important later, i.e., that no matter what the distribution function is, the following inequality (Boltzmann's inequality) holds:

$$\int \log f\, Q(f,f)\, d\boldsymbol{\xi} \leqslant 0, \tag{9.3}$$

and the equality sign applies if and only if f is given by Eq. (9.2). Now it is seen that the first statement is a simple corollary of the second one: in fact, if Eq. (9.1) is satisfied then multiplying it by $\log f$ and integrating gives Eq. (9.3) with the equality sign, which implies Eq. (9.2) if the second statement applies. *Vice versa*, if Eq. (9.2) holds, then, because of the results of the previous section applied to $\varphi = \log f$, $f'f'_* = ff_*$ and Eq. (9.1) is satisfied.

Let us prove, therefore, that Eq. (7.3) always holds for $f > 0$ and the equality sign implies, and is implied by, Eq. (9.2). If we use Eq. (8.13) with $\varphi = \log f$ we have

$$\begin{aligned} \int \log f\, Q(f,f)\, d\boldsymbol{\xi} &= \frac{1}{4m}\int (f'f'_* - ff_*) \log (ff_*/f'f'_*)\, B(\theta, V)\, d\boldsymbol{\xi}\, d\boldsymbol{\xi}_*\, d\theta\, d\varepsilon \\ &= \frac{1}{4m}\int f'f'_*(1-\lambda) \log \lambda\, B(\theta, V)\, d\boldsymbol{\xi}\, d\boldsymbol{\xi}_*\, d\theta\, d\varepsilon, \end{aligned} \tag{9.4}$$

where

$$\lambda = ff_*/(f'f'_*). \tag{9.5}$$

Now $f'f'_* > 0$, $B \geqslant 0$ (the equality sign applying only at $\theta = 0$); also, for any $\lambda \geqslant 0$ we have

$$(1-\lambda)\log\lambda \leqslant 0. \tag{9.6}$$

and the equality sign applies if and only if $\lambda = 1$ (note that $(1-\lambda)$ and $-\log\lambda$ are negative and positive together and both are zero if and only if $\lambda = 1$). If we use Eq. (9.6), Eq. (9.4) implies Eq. (9.3) and the equality sign applies if and only if $\lambda = 1$, i.e.

$$ff_* = f'f'_* \tag{9.7}$$

applies almost everywhere. But taking the logarithms of both sides of this equation, we obtain that $\varphi = \log f$ satisfies Eq. (8.14), i.e., $\varphi = \log f$ is given by Eq. (8.15); hence f is given by Eq. (9.2), as was to be shown.

We note that in Eq. (9.2) c must be negative since f must be integrable over the whole velocity space. If we put $c = -\alpha$, $\mathbf{b} = 2\alpha\mathbf{v}$, where $\mathbf{v}$ is another constant vector, Eq. (9.2) can be written as follows:

$$f(\boldsymbol{\xi}) = A\exp\left[-\alpha(\boldsymbol{\xi}-\mathbf{v})^2\right], \tag{9.8}$$

where A is a constant related to a, α, v^2 (α, $\mathbf{v}$, A constitute a new set of arbitrary constants). Equation (9.8) is the familiar Maxwellian distribution.

Chapter II

LINEAR TRANSPORT

1. The linearized collision operator

Because of the nonlinear nature of the collision term, the Boltzmann equation is very difficult to solve and to analyse. In Chapter I, Section 9 we met a very particular class of solutions, i.e., Maxwellians. The meaning of a Maxwellian distribution is clear: it describes equilibrium states (or slight generalizations of them, characterized by the fact that neither heat flux nor stresses other than isotropic pressure are present). If we want to describe more realistic nonequilibrium situations, when oblique stresses are present and heat transfers take places, we have to rely upon approximation methods.

Some of the most useful methods of solution are based upon perturbation techniques: we choose a paramater ε which can be small in some situations and expand f in a series of powers of ε [or, more generally, of functions $\sigma_n(\varepsilon)$, such that $\lim_{\varepsilon\to 0}\sigma_{n+1}(\varepsilon)/\sigma_n(\varepsilon) = 0$]. The resulting expansion, which in general cannot be expected to be convergent but only asymptotic to a solution of the Boltzmann equation, gives useful information for a certain range of small values of ε (sometimes larger than would be expected).

There are many different perturbation methods correspondings to different choices of ε; however, in this section we want to study the general features of any perturbation method with respect to the collision operator $Q(f,f)$. We shall restrict ourselves to power series in ε:

$$f = \sum_{n=0}^{\infty} \varepsilon^n f_n. \tag{1.1}$$

If we insert this expansion into $Q(f,f)$ and take into account both the quadratic nature of the collision operator and the Cauchy rule for the product of two series, we find

$$Q(f,f) = \sum_{n=0}^{\infty} \varepsilon^n \sum_{k=0}^{n} Q(f_k, f_{n-k}), \tag{1.2}$$

where $Q(f, g)$ is the bilinear operator defined by Eq. (I.8.2). The presence of a symmetrized expression is related to the fact that we can combine the terms with $k = k_0$ and $k = n-k_0$ (for any k_0, $0 \leqslant k_0 \leqslant n$).

Equation (1.2) shows that expanding f into a power series in the parameter ε implies an analogous expansion of the collision term the coefficients being as follows:

$$Q_n = \sum_{k=0}^{n} Q(f_k, f_{n-k}). \tag{1.3}$$

A significant number of perturbation expansions which are used in connection with the Boltzmann equation have the following feature: either as a consequence of the zeroth-order equation, or because of the assumptions underlying the perturbation method, the zeroth-order term in the expansion is a Maxwellian. We shall restrict our attention to this case at present: we note, however, that the parameters appearing in the Maxwellian (density, mass velocity, temperature) can depend arbitrarily upon the time and space variables (the Maxwellian is, in general, not required to satisfy the Boltzmann equation), but this will not concern us insofar as we deal with the collision operator which does not act on the space-time dependence of f.

Under the present assumptions we have (Chapter I, Sect. 9):

$$Q(f_0, f_0) = 0, \quad \text{i.e.,} \quad Q_0 = 0. \tag{1.4}$$

We observe now that Q_n $(n \geqslant 1)$ as given by Eq. (1.3) splits as follows:

$$Q_n = 2Q(f_0, f_n) + \sum_{k=1}^{n-1} Q(f_k, f_{n-k}) \quad (n \geqslant 1), \tag{1.5}$$

where the first term arises from $k = 0$, $k = n$ in Eq. (1.3) and the second term involves f_k with $k < n$ and is accordingly known at the nth step of perturbation procedure (in particular, it is zero for $n = 1$). As a consequence, the relevant operator to be considered is the linear operator $2Q(f_0, f_n)$ acting upon the unknown f_n, while the remainder can be written as a source term, say $f_0 S_n$. It is usual to put $f_n = f_0 h_n$ and consider h_n as unknown; then we can write Eq. (1.5) as follows:

$$Q_n = f_n L h_n + f_0 S_n \tag{1.6}$$

where, by definition, the *linearized collision operator* L is given by

$$Lh = 2f_0^{-1} Q(f_0, f_0 h) = \frac{1}{m}\int f_{0*}(h'_* + h' - h_* - h) B(\theta, V)\, d\xi_*\, d\varepsilon\, d\theta. \tag{1.7}$$

In order to obtain the second expression of Lh in Eq. (1.7) we used Eq. (I.8.2) and the fact that f_0, being Maxwellian, satisfies Eq. (I.9.7).

We can use now the properties of $Q(f, g)$ and the definition of L to deduce some basic properties of the latter. If we consider Eq. (I.8.12) with $f = f_0$, $g = f_0 h$ and $\varphi = \bar{g}$, we have:

$$\begin{aligned} \int f_0 \bar{g} L h\, d\xi &= 2\int \bar{g} Q(f_0, f_0 h)\, d\xi \\ &= -\frac{1}{4m}\int f_0 f_{0*}(h'_* + h' - h_* - h)(\bar{g}'_* + \bar{g}' - \bar{g} - \bar{g}_*) B(\theta, V)\, d\xi_*\, d\xi\, d\varepsilon\, d\theta. \end{aligned} \tag{1.8}$$

The last expression shows that interchange of h and g changes the integral into its complex conjugate (hence leaves the integral unchanged if g and h are real valued). This suggests the introduction of a Hilbert space $\mathscr{H}$ where the scalar product (g, h) and the norm $\|h\|$ are defined by

$$(g, h) = \int f_0(\xi) \bar{g}(\xi) h(\xi)\, d\xi; \qquad \|h\|^2 = (h, h). \tag{1.9}$$

Equation (1.8) then shows that

$$(g, Lh) = (Lg, h), \tag{1.10}$$

i.e., L is self-adjoint (here both g and h belong the domain of the operator L). If we put $g = h$ in Eq. (1.8) we obtain

$$(h, Lh) = -\frac{1}{4m}\int |h'_* + h' - h_* - h|^2 B(\theta, V)\, d\xi\, d\xi_*\, d\varepsilon\, d\theta \tag{1.11}$$

and since $B(\theta, V) > 0$, we have

$$(h, Lh) \leqslant 0 \tag{1.12}$$

and the equality sign holds if and only if the quantity which appears squared in Eq. (1.11) is zero, i.e., if h is a collision invariant. Eq. (1.11) says that L is a nonpositive operator in $\mathscr{H}$. We note that when the equality sign holds in Eq. (1.11), i.e., h is a collision invariant, then Eq. (1.7) gives

$$Lh = 0 \tag{1.13}$$

and, conversely, if we scalarly multiply this equation by h, according to the scalar product in $\mathscr{H}$ defined above, we obtain

$$(h, Lh) = 0 \tag{1.14}$$

which implies that h is a collision invariant. Therefore the collision invariants ψ_α are eigenfunctions of L corresponding to the eigenvalue $\lambda = 0$, and are the only eigenfunctions corresponding to such eigenvalue; all the other eigenvalues, if any, must be negative, because of Eq. (1.12). It can be verified that the latter equation is the linearized version of Eq. (I.9.3); in fact, if we write $f = f_0(1+h)$ in the latter equation and neglect terms higher than second (zeroth- and first-order terms cancel), we obtain Eq. (1.12). We note a further property of L which is a trivial consequence of the fact that the ψ_α's satisfy:

$$L\psi_\alpha = 0 \quad (\alpha = 0, 1, 2, 3, 4) \tag{1.15}$$

and of Eq. (1.10) (with $g = \psi_\alpha$):

$$(\psi_\alpha, Lh) = 0 \tag{1.16}$$

which is the linearized version of the following consequence of Eq. (I.8.13)

$$\int \psi_\alpha Q(f, f) = 0. \tag{1.17}$$

2. The linearized Boltzmann equation

As we mentioned in Section 1, a significant number of perturbation expansions which are used in connection with the Boltzmann equation have the form shown in Eq. (1.1). The result of inserting such expansion into the Boltzmann equation

$$\frac{\partial f}{\partial t} + \boldsymbol{\xi} \cdot \frac{\partial f}{\partial \mathbf{x}} = Q(f, f) \tag{2.1}$$

depends upon the meaning of ε. If ε does not appear directly in Eq. (2.1), then we must equate the coefficients of the various powers to obtain

$$\frac{\partial f_0}{\partial t} + \boldsymbol{\xi} \cdot \frac{\partial f_0}{\partial \mathbf{x}} = Q(f_0, f_0), \tag{2.2}$$

$$\frac{\partial f_n}{\partial t} + \boldsymbol{\xi} \cdot \frac{\partial f_n}{\partial \mathbf{x}} = Q_n, \tag{2.3}$$

where Q_n is given by Eq. (1.3). Equation (2.3) shows that f_0 must be a solution of the Boltzmann equation. Since we do not know any solutions except Maxwellians (with some irrelevant exceptions), we are practically forced to choose f_0 to be a Maxwellian; otherwise, making the zeroth-order step of the approximation procedure would be as hard as solving the original equation. Although there are Maxwellians with variable density, velocity and temperature which solve the Boltzmann equation, they are of limited use; accordingly, we shall choose our Maxwellian f_0 to have constant parameters. This choice is sufficiently broad for our purposes. Physically, it means that we study a situation in which there is little departure from an overall equilibrium.

