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The algebraic theory of compact Lawson semilattices
Applications of Galois connections to compact semilattices

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CONTENTS

Dedicated to Paul Dubreil
on the occasion of his 70th birthday
and his retirement

Introduction

Two aspects of this paper may attract the reader's interest – first, its objective and, second, its methods. The objective is to illuminate wider and wider classes of compact semilattices by demonstrating their intimate relationship to other well-studied areas in lattice theory and topological algebra. The method we employ in the pursuit of this objective is the extensive application of the theory of Galois connections. We have found this theory to be most useful when a Galois connection is thought of as a particular sort of adjoint situation.

Compact semilattices arise quite naturally in the theory of compact abelian monoids. To every such monoid S one associates the semilattice $E(S)$ of its idempotents as an important structural invariant. A thorough understanding of compact semilattices is thus as important in the theory of compact abelian monoids as is an understanding of compact groups. In fact, the theory of compact abelian monoids might be thought of as an amalgam of these two theories.

It is the consensus among workers in the field that a systematic study of $\mathscr{C}$, the category of compact semilattices, is rather hopeless at this time. However, two subcategories of $\mathscr{C}$ distinguish themselves as being most deserving of study. They are $\mathscr{CL}$, the category of compact Lawson semilattices and its subcategory $\mathscr{Z}$, the category of compact zero-dimensional (or pro-finite) semilattices. In a recent monograph, M. Mislove and the authors have demonstrated the exclusively algebraic character of $\mathscr{Z}$ by first showing that $\mathscr{Z}$ is isomorphic (which is a bit stronger than equivalent) to the category of complete algebraic lattices (which has for morphisms maps which preserve arbitrary infs and sups of upward directed nets) and, secondly, by systematically exploiting the Pontryagin duality which exists between $\mathscr{Z}$ and the category $\mathscr{S}$ of discrete semilattices.(1) For applications in topological algebra the study of $\mathscr{Z}$ will

(1) We avail ourselves of this opportunity to acknowledge a paper by Geissinger and Graves [6] which came to our attention after our duality memoir [8] appeared. In a purely algebraic framework, these authors present results which overlap and antedate a portion of the topological-algebraic theory developed in [8].

not be enough, particularly when the study of connected objects becomes imperative. Thus our attention moves to $\mathscr{CL}$.

At this point, $\mathscr{CL}$ might best be described by saying that $S \in \mathscr{C}$ belongs to $\mathscr{CL}$ if the characters $S \to I$ into the unit interval semilattice separate points. It was Lawson who contributed essentially to our understanding of $\mathscr{CL}$ (cf. [12], [13]). He showed that $\mathscr{CL}$ was a proper subclass of $\mathscr{C}$. Although much work has been done in $\mathscr{CL}$ we are still far from fully understanding it. This is illustrated by the ease with which significant open problems can be formulated (see e.g. Hofmann and Stralka [10], [11]) and with the comparatively sophisticated constructions which establish the existence of very complicated phenomena (cf. [11]). In addition, we do not have at this point a systematic theory for $\mathscr{CL}$ which links it to other well understood categories. It is true, as Hofmann and Mislove pointed out ([7]), that one can exhibit a category of operator algebras which is dual to $\mathscr{CL}$, even in such a way as to deserve to be called a Pontryagin duality. However, the dual category is probably more complicated than $\mathscr{CL}$ itself, so that little is gained for applications and a certain purely theoretical interest is all that remains of this duality.

In such a framework one can understand our present efforts to find close functorial links between the categories $\mathscr{Z}$ and $\mathscr{CL}$. It has been known for some time that a compact semilattice S belongs to $\mathscr{CL}$ if and only if it is the quotient of a $\mathscr{Z}$-semilattice. In this paper we propose a functor $P\colon \mathscr{CL} \to \mathscr{Z}$ such that for $L \in \mathscr{CL}$, PL is in many respects a canonical pre-image of L under a natural quotient map $r_L\colon PL \to L$. In essence the link is given by the fact that the Pontryagin dual of PL is the discrete opposite lattice underlying L.

The success of the reduction of $\mathscr{CL}$ to $\mathscr{Z}$ now depends largely on the detailed analysis of this situation. *First*, one wants to know which $\mathscr{Z}$-objects T can actually appear as canonical pre-images PL of some Lawson semilattice L. By what was said above, a necessary condition is that the character semilattice T be a complete lattice (since the underlying semilattice of any compact semilattice is always a complete lattice); but this is not sufficient. In the process of answering this question fully we give a complete algebraic characterization of compact Lawson semilattices. Indeed we provide a necessary and sufficient algebraic condition for a complete lattice L to carry a compact topology relative to which it belongs to $\mathscr{CL}$, namely, that for each $x \in L$ there is a smallest (lattice) ideal J such that $\sup J = x$. *Secondly*, one hopes to know the universal properties of the morphism $r_L\colon PL \to L$. Here we have two results. The first is expressed by saying that P defines a left reflector from the category $\mathscr{CL}$ into the subcategory $\mathscr{D}$ of $\mathscr{Z}$ containing all objects with complete dual and a class of morphisms, called compact morphisms,

because they preserve compact elements. In other words: If $T \in \mathscr{X}$ has a complete dual and if $f\colon T \to L$ is a morphism into some $L \in \mathscr{CL}$, then there is a morphism $f'\colon T \to PL$ with $f = r_L f'$, and there is only one such morphism which preserves compact elements. The second universality result is somewhat of a converse: If $\mathscr{E}$ denotes the class of all $\mathscr{CL}$-morphisms which are surjective and are lattice morphisms then PL is an $\mathscr{E}$-projective and $r_L \in \mathscr{E}$. Specifically, if $e\colon S \to T$ is in $\mathscr{E}$ and $r\colon PL \to T$ is any morphism, then there is a morphism $\bar{e}\colon PL \to S$ with $e\bar{e} = r$. (This applies, in particular, with $T = L$ and $r = r_L$.) *Thirdly*, one wishes to understand the internal geometric structure of PL. Concretely, PL is realized as the $\cap$-semilattice of all lattice ideals of L. An element $J \in PL$ is compact if and only if it is a principal ideal. Trivially, such a J is the unique largest lattice ideal with $\sup J = x$. By an earlier remark we associate with each $x \in L$ a unique smallest lattice ideal J with $\sup J = x$; we define $s_L\colon L \to PL$ by $s_L(x) = J$. Then s_L preserves arbitrary sups, and every compact subset C of PL with $r_L(C) = L$ contains $s_L(L)$. We denote with QL the smallest closed and multiplicatively closed subset of PL containing $s_L(L)$. Then QL is a Z-object which appears to be an important invariant of L even though the assignment $L \to QL$ is not functorial in any obvious fashion (as we exemplify by several examples). In certain special cases we are able to identify the structure of QL. We denote with AL the set of all elements x in L which are "openly accessible from above", i.e., for which $x \in (\text{interior } \uparrow x)^-$, where $x = \{y \in L\colon x \leqslant y\}$. Examples show that AL is neither closed relative to sup nor to inf. If, however, AL is a sublattice of L (which does occur not too infrequently), then QL is naturally isomorphic to the character semilattice of the supsemilattice underlying AL. These results are illustrated by examples.

We have explained the objective of the paper and its main results; now a few words on some aspects of the methods! A consistent theme in this paper is the continual application of Galois connections. In lattice theory and indeed the theory of partially ordered sets this is a classical tool which was introduced by Ore in 1944 and has been treated in the literature off and on during these past three decades. Through category theory it has been placed in its appropriate context of adjoint situations (which we choose as starting points). What is novel here is that for the first time, as far as we know, this tool is systematically introduced in the study of topological (notably compact) semilattices and lattices. At the risk of adhering to a minority notation, we call a Galois connection between two posets S, T, a pair of order preserving maps $g\colon S \to T$ and $d\colon T \to S$ such that, considered as functors, g is left adjoint to d. J. Schmidt denotes this situation a Galois connection of mixed type and remarks: "It is a curious fact that Galois connections of mixed type, in spite of

their frequent occurrence have not been paid much attention" [19]. In the same situation, the map d is called *residuated* by Derdérian [4], and under this terminology, residuation theory has now been developed in the book by Blyth and Janowitz [2]. If S and T are compact Lawson semilattices, then every morphism $q\colon S\to T$ is the left adjoint of a Galois connection. Its right adjoint $d\colon T\to S$ satisfies (i) d preserves sups, (ii) $d(\text{interior} \uparrow t) \subseteq \text{interior} \uparrow d(t)$ for all $t\in T$. Conversely, if a function $d\colon T\to S$ satisfies (i) and (ii), then it has a left adjoint g and g is a $\mathscr{CL}$-morphism. If S and T are in fact in $\mathscr{L}$, and if $K(S)$ is the sup-semilattice of compact elements in S then $K(g)\colon K(T)\to K(S)$ (as discussed in [8]) is precisely $d|K(T)$. Time and again Galois connections are used in the discussion; typically, the morphism $r_L\colon PL\to L$ is left adjoint to the map $s_L\colon L\to PL$.

The history of Galois connections between partially ordered sets comprises the names and works of Ore 1944 [14], Everett 1944 [5], Pickert 1952 [17], Aumann 1955 [1], Raney 1960 [18], Derdérian 1967 [4], Schmidt 1973 [19], Blyth and Janowitz 1973 [2]. The history of Lawson semilattices is indicated mainly by the papers of Lawson himself. The functor P links $\mathscr{CL}$ with the category of algebraic lattices whose history is marked by the work of Birkhoff, Frink, Nachbin, Dilworth, Crawley, and others; for more details we can refer to [8].

The content of this paper is arranged as follows. In Section 1 we present the theory of Galois connections for compact semilattices, including the basics in order to keep the presentation as self-contained as possible. Section 2 presents the theory of compact zero dimensional semilattices with complete dual; in this section we develop the algebraic theory of Lawson semilattices, and prepare much of the theory on which the following sections are based. Section 3 contains the discussion of the functor P and its properties, and Section 4 discusses the structure of PL and the significance of the set AL of elements which are openly accessible in L. The paper is fairly self-contained; however, reference to [8] as a source for the relevant duality theory is unavoidable.

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List of categories

General convention: "semilattice" means idempotent commutative monoid, and semilattice morphism means (identity preserving) monoid morphism between semilattices.

$\mathscr{S}$ — category of semilattices and semilattice morphisms.

$\mathscr{Z}$ — category of compact zero-dimensional semilattices and continuous semilattice morphisms.

$\mathscr{CL}$ — category of compact Lawson semilattices and continuous semilattice morphisms.

$\mathscr{C}$ — compact semilattices and continuous semilattice morphisms.

$\mathscr{V}$ — subcategory of $\mathscr{S}$ of all complete lattices and all lattice morphisms $f\colon S \to T$ preserving arbitrary infs.

$\mathscr{D}$ — subcategory of $\mathscr{Z}$ of all objects T with complete dual T and of all morphisms $f\colon S \to T$ with $f(K(S)) \subseteq K(T)$.

The subcategories $\mathscr{VL} \subseteq \mathscr{V}$ and $\mathscr{DL} \subseteq \mathscr{D}$ are defined in 2.28.

1. Galois connections

a. The basic theory of Galois connections. All of the elementary theory of Galois connections between partially ordered sets allows direct proof. The way we choose to think of Galois connections however is as a very special instance of a pair of adjoint functors. For easy reference we formulate the characteristic properties of adjoint functors since they will also play an important role in the later sections. We caution the reader that in our discussion a Galois connection involves a pair of isotone (i.e., order-preserving) functions as in [19] rather than as a pair of antitone functions as is the practice in [15] and [16].