We can put $f_n = f_0 h_n$ $(n \geqslant 1)$ and write Eq. (2.3) as follows

$$\frac{\partial h_n}{\partial t} + \boldsymbol{\xi} \cdot \frac{\partial h_n}{\partial \mathbf{x}} = L h_n + S_n \tag{2.4}$$

where, according to Eqs. (1.5) and (1.6),

$$S_1 = 0, \qquad S_n = f_0^{-1} \sum_{k=1}^{n-1} Q(f_0 h_k, f_0 h_{n-k}). \tag{2.5}$$

The sequence of Eqs (2.4) describes a successive approximation procedure for solving the Boltzmann equation. What is interesting is that at each step we have to solve the same equation, the only change being in the source term, which has to be evaluated in terms of the previous approximations. The equations to be solved involve a complicated integrodifferential operator and have a shape almost as complicated as that of the original Boltzmann equation, except for the fact that we got rid of the nonlinearity. The fact that the same operator appears at each step allows us to concentrate on the first step, i.e., to study the following equation:

$$\frac{\partial h}{\partial t} + \boldsymbol{\xi} \cdot \frac{\partial h}{\partial \mathbf{x}} = L h, \tag{2.6}$$

which is called the *linearized Boltzmann equation.*

The presence of a source term in the subsequent steps is hardly a complication in solving the equations, since well-known procedures allow us to solve an inhomogeneous linear equation once we are able to master the corresponding homogeneous equation. In practice, however, one usually makes only the first step, i.e., solves the linearized Boltzmann equation in place of the nonlinear one.

The study of the linearized Boltzmann equation is important for at least two reasons:

1. There are conditions (to be specified below) under which the results obtained from the linearized Boltzmann equation can be retained to faithfully represent the physical situation.

2. The fact that the linearized equation has the same structure (except for the nonlinearity in the collision term) as the full Boltzmann equation suggests that we can obtain a valuable insight into the features of the solutions of the full Boltzmann equation by studying the linearized one; these features are obviously not those related to nonlinear effects, but, e.g., those related to the behavior in the proximity of boundaries, for which the nonlinear nature of the collisions is expected to have little influence.

We now have to specify the conditions under which one can make use of the linearized Boltzmann equation to obtain physically significant results. Since the parameter ε was assumed not to appear in the Boltzmann equation, we must examine the initial and boundary conditions. Since we look for a solution in the form $f = f_0(1+h)$ with the condition that h can be regarded, in some sense, as a small quantity with respect to 1, a necessary condition is that h is small for $t = 0$ and at the boundaries.

As a consequence, a first condition is that the initial datum shows little departure from the basic Maxwellian f_0; this does not necessarily mean that h is small everywhere for $t = 0$, but that e.g., $\|h\| \ll 1$, where $\|h\|$ is given by Eq. (1.9). In particular, if $\bar{\varrho}$, $\bar{v}$, $\bar{T}$ are the initial density, velocity and temperature, ϱ_0, v_0, T_0 the corresponding parameters in f_0, the quantities $|\bar{\varrho}-\varrho_0|/\varrho_0$, $|\bar{T}-T_0|/T_0$, $|\bar{v}-v_0|/(RT_0)^{1/2}$ must be small with respect to 1.

The situation is similar but less obvious when we examine the boundary conditions if we put

$$f = f_0(1+h). \tag{2.7}$$

3. Model equations

One of the major shortcomings in dealing with the Boltzmann equation is the complicated structure of the collision integral, Eq. (3.1).

It is therefore not surprising that alternative, simpler expressions have been proposed for the collision term; they are known as *collision models*, and any Boltzmann-like equation where the Boltzmann collision integral is replaced by a collision model is called a *model equation* or a *kinetic model*.

The idea behind this replacement is that a large amount of detail of the two-body interaction (which is contained in the collision term) is not likely to influence significantly the values of many experimentally measured quantities; i.e., unless very refined experiments are devised, it is expected that the fine structure of the collision operator $Q(f,f)$ can be replaced by a blurred image, based upon a simpler operator $J(f)$ which retains only the qualitative and average properties of the true collision operator.

The most widely known collision model is usually called the *Bhatnagar, Gross and Krook* (BGK) *model*, although Welander proposed it independently at about the same time as the above-mentioned authors. The idea behind the BGK model (retained by more sophisticated models) is that the essential features of a collision operator are:

1) The true collision term $Q(f,f)$ satisfies Eq. (1.17); hence the collision model $J(f)$ must satisfy

$$\int \psi_\alpha J(f)\,d\xi = 0 \quad (\alpha = 0, 1, 2, 3, 4). \tag{3.1}$$

2) The collision term satisfies Eq. (7.3). Hence $J(f)$ must satisfy

$$\int \log f J(f)\,d\xi \leqslant 0; \tag{3.2}$$

equality holding if and only if f is a Maxwellian.

As shown in papers [2] and [3], this second property expresses the tendency of the gas to a Maxwellian distribution. The simplest way of taking this feature into account seems to assume that the average effect of collisions is to change the distribution function $f(\xi)$ by an amount proportional to the departure of f from a Maxwellian $\Phi(\xi)$. That is, if ν is a constant with respect to ξ, we introduce the following collision model

$$J(f) = \nu[\Phi(\xi) - f(\xi)]. \tag{3.3}$$

The Maxwellian $\Phi(\xi)$ has five disposable scalar parameters (ϱ, $\mathbf{v}$, T) according to Eq. (9.8); however, these are fixed by Eq. (3.1) which implies

$$\int \psi_\alpha \Phi(\xi)\,d\xi = \int \psi_\alpha f(\xi)\,d\xi, \tag{3.4}$$

i.e., at any space point and time instant $\Phi(\xi)$ must have exactly the density, velocity and temperature of the gas, given by the distribution function $f(\xi)$. Since the latter will in general vary with both time and space coordinates, the same will be true for the parameters of $\Phi(\xi)$ which is accordingly called the local Maxwellian. The "collision frequency" ν is not restricted at this level and has to be fixed by means of additional considerations; we note, however, that ν can be a function of the local state of the gas and hence vary with both time and space coordinates.

We still have to prove that the BGK model satisfies Eq. (3.2) and equality applies if and only if f is a Maxwellian. We find

$$\begin{aligned} \int \log f J(f)\,d\xi &= \int \log (f/\Phi) J(f)\,d\xi + \int \log \Phi J(f)\,d\xi \\ &= \int \nu\Phi[(1 - f/\Phi)\log(f/\Phi)]\,d\xi, \end{aligned} \tag{3.5}$$

where the integral involving $\log \Phi$ is zero, because the latter quantity is a linear combination of the ψ_α's and Eq. (3.1) applies. Equation (I.9.6) with $\lambda = f/\Phi$ then shows that the last integral in Eq. (3.5) is nonpositive and equal to zero if and only if $f = \Phi$, i.e., if and only if f is a Maxwellian, as required.

We observe that the nonlinearity of the proposed $J(f)$ is much worse than the nonlinearity of the collision term $Q(f,f)$; in fact, the latter is simply quadratic in f, while the former contains f in both the numerator and the denominator of an exponential (the $\mathbf{v}$ and α appearing in Φ are functionals of f, defined by Eq. (3.4)).

The main advantage in using the BGK collision term is that for any given problem one can deduce integral equations for the macroscopic variables A, $\mathbf{v}$, α; these equations are strongly nonlinear, but simplify some iteration procedures and make the treatment of interesting problems feasible on a high speed computer. Another advantage of the BGK model is offered by its linearized form (see next Section and Chapter III).

The BGK model contains the most basic features of the Boltzmann collision integral, but presents some shortcomings. Some of them can be avoided by suitable modifications, at the expense, however, of the simplicity of the model. A first modification can be introducted in order to allow the collision frequency to depend on the molecular velocity instead of being locally constant; this modification is suggested by the circumstance that a computation of the collision frequency for physical models of the molecules (rigid spheres, finite range potentials) shows that ν varies with the molecular velocity and this variation is expected to be important at high molecular velocities. Formally the modification is quite simple; we have only to allow ν to depend on $\boldsymbol{\xi}$ (more precisely on the magnitude c of the random velocity $\mathbf{c} = \boldsymbol{\xi} - \mathbf{v}$), while requiring that Eq. (3.1) still holds. All the basic formal properties, including Eq. (3.2), are retained, but the density, velocity and temperature which now appear in the Maxwellian Φ are not the local density, velocity and temperature of the gas, but some fictious local parameters related to five functionals of f; this follows from the fact that Eq. (3.1) now gives

$$\int \nu(c)\psi_\alpha \Phi \, d\boldsymbol{\xi} = \int \nu(c)\psi_\alpha f \, d\boldsymbol{\xi} \tag{3.6}$$

instead of Eq. (3.4).

A different kind of correction to the BGK model is obtained when we want to adjust the model to give the same Navier–Stokes equations as the full Boltzmann equation; in fact, the BGK model gives the value Pr = 1 for the Prandtl number, a value which is not in agreement with both the true Boltzmann equation and the experimental data for a monatomic gas (which agree in giving Pr $\simeq$ 2/3). In order to have a correct Prandtl number, a further adjustable parameter is required beside the already available ν; accordingly one is led to generalize the BGK model by substituting a local anisotropic threedimensional Gaussian in place of the local Maxwellian (which is an isotropic Gaussian):

$$\begin{gathered} \Phi = \varrho\pi^{-3/2}(\det \mathbf{A})^{1/2} \exp\Big[-\sum_{i,j=1}^{3} A_{ij}(\xi_j - v_j)(\xi_i - \mathrm{v}_i)\Big], \\ \mathbf{A} = \|A_{ij}\| = \|(2RT/\mathrm{Pr})\delta_{ij} - 2(1-\mathrm{Pr})\,p_{ij}/(\varrho\ \mathrm{Pr})\|^{-1}. \end{gathered} \tag{3.7}$$

If we let $\Pr = 1$, we recover the BGK model. A disadvantage of this model (called the ES, or *ellipsoidal statistical model*) is that it has not been possible to prove (or disprove) that Eq. (3.2) holds. Other models with different choices of Φ have been proposed but they are not interesting, except for linearized problems (see Chapter III), because they are extremely complicated from the point of view of obtaining solutions.

4. Linearized kinetic models

In the previous section we discussed the possibility of replacing the collision term in the Boltzmann equation by a simpler expression, called a *collision model*. The idea was that the large amount of detail of the two body interaction, contained in the collision term and reflected, e.g., in the details of the spectrum of the linearized operator is not likely to influence significantly the values of many experimentally measured quantities. This led to the BGK model and the variants which were discussed previously.

The same possibility arises in connection with the linearized and linear Boltzmann equations, for which satisfactory and systematic methods for constructing models have been devised.

The simplest model for the linearized collision operator is obtained by linearizing the BGK model. If we let $f = f_0\,(1+h)$ in Eq. (3.3) and neglect powers of h higher than first, the linearized BGK model turns out to be given by

$$Lh = \nu\Big[\sum_{\alpha=0}^{4} \psi_\alpha(\psi_\alpha, h) - h\Big] \tag{4.1}$$

where the collision invariants ψ_α are normalized in such a way that

$$(\psi_\alpha, \psi_\beta) = \delta_{\alpha\beta} \quad (\alpha, \beta = 0, 1, 2, 3, 4) \tag{4.2}$$

and (,) denotes the scalar product in $\mathscr{H}$, defined by Eq. (1.9). It is obvious that Eq. (4.1) exhibits an operator with a structure definitely simpler than the true linearized operator.

We remark that Eq. (4.1) can also be written as follows:

$$Lh = \nu(\Pi h - h) = -\nu(I-\Pi)h \tag{4.3}$$

where Π is the projection operator onto the five dimensional space $\mathscr{F}$ spanned by the ψ's, and I the identity operator (accordingly, $I-\Pi$ is the projector onto $\mathscr{W}$, the orthogonal complement of $\mathscr{F}$ in $\mathscr{H}$). Equation (4.1), or (4.2), implies that the properties expressed by Eqs (1.10), (1.12), (1.15) and, consequently, (1.16) are satisfied by the linearized BGK operator. This is almost evident for Eqs (1.10), (1.15) and (1.16) while Eq. (1.12) follows from Eq. (4.2), by noticing that

$$(h, Lh) = -\nu(h, (I-\Pi)h) = -\nu\,\|(I-\Pi)h\|^2 \leqslant 0,$$

where equality obviously holds if and only if $(I-\Pi)h = 0$ or $h = \sum_\alpha c_\alpha \psi_\alpha$.

A systematic procedure for improving the linearized BGK model and characterizing the latter as the first step in a hierarchy of models approximating the collision operator for Maxwell molecules with arbitrary accuracy (in a suitable norm) was proposed by Gross and Jackson. The idea is to start from and expansion of h into a series of eigenfunctions of the collision operator for Maxwell molecules, which form a complete set of orthogonal functions:

$$h = \sum_{\alpha=0}^{\infty} \psi_\alpha(\psi_\alpha, h), \tag{4.4}$$

where α is a single label which summarizes the triplet (n, l, m) in such a way that the collision invariants ψ_α correspond to $\alpha = 0, 1, 2, 3, 4$. Then, since, for Maxwell molecules, $L\psi_\alpha = \lambda_\alpha \psi_\alpha$ by definition,

$$Lh = \sum_{\alpha=0}^{\infty} L\psi_\alpha(\psi_\alpha, h) = \sum_{\alpha=0}^{\infty} \lambda_\alpha \psi_\alpha(\psi_\alpha, h). \tag{4.5}$$

A systematic procedure for approximating L consist in partially destroying the fine structure of the spectrum of L by collapsing all the eigenvalues corresponding to $\alpha > N$ into a single eigenvalue, which we shall denote by $-\nu_N$ (remember that $\lambda_\alpha \leqslant 0$).

This amounts to replacing L by an approximate operator L_N defined as follows:

$$L_N h = \sum_{\alpha=0}^{N} \lambda_\alpha \psi_\alpha(\psi_\alpha, h) - \nu_N \sum_{\alpha=N+1}^{\infty} \psi_\alpha(\psi_\alpha, h). \tag{4.6}$$

Equation (4.4) gives

$$\sum_{\alpha=N+1}^{\infty} \psi_\alpha(\psi_\alpha, h) = h - \sum_{\alpha=0}^{N} \psi_\alpha(\psi_\alpha, h) \tag{4.7}$$

and Eq. (4.6) becomes

$$L_N h = \sum_{\alpha=0}^{N} (\lambda_\alpha + \nu_N)\psi_\alpha(\psi_\alpha, h) - \nu_N h. \tag{4.8}$$

In particular, if $N = 4$, $\lambda_\alpha = 0$ for $0 \leqslant \alpha \leqslant N$ and, consequently, Eq. (4.8) reduces to Eq. (4.1) (with $\nu_4 = \nu$); by taking N larger and larger, we include more and more details of the spectrum of L into the model. If we take $N = 9$ by including the five eigenfunctions corresponding to $n = 0$, $l = 2$ in Eq. (4.10), we obtain the linearized version of the ES model, Eq. (3.7).

The above procedure applies only to the case of Maxwell's molecules. However, a slight generalization of the expansion (4.5) is capable of producing collision models in correspondence with any kind of linearized collision operator. In fact, nothing prevents using Eq. (4.4) even if the ψ_α (eigenfunctions of the Maxwell collision operator) are not the eigenfunctions of the operator under consideration. Applying L to both sides of Eq. (9.4) gives

$$Lh = \sum_{\beta=0}^{\infty} L\psi_\beta(\psi_\beta, h) \tag{4.9}$$

but we cannot perform the second step in Eq. (4.5). We can, however, expand the right-hand side of Eq. (4.9) into a series of the ψ_α and obtain

$$Lh = \sum_{\alpha,\beta=0}^{N} \lambda_{\alpha\beta}(\psi_\beta, h)\psi_\alpha, \tag{4.10}$$

where

$$\lambda_{\alpha\beta} = (\psi_\alpha, L\psi_\beta) = \lambda_{\beta\alpha}. \tag{4.11}$$

Equation (4.10) generalizes Eq. (4.5) and reduces to the latter when $\lambda_{\alpha\beta} = \lambda_\alpha \delta_{\alpha\beta}$. If we now introduce the approximation $\lambda_{\alpha\beta} = -\nu_N \delta_{\alpha\beta}$ for $\alpha, \beta > N$, we obtain the model

$$L_N h = \sum_{\alpha,\beta=0}^{N} (\nu_N \delta_{\alpha\beta} + \lambda_{\alpha\beta})\psi_\alpha(\psi_\beta, h) - \nu_N h \tag{4.12}$$

which generalizes Eq. (4.8) to operators other than Maxwell's. Taking $N = 4$ gives the BGK model again.

We remark that L_N can be written in the form $L_N = K_N - \nu_N I$, where ν_N is a constant, I the identity operator, and K_N maps any function onto the finite dimensional space $\mathscr{H}_N$ spanned by the ψ_α ($\alpha \leqslant N$). If one writes the eigenvalue equation for L_N and takes the projections of this equation onto $\mathscr{H}_N$ and its orthogonal complement, one finds that the two equations are uncoupled; it is then easy to show that the spectrum is made of N eigenvalues between $-\nu_N$ and 0 (with polynomial eigenfunctions; in particular, the ψ_α in the case the model defined by Eq. (4.8)) and an eigenvalue $-\nu_N$, infinitely many times degenerate.

We remark that the collision operators for hard spheres and hard potentials with angular cutoff are unbounded and display a continuous spectrum; the operators for hard potentials without cutoff are also unbounded. If these features of the operator have any influence on the solution of particular problems, this influence is lost when we adopt the models described by Eq. (4.12) (or (4.8) as a particular case). It is therefore convenient to introduce and investigate models which retain the abovementioned features of the linearized collision operator; this can be done in several ways.

Conceptually, the simplest procedure is based upon exploiting the fact that we either know (rigid spheres and angular cutoff) or conjecture (potentials with finite range) that $L = K - \nu I$ where the operator $K^* = \nu^{-1/2} K \nu^{-1/2}$ is self-adjoint and completely continuous in $\mathscr{H}$; accordingly, the kernel of K^*

can be expanded into a series of its square summable eigenfunctions $\varphi_\alpha \nu^{1/2}$ (such that $K\varphi_\alpha = \mu_\alpha \nu\varphi_\alpha$). In other words, we can write

$$Kh = \nu(\xi) \sum_{\alpha=0}^{\infty} \mu_\alpha \varphi_\alpha(\nu\varphi_\alpha, h). \tag{4.13}$$

Truncating this series (degenerate kernel approximation), we obtain the model:

$$L_N h = \nu(\xi) \sum_{\alpha=0}^{N} \mu_\alpha \varphi_\alpha(\nu\varphi_\alpha, h) - \nu(\xi)h. \tag{4.14}$$

This operator automatically satisfies the basic requirements, expressed by Eqs (1.10), (1.12), (1.15), (1.16). Since the first five φ_α are the collision invariants ψ_α and correspond to $\mu_\alpha = 1$ $(0 \leqslant \alpha \leqslant 4)$, if we take $N = 4$ in Eq. (9.14), we have

$$L_4 h = \nu(\xi) \sum_{\alpha=0}^{4} \psi_\alpha(\nu\psi_\alpha, h) - \nu(\xi)h, \tag{4.15}$$

where the collision invariants are normalized according to

$$(\psi_\alpha, \nu\psi_\beta) = \delta_{\alpha\beta}. \tag{4.16}$$

Equation (4.15) is nothing other than the linearized version of the nonlinear model with velocity-dependent collision frequency which was briefly discussed in Section 3.

Chapter III

ANALYTICAL SOLUTIONS OF MODELS

1. The method of elementary solutions

The theory developed in Chapter II shows that the study of the linearized Boltzmann equation is worthwhile undertaking and that many of the features of its solutions can be retained by using model equations (Chapter II, Sect. 4). We can say more, i.e., that practically all the features are retained by a properly chosen model. The advantages offered by the models consist essentially in simplifying both the analytical and numerical procedures for solving boundary value problems of special interest.

In particular, the use of models is invaluable in those cases when the solution of the latter is explicit (in terms of quadratures of functions, whose qualitative behavior can be studied by analytical means). Accordingly we shall devote this chapter to the analytical manipulations which can be used to obtain interesting information from the model equations. The method used throughout is the method of separation of variables.

The first step is to construct a complete set of separated variable solutions ("elementary solutions") and then represent the general solution as a superposition of the elementary solutions; the second step is to use the boundary and initial conditions to determine the coefficients of the superposition. While the first problema can be solved for the model equations discussed in Chapter II, Section 4, the second problem can be solved exactly in only a few cases. The method retains its usefulness, however, even when the second problem is not solvable, or is only approximately solvable because it is capable of providing an analytical representation of the solution and hence a picture of its qualitative behavior (see Sect. 5).

It must be stated that the method of separation of variables is not the only one capable of solving these problems; transform techniques of the Wiener–Hopf type are completely equivalent to the method of elementary solutions.

2. Splitting of a one-dimensional model equation

We begin by considering the simplest kind of problems, i.e., steady problems in one-dimensional geometry, and the simplest collision model, i.e., the linearized Krook model with velocity dependent frequency, given by Eq. (II.4.15).

Accordingly we consider the equation

$$\xi_1 \frac{\partial h}{\partial x} = Lh, \tag{2.1}$$

where x is the Cartesian coordinate upon which h is assumed to depend, ξ_1 the x component of the molecular velocity $\boldsymbol{\xi}$ and

$$Lh = \nu(\boldsymbol{\xi})\Big[\sum_{\alpha=0}^{4} (\nu\psi_\alpha, h) - h\Big]; \qquad (\psi_\alpha, \nu\psi_\beta) = \delta_{\alpha\beta}. \tag{2.2}$$

The unknown h can be split as follows

$$h = h_1 + h_2 + h_3, \tag{2.3}$$

where

$$\begin{aligned} h_1 &= \Pi_1 h \equiv \tfrac{1}{2}(I + P_2 P_3)h, \\ h_2 &= \Pi_2 h \equiv \tfrac{1}{4}(I + P_2)(I - P_3)h, \\ h_3 &= \Pi_3 h \equiv \tfrac{1}{4}(I + P_3)(I - P_2)h, \end{aligned} \tag{2.4}$$

P_k denoting as usual the operator which reflects the kth component of $\boldsymbol{\xi}$ $[P_3 f(\xi_1, \xi_2, \xi_3) \equiv f(\xi_1, \xi_2, -\xi_3)]$. In such a way the Hilbert space $\mathscr{H}$ where h can be located is split into three mutually ortogonal subspaces $\mathscr{H}_1$, $\mathscr{H}_2$, $\mathscr{H}_3$ ($P_k^2 = I$ and $P_k P_h = P_h P_k$ imply that the operators Π_k satisfy $\Pi_k \Pi_h$

$= \delta_{kh}\Pi_k$, $\sum_{k=1}^{3} \Pi_k = I$). The collision frequency ν will be assumed to be even in all the components of $\boldsymbol{\xi}$ (usually, it depends only upon the speed ξ). ψ_0, ψ_1, ψ_4 being linear combinations of 1, ξ_1, ξ^2 belong to $\mathscr{H}_1$, ψ_2 to $\mathscr{H}_2$, ψ_3 to $\mathscr{H}_3$. Hence if we apply Π_1, Π_2, Π_3 to Eq. (2.1), we obtain

$$\xi_1 \frac{\partial h_1}{\partial x} = \nu[(\nu\psi_0, h_1)\psi_0 + (\nu\psi_1, h_1)\psi_1 + (\nu\psi_4, h_1)\psi_4 - h_1], \tag{2.5}$$

$$\xi_1 \frac{\partial h_2}{\partial x} = \nu[\psi_2(\nu\psi_2, h_2) - h_2], \tag{2.6}$$

$$\xi_1 \frac{\partial h_3}{\partial x} = \nu[\psi_3(\nu\psi_3, h_3) - h_3]. \tag{2.7}$$

The remarkable fact is that Eqs (2.5), (2.6), (2.7) are uncoupled; in addition, Eqs (2.6) and (2.7) contain just one "moment", $(\nu\psi_k, h_k)$ $(k = 2, 3)$, and Eqs (2.5) contain three of such terms, instead of the five "moments" in Eq. (2.1).