PROPOSITION 1.1. *Let $G\colon \mathcal{S}\to\mathcal{T}$ and $D\colon \mathcal{T}\to\mathcal{S}$ be a pair of functors. Then the following statements are equivalent:*

(1) *There is a natural transformation $\eta\colon 1_{\mathcal{S}}\to DG$ such that for every morphism $f\colon A\to DB$ in $\mathcal{S}$ there is a unique morphism $\varphi\colon GA\to B$ in $\mathcal{T}$ such that $f=(D\varphi)\eta_A$.*

(2) *There is a natural transformation $\varepsilon\colon GD\to 1_{\mathcal{T}}$ such that for every morphism $\varphi\colon GA\to B$ in $\mathcal{T}$ there is a unique $f\colon A\to DB$ in $\mathcal{S}$ such that $\varphi=\varepsilon_B(Gf)$.*

(3) *There is a natural isomorphism of sets $\alpha_{A,B}\colon \mathcal{T}(GA,B)\to\mathcal{S}(A,DB)$.*

(4) *There are natural isomorphisms $\eta\colon 1_{\mathcal{S}}\to DG$* and *$\varepsilon\colon GD\to 1_{\mathcal{T}}$ such that $(D\varepsilon_B)\eta_{DB}=1_{DB}$ and $\varepsilon_{GA}(G\eta_A)=1_{GA}$ for all objects A and B.*

The relations among the various natural transformations are as follows: $\eta_A=\alpha_{A,B}(1_{GA})$, $\varepsilon_B=\alpha^{-1}_{A,B}(1_{DB})$, $\alpha_{A,B}=D(?)\eta_A$ (i.e., $\alpha_{AB}(f)=(D\varphi)\eta_A$ for $\varphi\colon GA\to B$), $\alpha^{-1}_{A,B}=\varepsilon_B G(?)$. The φ in (1) is given by $\varepsilon_B(Ff)$, the f in (2) by $(D\varphi)\eta_A$. ■

DEFINITION 1.2. If the conditions of 1.1 are satisfied, then G is said to be a *left adjoint* of D, and D a *right adjoint* of G.

The most crucial property of adjoint functors is their preservation behavior.

PROPOSITION 1.3. *If $G\colon \mathcal{S}\to\mathcal{T}$ is left adjoint to $D\colon \mathcal{T}\to\mathcal{S}$, then D preserves limits and G preserves colimits.*

We specialize this framework immediately to a very simple situation. Every partially ordered set S (with a partial order $\geqslant$) gives rise to a category, whose objects are the elements of S and whose morphisms $f\colon x\to y$

are those pairs (x, y) with $x \geqslant y$, composition being $(y, x)(z, y) = (z, x)$. We will identify S with this category. Thus card $S(x, y) = 1$ if and only if $x \geqslant y$ and $S(x, y) = \emptyset$ otherwise. With this understanding a function $f\colon S \to T$ between partially ordered sets is a morphism of partially ordered sets (i.e., is isotone) if and only if f is a functor.

The set-up of adjoint functors is now reformulated for posets as:

PROPOSITION 1.4. *Let $g\colon S \to T$ and $d\colon T \to S$ be two morphisms of posets. Then the following conditions are equivalent:*

(1) *For all $s \in S$, $t \in T$ one has $s \geqslant dg(s)$ and if $s \geqslant d(t)$ then $g(s) \geqslant t$.*

(2) *For all $s \in S$, $t \in T$ one has $gd(t) \geqslant t$, and if $g(s) \geqslant t$ then $s \geqslant d(t)$.*

(3) *For all $s \in S$, $t \in T$ one has $g(s) \geqslant t$ if and only if $s \geqslant d(t)$.*

(4) $1_S \geqslant dg$ *and* $gd \geqslant 1_T$ [*with* $\geqslant$ *defined in the expected way*].

These conditions imply

(4′) $d = dgd$ *and* $g = gdg$.

Proof. (4) implies (4′): From (4) one obtains $d \geqslant (dg)d$ on one hand and $d(gd) \geqslant d$ on the other. Thus $d = dgd$. Similarly, $g = gdg$. After this remark one notes that the conditions of 1.4 are precisely the specializations of the corresponding conditions of (1.4). ■

As was noted previously, 1.4 also has a direct proof.

DEFINITION 1.5. If $g\colon S \to T$ and $d\colon T \to S$ are two morphisms of posets satisfying the equivalent conditions of 1.4, then (g, d) is called a *Galois connection* between S and T. We say that g is a *left adjoint* of d and d is a *right adjoint* of g. (For typographical reasons we choose d for droite and g for gauche.)

If S is a poset, let S^{op} be the poset corresponding to the opposite category, i.e., $x \leqslant^{\mathrm{op}} y$ in S^{op} if and only if $y \leqslant x$ in S. If (g, d) is a Galois connection between S and T, then $d = d^{\mathrm{op}}\colon T^{\mathrm{op}} \to S^{\mathrm{op}}$ is left adjoint to $g = g^{\mathrm{op}}\colon S^{\mathrm{op}} \to T^{\mathrm{op}}$.

For easy reference, the following propositions present the basic facts on Galois adjunctions.

PROPOSITION 1.6. *In a Galois connection one member determines the other uniquely* (i.e., *if* (g, d), (g', d) *and* (g, d') *are Galois connections then* $d = d'$ *and* $g = g'$).

Proof. Adjoint functors determine each other uniquely up to natural isomorphisms of functors. In the case of morphisms of posets, natural isomorphisms of functors means equality of morephisms. ■

PROPOSITION 1.7. *If (g, d) is a Galois connection, then g preserves all* (*existing*) *infs and d preserves all* (*existing*) *sups.*

Proof. If S is a poset and $X \subseteq S$, then $\inf X = \operatorname{colim} X$ and $\sup X = \lim X$. But left adjoint functors preserve colimits and right adjoint functors preserve limits (1.3). ■

Recall that for an element x of a poset S, $\uparrow x = \{s \in S\colon s \geqslant x\}$ and $\downarrow x = \{s \in S\colon s \leqslant x\}$.

PROPOSITION 1.8. *If (g, d) is a Galois connection between the posets S and T, then*

(i) $d(t) = \min g^{-1}(\uparrow t)$ *for all* $t \in T$.

(ii) $g(x) = \max d^{-1}(\downarrow s)$ *for all* $s \in S$.

Proof. We indicate the proof for (i). If $s \in g^{-1}(\uparrow t)$, then $g(s) \geqslant t$. Hence, $s \geqslant d(t)$ by 1.4.2. Also $g(d(t)) \geqslant t$ by 1.4.2, hence $d(t) \in g^{-1}(\uparrow t)$. This proves (i). ■

PROPOSITION 1.9. (i) *If S is a complete lattice and $g\colon S \to T$ is a function preserving arbitrary infs, then g has a right adjoint $d\colon T \to S$ defined by* $d(t) = \inf g^{-1}(\uparrow t) = \min g^{-1}(\uparrow t)$.

(ii) *If T is a complete lattice and $d\colon T \to S$ is a function preserving arbitrary sups, then d has a left adjoint $g\colon S \to T$ defined by* $g(s) = \sup d^{-1}(\downarrow s) = \max d^{-1}(\downarrow s)$.

Proof. (i) Since S is a complete lattice, $\inf g^{-1}(\uparrow t)$ exists for all t. Since g preserves infs, we have $g(d(t)) = g(\inf g^{-1}(\uparrow t)) = \inf g(g^{-1}(\uparrow t)) \geqslant y$, because $g(g^{-1}(\uparrow t)) \subseteq \uparrow t$. Suppose that $g(s) \geqslant t$, then $s \in g^{-1}(\uparrow t)$. Hence $d(t) = \inf g^{-1}(\uparrow t) \leqslant s$. Thus 1.4.2 is satisfied. The proof of (ii) is similar.

We remark that a poset S is a complete lattice iff it is complete as a category iff it is cocomplete as a category. In the light of this note, 1.9 is once more only a special case of general existence theorems of adjoint functors.

PROPOSITION 1.10. *Let (g, d) be a Galois connection between S and T.*

(i) *If $Y \subseteq T$ and if* $\inf Y$ *and* $\inf d(Y)$ *exist, then* $d(\inf Y) \leqslant \inf d(Y)$.

(ii) *If $X \subseteq S$ and if* $\sup X$ *and* $\sup g(X)$ *exist, then* $g(\sup X) \geqslant \sup g(X)$. *In particular, if S and T are lattices, then* $d(t_1 t_2) \leqslant d(t_1) d(t_2)$ *and* $g(s_1 \vee s_2) \geqslant g(s_1) \vee g(s_1)$.

Proof. We prove (i) and leave the rest as an exercise. By 1.4.4 one has $y \leqslant gd(y)$, hence $\inf Y \leqslant \inf gd(Y)$. But g preserves inf's by 1.7, hence $\inf gd(Y) = g(\inf d(Y))$. By 1.4.3, $\inf Y \leqslant g(\inf d(Y))$ is equivalent to $d(\inf Y) \leqslant \inf d(Y)$. ■

PROPOSITION 1.11. *If (g_1, d_1) is a Galois connection between S_1 and S_2 and (g_2, d_2) is a Galois connection between S_2 and S_3, then $(g_2 g_1, d_1 d_2)$ is a Galois connection between S_1 and S_3.*

Proof. Immediate from 1.4.3. ■

This proposition is also a special case of a more general proposition on adjoint functors.

PROPOSITION 1.12. *Let (g, d) be a Galois connection between S and T. Then the following statements are equivalent:*

(1) *g is surjective.*

(2) $gd = 1_T$.

(3) *d is injective.*

Likewise, the following are also equivalent:

(I) *g is injective.*

(II) $1_S = dg$.

(III) *d is surjective.*

Proof. If g is surjective, then $g = gdg$. (1.4.4′) implies $1_T = gd$. If $1_T = gd$, then d is a coretraction, hence injective. If d is injective, then $d = dgd$. (1.4.4′) implies $1 = gd$, which in turn implies that g is a retraction, hence surjective. Thus (1), (2) and (3) are equivalent. The equivalence of (I), (II) and (III) is proved analogously. ■

Summarizing some of these results we formulate

THEOREM 1.13. *For two complete lattices S and T, let* $\mathrm{INF}(S, T)$ *(resp.* $\mathrm{SUP}(T, S)$*) denote the set of all inf-preserving (resp. sup-preserving) functions from S into T (resp. from T into S). The function which associates with $g \in \mathrm{INF}(S, T)$ its unique right adjoint $d \in \mathrm{SUP}(T, S)$ is a canonical bijection mapping the set* $\mathrm{INF}_{\mathrm{sur}}(S, T)$ *of all surjective members of* $\mathrm{INF}(S, T)$ *onto the set* $\mathrm{SUP}_{\mathrm{inj}}(T, S)$ *of all injective members of* $\mathrm{SUP}(T, S)$; *likewise it establishes a bijection between* $\mathrm{INF}_{\mathrm{inj}}(S, T)$ *and* $\mathrm{SUP}_{\mathrm{sur}}(T, S)$. *The categories* INF *and* SUP, *whose objects are complete lattices and whose morphisms are inf-preserving (resp. sup-preserving) functions are dual to each other.* ■

b. Applications of Galois connections to compact semilattices. After these preliminaries we now consider compact topological semilattices. Recall that a *semilattice* is a commutative idempotent monoid while a *compact semilattice* has a compact Hausdorff underlying space such that the multiplication is jointly continuous. On a semilattice there is a natural partial order given by $x \leqslant y$ if and only if $xy = x$. A compact semilattice is always a complete lattice [8]. The main purpose of the following discussion is to show that the functions of a Galois connection between compact semilattices automatically have certain continuity properties.