Equation (2.5) describes the heat transfer processes taking place along the x-axis, Eq. (2.6) and (2.7) the shear effects due to motions in the y- and z-direction, respectively. We shall begin from considering the simplest case, i.e., Eq. (2.6) (or Eq. (2.5), which differs from Eq. (2.6) for the name of the axes). By letting $h_3 = \psi_3 Y$ we obtain:

$$\xi_1 \frac{\partial Y}{\partial x} = \nu[(\nu\psi_3^2, Y) - Y]. \tag{2.8}$$

3. Elementary solutions of the simplest transport equation

According to the results of Section 2, the simplest problems of rarefied gas dynamics lead to the following equation

$$\xi_1 \frac{\partial Y}{\partial x} + \nu(\xi) Y_1(x, \boldsymbol{\xi}) = \nu(\xi)\int g_0(\xi)\, Y(x, \boldsymbol{\xi})\, d\boldsymbol{\xi}, \tag{3.1}$$

where

$$g_0(\boldsymbol{\xi}) = \nu f_0 \psi_3^2. \tag{3.2}$$

Let us use the variable w defined as follows:

$$w = \frac{\xi_1}{\nu(\xi_1, \xi_2, \xi_3)} \tag{3.3}$$

provided this relation is uniquely invertable by

$$\xi_1 = \xi_1(w, \xi_2, \xi_3) \tag{3.4}$$

which is clearly possible if $\partial[\xi_1/\nu(\xi_1, \xi_2, \xi_3)]/\partial\xi_1$ is different from zero for any ξ_2, ξ_3. In such a case, Eq. (3.1) becomes

$$w\frac{\partial Y}{\partial x} + Y = \int_{-k}^{k} Z(x, w_1)\,dw_1 \tag{3.5}$$

where

$$Z(x, w) = \int g_0[\xi_1(w, \xi_2, \xi_3), \xi_2, \xi_3]\,Y(x, w, \xi_2, \xi_3)\,\frac{\partial\xi_1}{\partial w}\,d\xi_2\,d\xi_3. \tag{3.6}$$

and

$$k = \lim_{\xi_1\to\infty} \frac{\xi_1}{\nu(\xi_1, \xi_2, \xi_3)} \tag{3.7}$$

is assumed to be independent of ξ_2, ξ_3 (which is the case if $\nu = \nu(\xi)$).

Let us put

$$Z_0(w) = \int g_0[\xi_1(w, \xi_2, \xi_3), \xi_2, \xi_3]\,\frac{\partial\xi_1}{\partial w}\,d\xi_2\,d\xi_3; \tag{3.8}$$

multiply Eq. (3.5) by $g_0(\partial\xi_1/\partial w)$ and integrate with respect to ξ_2, ξ_3. We obtain:

$$w\frac{\partial Z}{\partial x} + Z(x, w) = Z_0(w)\int_{-k}^{k} Z(x, w_1)\,dw_1. \tag{3.9}$$

Let us begin by separating the variables. Putting

$$Z(x, w) = X(x)g(w)Z_0(w), \tag{3.10}$$

it is easily seen that, either $Z = A_0$ (A_0 arbitrary constant) or

$$Z_u(x, w) = e^{-x/u}g_u(w)Z_0(w), \tag{3.11}$$

where $g_u(\xi)$ satisfies

$$(-w/u+1)g_u(w) = \int_{-k}^{k} g_u(w_1)Z_0(w_1)\,dw_1 \tag{3.12}$$

and u the separation parameter, has been used to label the elementary solutions.

Though, *a priori*, u may assume any complex value it is easily seen that, since $\int_{-k}^{k} Z_0\,dw = 1$, u is a real number. This follows from a direct argument or from general results.

When $k < \infty$ it is also easy to show that there are no real eigenvalues outside the interval $(-k, k)$. This result can be proved by the same technique to be employed in Section 6 for discussing the discrete spectrum in the timedependent case.

Then the values of u must be real and comprised between $-k$ and k. This requires some care, because one cannot divide by $u-w$ in Eq. (3.12). This difficulty is overcome by letting $g_u(w)$ be a generalized function.

If we disregard a multiplicative constant (i.e., normalize g_u in such a way that the right-hand side of Eq. (3.12) is equal to 1) $g_u(w)$ will satisfy

$$\left(\frac{u-w}{u}\right)g_u(w) = 1. \tag{3.13}$$

For $w \neq u$ we find $g_u(w) = u/(u-w)$ but this has no meaning at $w = u$ (in particular, the integral appearing in Eq. (3.12) does not exist in the ordinary sense). It is possible, however, to define a generalized function, e.g., as limit of the sequence

$$g_m(w) = m^2 u(u-w)/[m^2(u-w)^2+1]. \tag{3.14}$$

This limit, to be understood in the sense of convergence of a sequence of generalized functions. The limit is a generalized function denoted by

$$P\frac{u}{u-w} \equiv \lim_{m\to\infty} \frac{m^2 u(u-w)}{1+m^2(u-w)^2}, \tag{3.15}$$

where P can be read "principal part of". The integrals involving the generalized function, which has been just defined, are to be interpreted as Cauchy principal value integrals

$$\begin{aligned}\int P\frac{u}{u-w}\,\varphi(w)\,dw &= P\int\frac{u\varphi(w)}{u-w}\,dw \\ &= \lim_{\varepsilon\to 0}\int_{|w-u|>\varepsilon}\frac{u\varphi(w)}{u-w}\,du.\end{aligned} \tag{3.16}$$

We can ask now whether or not $P[u/(u-w)]$ is the only solution of Eq. (3.13). The answer is no. As a matter of fact, the most general solution will be the sum of $P[u/(u-w)]$ and the general solution of the homogeneous equation

$$(u-w)\,T(w) = 0. \tag{3.17}$$

Now, the most general solution of Eq. (3.17) is a multiple of $\delta(u-w)$ where δ denotes the Dirac delta function. Therefore the general solution of Eq. (3.13) reads as follows:

$$g_u(w) = P\frac{u}{u-w} + p(u)\,\delta(u-w) \tag{3.18}$$

where the factor in front of the delta function can depend upon u and has been called $p(u)$. In order that Eq. (3.12) be satisfied by Eq. (3.18) the normalization condition for $g_u(w)$ must be satisfied, i.e. the right-hand side of Eq. (3.12) must be equal to 1. This condition can be satisfied for any real u and serves for determining $p(u)$

$$\begin{aligned} p(u) &= [Z_0(u)]^{-1}\left[1 - P\int_{-k}^{k}\frac{uZ_0(w)}{u-w}\,dw\right] \\ &= [Z_0(u)]^{-1}\int_{-k}^{k}\frac{wZ_0(w)}{w-u}\,dw \end{aligned} \tag{3.19}$$

where the fact that $\int_{-k}^{k} Z_0(w)dw$ has been used.

The generalized eigenfunctions $g_u(w)$ have many properties of orthogonality and completeness. The properties of orthogonality and completeness in partial ranges (notably $0 < w < k$) are far from trivial to prove, since they require solving singular integral equations. However, standard techniques are available for treating such problems and the following results can be obtained:

THEOREM I. *The generalized functions* $Z_0(w)g_u(w)$ $(-k < u < k)$ *and* $g_\infty = Z_0(w)$, *complemented with* $g_* = wZ_0(w)$, *form a complete set for the functions* $Z(w)$ *defined on the real axis, satisfying a Hölder condition in any open interval contained in* $(-k, k)$ *and such that*

$$\int_{-k}^{k} w^2 Z(w)dw < \infty. \tag{3.20}$$

Also, the coefficients of the generalized expansion:

$$Z(w) = \left[A_0 + A_1 w + \int_{-k}^{k} A(u)g_u(w)du\right]Z_0(w) \tag{3.21}$$

are uniquely and explicitly determined by

$$A_0 = \left[\int_{-k}^{k} w^2 Z_0(w)dw\right]^{-1}\int_{-k}^{k} w^2 Z(w)dw, \tag{3.22}$$

$$A_1 = \left[\int_{-k}^{k} w^2 Z_0(w)dw\right]^{-1}\int_{-k}^{k} wZ(w)dw, \tag{3.23}$$

$$A(u) = [C(u)]^{-1}\int_{-k}^{k} wZ(w)g_u(w)dw, \tag{3.24}$$

where

$$C(u) = uZ_0(u)\{[p(u)]^2+\pi^2 u^2\}. \tag{3.25}$$

THEOREM II. *The generalized eigenfunctions* $Z_0(w)g_u(w)$ $(0 < u < k)$ *and* $g_\infty = Z_0(w)$ *form a complete set for the functions* $Z(w)$ *defined on* $0 < w < k$, *satisfying a Hölder condition in any open interval contained in* $(0, k)$ *and integrable with respect to the weight* w^2. *Also, the coefficients of the generalized expansions*

$$Z(w) = \Big[A_0+ \int_0^k A(u)g_u(w)\,du\Big]Z_0(w) \tag{3.26}$$

are uniquely and explicitly determined by:

$$A_0 = \Big[\int_{-k}^{k} w^2 Z_0(w)\,dw\Big]^{-1}\int_0^k wP(w)Z(w)\,dw, \tag{3.27}$$

$$A(u) = [C(u)P(u)]^{-1}\int_0^k wP(w)g_u(w)Z(w)\,dw. \tag{3.28}$$

Here we have put

$$P(w) = w \exp\left\{-\frac{1}{\pi}\int_0^k \tan^{-1}[\pi t/p(t)]\,\frac{dt}{t+w}\right\} \qquad (w > 0), \tag{3.29}$$

where the inverse tangent varies from $-\pi$ *to* 0 *when* t *varies from* 0 *to* k.

By means of the completeness property expressed by Theorem I one can also show that the general solution of Eq. (3.9) is given by

$$Z(x, w) = \Big[A_0+A_1(x-w)+\int_{-k}^{k} A(u)e^{-x/u}g(w)\,du\Big]Z_0(w). \tag{3.30}$$

Theorem II is equally or, perhaps, more important, because it allows us to solve boundary value problems. This theorem shows that the generalized eigenfunctions are orthogonal on $(0, k)$ with respect to the weight $wZ_0(w)P(w)$. This orthogonality property is more standard than the full range orthogonality, because the weight function is positive. The only trouble now is the complicated expression of $P(w)$; it is to be noted, however, that $P(w)$, though far from being an elementary function, satisfies two important identities which make the manipulation of integrals involving $P(w)$ much easier than would be expected. These identities are:

$$\left[\int_{-k}^{k} w^2 Z_0(w)\,dw\right]^{-1}\int_0^k \frac{tZ_0(t)P(t)}{t+u}\,dt = [P(u)]^{-1}, \tag{3.31}$$

$$(3.32)\quad u-\frac{1}{\pi}\int_0^k \tan^{-1}[\pi t/p(t)]\,dt-\left[\int_{-k}^{k} w^2 Z_0(w)\,dw\right]\int_0^k \frac{t\,[Z_0(t)\,P(t)]^{-1}}{[p(t)]^2+\pi^2 t^2}\,\frac{dt}{t+u} = P(u)$$

Also

$$(3.33)\qquad P(0)=\Big[\int_{-k}^{k} u^2 Z_0(u)\,du\Big]^{1/2}.$$

Once Eq. (3.9) is solved, the general solution of Eq. (3.5), and hence (3.1) is easily written down since it is matter of solving an ordinary differential equation with given source term. We obtain

$$(3.34)\qquad Y(x,\boldsymbol{\xi})=A_0+A_1\,\{x-[\xi_1/\nu(\xi)]\}+ \\ +\int_{-k}^{k} A(u)\,e^{-x/u}\,g_u\left(\frac{\xi_1}{\nu(\xi)}\right)du+B(\boldsymbol{\xi})\,e^{-x\nu(\xi)/\xi_1},$$

where $B(\boldsymbol{\xi})$ is an arbitrary function, provided only it satisfies:

$$(3.35)\qquad \int\left\{g_0(\boldsymbol{\xi})\,B(\boldsymbol{\xi})\left[\frac{\partial}{\partial\xi_1}\left[\frac{\xi_1}{\nu(\xi)}\right]^{-1}\right]_{\xi_1=\xi_1(w,\xi_2,\xi_3)}\right\}d\xi_2\,d\xi_3=0.$$