A net (s_i) in a partially ordered set is increasing (decreasing) if for $i \leqslant j$ in the index set we must have $s_i \leqslant s_j$ $(s_i \geqslant s_j)$.

DEFINITION 1.14. A function $f\colon S \to T$ between partially ordered topological spaces is said to be *continuous from below* (resp. *above*) if for every converging increasing (decreasing) net (s_j) in S we have $f(\lim s_j) = \lim f(s_j)$.

In a compact semilattice the monotone convergence theorem is available. Thus, any monotone net (s_i) converges to $\sup\{s_j\}$ if it is increasing

and to $\inf\{s_j\}$ if it is decreasing (cf. [8]). From this remark and 1.7 we obtain immediately:

PROPOSITION 1.15. *Let (g, d) be a Galois connection between compact semilattices. Then g is continuous from above and d is continuous from below.* ■

DEFINITION 1.16. If (s_j) is a net in a partially ordered topological space, then $\overline{\lim} s_j$ (resp. $\underline{\lim} s_j$) denotes the least upper bound (resp. greatest lower bound) of the set of cluster points of (s_j) (should any exist). A function $f\colon S \to T$ between partially ordered topological spaces is said to be *upper semicontinuous* (resp. *lower semicontinuous*) if for any converging net (s_j) one has $t \leqslant f(\lim s_j)$ (resp. $t \geqslant f(\lim s_j)$) for all cluster points t of $(f(s_j))$. If T is compact, this is equivalent to $\overline{\lim} f(s_j) \leqslant f(\lim s_j)$ (resp. $\underline{\lim} f(s_j) \geqslant f(\lim s_j)$).

DEFINITION 1.17. A semilattice S is a *Lawson semilattice* if it is topological and if for every pair of elements $a, b \in S$ with a not $\geqslant b$ there is an $s \in S$ such that $a \notin \uparrow s$, but $b \in \operatorname{int} \uparrow s$.

In the case of a compact semilattice this means that every principal filter $\uparrow b$ has a neighborhood basis of sets $\uparrow s$. It will then follow by a standard argument that the point b has a basis of neighborhood subsemilattices $\downarrow V \cap \uparrow s$ where $b \in \operatorname{int} \uparrow s$ and V is a neighborhood of b.

PROPOSITION 1.18. *Let (g, d) be a Galois connection between two compact semilattices S and T of which T is a Lawson semilattice. Then g is upper semicontinuous and d is lower semicontinuous.*

Proof. First we treat g. Suppose that $s = \lim s_j$ in S. If we suppose, by way of contradiction, that $g(s)$ not $\geqslant \overline{\lim} g(s_j)$, then the net $(g(s_j))$ in T must have at least one cluster point $t \not\leqslant g(s)$. Since T is a Lawson semilattice, there exists a $t_0 \in T$ such that $g(s) \notin \uparrow t_0$, but $t \in \operatorname{int} \uparrow t_0$. From the latter relation we conclude the existence of a subnet $(s_{j(p)})$ such that $g(s_{j(p)}) \in \uparrow t_0$ for all p, i.e. $g(s_{j(p)}) \geqslant d(t_0)$ for all p. This implies $s = \lim s_{j(p)} \geqslant d(t_0)$, and thus $g(s) \geqslant t_0$ (again by 1.4.3) which is a contradiction to $g(s) \notin \uparrow t_0$.

Next we turn to d. Let us assume, contrary to the assertion, that for some net (t_j) in T with $t = \lim t_j$ we had $\underline{\lim}\, d(t_j)$ not $\geqslant dt$. Then there must be at least one cluster point s of the net $(d(t_j))$ with s not $\geqslant d(t)$. We have $s = \lim d(t_{j(p)})$ for a suitable subnet; hence by the upper semicontinuity of g (just proved) we conclude $g(s) \geqslant \overline{\lim}\, gd(t_{j(p)})$. By 1.4.4 we know $gd(t_{j(p)}) \geqslant t_{j(p)}$, and since $\lim t_{j(p)} = t$, we may conclude $\overline{\lim}\, gd(t_{j(p)}) \geqslant t$. Thus $g(s) \geqslant t$, hence $s \geqslant d(t)$ by 1.4.3, and this is a contradiction. ■

One is naturally led to ask how much in 1.18 is missing for the continuity of g, say. This question is answered in the following result:

PROPOSITION 1.19. *Let (g, d) be a Galois connection between two compact semilattices S and T of which T is a Lawson semilattice. Then the following statements are equivalent:*

(1) *g is continuous.*

(2) *$d(\mathrm{int}\uparrow t) \subseteq \mathrm{int}\uparrow d(t)$ for all $t \in T$.*

Proof. By 1.18, condition (1) is equivalent to

(1′) *$g(s) \leqslant \underline{\lim}\, g(s_j)$ for all nets (s_j) in S with limit s.*

We first prove (2) $\Rightarrow$ (1′): Assume (2) and not (1′); then there is a net (s_j) with limit s and a cluster point t of $g(s_j)$ with $g(s) \not\leqslant t$. Since T is a Lawson semilattice, there is a t_0 such that $t \notin \uparrow t_0$ but that $g(s) \in \mathrm{int}\uparrow t_0$. Then by (2) we conclude $dg(s) \in d(\mathrm{int}\uparrow t_0) \subseteq \mathrm{int}\uparrow d(t_0)$. Since $s \geqslant dg(s)$ by 1.4.4, we have also $s \in \mathrm{int}\uparrow d(t_0)$. Hence there is a subnet $(s_{j(p)})$ with $t = \lim g(s_{j(p)})$ and $s_{j(p)} \in \uparrow d(t_0)$. Thus $s_{j(p)} \geqslant d(t_0)$, hence $g(s_{j(p)}) \geqslant t_0$ (1.4.3) for all p. This implies $t \geqslant t_0$, contradicting $t \notin \uparrow t_0$.

Secondly, we prove (1) $\Rightarrow$ (2): Let U be open in T with $U \subseteq \uparrow t$. Then $V = \uparrow U$ is also open in T (indeed, $y \in V$ if $y \geqslant u$ for some $u \in U$; and if $uy = y$ with $u \in U$, then we find an open neighborhood W of y with $uW \subseteq U$; but then any $w \in W$ satisfies $w \geqslant uw \in U$, hence $w \in V$). If $z \in gd(V)$, then there is some $v \in V$ with $z = gd(v) \geqslant v$, (1.4.4) hence $z \in V$. Thus $gd(V) \subseteq V$, whence $d(V) \subseteq g^{-1}(V)$. If $x \in g^{-1}(V)$, then $g(x) \geqslant u$ for some $u \in U$, hence $x \geqslant d(u)$ (1.4.3) and $u \geqslant t$, hence $x \geqslant d(t)$; thus $g^{-1}(V) \subseteq \uparrow d(t)$. Hence $d(U) \subseteq d(V) \subseteq g^{-1}(V) \subseteq \uparrow d(t)$. Since g is continuous by (1), $g^{-1}(V)$ is open, and thus $d(U) \subseteq \mathrm{int}\uparrow d(t)$. ■

In the light of 1.13 we can conclude

THEOREM 1.20. *Let S, T be compact semilattices and T a Lawson semilattice. Then the function which associates with an inf-preserving map $S \to T$ its adjoint yields a natural bijection between the set of all (continuous) morphisms $S \to T$ of compact semilattices and the set of all functions $d\colon T \to S$ satisfying the following conditions:*

(i) *d preserves sups.*

(ii) *$d(\mathrm{int}\uparrow t) \subseteq \mathrm{int}\uparrow d(t)$ for all $t \in T$.*

If d satisfies these conditions, then the following condition is automatically satisfied:

(iii) *d is continuous from below and lower semicontinuous.*

The class of all compact Lawson semilattices together with all functions $d\colon T \to S$ satisfying (i)–(iii) *is a category, and this category is dual to the category $\mathscr{CL}$ of compact Lawson semilattices (and continuous semilattice morphisms).* ■

The duality mentioned in the previous theorem is not strictly a duality of topological semilattices because of the special nature of the morphisms in the dual category. We note now, however, that in the case of the cate-

gory $\mathscr{Z}$ of zero dimensional compact semilattices it reduces to the duality which is described in detail in [8]. For the purpose of the record let us recall that in a compact zero dimensional semilattice S the subset $K(S)$ of all local minima is, at the same time, the set of all compact elements of the underlying complete lattice; it is a sup-semilattice which is naturally isomorphic to the (discrete) character semilattice S under the map assigning to a character $\chi\colon S\to 2$ (where 2 is the two element semilattice) the compact element $k = \min\chi^{-1}(1)$. (We observe in passing that the morphism $\tilde{\chi}\colon 2\to S$ with $\tilde{\chi}(0) = 0$, $\tilde{\chi}(1) = k$ is the right adjoint of χ.) If $f\colon S\to T$ is a $\mathscr{Z}$-morphism, then a morphism $K(f)\colon K(T)\to K(S)$ was defined by $K(f)(k) = \inf f^{-1}(\uparrow k)$, making K a functor $\mathscr{Z}\to\mathscr{S}^{\mathrm{op}}$ yielding an equivalence (i.e. placing $\mathscr{Z}$ into duality with $\mathscr{S}$ together with a suitably chosen back functor $\mathscr{S}\to\mathscr{Z}$) (see [8], II-3.20, p. 47). In the framework of Galois connections these facts are rephrased as follows:

COROLLARY 1.21. *Let $g\colon S\to T$ be a $\mathscr{Z}$-morphism and $d\colon T\to S$ its right adjoint. Then $d(K(T)) \subseteq K(S)$, and the restriction and corestriction $d|K(T)\colon K(T)\to K(S)$ is precisely $K(g)$.* ∎

Reformulating once again we can say that $K(g)\colon K(T)\to K(S)$ has a unique extension to a function $d\colon T\to S$ satisfying (i)–(iii) of 1.19, and (g, d) is a Galois connection between S and T.

c. Supplementary results on Lawson semilattices. In order to complement the characterization theorem 1.20 for morphisms we record some results which are, in essence, due to Lawson (cf. [13]).

DEFINITION 1.22. Let S be a compact semilattice and let $\mathscr{F}$ be a filter base (of subsets of S) on S. For each $F\in\mathscr{F}$, then $\inf F\in S$ exists (cf. e.g. [8], II-1) and $G\subseteq F$ in $\mathscr{F}$ implies that $\inf F\leqslant\inf G$. Hence $(\inf F)_{F\in\mathscr{F}}$ is an increasing net which has a limit $\liminf\mathscr{F} = \lim(\inf F)_{F\in\mathscr{F}} = \sup\{\inf F\colon F\in\mathscr{F}\}$ (*loc. cit.*). We call this element the *limit inferior* or simply the lim inf of $\mathscr{F}$. (Note that it is defined in any complete lattice via $\sup\{\inf F\colon F\in\mathscr{F}\}$.)

PROPOSITION 1.23. *Let S be a compact semilattice. Then the following statements are equivalent:*

(1) *S is a Lawson semilattice.*

(2) *For each converging filter base $\mathscr{F}$ on S one has* $\lim\mathscr{F} = \liminf\mathscr{F}$.