We end this section by noticing the form taken by the previous results in particular cases. If $\nu(\xi)=\nu$ is constant (BGK model) and we take $(2RT_0)^{1/2}$ as speed unit and $(2RT_0)^{1/2}\nu^{-1}=2\pi^{-1/2}l$ as length unit (T_0 being the unperturbed temperature, l a suitably defined mean free path), then $\psi_3=\sqrt{2}\,\xi_3$, $g_0=2\xi_3^2 f_0$, $w=\xi_1$, $k=\infty$ and

$$(3.36)\qquad Z_0(w)=\pi^{-1/2}e^{-w^2}.$$

Accordingly

$$(3.37)\qquad p(u)=e^{u^2}P\int_{-\infty}^{\infty}\frac{we^{-w^2}}{w-u}\,dw=\pi^{1/2}\left(e^{u^2}-2u\int_0^u e^{t^2}\,dt\right).$$

Thus $p(u)$ can be expressed in terms of tabulated functions. Equations (3.22), (3.23), (3.27), (3.31), (3.32), (3.33) slightly simplify since

$$(3.38)\qquad \int_{-k}^{k} w^2 Z_0(w)\,dw=1/2.$$

If $\nu(\xi) = \sigma\xi$ (constant mean free path), $\psi_3 = [3/(4\sigma)]^{1/2}$, $w = \xi_1/(\xi\sigma)$, $k = 1/\sigma$ and, if we take $(2RT_0)^{1/2}$ as speed unity, $1/\sigma$ as length unity:

$$Z_0(w) = \tfrac{3}{4}(1-w^2), \tag{3.39}$$

$$\begin{aligned} p(u) &= \frac{1}{1-u^2}\, P \int_{-1}^{1} \frac{w(1-w^2)}{w-u}\, du \\ &= \frac{2(3u^2-1)}{3(1-u^2)} + u \log\left(\frac{1-u}{1+u}\right). \end{aligned} \tag{3.40}$$

4. Application of the general method to the Kramers and Milne problems

In this section we shall apply the above results to two typical boundary value problems of transport theory, the Kramers and Milne problems.

The Kramers problem consists in finding the molecular distribution function of a gas in the following situation: the gas fills the half space $x > 0$ bounded by a physical wall in the plane $x = 0$ and is nonuniform because of a gradient along the x-axis of the z-component of the mass velocity; this gradient tends to a constant a when x goes to infinity. It is seen that this problem can be considered as the limiting case of plane Couette flow (shear flow between two parallel plates), when one of the plates is pushed to infinity, while keeping a fixed ratio between the relative speed of the plates and their mutual distance. More generally, the Kramers problem can be interpreted as a connection problem through the kinetic boundary layer; in this case "infinity" represents the region where the viscous gas theory holds and the velocity gradient "at infinity" can be regarded as constant because it does not vary sensibly on the scale of the mean free path.

Both of these interpretations of the Kramers problem suggest that a convenient linearization is about a Maxwellian endowed with a mass velocity ax in the z-direction. Because of the non-uniformity of the Maxwellian, linearization gives an inhomogeneous Boltzmann equation

$$2ac_1c_3 + c_1\frac{\partial h}{\partial x} = Lh \tag{4.1}$$

where $\mathbf{c} = (c_1, c_2, c_3) = (\xi_1, \xi_2, \xi_3 - ax)$. Equation (4.1) can be reduced to the homogeneous Boltzmann equation by subtracting a particular solution. One particular solution, independent of x, is suggested by the kinetic theory of viscosity; this solution, $L^{-1}(2ac_1c_3)$ is given by $-2ac_1c_3/\nu(c)$ for the collision model given by Eq. (2.2). Therefore we have:

$$h = -2ac_1c_3/\nu(c) + 2c_3 Y(x, \mathbf{c}) \tag{4.2}$$

where $Y(x, \xi)$ satisfies Eq. (3.1). The mass velocity is given by

$$v_3 = ax + 2\varrho_0^{-1} \int \xi_3^2 Y(x, \xi) f_0(\xi) d\xi, \tag{4.3}$$

the first term being the contribution from the Maxwellian $f_0(\mathbf{c})$.

Concerning the boundary conditions, we shall assume that the molecules are re-emitted from the wall according to a Maxwellian distribution completely accommodated to the state of the wall. Therefore the boundary condition for h reads as follows:

$$h(0, \mathbf{c}) = 0 \quad (c_1 > 0) \tag{4.4}$$

and this in terms of Y, becomes:

$$Y(0, \xi) = a\xi_1/\nu(\xi). \tag{4.5}$$

In addition, Y must satisfy the condition of boundedness at infinity.

According to the discussion in Section 3, the general solution of Eq. (3.1) which also satisfies the condition of boundedness at infinity is given by

$$Y(x, \xi) = A_0 + \int_0^k A(u) e^{-x/u} g_u\left(\frac{\xi_1}{\nu(\xi)}\right) du + B(\xi) e^{-x\nu(\xi)/\xi_1}, \tag{4.6}$$

where $B(\xi)$ satisfies Eq. (3.39). The condition to be satisfied at the plate gives:

$$a\xi_1/\nu(\xi) = A_0 + \int_0^k A(u) g_u\left(\frac{\xi_1}{\nu(\xi)}\right) du + B(\xi). \tag{4.7}$$

Equations (4.7) and (3.35) easily give $B(\xi) = 0$. Thus solving Eq. (4.7) means expanding $Z(w) = awZ_0(w)$ according to Theorem II of Section 3; therefore, A_0 and $A(u)$ are immediately obtained through Eqs (3.27) and (3.28). The result is as follows:

$$A_0 = -a\pi^{-1} \int_0^k \tan^{-1} [\pi w/p(w)] dw, \tag{4.8}$$

$$A(u) = -a[Z_0(u) P(u)]^{-1} \{[P(u)]^2 + \pi^2 u^2\}^{-1} \int_{-k}^{k} w^2 Z(w) dw, \tag{4.9}$$

where use has been made of Eq. (3.31) which yields Eq. (4.9) directly and the following identity by asymptotically expanding for large values of u and comparing with Eq. (3.32) or Eq. (3.29):

$$\left[\int_{-k}^{k} w^2 Z_0(w) dw\right]^{-1} \int_0^k w^2 Z_0(w) P(w) dw = -\frac{1}{\pi} \int_0^k \tan^{-1} [\pi t/p(t)] dt \tag{4.10}$$

Equation (4.10) yields Eq. (4.8).

Substituting Eqs (4.8) and (4.9) into Eq. (4.6) gives the solution of the Kramers problem. The mass velocity is readily obtained from Eqs. (4.3) and (4.6):

$$v_3(x) = ax + A_0 + \int_0^k A(u)\, n(u)\, e^{-x/u}\, du, \tag{4.11}$$

where

$$n(u) = 2\varrho_0^{-1} \int f_0\, \xi_3^2\, g_u\left(\frac{\xi_1}{\nu(\xi)}\right) d\boldsymbol{\xi} \tag{4.12}$$

and A_0 and $A(u)$ are given by Eq. (4.8) and (4.9). From Eq. (4.11) we recognize that A_0 is the macroscopic slip of the gas on the plate; it has the form ζa where ζ is the slip coefficient:

$$\zeta = -\pi^{-1} \int_0^k \tan^{-1}[\pi w/p(w)]\, dw \tag{4.13}$$

Thus the evaluation of the slip coefficient has been reported to quadratures for an arbitrary $\nu(\xi)$. In particular, if $\nu(\xi) = \text{const}$ (BGK model), use of Eq. (3.40) and partial integration yields

$$\zeta = \theta\pi^{1/2} \int_0^\infty \frac{w e^{w^2}\, dw}{[p(w)]^2 + \pi^2 w^2} = 2l \int_0^\infty \frac{w e^{w^2}\, dw}{[p(w)]^2 + \pi^2 w^2}. \tag{4.13a}$$

Here $p(w)$ is given by Eq. (3.37) and l is the mean free path related to $\theta = \nu^{-1}$ by $\theta = 2\pi^{-1/2} l$ $(2RT_0 = 1)$. The integral appearing in Eq. (4.13) can be evaluated numerically with the following result

$$\zeta = (1.01615)\theta = (1.1466)l. \tag{4.14}$$

If $\nu(\xi)$ is not constant, one must evaluate ζ through Eq. (4.13). Some specific cases can be easily worked out numerically; thus for $\nu(\xi)$ increasing linearly when $\xi \to \infty$ the value of ζ is somewhat lower (3 %) than for the BGK model (for a fixed value of the viscosity coefficient).

Henceforth in this section we shall restrict our considerations to the case of the BGK model (constant collision frequency). In this case, Eq. (4.11) can be written as follows:

$$v_3(x) = a[x + \zeta - (\pi^{1/2}\theta/2)\, I(x/\theta)] \tag{4.15}$$

where $I(x/\theta)$ is practically zero outside the kinetic layer and can be easily evaluated.

A direct evaluation of the microscopic slip, i.e., the velocity of the gas at the wall, results without any numerical integration. As a matter of fact, we have:

$$v_3(0) = a\left(\zeta - l \int_0^\infty \frac{e^{w^2}[P(w)]^{-1}}{[p(w)]^2 + \pi^2 w^2}\, dw\right) = (2/\pi)^{1/2} al \tag{4.16}$$

where the last result is obtained by letting $u = 0$ in Eq. (3.32) and taking into account Eqs. (3.33), (4.13) and (3.38).

Analogously we can evaluate the distribution function of the molecules arriving at the plate. We obtain:

$$Y(0, \boldsymbol{\xi}) = Y(0, \xi_1) = 2\pi^{-1/2} al\xi_1 + 2\pi^{-1/2} alP(-\xi_1) \qquad (\xi_1 < 0) \tag{4.17}$$

where Eq. (3.32) has been used. Then $h(0, \mathbf{c})$ (the perturbation of the Maxwellian distribution at the plate) is given by

$$h(0, \mathbf{c}) = 4\pi^{-1/2} alc_3 P(|c_1|) \qquad (c_1 < 0) \tag{4.18}$$

and the function $P(w)$ $(w > 0)$ receives a physical interpretation in terms of the distribution function of the molecules arriving at the wall. From Eqs (3.32) it is easily inferred that

$$|c_1| + 0.7071 < P(|c_1|) < |c_1| + 1.01615. \tag{4.19}$$

Hence the distribution function of the arriving molecules is rather close to a Hilbert distribution; in fact, a Hilbert expansion would predict Eq. (4.16) with $P(|c_1|)$ linear in $|c_1|$ (such is the distribution holding outside the kinetic layer, see Eqs (4.2) and (4.6)). The fact that the distribution of the molecules arriving at the plate is close to the one prevailing outside the kinetic boundary layer is not surprising; in fact each molecules has the velocity acquired after its last collision, which, on the average, happened a mean free path from the wall, i.e., in a region where the distribution function is of the Hilbert type. It is interesting to note that Maxwell assumed that the distribution function of the arriving molecules was exactly the one prevailing far from the wall; by using this assumption and conservation of momentum, he was able to evaluate the slip coefficient, without solving the Kramers problem. He found $\zeta = l$ (with an error of 15 %) and

$$h(0, \mathbf{c}) = 4\pi^{-1/2} alc_3(|c_1| + 0.8863) \qquad (c_1 < 0), \tag{4.20}$$

i.e. a good approximation to the correct result given by Eqs. (4.18) and (4.19).

5. Application to the flow between parallel plates

We have just seen that half-space problems connected with Eq. (3.1), or equivalently Eq. (3.9), can be solved by analytical means. This is not true for gas flows between parallel plates, like Couette and Poiseuille flows. The method of elementary solutions, however, can be used to obtain series solution and gain insight into the qualitative behavior of the solution.

Let us consider the flow problems first and restrict our attention to the case of constant collision frequency $v = \theta^{-1}$.