Proof. (1) ⇒ (2). Set $b = \lim\mathscr{F}$. If a not $\geqslant b$, then there is $s\in S$ with $a\notin\uparrow s$ and $b\in\operatorname{int}\uparrow s$ (1.17). Since $b = \lim\mathscr{F}$, there is an $F\in\mathscr{F}$ with $F\subseteq\uparrow s$, whence $s\leqslant\inf F$. Hence a not $\geqslant b$ implies a not $\geqslant\liminf\mathscr{F}$, i.e. $b\leqslant\liminf\mathscr{F}$. Suppose that $\liminf\mathscr{F}\not\leqslant b$; then we find $t\in S$ such that $b\notin\uparrow t$ and $\liminf\mathscr{F}\in\operatorname{int}\uparrow t$ (1.17). Then we know that for all sufficiently small $F\in\mathscr{F}$, $F\cap\uparrow t = \emptyset$. Hence $\inf F\notin\uparrow t$. Then $\liminf\mathscr{F}\notin\operatorname{int}\uparrow t$ and this is a contradiction.

(2) $\Rightarrow$ (1). Suppose that a not $\geqslant b$. Let $\mathscr{F} = \mathscr{U}(b)$ be the neighborhood filter of b. By (2) we have $\sup\{\inf U\colon U \in \mathscr{U}(b)\} = \liminf \mathscr{F} = \lim \mathscr{U}(b) = b$. Hence $\inf U \not\leqslant a$ for some $U \in \mathscr{U}(b)$. Set $s = \inf U$; then $a \notin \uparrow s$ and $b \in \operatorname{int} U \subseteq U \subseteq \uparrow s$. Thus $b \in \operatorname{int} \uparrow s$. ■

COROLLARY 1.24. *If S is a complete lattice, then there is at most one topology relative to which S is a compact Lawson semilattice.*

Proof. By 1.23, topological convergence relative to a Lawson topology is uniquely determined by lattice theoretical data. Note that here one only uses (1) $\Rightarrow$ (2). ■

COROLLARY 1.25. *If $f\colon S \to T$ is a function between Lawson semilattices preserving arbitrary infs and sups of upward directed sets, then f is a (continuous) morphism.*

Proof. Let $\mathscr{F}$ be an ultrafilter of sets on S. Then $\mathscr{F}$ converges and $f(\lim \mathscr{F}) = f(\sup\{\inf F\colon F \in \mathscr{F}\}) = \sup\{f(\inf F)\colon F \in \mathscr{F}\}$ since f respects up-directed sups. But $f(\inf F) = \inf(f(F))$, since f preserves infs. Hence $f(\lim \mathscr{F}) = \sup\{\inf f(F)\colon F \in \mathscr{F}\} = \liminf f(\mathscr{F})$. Since $f(\mathscr{F})$ is an ultrafilter base, $\lim f(\mathscr{F})$ exists, and hence equals $\lim f(\mathscr{F})$ by 1.23. But a function between compact spaces is continuous if and only if it preserves limits of ultrafilters. ■

Combining these results with 1.20 (and recalling that the sufficient conditions of 1.25 are also necessary) we obtain

THEOREM 1.26. *Let S and T be compact Lawson semilattices and let $g\colon S \to T$ be a function. Then the following statements are equivalent:*

(1) *g is a (continuous) morphism of compact semilattices.*

(2) *g preserves arbitrary infs and sups of upward directed sets.*

(3) *g has a right adjoint $d\colon T \to S$ satisfying $d(\operatorname{int} \uparrow t) \subseteq \operatorname{int} \uparrow d(t)$ for all $t \in T$.* ■

It is noteworthy that condition (2) can be expressed in a seemingly weaker form.

PROPOSITION 1.27. *Condition* (2) *in* 1.26 *is equivalent to*

(2′) *g preserves arbitrary infs and sups of chains.*

Proof. In order to show that (2′) $\Rightarrow$ (2) we formulate a lemma.

LEMMA 1.28. *An isotone function $f\colon S \to T$ between two complete lattices preserves sups of upward directed sets if and only if it preserves sups of chains.*

Proof. In order to establish sufficiency, let A be an upward directed set in S. Define a set X of S by $X = \{x \in S\colon f(x) \leqslant \sup f(A)\}$. Then $A \subseteq X$, and if $a, b, c \in X$ with $a \leqslant c$ and $b \leqslant c$ then $a \vee b \in X$.

If $C \subseteq X$ is a chain, then $f(\sup C) = \sup f(C) \leqslant \sup f(A)$ if f preserves sups of chains. Hence $\sup C \in X$. By Bruns' Lemma [3], then $\sup I \in X$

for every directed subset I of X. In particular, $f(\sup A) \leqslant \sup f(A)$. The converse inequality is always correct. ■

DEFINITION 1.29. A function $f\colon S \to T$ between posets is called *normal* if it preserves sups of upward directed sets.

With this terminology we can rephase a portion of 1.26 as follows: *A function between compact Lawson semilattices is a morphism if and only if it preserves arbitrary infs and is normal.* ■

2. Compact zero-dimensional semilattices with complete dual

a. Dual completeness. In our discussion of the category $\mathscr{Z}$ of compact zero-dimensional semilattices in [8] we had occasion to consider various classes of special morphisms which were characterized by different preservation properties (e.g. the preservation of prime elements, of the sup operation, etc.). We complement this line of investigation by introducing morphisms which preserve compact elements.

DEFINITION 2.1. A morphism $f\colon S\to T$ in $\mathscr{Z}$ is said to be *compact* if $f(K(S))\subseteq K(T)$, i.e. if it preserves compactness of elements. If f is compact, then f induces, by restriction and corestriction, a poset morphism $f_K\colon K(S)\to K(T)$. ■

We now characterize compact morphisms in the spirit of duality:

PROPOSITION 2.2. *Let $f\colon S\to T$ be a $\mathscr{Z}$-morphism. Then the following statements are equivalent*:

(1) *f is compact.*

(2) *For every compact open filter F of S the filter $\uparrow f(F)$ is compact open.*

(3) *The function $K(f)\colon K(T)\to K(S)$ has a left adjoint (namely, f_K).*

(4) *The dual morphism $\hat{f}\colon \hat{T}\to\hat{S}$ of semilattices has a right adjoint.*

(5) *If F is a principal filter of $\hat{S}$, then the filter $f^{-1}(F)$ is principal for all F, i.e. $\min\hat{f}^{-1}(\uparrow\sigma)$ exists for all $\sigma\in\hat{S}$.*

Proof. (1) $\Leftrightarrow$ (2) is immediate from the definition insofar as first, a filter F of S is compact open iff it is of the form $\uparrow k$ with some $k\in K(S)$ and as, secondly, $\min\uparrow f(F)=f(\min F)$ for every closed filter F.

(1) $\Leftrightarrow$ (5) likewise is a translation of the definition in view of the results in ([8], II-2, II-3.3), where it was shown that S could be identified with the semilattice $\mathscr{F}(\hat{S})$ of filters of S in such a way that the principal filters are precisely the compact elements; under this identification the morphism f corresponds precisely to the map $F\to\hat{f}^{-1}(F)\colon \mathscr{F}(\hat{S})\to\mathscr{F}(\hat{T})$.

(1) $\Rightarrow$ (3): Let $d\colon T\to S$ be the right adjoint of $f\colon S\to T$, and let $j\in K(S)$ and $k\in K(T)$. Then $j\geqslant d(k)$ iff $f(j)\geqslant k$ by 1.4.3. Since $d(k)$

$= K(f)(k)$ by 1.20, and $f(j) = f_K(j)$ by (1) above, $j \geqslant K(f)(k)$ iff $f_K(j) \geqslant k$. Hence $(f_K, K(f))$ is a Galois connection between $K(S)$ and $K(T)$.

(3) $\Rightarrow$ (4). By [8], II-3.20, $K(S)$ and $\hat{S}$ are naturally isomorphic when $K(S)$ is given the opposite order, i.e. if $K(S)$ is considered as a sup-semilattice. Hence the implication is trivial.

(4) $\Rightarrow$ (5): Clear from 1.8. ■

The conceptually most interesting characterization is the equivalence (1) $\Leftrightarrow$ (4); the others are of a more technical nature.

COROLLARY 2.3. *Let f: $S \to T$ be a $\mathscr{Z}$-morphism. If $\hat{T}$ is complete, then the following are equivalent statements*:

(1) *f is compact.*

(2) *$\hat{f}$ preserves arbitrary (not only finite) infs.*

Proof. For (1) $\Rightarrow$ (2) use 2.2 ((1) $\Rightarrow$ (4)) and 1.7. For (2) $\Rightarrow$ (1) use 1.9. i and 2.2 ((4) $\Rightarrow$ (1)). ■

It appears after these considerations that the right type of morphisms for the category of complete semilattices are the morphisms preserving arbitrary infs. Thus we make the following definition:

DEFINITION 2.4. (i) The *category $\mathscr{V}$ of complete semilattices* is the subcategory of $\mathscr{S}$ containing all objects S which are complete semilattices (hence are complete lattices) and all semilattice morphisms preserving arbitrary infs (hence are characterized also as having right adjoints).

(ii) An object $T \in \mathscr{Z}$ is said to be *dual complete* iff $\hat{T} \in \mathscr{V}$. The category $\mathscr{D}$ of dual complete $\mathscr{Z}$-objects contains all dual complete $T \in \mathscr{Z}$ and all compact morphisms between them.

Note that $\mathscr{V}$ is not a full subcategory of $\mathscr{S}$ and $\mathscr{D}$ is not a full subcategory of $\mathscr{Z}$.

PROPOSITION 2.5. *The categories $\mathscr{V}$ and $\mathscr{D}$ are dual under the restriction and corestriction of the functors giving the duality between $\mathscr{S}$ and $\mathscr{Z}$* [8]. *Moreover, let $S \in \mathscr{V}$ and let $T \in \mathscr{D}$ be its dual. Then we have the following three groups of statements which are equivalent within each group*:

(1) *S is a distributive lattice.*

(2) *T is a distributive lattice.*

(3) *T is a Brouwerian arithmetic lattice.*

(I) *S is primally generated* (every element is a finite product (inf) of primes).

(II) *T is a Brouwerian topological lattice.*

(III) *T is completely distributive.*

(i) *S is projective* (in $\mathscr{S}$).

(ii) *T is a Brouwerian arithmetic topological lattice in which Tk is finite for each $k \in K(T)$.*

Proof. The duality follows from the definitions of $\mathscr{V}$ and $\mathscr{D}$ in 2.4 and from 2.3. The results on distributivity and projectivity are clear from III-1 and III-3 in [8]. ■

The duality which was exploited in [8] and which is being applied in this discussion is by its very nature a duality of (inf-) semilattice. It so happens that, in our present context, most objects which arise are in fact lattices due to completeness. It is important that we recall all the time that we consider the finite inf operation as the primary operation and that the term "morphism" refers to the preservation of that operation —unless the contrary is explicitly stated. This agreement singles out a partial order from among the two obvious partial orders. On the other hand, the importance and usefulness of the sup-semilattice $K(S)$ forces us to consider, occasionally, the sup operation and the converse order as primary. In order to avoid confusion we establish the

CONVENTION 2.6. Let S be a complete lattice. Then S^{op} is the complete semilattice whose primary operation is $(s, t) \to s \vee t = \sup\{s, t\}$. In particular, if $S \in \mathscr{Z}$, then the sup-semilattice of compact elements will be denoted by $K(S)^{\mathrm{op}}$. As such it is a member of $\mathscr{S}$. This is minutely at variance with the notation adopted in [8], but should be easily understood. If S and T are complete (semi-) lattices, then a function $f\colon S \to T$ is characterized as preserving finite sups by saying that $f\colon S^{\mathrm{op}} \to T^{\mathrm{op}}$ is a morphism of semilattices. ■

b. The compact closure operator. We now introduce an important Galois connection in the context of dual complete lattices. We prepare for this definition with a simple lemma.

LEMMA 2.7. *Let B be a poset and let A be an order-dense subset of B* (i.e. $b = \sup(\downarrow b \cap A)$ for all $b \in B$). *Then*

(i) *Let $X \subseteq A$. Then $\inf_A X$ exists if and only if $\inf_B X$ exists and is contained in A.*

(ii) *The inclusion map $j\colon A \to B$ preserves all existing infs.*

(iii) *If A is a complete lattice, then j has a right adjoint $c\colon B \to A$.*

(iv) *If B is a complete lattice and j has a left adjoint d, then A is a complete lattice.*

Proof. (i) If $\inf_B X$ exists and is in A, then clearly $\inf_A X = \inf_B X$. Conversely, let Y be the set of lower bounds of X in B. If $b \in Y$, then $b = \sup_B(\downarrow b \cap A)$ by hypothesis. However, $\downarrow b \cap A \subseteq Y \cap A$, whence $b \leqslant \max(Y \cap A) = \inf_A X$.

(ii) is a reformulation of (i).

(iii) follows from 1.9.i.

(iv) If $X \subseteq A$, then $d(\sup_B X) = \sup_A X$ by 1.12(II) and 1.7. ■

Now we recall that $K(T)$ is dense in T, an object of $\mathscr{Y}$ ([8], II-1.12, p. 32). Thus, if $K(T)$ is complete, then the inclusion map $j_T\colon K(T)\to T$ has a right adjoint by 2.7.iii. Since $K(T)^{\mathrm{op}} \cong \hat{T}$, we are led to the following definition:

DEFINITION 2.8. Let T be an object of $\mathscr{D}$, and let $j_T\colon K(T)\to T$ be the inclusion map. The right adjoint $c_T\colon T\to K(T)$ of j_T is called the *compact closure operator* (*of* T), and $c_T(t)$ is called the *compact closure of* t.

This notation is motivated by the following facts.

PROPOSITION 2.9. *Let $T\in\mathscr{D}$ be a dual complete semilattice. The compact closure operator $c_T\colon T\to K(T)$ has the following properties*:

(1) (a) $c_T(t) = \min j_T^{-1}(\uparrow t) = \min_{K(T)}(\uparrow t\cap K(T)) = \min(\uparrow t\cap K(T))$.

(b) $c(t) = \sup_{K(T)}(\downarrow t\cap K(T))$.

(2) $t\leqslant jc(t) = c(t)$ *for all* $t\in T$, *and* $t = c(t)$ *if and only if* $t\in K(T)$.

(3) $c^2 = c$, *i.e.* c *is a retraction.*

(4) c *preserves arbitrary sups.*

(5) $c(\inf_T X)\leqslant \inf_{K(T)} c(X) = \inf_T c(X)$ *for all* $X\subseteq T$.

Proof. (1) The first equality in (a) follows from 1.8.i, and the others are immediate from the definitions and the remark that $\inf_{K(T)}C = \inf_T C$ for $C\subseteq K(T)$. For (b) we observe first that $c(t) = \min_{K(T)}(\uparrow t\cap K(T))$ is obviously an upper bound in $K(T)$ of $\downarrow t\cap K(T)$. Now let $b\in K(T)$ be an upper bound for $\downarrow t\cap K(T)$. Then $t = \sup_T(\downarrow t\cap K(T))\leqslant b$. Hence $b\in\uparrow t\cap K(T)$. Thus $c(t) = \min_{K(T)}(\uparrow t\cap K(T))\leqslant b$.

(2) The first part follows from 1.4.4, while the second part follows from (1a) or (1b).

(3) We have $c^2 = cjc = c$ by 1.4.4′ (or by (2) above).

(4) follows from 1.7.

(5) follows from 1.10. ∎

The properties (2), (3), (4) above justify the name "closure operator." It is clear that the existence of a closure operator c (right adjoint for j_T) in a $\mathscr{Y}$-object T implies that $T\in\mathscr{D}$, since $c(\sup_T A) = \sup_{K(T)} A$ for $A\subseteq K(T)$. The remainder of this section is devoted to further inspection of the compact closure operator. The first question one raises is how those semilattices T for which the function $c_T\colon T\to K(T)$ is a morphism of semilattices might be characterized (2.6!).

LEMMA 2.10 (Bruns). *Let L be a complete lattice. The following statements are equivalent*:

(1) *For each chain J and each $x\in L$ with $x\leqslant \sup J$ we have $x\leqslant \sup xJ$.*

(2) *For each upward directed set J and each $x\in L$ with $x\leqslant \sup J$ we have $x\leqslant \sup xJ$.*

(3) *For each upward directed set J and each $x \in L$ we have* $x(\sup J) = \sup xJ$ (in other notation: $x \wedge (\vee J) = \vee (x \wedge J)$).

(4) *For each pair J_1, J_2 of upward directed sets one has* $(\sup J_1)(\sup J_2) = \sup J_1 J_2$.

Proof. (1) $\Rightarrow$ (3). Let x and J be as in (3). We define the set $S = \{y \in L\colon xy \leqslant \sup xJ\}$. If X is a chain in S, then by hypothesis (1) $\sup X \in S$ (and if $a, b \in S$ has an upper bound in L then $a \vee b \in S$). Then by Bruns' lemma [3], for every directed subset $J' \subseteq S$ we have $\sup J' \in S$. Take $J = J'$. Hence $x \sup J \leqslant \sup xJ$. The converse equality holds for any subset J.

(4) $\Rightarrow$ (2) $\Rightarrow$ (1) are obvious.

(3) $\Rightarrow$ (4). Let J_1, J_2 be as given. If $s = \sup J_2$, then $xs = \sup xJ_2$ by (3), whence $(\sup J_1)s = \sup J_1 s$ by (3) once again. Hence $(\sup J_1) \times \times (\sup J_2) = \sup\{xs\colon x \in J_1\} = \sup\{\sup xJ_2\colon x \in J_1\} = \sup\{xy\colon x \in J_1, y \in J_2\} = \sup J_1 J_2$. ■

DEFINITION 2.11. A complete lattice L is called *upper continuous* if it satisfies the equivalent conditions of 2.10. The term *lower continuous* is defined dually.

PROPOSITION 2.12. *Every compact (semi-) topological semilattice S is lower continuous.*

Proof. If J is an upward directed subset of S then $\lim J = \sup J$ ([8], II-1.1, p. 29). Hence by the continuity of translations, $x(\sup J) = x(\lim J) = \lim xJ = \sup xJ$. ■

PROPOSITION 2.13. *Let $T \in \mathscr{D}$ be a dual complete lattice. Then the following statements are equivalent:*

(1) *The compact closure operator $c_T\colon T \to K(T)$ is a morphism of* (inf-) *semilattices.*

(2) *$K(T)$ is upper continuous.*

(3) *$\hat{T}$ is lower continuous.*

Proof. (1) $\Rightarrow$ (2). Let $k \in K(T)$ and let $A \subseteq K(T)$ be an upward directed set with $t = \sup_T A$. Then we have $\sup_{K(T)} A = \sup_{K(T)} {\downarrow} A = \sup_{K(T)} \big({\downarrow} t \cap K(T)\big) = c_T(t) = c(t)$. ${\downarrow} C$ is a filter in $K(T)^{\mathrm{op}}$, and by [8] there is a bijection between the elements of T and the filters of $K(T)^{\mathrm{op}}$ given by $t \to {\downarrow} t \cap K(T)$. By 2.9.2 we have $c(k) = k$. Hence $k \sup A = c(k)c(t) = c(kt) = \sup_{K(T)}\big({\downarrow} kt \cap K(T)\big) = \sup_{K(T)} kA$ since ${\downarrow} kt \cap K(T) = k\big({\downarrow} t \cap K(T)\big) = k({\downarrow} C) = {\downarrow} kC$.

(2) $\Rightarrow$ (1): Let $t_1, t_2 \in T$; then $J_i = {\downarrow} t_i \cap K(T)$, $i = 1, 2$, are upwards directed sets in $K(T)$. Hence $c(t_1)c(t_2) = \sup_{K(T)} J_1 \sup_{K(T)} J_2 = \sup_{K(T)} J_1 J_2$ by 2.10.4. But $J_1 J_2 = \big({\downarrow} t_1 \cap K(T)\big)\big({\downarrow} t_2 \cap K(T)\big) = {\downarrow} t_1 t_2 \cap K(T)$, whence $\sup_{K(T)} J_1 J_2 = c(t_1 t_2)$.

(2) $\Leftrightarrow$ (3) by duality [8] II-3. ■

c. Algebraic and order theoretic characterization of Lawson semilattices. We have now clarified the question of when equality holds in 2.9.5 for finite sets X. Now we discuss the question of equality in 2.9.5 for arbitrary sets X.

PROPOSITION 2.14. *Let $T \in \mathscr{D}$ be a dual complete semilattice. Then the following statements are equivalent:*

(1) *The compact closure operator c_T preserves arbitrary infs* (i.e. equality holds in 2.9.5).

(2) *c_T has a right adjoint m_T: $K(T) \to T$.*

(3) *For each compact element $k \in K(T)$, the set of all $t \in T$ whose compact closure $c_T(t)$ dominates k has a smallest element.*

If these conditions are satisfied, then m_T preserves arbitrary sups and

(i) $m_T(k) = \min\{t \in T\colon c_T(t) = k\} = \min c_T^{-1}(\uparrow k) \leqslant k$.

Proof. (1) $\Rightarrow$ (2): By 1.8, c_T has a right adjoint given by $m_T(k) = \min c_T^{-1}(\uparrow k) = \min c_T^{-1}(k)$.

(2) $\Rightarrow$ (1): 1.7.