The general solution, Eq. (3.34), of Eq. (3.1) can be rewritten as follows:

$$(5.1) \qquad Y(x, \xi) = A_0 + A_1 (x - \xi) + \int_{-\infty}^{+\infty} A(u) \exp\left(-\frac{x}{u} - \frac{\delta}{2|u|} \right) g_u(\xi)\, du$$

where the last term in Eq. (3.34) has been omitted because is usually absent, ξ_1 has been denoted by ξ since no confusion arises and δ is the distance between the plates ($\theta = 1$ and $2RT_0 = 1$, as usual). $A(u)$ has been redefined by inserting a factor $\exp[-\delta/2|u|]$ for convenience, and the plates are assumed to be located at $x = \pm\delta/2$. Equation (5.1) shows that for sufficiently large δ the picture is the following: a core, where a continuum description (based on the Navier–Stokes equations) prevails, surrounded by kinetic boundary layers, produced the interaction of the molecules with the walls and described by the integral term in Eq. (5.1).

As δ becomes smaller, however, the exponentials in the latter term are never negligible, i.e., the kinetic layers merge with the core to form a flow field which cannot be described in simple terms. Finally, when δ is negligibly small, $Y(x, \xi)$ does not depend sensibly on x, and the molecules retain the distribution they had just after their last interaction with a boundary. In the case of Couette flow (i.e., when there are two plates at $x = \pm\delta/2$ moving with velocities $\pm V/2$ in the z-direction) the situation is well described by the above short discussion, although it is possible to' obtain a more detailed picture by finding approximate expressions for A_1 and $A(u)$ (A_0 is zero and $A(u)$ is odd in because of the antisymmetry inherent in the problem). Accordingly we shall consider in more detail the case of Poiseuille flow between parallel plates, which lends itself to more interesting considerations.

Plane Poiseuille flow is the flow of a fluid between two parallel plates induced by a pressure gradient parallel to the plates. In the continuum case no distinction is made between a pressure gradient arising from a density gradient and one arising from a temperature gradient. This distinction, on the contrary, is to be taken into account when a kinetic theory description is considered. We shall restrict ourselves to the former case; similar remarks apply to the case of a temperature gradient.

The basic linearized Boltzmann equation for Poiseuille flow in a channel of arbitrary cross-section (including the slab as a particular case) will now be derived.

We assume that the walls re-emit the molecules with a Maxwellian distribution f_0 with constant temperature and an unknown density $\varrho = \varrho(z)$ (z being the coordinate parallel to the flow). If the length of the channel is much larger than any other typical length (mean free path, distance between the walls), then we can linearize about the above-mentioned Maxwellian f_0;

in fact, $\varrho(z)$ is slowly varying and f_0 would be the solution in the case of a rigorously constant ϱ. Accordingly we have

$$\xi_1 \frac{\partial h}{\partial x}+\xi_2 \frac{\partial h}{\partial y}+\xi_3 \frac{\partial h}{\partial z}+\frac{1}{\varrho}\frac{d\varrho}{dz}\xi_3 = Lh. \tag{5.2}$$

Because of the assumption of a slowly varying ϱ (long tube), we can regard $\frac{1}{\varrho}\frac{d\varrho}{dz}$ as constant (i.e., we disregard higher order derivatives of ϱ as well as powers of first order derivatives). If $\frac{1}{\varrho}\frac{d\varrho}{dz}$ is constant, it follows that $\partial h/\partial z = 0$, since z does not appear explicitly in the equation nor in the boundary conditions. The latter can be written

$$h(x, y, z, \boldsymbol{\xi}) = 0 \qquad ((x, y)\in \partial\Sigma;\ xn_1+yn_2 > 0), \tag{5.3}$$

where $\partial\Sigma$ is the contour of the cross-section and $\mathbf{n} = (n_1, n_2)$ the normal unit vector pointing into the channel. Therefore, we can write

$$\xi_1 \frac{\partial h}{\partial x}+\xi_2 \frac{\partial h}{\partial y}+k\xi_3 = Lh, \tag{5.4}$$

where $k = \frac{1}{\varrho}\frac{d\varrho}{dz}$. Equation (5.4) governs linearized Poiseuille flows in a very long tube of arbitrary cross-section. If we specialize to the case of a slab and use the BGK model, we have:

$$h = 2\xi_3 W(x, \xi_1) \tag{5.5}$$

where $W(x, \xi)$ satisfies

$$\xi \frac{\partial W}{\partial x}+\frac{k}{2} = \pi^{-1/2} \int_{-\infty}^{+\infty} e^{-\xi_1^2} W(x, \xi_1)d\xi_1 - W(x, \xi_1), \tag{5.6}$$

$$W\left(-\frac{\delta}{2}\operatorname{sgn}\xi, \xi\right) = 0, \tag{5.7}$$

provided x is measured in θ units. The above equations follow from a splitting analogous to the one considered in Section 2.

Since $W(x, \xi)$ does not depend on the y and $-z$ components of $\boldsymbol{\xi}$ and $\nu = 1$, Eq. (5.6) differs from Eq. (3.1) because of the inhomogeneous term $k/2$. If we find a particular solution of Eq. (5.6), then we can add it to the general solution (5.1) of the homogeneous equation in order to have the general solution of Eq. (5.6). By differentiation of the latter equation, we deduce that $\partial W/\partial x$ satisfies Eq. (3.1) with $\nu = 1$; since the general solution of Eq. (3.1) contains exponentials (which reproduce themselves by integration and differentiation) and a linear function of x, we try a particular solution of Eq. (5.6) in the form of a quadratic function of x (with coefficients depending upon ξ). It is verified that solutions of this form exist and one of them is

$$W_0(x,\xi) = \frac{k}{2}\left[x^2 - \frac{\delta^2}{4} - 2x\xi - (1-2\xi^2)\right]. \tag{5.8}$$

Therefore

$$W(x,\xi) = W_0(x,\xi) + Y(x,\xi), \tag{5.9}$$

where $Y(x,\xi)$ is given by Eq. (5.1). Equation (5.7) gives the following boundary condition for $Y(x,\xi)$:

$$Y\left(-\frac{\delta}{2}\operatorname{sgn}\xi,\ \xi\right) = -\left[|\xi| - (1-2\xi^2)\frac{1}{\delta}\right]\frac{k\delta}{2}. \tag{5.10}$$

Since the symmetry inherent in our problem implies that

$$Y(x,\xi) = Y(-x,-\xi), \tag{5.11}$$

$A_1 = 0$ and $A(u) = A(-u)$. If we take this into account, Eqs (5.1) and (5.10) give

$$A_0 + \int_0^\infty A(u)g_u(\xi)\,du = -\frac{k\delta}{2}\left[\xi - (1-2\xi^2)\frac{1}{\delta}\right] - \int_0^\infty \frac{uA(u)}{u+\xi}e^{-\delta/u}\,du \qquad (\xi > 0), \tag{5.12}$$

and the equation for $\xi < 0$ is not required because $A(u) = A(-u)$. If we call the right-hand side of this equation $\pi^{1/2}e^{\xi^2}Z(\xi)$ and use Eq. (3.40), Eq. (5.12) becomes Eq. (3.30) ($k = \infty$) and we can apply Eqs (3.31) and (3.32), thus obtaining:

$$A_0 = -\left\{\sigma + \frac{1}{\delta}(\tfrac{1}{2}+\sigma^2) - \int_0^\infty ue^{-\delta/u}[P(u)]^{-1}A(u)\,du\right\}\frac{k\delta}{2}, \tag{5.13}$$

$$A(u) = \frac{k}{2}\pi^{1/2}\left(u + \frac{\delta}{2} + \sigma\right)e^{u^2}[P(u)]^{-1}\{[p(u)]^2 + \pi^2u^2\}^{-1} + \frac{\pi^{1/2}}{2}e^{u^2}[P(u)]^{-1}\{[p(u)^2 + \pi^2u^2\}^{-1}\int_0^\infty \frac{\xi[P(\xi)]^{-1}}{u+\xi}e^{-\delta/\xi}A(\xi)\,d\xi \qquad (\xi > 0) \tag{5.14}$$

where permissible inversions of the order of integrations have been performed and Eq. (3.31) used. Here $\sigma = \zeta/\theta$ (see Eqs (4.13) and (4.13a). Thus the problem has been reduced to the task of solving an integral equation in the unknown $A(u)$ Eq. (5.14). This equation is a classical Fredholm equation of the second kind with symmetrizable kernel. The corresponding Neumann–Liouville series can be shown to converge for any given positive value of δ. It is also obvious that the larger δ, the more rapid is the convergence. This allows the ascertainement of some results in the near-continuum regime. In particular, if terms of order $\exp[-3(\delta/2)^{3/2}]$ are negligible, only the zero order term of the series need be retained:

$$A(u) = \frac{k}{2}\pi^{1/2}\left\{\left(\frac{\delta}{2}+\sigma\right)+u\right\} e^{u^2}[P(u)]^{-1}\{[p(u)]^2+\pi^2u^2\}^{-1}. \tag{5.15}$$

Within the same limits of accuracy, A_0 is given by:

$$A_0 = -\frac{k\delta}{2}\left[\sigma+\frac{1}{\delta}(\tfrac{1}{2}+\sigma^2)\right]. \tag{5.16}$$

We note that this zero order approximation is by far more accurate than a continuum treatment (even if slip boundary conditions are used in the latter). In fact, even in the zero-order approximation:

1) Kinetic boundary layers are present near the walls.

2) In the main body of the flow the mass velocity satisfies the Navier–Stokes momentum equation; however, the corresponding extrapolated boundary conditions show the presence not only of the first order slip but also of a second order slip:

$$v_3\left(\pm\frac{\delta}{2}\right) = \pm\sigma\left(\frac{\partial v_3}{\partial x}\right)_{x=\pm\delta/2} - \tfrac{1}{2}(\tfrac{1}{2}+\sigma^2)\left(\frac{\partial^2 v_3}{\partial x^2}\right)_{x=\pm\delta/2}. \tag{5.17}$$

In order to obtain these results, we observe that the mass velocity is given by

$$\begin{aligned} v_3(x) &= \pi^{-1/2}\int_{-\infty}^{\infty} W(x,\xi)e^{-\xi^2}d\xi \\ &= \frac{k}{2}(x^2-\delta^2/4)+A_0+\int_{-\infty}^{\infty} A(u)\exp\left[-\frac{x}{u}-\frac{\delta}{2|u|}\right]du \end{aligned} \tag{5.18}$$

where Eqs (5.8) and (5.9) have been taken into account. Equation (5.20) is exact; if terms of order $\exp[-3(\delta/2)^{3/2}]$ can be neglected, A_0 and $A(u)$ are given by Eqs (5.16) and (5.15). In particular, the integral term in Eq. (5.18) describes the space transients in the kinetic boundary layers; in the main body of the flow the integral term is negligible and we have

$$v_3(x) = \frac{k}{2}[x^2-\delta^2/4-\sigma\delta-(\tfrac{1}{2}+\sigma^2)] \qquad (\delta \gg 1). \tag{5.19}$$

It is easily checked that this expression solves the Navier–Stokes momentum equation for plane Poiseuille flow and satisfies the boundary conditions (5.17).

We can also easily write down the distribution function in the main body of the flow. As a matter of fact, $Y(x, \xi)$ reduces here to A_0, so that Eqs (5.8) and (5.9) give

$$W(x, \xi) = \frac{k}{2}\left[x^2 - \frac{\delta^2}{4} - 2x\xi - (1-2\xi^2) - \sigma\delta - (\tfrac{1}{2}+\sigma^2)\right] \quad (\delta \gg 1). \tag{5.20}$$

By taking Eq. (5.19) into account, Eq. (5.20) can be rewritten as follows:

$$W(x, \xi) = v_3(x) - \theta \frac{\partial v_3}{\partial x}\xi + (\xi^2 - \tfrac{1}{2})\theta^2 \frac{\partial^2 v_3}{\partial x^2} \quad (\delta \gg 1) \tag{5.21}$$

where general units for x have been restored (i.e., we have written x/θ in place of x). Equation (5.21) clearly shows that, in the main body of the flow the distribution function is the Hilbert–Chapman–Enskog type (power series in θ), as was to be expected. It is a truncated series, but the truncation does not occur at the Navier-Stokes level of description. As a matter of fact, Eq. (5.21) gives a Burnett distribution function and this explains the appearance of a second order slip from a formal point of view. From an intuitive standpoint, the second order slip can be attributed to the fact that molecules with nonzero velocity in the z-direction move into a region with different density before having any collision and there is a net transport of mass because of the density gradient; i.e. molecules move preferentially toward smaller densities even before suffering any collision, and, therefore, at a mean free path from the wall an effect of additional macroscopic slip appears.