(3) is clearly equivalent to the existence, for each $k \in K(T)$, of $m_T(k) = \min\{t \in T\colon c_T(t) \geqslant k\}$. Then, by (1.4.3), $c_T(t) \geqslant k$ is equivalent to $t \geqslant m_T(k)$, hence m_T is right adjoint to c_T. Thus (3) $\Rightarrow$ (2). By 1.7, m_T preserves sups. ■

The preceding characterization does not complete the story, however. From the viewpoint of compact semigroups it is important to know when the kernel relation $R = \{(x, y)\colon x, y \in T\ c_T(x) = c_T(y)\}$ of the compact closure operator is closed relative to the natural topology of T (in the sense of [8], II-3). For if this is the case, then $K(T)$ can be equipped with a semilattice topology such that T/R and $K(T)$ are isomorphic under the canonical morphism induced by c_T. In this direction we have the following result:

PROPOSITION 2.15. *Let $T \in \mathscr{D}$ be dual complete. Then the following statements are also equivalent to* (1)–(3) *of* 2.14:

(4) *On $K(T)$ there is a semilattice topology such that c_T: $T \to K(T)$ is a continuous morphism relative to this topology.*

(5) *The kernel relation $R = \{(x, y)\colon c_T(x) = c_T(y)\}$ of c_T is a closed congruence in T.*

Proof. (4) and (5) are clearly equivalent. If (5) is satisfied, then $R(X)^- = R(X)$ if $\bar{X} = X$; let $k \in K(T)$; then $R(\uparrow k) = \{t \in T\colon c_T(t) \geqslant k\}$ is a closed subsemilattice which then has a minimum. Thus 2.14 (3) holds. Finally suppose that the conditions of 2.14 are satisfied. Then R is a congruence as the kernel relation of a morphism; we have to show that R is closed. Let $(x, y) \notin R$. Then $c(x) \neq c(y)$. We may assume that $c(y)$not $\geqslant c(x)$. Then there is a $k \in K(T)$ such that $k \leqslant x$ and $c(y)$ not $\geqslant k$; for, if not, then $c(y)$

would be an upper bound for $\downarrow x \cap K(T)$, implying $c(x) \leqslant c(y)$ after 2.9.1.b. We define $U = \uparrow k$ and $V = I(m(k))$ (which, by definition, is $T \setminus \uparrow m(k)$, see [8]). Then U and V are open; clearly, $x \in U$ since $k \leqslant x$; and $c(y)$ not $\geqslant k$ is equivalent to y not $\geqslant m(k)$ by 1.4.3, hence to $y \in V$. Now take $(u, v) \in U \times V$. We claim that $(u, v) \notin R$; indeed, $v \in V$ means v not $\geqslant m(k)$, hence $c(v)$ not $\geqslant k$ by 1.4.3, hence $c(v) \notin U$, whereas $u \in U$ means $u \geqslant k$; hence $c(u) \geqslant c(k) = k$ (2.9.2), whence $c(u) \in U$. Thus we have shown that every point $(x, y) \notin R$ has an open neighborhood $U \times V$ in $T \times T$ with $(U \times V) \cap R = \emptyset$. This proves that R is closed. ■

Something is still missing. The characterizations in 2.14 and 2.15 all involve the embedding of $K(T)$ in T. One clearly seeks a characterization which is expressible in terms of $K(T)$ (i.e. in terms of the dual $\hat{T}$) alone. In order to respond to this demand, we make the following definitions, which will facilitate the formulations:

DEFINITION 2.16. A lattice will be called an AL-*lattice* (or, *algebraically of the Lawson type*) if it is complete and if the following condition is satisfied:

(i) For each lattice ideal J_1 and each element $x \not\leqslant \sup J_1$, there is an element $y \notin J_1$ such that for every lattice ideal J with $y \notin J$ one has in fact $x \not\leqslant \sup J$.

We say that a lattice L is a VL-*lattice* (or, *dually of the Lawson type*) if L^{op} is an AL-lattice, i.e. if it is complete and the following condition is satisfied:

(ii) For each filter F_1 and each element x not $\geqslant \inf F_1$, there is an element $y \notin F_1$ such that for every filter F with $y \notin F$ one has in fact x not $\geqslant \inf F$.

PROPOSITION 2.17. *Let $T \in \mathscr{D}$ be the dual of a complete semilatice $S \in \mathscr{V}$. Then each of the following conditions is equivalent to the conditions of* 2.14 *and* 2.15:

(6) *$K(T)$ is an* AL-*lattice.*

(7) *S is a* VL-*lattice.*

Proof. Clearly, (6) $\Leftrightarrow$ (7) by duality ([8], II-3.20, p. 47).

(2) $\Rightarrow$ (6): Suppose that $k_1 \in K(T)$ and that J_0 is a lattice ideal of $K(T)$ such that $k_1 \not\leqslant \sup_{K(T)} J_0$. Set $t = \sup_T J_0$. Then $\sup_{K(T)} J_0 = c(t)$ (2.9.1.b). We are given $c(t)$ not $\geqslant k_1$; hence we know t not $\geqslant m(k_1)$ by (2) and 1.4. Since $m(k_1) = \sup_T(\downarrow m(k_1) \cap K(T))$ ([8], II-1.12, p. 32), we find a $k \in \downarrow m(k_1) \cap K(T)$ with t not $\geqslant k$. Now suppose that J is a lattice ideal of $K(T)$ with $k \notin J$. Set $s = \sup_T J$. Then $k \not\leqslant s$, whence in particular, s not $\geqslant m(k_1)$. Then $c(s)$ not $\geqslant k_1$ by 1.4. But since $c(s) = \sup_{K(T)} J$, we have shown $k_1 \notin J$, thereby establishing (6).

Now we prove (6) $\Rightarrow$ 2.14(3). We recall that $J \to \sup_T J$ is a bijective morphism from the semilattice of lattice ideals of $K(T)$ (i.e. the semilattice

$\mathscr{F}(K(T)^{\mathrm{op}})$) onto T ([8], II-2, II-3); and by 2.9.1.b we have $\sup_{K(T)} J = c(\sup_T J)$. We therefore conclude that 2.14(3) is equivalent to

(3′) *For each element $k \in K(T)$ there is a smallest lattice ideal J of $K(T)$ such that $k \geqslant \sup_{K(T)} J$.*

The following lemma then completes the proof:

LEMMA 2.18. *Each* AL-*lattice* (2.16.i) *satisfies the following condition*:

(8) *For each $x \in L$ there is a smallest lattice ideal J such that $x \geqslant \sup J$.*

Remark. Condition (8) may be clearly rephrased as follows:

(8′) *For each $x \in L$ one has*

$$x \geqslant \sup \bigcap \{I\colon I \text{ is a lattice ideal with } x \geqslant \sup I\}.$$

Proof. We set $J_0 = \bigcap \{I\colon I \text{ a lattice ideal with } x = \sup I\}$. Obviously, $\sup J_0 \leqslant x$. We must show $x \leqslant \sup J_0$. By way of contradiction, suppose $x \nleqslant \sup J_0$. By 2.16.i there is a $y \in L \setminus J_0$ such that for every lattice ideal I with $y \notin I$ one has $x \nleqslant \sup I$. The definition of J_0 and $y \notin J_0$ imply the existence of a lattice ideal I with $x = \sup I$ and $y \notin I$. But then one has $x \nleqslant \sup I$ by the choice of y, and this is a contradiction. ∎

NOTATION 2.19. If L is an AL-lattice [resp., VL-lattice] and $J \subseteq L$ lattice ideal [resp., $F \subseteq L$ a filter], then J_0 [resp., F_0] denotes the amallest ideal [filter] with $\sup J_0 = \sup J$ [resp., $\inf F_0 = \inf F$]. ∎

Since we have proved the equivalence of (6) and (3′) and since every complete lattice L is isomorphic to $K(L^{\mathrm{op}})$ (for $K(T)^{\mathrm{op}} = \hat{T}$ by [8], II-3.20) with $L^{\mathrm{op}} \in \mathscr{D}$, we have in fact proved a characterization of AL-lattice and, by duality, of VL-lattices:

COROLLARY 2.20. (a) *A lattice L is an* AL-*lattice iff it is complete and satisfies* (8) *of* 2.18. (b) *A lattice L is a* VL-*lattice iff it is complete and satisfies*:

(9) *For every $x \in L$ there is a smallest filter F such that $x \leqslant \inf F$.* ∎

Also, if (8) *and* (9) *hold, then the same conditions are true with* $=$ *replacing* $\geqslant$, *resp.* $\leqslant$.

With 2.17 and 2.20 we have a completely intrinsic characterization in terms of $K(T)$ above of the property 2.14 (1).

It is clear that condition 2.15(4) determines the topology on $K(T)$ uniquely; for the quotient topology (relative to c_T) on $K(T)$ is the finest one such that c_T is continuous, and it is a compact topology. Moreover, compact Lawson topologies as such are unique (1.24). It is known that quotients of Lawson semilattices are Lawson (see 5.1 below), but we will give an independent proof of this fact in the present context:

LEMMA 2.21. *The topology defined on $K(T)$ by* 2.15 (4) *makes $K(T)$ into a Lawson semilattice* (1.17), *and this topology is the only one with this property.*

Proof. Let $a, b \in K(T)$ with a not $\geqslant b$. We must find a $k \in K(T)$ such that a not $\geqslant k$ but $b \in \uparrow_{K(T)} k$. If a not $\geqslant b$ (i.e. $c(a)$ not $\geqslant b$), then a not $\geqslant m(b)$ (1.4.3 and 2.14.(2), 2.15). Hence there is a $k \in K(T)$ such that $k \leqslant m(b)$, but a not $\geqslant k$. Now $I(k) = T \setminus \uparrow k$ is compact, hence $c_T(I(k))$ is closed, and because of $K(T) = c(I(k)) \cup (\uparrow k \cap K(T))$ we know that $K(T) \setminus c(I(k)) \subseteq \operatorname{int} \uparrow_{K(T)} k$. It suffices now to show that $b \notin c(I(k))$. But if there were a t not $\geqslant k$ such that $b = c(t)$, then $m(b) \leqslant t$; and since $k \leqslant m(b)$ this is impossible. The uniqueness of the topology follows from 1.24. ■

On the other hand, we note

LEMMA 2.22. *Every compact Lawson semilattice is an* AL-*semilattice*.

Proof Let L be a Lawson semilattice, and suppose that J_1 is a lattice ideal with $x \not\leqslant \sup J_1$. By 1.17 there is an $s \in L$ such that $\sup J_1$ not $\geqslant s$ but $x \in \operatorname{int} \uparrow s$. Now let J be an arbitrary lattice ideal with $s \notin J$, i.e. $J \subseteq I(s)$. Then $\sup J = \lim J_1$, since J is upwards directed ([8], II-1). Hence $\sup J \in \overline{I(s)} = L \setminus \operatorname{int} \uparrow s$, and thus $x \not\leqslant \sup J$. Since L is compact, L is a complete lattice ([8], II-1), whence L is an AL-lattice. ■

We have now seen that a lattice carries a (unique) compact topology making it into a Lawson semilattice if and only if it is complete and satisfies $x = \sup\{y$: every lattice ideal I with $\sup I \geqslant x$ contains $y\}$ by 2.17 and 2.20. It is now notationally convenient to call an element y in a lattice L *relatively compact under* x iff it is contained in every lattice ideal I of L with $\sup I \geqslant x$. This is ostensibly equivalent to saying that for all subsets $X \subseteq L$ with $\sup X \geqslant x$ there is a finite subset $Y \subseteq X$ with $y \leqslant \sup Y$. Let us call a lattice *relatively algebraic* if it is complete and every element in it is the l.u.b. of all relatively compact elements under it. Thus a lattice is relatively algebraic if and only if it is an AL-lattice (2.16).

It is now probably worthwhile to collect our principal results in the paragraphs following 2.14 in the following theorem, the core result of this section. It could be called the algebraic and order theoretic characterization of compact Lawson semilattices.

THEOREM 2.23. *Let $S \in \mathscr{V}$ be a complete lattice and $T \in \mathscr{D}$ its dual. Then the following conditions are equivalent*:

(i) *The compact closure operator c_T: $T \to K(T)$ preserves arbitrary infs.*

(ii) *c_T has a right adjoint m_T: $K(T) \to T$.*

(iii) *On $K(T)$ there is a unique topology which makes $K(T)$ into a compact Lawson semilattice such that c_T: $T \to K(T)$ is a morphism of compact semilattices.*

(iv) *For each $k \in K(T)$ there exists a smallest lattice ideal J in $K(T)$ such that $k \leqslant \sup_{K(S)} J$* (i.e. *$K(T)$ is an* AL-*lattice*).

(v) *For each* $s \in S$ *there is a smallest filter* F *such that* $s = \inf F$ (i.e. S *is a* VL-*lattice*).

If these conditions hold, then (iv) *and* (v) *hold with* $=$ *replacing* $\leqslant$, *resp.* $\geqslant$.