The presence of an additional slip means that, for a given pressure gradient and plate distance, more molecules pass through a cross section that predicted by Navier–Stokes equations with first order slip. This is easily checked, for sufficiently large δ, by using Eqs. (5.18), (5.15), (5.16), which give for the flow rate:

$$F = \int_{-d/2}^{d/2} \varrho v_3(x)\,dx \simeq -\tfrac{1}{2}\frac{d\varrho}{dx} d^2 \left\{\tfrac{1}{6}\delta + \sigma + \frac{2\sigma^2 - 1}{\delta}\right\} \quad (\delta \gg 1). \tag{5.22}$$

Here x and z are in general units and $d = \delta\theta$ is the distance between the plates in the same units. Therefore, for given geometry and pressure gradient, the nondimensional flow rate is

$$Q(\delta) = \tfrac{1}{6}\delta + \sigma + \frac{2\sigma^2 - 1}{\delta} \quad (\delta \gg 1). \tag{5.23}$$

The last term is the correction to the first order slip theory; it arises in part from the second order slip and in part from the kinetic boundary layers. In fact the gas near the walls moves more slowly than predicted by an extrapolation of Eq. (5.19); this brings in a contribution to $Q(\delta)$ of the same order as the second order slip, thus reducing the effect of the latter (without eliminating it completely, however). It is also clear that, although Eq. (5.23) is valid for large values of δ, the increase in $Q(\delta)$ with respect to the prediction of the first order slip theory persists for small values of δ, because the molecules with velocity almost parallel to the wall give a sensible contribution to the motion by travelling downstream for a mean free path. In particular, in the limiting case of free molecular flow, Eq. (5.8) formally reduces to

$$\xi\, \partial W/\partial x + k/2 = 0 \tag{5.24}$$

or

$$W = -(k/2)(x/\xi + d/2|\xi|) \tag{5.25}$$

where general length units are used. Equation (5.25) clearly shows that molecules travelling almost parallel to the wall ($\xi \simeq 0$) cannot be in free molecular flow. Equation (5.25) can be assumed to hold for $|\xi| < \delta$ (recall that $|\xi|$ is nondimensional). Hence for $\delta \to 0$

$$v_3 \simeq -\frac{k}{2}\frac{d}{2}\pi^{-1/2}\int_{|\xi|>\delta}\frac{1}{|\xi|}e^{-\xi^2}\,d\xi \simeq -\frac{k\pi^{-1/2}}{2}\,d\log\delta \qquad (\delta \ll 1) \tag{5.26}$$

and

$$Q(\delta) \simeq -\pi^{-1/2}\log\delta \qquad (\delta \ll 1). \tag{5.27}$$

This approximate argument is confirmed by a study of the nearly-free molecular regime ($\delta \to 0$). This study can be based either on iteration procedures or on a different use of the method of elementary solutions. In both cases the conclusion is that Eq. (5.27) is correct and this means that, for $\delta \to 0$, higher order contributions from kinetic layers destroy the $1/\delta$ term in Eq. (5.23) but leave a weaker divergence for $\delta \to 0$ (essentially related to the molecules travelling parallel to the plates). The behavior for large values of δ, Eq. (5.23), and for small values of δ, Eq. (5.27), imply the existence of at least one minimum in the flow rate. This minimum was experimentally found a long time ago by Knudsen and then by different authors for long tubes of various cross-section. The above discussion gives a qualitative explanation of the presence of the minimum, although its precise location for slabs and more complicated geometries must be found by appropriate techniques of solution.

6. Unsteady solutions of kinetic models with constant collision frequency

If one considers the time dependent BGK equation in one-dimensional plane geometry, shear effects can be separated from effects related to normal stresses and heat transfer in the same way as for steady situations. The relevant equation for shear flows problems is as follows:

$$\frac{\partial Y}{\partial t}+\xi\frac{\partial Y}{\partial x}+Y(x,\xi)=\pi^{-1/2}\int_{-\infty}^{\infty}e^{-\xi_1^2}Y(x,\xi_1)d\xi_1, \tag{6.1}$$

i.e. the time-dependent analogue of Eq. (3.1) (when we let $v=1$ and assume that Y does not depend upon ξ_2 and ξ_3, which is usually the case). Here both x and t are expressed in θ units, since we have let $v=\theta^{-1}=1$.

The method to be employed to study Eq. (6.1) can be described as follows. A Laplace transform is taken with respect to time and accordingly the time-dependent problem is reduced to a steady one. The solution of the problem depends now on a complex parameter s. After separating the space and velocity variables, the spectrum of values of the separation parameter u must be studied in its dependence on s. This study is essential in order to treat the problem of inversion.

Let us take the Laplace transform of Eq. (6.1). Without any loss of generality, a zero initial value for Y will be assumed. In fact, a particular solution of the inhomogeneous transformed equation, which would result from a nonzero initial condition, can be constructed by using the Greens' function and the latter can easily be obtained when the general solution of the homogeneous equation is known. Accordingly, we shall restrict ourselves to the homogeneous transformed equation:

$$(s+1)\tilde{Y}+\xi\frac{\partial\tilde{Y}}{\partial x}=\pi^{-1/2}\int_{-\infty}^{\infty}e^{-\xi_1^2}\tilde{Y}(x,\xi_1)d\xi_1, \tag{6.2}$$

where $\tilde{Y}$ is the Laplace transform of Y. The same equation (with $s=i\omega$) governs the state of a gas forced to undergo steady transverse oscillations with frequency ω.

Separating the variables in Eq. (6.2) gives

$$\tilde{Y}_u(x,\xi;s)=g_u(\xi;s)\exp\left[-(s+1)x/u\right], \tag{6.3}$$

where u is the separation parameter and $g_u(\xi;s)$ satisfies

$$(s+1)(1-\xi/u)g_u(\xi;s)=\pi^{-1/2}\int_{-\infty}^{\infty}g_u(\xi_1;s)e^{-\xi_1^2}d\xi_1. \tag{6.4}$$

The right-hand side does not depend on ξ and can be normalized to unity. Accordingly, we are led to a typical division problem in complete analogy with the steady case. If the factor $(u-\xi)$ cannot be zero, i.e., u is not a real number, $g_u(\xi; s)$ is an ordinary function given by

$$g_u(\xi; s) = u/(u-\xi) \tag{6.5}$$

with the normalization condition:

$$\pi^{-1/2} \int_{-\infty}^{\infty} \frac{ue^{-\xi^2}}{u-\xi}\, d\xi = s+1. \tag{6.6}$$

If, on the contrary, u is a real number, $g_u(\xi)$ must be treated as a generalized function and Eq. (6.4) gives:

$$g_u(\xi; s) = P\frac{u}{u-\xi} + p(u; s)\delta(u-\xi), \tag{6.7}$$

where $p(u; s)$ fixed by Eq. (6.6), is given by

$$p(u; s) = \pi^{1/2} e^{u^2} s + p(u), \tag{6.8}$$

$p(u)$ being given by Eq. (3.41). Equation (6.7) gives the generalized eigensolutions corresponding to the continuous spectrum $(-\infty < u < \infty)$. The essential point, now, is to study the possible values of u which satisfy Eq. (6.6) and therefore form the discrete spectrum. Such values coincide, according to Eq. (6.6) with the zeroes of the following function of the complex variable z:

$$M(z; s) = 1 - \pi^{-1/2}(s+1)^{-1} \int_{-\infty}^{\infty} \frac{ze^{-t^2}}{z-t}\, dt. \tag{6.9}$$

This function is analytic in the complex z-plane with a cut along the real axis where $M(z; s)$ suffers a discontinuity. In fact, the Plemelj formulas (see [2]) give the following result for the limiting values $M^{\pm}(u; s) = \lim_{\varepsilon\to 0} M(u \pm i\varepsilon; s)$ (u real, $\varepsilon > 0$):

$$M^{\pm}(u; s) = 1 - \pi^{-1/2}(s+1)^{-1}\left[P \int_{-\infty}^{\infty} \frac{ue^{-t^2}}{u-t}\, dt \pm \pi i u e^{-u^2}\right]. \tag{6.10}$$

Equation (6.10) can also be written as follows:

$$M^{\pm}(u; s) = e^{-u^2}(s+1)^{-1}\pi^{-1/2}[p(u; s) \pm \pi i u]. \tag{6.11}$$

In the limiting case of s such that:

$$p(u; s) \pm \pi i u = 0 \quad \text{(real } u\text{)}, \tag{6.12}$$

the discrete spectrum merges into the continuous one. Equations (6.12) are satisfies on a closed heart-shaped curve of the complex s-plane. We have the following parametric representation for such curve (to be called γ):

$$\begin{aligned} \operatorname{Re} s &= -\pi^{-1/2} e^{-u^2} p(u) \\ \operatorname{Im} s &= -\pi^{1/2} u e^{-u^2} \qquad (-\infty < u < \infty). \end{aligned} \tag{6.13}$$

These equations are obtained from Eq. (6.12) and (6.8), taking into account that $p(u)$ is real. The equation $M(z; s) = 0$ defines a mapping from the z-plane to the s-plane: in fact this equation gives unambiguously a point in the s-plane once a point z off the real axis has been fixed. When z goes to a real value u, $M(z; s) = 0$ becomes Eq. (6.12), because of Eq. (6.11); the double sign of course, is connected with going from above or from below. Therefore, when u ranges through the real axis, s describes the curve γ counterclockwise if we think of the real axis as the boundary of the upper half plane, clockwise if we think of the real axis as the boundary of the lower half plane. In both cases Eq. (6.13) establishes a one-to-one correspondence between the curve γ of the s-plane and the real axis of the z-plane. From this fact and the argument principle it follows that both the lower and the upper half plane are conformally mapped into the region inside γ by the mapping $M(z; s) = 0$ and for each half plane the mapping is one-to-one. It follows that for any s in the region inside γ there are two complex values of u which satisfy Eq. (6.6) while there none outside. It is easily seen, from Eq. (6.6), that these values are negatives of each other. We shall denote them by $\pm u_0(s)$.

It is now possible to extend the results of the steady case $s = 0$ and, in particular, Theorem I and II of Section 3. The completeness results remain true and there are only slight changes in the equation. Thus, in the case of full range $(-\infty < \xi < \infty)$ we can expand any function $g(\xi)$ such that $Z(\xi) = e^{-\xi^2} g(\xi)$ satisfies the assumptions of theorem I as follows:

$$g(\xi) = A_+ g_+(\xi; s) + A_- g_-(\xi; s) + \int_{-\infty}^{\infty} A(u) g_u(\xi; s)\, du \tag{6.14}$$

where $g_u(\xi; s)$ is given by Eq. (6.7) and $g_\pm$ by Eq. (6.5) with $u = \pm u_0$. The coefficients A_+ and A_- are zero outside γ (g_+ and g_- do not exist there) while are given by

$$A_\pm = \pi^{-1/2} [s(1 - 2u_0^2) + 1]^{-1} \int_{-\infty}^{\infty} \frac{\xi e^{-\xi^2}}{\xi \pm u_0} g(\xi)\, d\xi \tag{6.15}$$

for s inside γ. For any s, $A(u)$ is given by Eq. (3.28), provided that $p(u)$ is replaced by $p(u; s)$ throughout, $Z(w) = Z_0(w) g(w)$ and $Z_0(w)$ is given by Eq. (3.36). It is obvious that $A_\pm$ and $A(u)$ and possibly $g(\xi)$, depend upon s, though this dependence has not been exhibited in the equations. In the case of the half range $0 < \xi < \infty$, a function $g(\xi)$ can be expanded as follows

$$g(\xi) = A_+ g_+(\xi; s) + \int_0^{\infty} A(u) g_u(\xi; s)\, du \tag{6.16}$$

where A_+ is zero for s outside γ and is given by

$$A_+ = [2u_0^2 s\pi^{1/2} P(u_0)]^{-1} \int_0^\infty \xi g(\xi) g_+(\xi; s) P(\xi; s) e^{-\xi^2} d\xi \tag{6.17}$$

for s inside γ. Here

$$P(u; s) = \frac{u}{u_0+u} \exp\left\{ -\frac{1}{\pi} \int_0^\infty \tan^{-1}[\pi t/p(t; s)] \frac{dt}{t+u} \right\} \tag{6.18}$$

For any s, $A(u)$ is given by Eq. (3.32), where $Z(w) = Z_0(w)g(w)$ and $Z_0(w)$ is given by Eq. (3.40), while $P(u)$ is given by Eq. (6.18) when s is inside γ and by the same equation with $u_0 = 0$ when s is outside γ. The function $P(u; s)$ again satisfies certain identities which make its manipulation simpler than would be expected.