Moreover, every compact Lawson semilattice L *is isomorphic* (*as a compact semilattice*) *to some* $K(T)$ *with the topology described in* (iii). *In particular, a lattice carries a compact Lawson topology if and only if it is relatively algebraic.* ■

In 1.24 we proved the uniqueness of a compact Lawson topology on a complete lattice L; in 2.23 we have now an explicit characterization of the existence of such topologies.

d. The functoriality of j, c, m. We finish this section by discussing the relationship between morphisms and the maps j_T, c_T, m_T.

PROPOSITION 2.24. *Let* $g\colon S \to T$ *be a morphism in* $\mathscr{D}$ *and let* $d\colon T \to S$ *be its right adjoint. Let* $g_K\colon K(S) \to K(T)$ *be the map induced according to* 2.1. *Then the following conditions are equivalent*:

(1) *The diagram*

$$\begin{tikzcd} S \arrow[r, "g"] \arrow[d, "c_S"'] & T \arrow[d, "c_T"] \\ K(S) \arrow[r, "g_K"'] & K(T) \end{tikzcd}$$

commutes.

(2) *The map* g_K *preserves sups of upward directed sets.*

(2′) *The map* g_K *preserves sups of chains.*

If the maps $c_S\colon S \to K(S)$ *and* $c_T\colon T \to K(T)$ *preserve arbitrary infs then these conditions are also equivalent to the following*:

(3) *The diagram*

$$\begin{tikzcd} S & T \arrow[l, "d"'] \\ K(S) \arrow[u, "m_S"] & K(T) \arrow[l, "K(g)"] \arrow[u, "m_T"'] \end{tikzcd}$$

is commutative.

(4) *If* $K(S)$ *and* $K(T)$ *are given the topologies which induce the structure of a Lawson semilattice according to* 2.23.iii, *then* $g_K\colon K(S) \to K(T)$ *is a morphism of Lawson semilattices.*

Proof. (1) $\Rightarrow$ (2). Let $J \subseteq K(S)$ be an upward directed set. We define $I \subseteq S$ by $I = c_S^{-1}(J) = \{s \in S\colon c_S(s) \in J\}$. Then I is also upward directed; for if $s_j \in I$, $j = 1, 2$, then there is $k \in J$ with $c_S(s_j) \leqslant k$; since $k = c_S(k)$, then $k \in I$, and $s_j \leqslant c_S(s_j) \leqslant k$. Moreover, $\sup_S I = \sup_S J$, whence

(i) $c_S(\sup_S I) = c_S(\sup_S J) = \sup_{K(S)} J$, by 2.9.4 considering $c_S(J) = J$.

Now, $\sup_{K(T)} g_k(J) = \sup_{K(T)} g_K c_S(J) = \sup_{K(T)} c_T g(J)$ by (1). But g preserves sups of upward directed sets as a $\mathscr{Z}$-morphism ([8], II-3.22, p. 48), and c_T preserves sups (2.9.4). Hence $\sup_{K(T)} c_T g(I) = c_T g(\sup_S I) = g_K c_S(\sup_S I) = g_K(\sup_{K(S)} J)$ by (1) and (i) above. Thus $\sup_{K(T)} g_K(J) = g_K(\sup_{K(S)} J)$ which we had to show.

(2) $\Rightarrow$ (1): If (2) holds then all maps in the diagram (1) preserve sups of upwards directed sets. In S every element s is the sup of the upwards directed family $\downarrow s \cap K(S)$ of compact elements ([8], II-1.12, p. 35). It suffices therefore to observe that for $k \in K(S)$ one has $g_K c_S(k) = g_K(k) = c_T g_K(k) = c_T g(k)$.

(2) $\Leftrightarrow$ (2′): 1.28. Suppose that c_S: $S \to K(S)$ and c_T: $T \to K(T)$ preserve arbitrary infs. Then (1) $\Leftrightarrow$ (3) is immediate from the fact that the diagram in (3) is obtained from that in (1) by passing to right adjoints (recall 1.6 and 1.11).

(4) $\Rightarrow$ (2): Clear (1.26).

(1) $\Rightarrow$ (4): This implication will follow from the subsequent general remarks:

LEMMA 2.25. (a) *Let X, Y, Z be topological spaces and suppose that Y is compact. If $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ are closed subsets, then $S \circ R = \{(x, z)$: there is a $y \in Y$ with $(x, y) \in R$ and $(y, z) \in S\}$ is closed in $X \times Z$.*

(b) *A function f: $X \to Y$ between compact spaces is continuous if and only if graph f is closed in $X \times Y$.*

(c) *Let X, Y, Z be monoids, $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ submonoids. Then $S \circ R$ is a submonoid of $X \times Z$.*

(d) *A function f: $X \to Y$ between monoids is a morphism if and only if graph f is a submonoid in $X \times Y$.* ■

Indeed, if (1) of 2.24 is satisfied, then by the surjectivity of c_S we have graph $g_K = (\text{graph}\, c_T g) \circ (\text{graph}\, c_S)^{-1}$, and this is a closed submonoid of $K(T) \times K(S)$. ■

PROPOSITION 2.26. *Let S, $T \in \mathscr{D}$ be dual complete objects, and f: $K(S) \to K(T)$ a function.*

(i) *Suppose that f preserves (arbitrary) infs. Let h: $K(T) \to K(S)$ be the right adjoint of f and consider h^{op}: $K(T)^{\mathrm{op}} \to K(S)^{\mathrm{op}}$. By duality* ([8], II-3) *there is a unique $\mathscr{Z}$-morphism g: $S \to T$ such that $K(g)^{\mathrm{op}} = h^{\mathrm{op}}$* (i.e. g is (essentially) the dual of h^{op}). *Then $g_K = f$.*

(ii) *If f preserves arbitrary infs and sups of upwards directed sets, then the diagram*

$$\begin{tikzcd} S \arrow[r, "g"] & T \arrow[d, "c_S"] \\ K(S) \arrow[u, "j_S"] \arrow[r, "f"'] & K(T) \end{tikzcd}$$

commutes.

Proof. (i) After the developments of [8], II-3, the function g is given by the formula

$$g(s) = \sup_T h^{-1}(\downarrow s \cap K(S)) = \sup_T \{k \in K(T) \colon h(k) \leqslant s\}.$$

Let $k_1 \in K(S)$. Then $k_1 \geqslant h(k)$ is equivalent to $f(k_1) \geqslant k$ by 1.4.3. Hence

$$g(k_1) = \sup_T \{k \in K(T) \colon k \leqslant f(k_1)\} = f(k_1).$$

(ii) follows from (i) and 2.24. ■

PROPOSITION 2.27. *Let $g\colon S \to T$ be a morphism in $\mathscr{D}$ and $d\colon T \to S$ its right adjoint. Then the diagram*

$$\begin{tikzcd} S \arrow[d, "c_S"'] & T \arrow[l, "d"'] \arrow[d, "c_T"] \\ K(S) & K(T) \arrow[l, "K(g)"] \end{tikzcd}$$

commutes.

Proof. The diagram is obtained from the following commuting diagram by passing to the right adjoints:

$$\begin{tikzcd} S \arrow[r, "g"] & T \\ K(S) \arrow[u, "j_S"] \arrow[r, "g_K"'] & K(T) \arrow[u, "j_T"'] \end{tikzcd} \quad ■$$

DEFINITION 2.28. (a) We define $\mathscr{V}\mathscr{L}$ in $\mathscr{V}$ (2.4.i) to be the subcategory of all VL-lattices (2.16.ii) and all morphisms in $\mathscr{V}$ which, in addition, have right adjoints preserving infs of downward directed sets.

(b) Let $\mathscr{D}\mathscr{L}$ in $\mathscr{D}$ be the subcategory of all $T \in \mathscr{D}$ satisfying 2.23 (i) (or 2.14.1) and all morphisms g such that g_K preserves sups of upward directed sets.

(c) Let $\mathscr{C}\mathscr{L}$ be the category of all compact Lawson semilattices with continuous semilattice morphisms.

If one considers 2.5, 2.23, and the preceding results, one concludes the following duality and equivalence theorem for compact Lawson semilattices:

THEOREM 2.29. *The categories $\mathscr{V}\mathscr{L}$ and $\mathscr{D}\mathscr{L}$ are dual (under the standard duality of $\mathscr{S}$ and $\mathscr{Z}$), and $\mathscr{D}\mathscr{L}$ and $\mathscr{C}\mathscr{L}$ are equivalent under the functor which associates with a $\mathscr{D}\mathscr{L}$-object T the semilattice $K(T)$ with its Lawson topology and with a $\mathscr{D}\mathscr{L}$-morphism $g\colon S \to T$ the morphism $g_K\colon K(S) \to K(T)$ of Lawson semilattices.* ■

Note that the category mentioned in 1.20 is dual to $\mathscr{C}\mathscr{L}$, hence equivalent to $\mathscr{V}\mathscr{L}$.

It is probably useful to see in an example that the conditions of 2.24 are not automatically satisfied (see also 2.33):

EXAMPLE 2.30. Let C be the complete lattice $\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, 1\}$ relative to its natural order. Let $S = C \times \{0, 1\}$ and $T = C \times \{0, \frac{1}{2}, 1\}$ with the lexicographic order and let $g\colon S \to T$ be the inclusion. Then g is a $\mathscr{Z}$-morphism and $S, T \in \mathscr{D}$ (since $K(S) = S \setminus \{(1, 0)\}$ and $K(T) = T \setminus \{(1, 0)\}$ are complete). Let $J \subseteq S$ be the chain $(C \setminus \{1\}) \times \{1\}$. Then $g_K(\sup_{K(S)} J) = (1, 1)$, but $\sup_{K(T)} g_K(J) = (1, \frac{1}{2})$. ■

COROLLARY 2.31. *Let $g\colon S \to T$ be a $\mathscr{D}$-morphism and let $d\colon T \to S$ be its right adjoint. Then the following conditions are equivalent:*

(i) *g is surjective;*

(ii) *g_K is surjective;*

(iii) *d is injective;*

(iv) *$K(g)$ is injective.*

If these conditions are satisfied then the diagram (without the dotted

$$\begin{tikzcd}
 & & T \arrow[dl, "d"'] \arrow[dd, "1"] \arrow[ddr, "c_T"] & \\
S \arrow[rr, "g"] \arrow[dd, "c_S"'] & & & \\
 & & T \arrow[dd, dashed] & K(T) \arrow[dl, "K(g)"'] \arrow[dd, "1"] \\
K(S) \arrow[rr, "g_K"'] & & & \\
 & & K(T) &
\end{tikzcd}$$

arrow) commutes. In particular $g_K c_s d = c_T$. If g_K is normal (1.27), *then the diagram with the dotted arrow commutes, too.*

Proof. We have (i) $\Leftrightarrow$ (iii) and (ii) $\Leftrightarrow$ (iv) by 1.12. (i) $\Leftrightarrow$ (iv) by duality ([8], I-4.1, p. 21 and II-3.20, p. 47). The remainder follows from 2.30. ■

COROLLARY 2.32. *Let $g\colon S \to T$ be a $\mathscr{D}$-morphism and $d\colon T \to S$ its right adjoint. Then the following conditions are equivalent:*

(i) *g is injective,*

(ii) *g_K is injective,*

(iii) *d is surjective,*

(iv) *$K(g)$ is surjective.*

If these conditions are satisfied, then the following diagram commutes (without the dotted arrow).

$$\begin{tikzcd}
 & & S \arrow[dd, "c_S"] \\
S \arrow[urr, "1"] \arrow[dr, "g"] \arrow[dd, dashed, "c_S"'] & & \\
 & T \arrow[uur, "d"] \arrow[dd, "c_T"] & \\
 & & K(S) \\
K(S) \arrow[uurr] \arrow[dr, "g_K"'] & & \\
 & K(T) \arrow[uur, "K(g)"'] &
\end{tikzcd}$$

In particular, $K(g)c_T g = c_S$.