7. Analytical solutions of specific problems

The theory sketched in Section 6 can be used to solve analytically problems of shear flows when the region filled by gas is the whole space or a half space. One can, e.g., solve the following problem: let two half-spaces be separated by the plane $x = 0$, and assume that initially the gas has the same density ϱ_0 and temperature T_0 in both regions, while the gas in the region $x > 0$ flows uniformly in the z-direction with velocity U and the gas in the region $x < 0$ flows uniformly in the same direction with velocity $-U$; we want to find the evolution of the gas, i.e., the smoothing out and diffusion of the velocity discontinuity. The problem can be solved by using the theorem of full range completeness to construct the Laplace transform of the solution. One can even obtain an analytic inversion of the Laplace transform and write the solution for the mass velocity as follows:

$$\begin{aligned} v(x, t) = U \operatorname{sgn} x \Bigg[1 - 2\pi^{-1/2} \int_{-\infty}^{\infty} & H[(u/x)-(t/t)] \exp\{-x/(\vartheta u) + \\ & + q(u)[(x/u)-t]/\vartheta\} \left\{ e^{-u^2} \cos\left[\frac{\pi^{1/2} u}{\vartheta} (t-x/u) e^{-u^2} \right] + \right. \\ & \left. + \frac{1-q(u)}{u\pi^{1/2}} \sin\left[\frac{\pi^{1/2} u}{\vartheta} (t-x/u) e^{-u^2} \right] \right\} du \Bigg] \end{aligned} \tag{7.1}$$

where H is the Heaviside step function $\operatorname{sgn} x = H(x) - H(-x)$ and

$$q(u) = \pi^{-1/2} e^{-u^2} p(u). \tag{7.2}$$

One can use the exact solution to obtain asymptotic expansions for both short and long times and numerical tabulation of the space-time behavior of the gas. The solution shows that the velocity profile becomes more and more flattened as time goes on, but a disagreement of about 10% from the Navier–Stokes equations is still present after 12 collision times.

Half-space problems are more difficult to solve, since they require using the half-range completeness theorem and, consequently, equations involving $P(u; s)$. The solution can, however, be always reduced to a double quadrature for initial value problems and a single quadrature for problems of steady oscillations, provided the boundary conditions give an explicit expression of the distribution function of the molecules entering the half-space (as in the case of complete diffusion from the wall).

As an example of half-space problem, we consider the propagation of Rayleigh waves in a half-space: let a half-space be filled with a gas of density ϱ_0 and temperature T_0 and bounded by an infinite plane wall which is oscillating in its own plane with frequency ω. We shall consider the system in a steady state when the transients have disappeared. Therefore, if the velocity of the wall is the real part of $Ue^{i\omega t}$, U being a constant, the solution of the linearized problem will be the real part of a function h having time dependence $e^{i\omega t}$ and satisfying

$$i\omega h + \xi_1\, \partial h/\partial x = Lh. \tag{7.3}$$

The linearized boundary condition at the wall, which is assumed to diffuse the molecules according to the basic Maxwellian f_0, can be written as follows:

$$h(0, \boldsymbol{\xi}, t) = 2Ue^{i\omega t}\xi_3 \qquad (\xi_1 > 0). \tag{7.4}$$

We require also that the solution be bounded at infinity. If the BGK model is assumed to describe collisions, we can write

$$h(x, \boldsymbol{\xi}, t) = 2Ue^{i\omega t}\xi_3\, \tilde{Y}(x, \xi_1), \tag{7.5}$$

where $\tilde{Y}(x, \xi)$ satisfies Eq. (6.2) (with $s = i\omega$) and the boundary condition:

$$\tilde{Y}(0, \xi) = 1 \qquad (\xi > 0). \tag{7.6}$$

We shall write s in place of $i\omega$, because the following results are valid for any complex a ($\operatorname{Re} s \geqslant -1$) and the more general form will be useful later. Using the boundary condition, Eq. (7.6), and half-range completeness (see Sect. 6) together with boundedness at infinity in space, we find:

$$\tilde{Y}(x, \xi) = -\pi^{1/2} s \int_0^\infty \frac{g_u(\xi; s)[P(u; s)]^{-1}}{[p(u; s)]^2 + \pi^2 u^2} \exp\left[-(s+1)\frac{x}{u} + u^2\right] du, \tag{7.7}$$

when s is outside γ (then $P(u; s)$ is given by Eq. (6.18) with $u_0 = 0$). Analogously, when s is inside γ we find:

$$(7.8)\quad \tilde{Y}(x,\xi) = -2g_+(\xi)\exp\left[-(s+1)x/u_0\right]\left[P(u_0;s)\right]^{-1} - \\ -\pi^{1/2}s\int_0^\infty \frac{\left[P(u;s)\right]^{-1}g_u(\xi;s)\exp\left[-(s+1)x/u+u^2\right]}{\left[p(u;s)\right]^2+\pi^2u^2}\,du,$$

where $u_0 = u_0(s)$ is selected between the two possible values in such a manner that

$$(7.9)\qquad \operatorname{Re}\left[\frac{s+1}{u_0(s)}\right] > 0$$

and $P(u;s)$ is given by Eq. (6.18).

Let us now briefly discuss the solution. First of all, we note that there is a limiting frequency ω_0 ($i\omega_0 \in \gamma$) such that for $\omega > \omega_0$ we have only the eigensolutions of the continuous spectrum. It seems, therefore, that for $\omega > \omega_0$ no plane shear wave exists. However, we are able to exhibit a discrete term for $\omega > \omega_0$; as a matter of fact, we can rotate the path of integration in Eq. (7.8) downward, provided that we add the contribution from any poles of the integrand between this half straight line and the real semi-axis. Now, it is easily seen that, at least for frequencies larger than, but still close to ω_0 there is one such pole u_0 which satisfies

$$(7.10)\qquad p(u_0;s) - \pi i u_0 = 0,$$

where $p(u;s)$ is given by Eq. (6.8) and the second expression in Eq. (3.41), which has a meaning for any complex u. Equation (7.10) is the analytical continuation of Eq. (6.6) for $\omega > \omega_0$. From this point of view, ω_0 loses its character of critical frequency. Another feature of our results is that for any fixed frequency, if we go sufficiently far from the wall, the contribution from the continuous spectrum dominates the discrete term, since the former is less than exponentially damped. This feature is strictly related to the fact that the spectrum of values of v extends to ∞ and would not be present if the collision frequency increased at least linearly with molecular speed for large values of the latter. The experimental verification of this asymptotic behavior seems to be outside the available techniques. As a matter of fact, the physically relevant region (say 1/10 to 10 mean free paths) appears to be characterized by the fact that the discrete term (either the genuine one or its analytical continuation) dominates, according to estimates made by Dorning and Thurber for a similar problem concerning neutron waves.

Another solvable problem is the following: Let a half-space be filled with a gas of density ϱ_0 and temperature T_0 and bounded by a plate; the gas is initially in absolute equilibrium and the wall is at rest; then the plate is set impulsively into motion in its own plane with constant velocity U; the propagation into the gas of the disturbance produced by the motion of the plate is to be studied. This problem is known under the name of Rayleigh's problem; we want to solve it analytically by using the linearized BGK model.

The perturbed distribution function satisfies the linearized Boltzmann equation and the following initial and boundary conditions:

$$h(x, 0, \boldsymbol{\xi}) = 0, \tag{7.11}$$

$$h(0, t, \boldsymbol{\xi}) = 2U\xi_3 \quad (\xi_3 > 0). \tag{7.12}$$

Also, when $x \to \infty$, $h(x, t, \boldsymbol{\xi})$ must be bounded for any fixed t and if we use the BGK model, we have

$$h(x, t, \boldsymbol{\xi}) = 2U\xi_3\, Y(x, t, \xi_1), \tag{7.13}$$

where $Y(x, t, \xi)$ satisfies Eq. (6.1) and the following initial and boundary conditions:

$$Y(x, 0, \xi) = 0, \tag{7.14}$$

$$Y(0, t, \xi) = 1 \quad (\xi > 0). \tag{7.15}$$

By introducing the Laplace transform of Y, $\tilde{Y}(x, s, \xi)$, Eq. (6.1) reduces to Eq. (6.2) while the boundary condition at the wall becomes

$$\tilde{Y}(0, s, \xi) = 1/s \quad (\xi > 0). \tag{7.16}$$

Accordingly, $\tilde{Y}$ is obtained from the equations for the oscillating wall by multiplying the right-hand side of the equations by $1/s$ the same is also true for the mass velocity and the stress. It is important to note that $\tilde{Y}$ defined by Eqs (7.7) and (7.8) for s outside and inside γ, is an analytic function of s not only outside and inside γ but also through this curve; in other words, Eq. (7.8) is the analytic continuation of Eq. (7.7) inside γ. This follows from the fact that the expression appearing in Eq. (7.7) undergoes a discontinuity when s crosses γ and these discontinuity is equal to the limiting value of the discrete term in Eq. (7.8). Hence, when inverting the Laplace transform, the path of integration can be moved trough γ, provided that in every region the appropriate expression is used. On the other hand, the segment $(-1, 0)$ of the real axis is easily seen to be a discontinuity line because of the choice of u_0, Eq. (7.9). According to well-known theorems on the Laplace transform, $Y(x, t, \xi)$ is given by

$$Y(x, t, \xi) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{s}\, \tilde{Y}(x, s, \xi)\, ds, \tag{7.17}$$

where $\tilde{Y}$ is given by Eq. (7.7) and the path of integration is a vertical straight line to the right of γ. Owing to the analyticity properties of $\tilde{Y}$, the this integration path is the s-plane can be deformed to a path indented on the segment $(-1, 0)$ of the real axis, and along the vertical line $\mathrm{Re}\,(s+1) = 0$. The resulting integrals can be put into a completely real form.

Thus the problem is solved in terms of quadratures, which can, in principle, be performed with any desired accuracy. But we can use our results also for obtaining interesting information by analytical manipulations. We can, e.g., expand our results for short and long times.

For long times, e.g., the mass velocity is given by:

$$v(x, t)/U \simeq 1-(\pi\nu t)^{-1/2} v_3(x), \qquad t \to \infty, \tag{7.18}$$

where ν is the kinematic viscosity ($\theta = 2\nu$ if $2RT_0 = 1$), and $v_3(x)$ the mass velocity corresponding to a unit gradient at infinity in the Kramers problem (Eq. (4.15) with $a = 1$). Therefore the flow shows the same structure of the kinetic boundary layer both in the steady and in the time-dependent flows (for large values of the time variable); this is not surprising, because that the equations which describe the kinetic layer do not depend upon the particular problem. In particular, the slip coefficient retains, in this time-dependent flow and for sufficiently large values of t, the same values as in the steady flows.

The analytical solution also leads to simple expressions of the velocity and the stress at the plate. As an example, we quote

$$\begin{aligned} v(0, t)/U &= 1-\frac{1}{\pi}\int_0^1 (u^{-1}-1)^{1/2} e^{-ut}\,du \\ &= \tfrac{1}{2}+\tfrac{1}{2}\int_0^{t/2} e^{-v} v^{-1} I_1(v)\,dv \end{aligned} \tag{7.19}$$

where I_1 denotes the modified Bessel function of the first kind and order one.

References

A full bibliography covering the matter treated in the previous pages would take several pages. Accordingly mention is restricted here to two books of the author, where one can find a detailed list of references as well as a more extended treatment of the topics presented here:

[1] C. Cercignani, *Theory and Application of the Boltzmann Equation*, Scottish Academic Press, Edinburgh, and Elsevier New York 1975.

[2] – *Mathematical Methods in Kinetic Theory*, Plenum Press, New York and McMillan, London 1969.

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