The diagram commutes with the dotted arrow iff g_K *is normal* (1.27).

Proof. As before in 2.31, (i) $\Leftrightarrow$ (iii) and (ii) $\Leftrightarrow$ (iv) by 1.12. (i) $\Rightarrow$ (ii) is trivial. (ii) $\Rightarrow$ (i): If g_K is injective, then $K(g)$ is surjective, (iv) $\Leftrightarrow$ (i) by duality (I-4.1, p. 21 and II-3.20, p. 47 [8]). The remainder follows from 1.12, 2.27 and 2.24. ■

The following example is an application of some theoretical importance (see [9]).

EXAMPLE 2.33. Let $S \in \mathscr{D}$ be a dual complete object and K a chain in $K(T)$ which is maximal relative to $\subseteq$ and set $T = \bar{K}$ in S. Let $d\colon T \to S$ be the inclusion map. Then the left adjoint $g\colon S \to T$ is a $\mathscr{D}$-morphism with $K(T) = K$, $K(g) = d|K$ such that 2.31 applies. In particular, $g|T = 1_T$, $g|K = 1_K$. The map g_K need not preserve sups of chains.

Proof. Since T is a closed chain in S, then $\sup_T Y = \sup_S Y$ for $Y \subseteq T$, hence d preserves sups, thus has a left adjoint g. Since K is maximal in $K(T)$, we have $\sup_K X = \sup_{K(S)} X$ for $X \subseteq K$, hence $d|K$ preserves sups and has a left adjoint $f\colon K(S) \to K$. We show next that $K(T) = K$; trivially, $K \subseteq K(T)$. Suppose that $k \in K(T)$. Then $k = \inf_T(\uparrow k \cap K)$ because $k \in T = \bar{K}$ and $k \in K(T)$. But then $k = c_S(k)$ by 2.9.1. Hence $k \in K(S)$. Since K is maximal, we then have $k \in K$. Thus $K(T) = K$, and T is a $\mathscr{D}$-object since K is complete. Now we apply 2.26(i) and find a g as asserted.

As a concrete example let S be the semilattice of lattice ideals of $I \times I$; then $I \times I$ is identified with $K(S)$ under $(x, y) \to \downarrow(x, y)$. Take $K = (\{0\} \times I) \cup (I \times \{1\})$. Then $g_K = f\colon K(S) \to K$ is given by $f(x, y) = \max\{(u, v) \in K\colon (u, v) \leqslant (x, y)\}$; this function does not preserve the sup of the chain $\{1\} \times I$. ■

3. The (right) reflector $P\colon \mathscr{CL} \to \mathscr{D}$

We change the viewpoint: In Section 2 we departed from the category $\mathscr{D}$ of dual complete $\mathscr{X}$ objects and compact morphisms and investigated in detail how this yielded an access to the category $\mathscr{CL}$ of compact Lawson semilattices. Now we start from compact semilattices and produce a functor which will map $\mathscr{CL}$ into $\mathscr{D}$; this will illustrate once more the very algebraic character of compact Lawson semilattices. The results of Section 2 will be freely used in the present section.

a. The ideal lattice.

DEFINITION 3.1. Let L be a compact semilattice. We denote with PL the semilattice of all lattice ideals $J \subseteq L$ relative to intersection. Recall that J is a lattice ideal iff $JL \subseteq J$ and $J \vee J \subseteq J$ and that $J \cap I = JI$ for two lattice ideals J and I. The morphism $L \to PL$ which associates with an $x \in L$ the principal ideal $Lx = \downarrow x$ will be denoted by $i_L\colon L \to PL$. The function which associates with a $J \in PL$ the element $\sup J \in L$ will be called $r_L\colon PL \to L$. ■

The following paragraph gives a first glimpse of the relation between this setup and Section 2.

PROPOSITION 3.2. *The semilattice PL has a unique $\mathscr{X}$-topology such that $PL \in \mathscr{D}$. In fact one has $K(PL) = i_L(L)$. The diagram of lattice morphisms*

(i)

$$\begin{tikzcd} & PL \arrow[dr, "j_{PL}"] & \\ L \arrow[ur, "i_L"] \arrow[rr, "i'_L"'] & & K(PL) \end{tikzcd}$$

commutes with a lattice isomorphism i'_L. The semilattice PL is the semilattice of filters $\mathscr{F}(L^{\mathrm{op}})$ of L^{op} ([8], II-2) and thus is the dual of the discrete semilattice L^{op} in the sense of [8]; conversely, L^{op} may be identified with the dual of PL. The function $r_L\colon PL \to L$ is an $\mathscr{S}$-morphism and preserves arbitrary sups and the diagram

(ii)

$$\begin{tikzcd} & PL \arrow[dl, "r_L"'] \arrow[dr, "c_{PL}"] & \\ L \arrow[rr, "i'_L"'] & & K(PL) \end{tikzcd}$$

commutes. If $L \in \mathscr{CL}$, then r_L is continuous, hence a $\mathscr{CL}$-morphism and conversely.

Proof. Practically everything is immediate from the definitions and the results of Section 2. Indeed, it is clear that a subset J of L is a lattice ideal iff it is a filter of L^{op}, hence $PL = \mathscr{F}(L^{\mathrm{op}}) \in \mathscr{Z}$; the compact elements of a filter semilattice $\mathscr{F}(S)$ are precisely the principal filters ([8], II-3.8, p. 41), whence $K(PL) = i_L(L)$. Since L is a complete lattice, $PL \in \mathscr{D}$. If we let i'_L be the corestriction of i_L, then trivially, $j_{PL} i'_L = i_L$ and i'_L is evidently bijective. The morphism c_{PL}: $PL \to K(PL)$ associates with each lattice ideal J the smallest compact element (i.e. principal ideal) containing J; this is obviously $\sup L = i'_L(r_L(J))$. It follows that (ii) commutes. By 2.12, 2.13 we know that r_L is an $\mathscr{S}$-morphism. By (i) and (ii) and 2.8, r_L is the right adjoint of i_L, hence preserves sups. Suppose now that $L \in \mathscr{CL}$. Since i'_L is an isomorphism of lattices, $K(PL)$ is an AL-lattice by 2.22, hence PL satisfies the equivalent conditions of 2.23. In particular, if $K(PL)$ is equipped with its unique Lawson topology according to 2.23.iii, then c_{PL} is a $\mathscr{CL}$-morphism; since i'_L is now a $\mathscr{CL}$-isomorphism by 1.24, then also $r_L \in \mathscr{CL}$. The converse follows from 2.21. ■

Proposition 3.3. *Let f: $L_1 \to L_2$ be a morphism of compact semilattices. Then there exists a unique $\mathscr{D}$-morphism g: $PL_1 \to PL_2$ such that*

$$\begin{tikzcd} PL_1 \arrow[r, "g"] \arrow[d, "r_{L_1}"'] & PL_2 \arrow[d, "r_{L_2}"] \\ L_1 \arrow[r, "f"'] & L_2 \end{tikzcd}$$

commutes, and one has the following statements:

(i) $g(J) = \downarrow f(J)$ *for all* $J \in PL_1$,

(ii) *Let h: $L_2 \to L_1$ be the right adjoint of f; the ng and h^{op}: $L_2^{\mathrm{op}} \to L_1^{\mathrm{op}}$ are dual* (see 3.2).

Proof: In the light of 3.2, the proposition is just a reinterpretation of 2.26. (Notably (i): By 2.26 (i) we have $g(\downarrow x) = \downarrow f(x)$ for all $x \in L_1$; since g preserves sups of upwards directed sets, $g(J) = g(\bigcup\{\downarrow x\colon x \in J\}) = \bigcup\{\downarrow f(x)\colon x \in J\} = \downarrow f(J)$.) ■

Definition 3.4. Let $\mathscr{C}$ be the category of compact semilattices and continuous morphisms. We denote the unique morphism g in 3.3 with

Pf and observe immediately that $P1_L = 1_{PL}$ and $P(f_2f_1) = (Pf_2)(Pf_1)$, i.e. that $P\colon \mathscr{C}\to\mathscr{D}$ is a functor. We note that the functor takes $\mathscr{CL}$ into $\mathscr{DL}$ by 2.29 and 3.3. ■

DEFINITION 3.5. If $L\in\mathscr{CL}$, then the $\mathscr{CL}$-morphism $r_L\colon PL\to L$ has a right adjoint (2.14) which we denote by $s_L\colon L\to PL$. ■

b. The morphism $s_L\colon L\to PL$.

PROPOSITION 3.6. *If $L\in\mathscr{CL}$, then the function $s_L\colon L\to PL$ has the following properties*: (i) $r_Ls_L = 1_L$; *in particular, s_L is injective.* (ii) *s_L preserves arbitrary sups and satisfies $s(xy)\leqslant s(x)s(y)$. The function s_L is a $\mathscr{S}$-morphism (preserves infs) iff $s_L(L)$ is a sublattice* (i.e. is closed under the multiplication of PL). (iii) *s_L is continuous from below* (1.14) *and lower semicontinuous* (1.16). (iv) $s_L(\operatorname{int}\uparrow x)\subseteq \operatorname{int}\uparrow s_L(x)$.

Proof. (i): 1.12 (ii): 1.7 and 1.10, and the fact that an order preserving bijection between lattices is an isomorphism. (iii) and (iv): 1.20. ■

The following characterization of s_L will be rather crucial:

PROPOSITION 3.7. *Let $L\in\mathscr{CL}$ be a Lawson semilattice, and $J\in PL$ a lattice ideal with* $\sup J = x\in L$. *Then the following statements are equivalent*:

(1) *J is the smallest lattice ideal satisfying* $\sup J = x$ (i.e. $J = (\downarrow x)_0$ (2.19)).

(2) $J = s_L(x)$.

(3) *$y\in J$ if and only if $x\in\operatorname{int}\uparrow y$.*

Proof. (1) $\Leftrightarrow$ (2) by 1.8 and 3.1. (See also 2.14.) (1) $\Leftrightarrow$ (3): let $I = \{y\in L\colon x\in\operatorname{int}\uparrow y\}$. Obviously, $IL\subseteq I$, and if $y, z\in I$ then $x\in\operatorname{int}\uparrow y\cap\cap\operatorname{int}\uparrow z\subseteq \operatorname{int}(\uparrow y\cap\uparrow z) = \operatorname{int}\uparrow(y\vee z)$, thus $y\vee z\in I$. Hence $I\in PL$. Clearly, $\sup I\leqslant x$; by 1.17, $\sup I = x$. Suppose that $J\neq I$. Then there is a $y\in I\setminus J$, i.e. $\uparrow y\cap J = \emptyset$; but $x\in\operatorname{int}\uparrow y$ on one hand and $x = \sup J = \lim J\in\bar{J}$ on the other, and this is a contradiction. ■

This observation gives rise to various results on the morphism s_L.

COROLLARY 3.8. *If $T\in\mathscr{L}$, then $s_T(x) = \downarrow(\downarrow x\cap K(T)) = \{y\in T$: there is a $k\in K(T)$ with $y\leqslant k\leqslant x\}$.*

Proof. By 3.7 we have $y\in s_T(x)$ iff $x\in\operatorname{int}\uparrow y$; since $x = \sup(\downarrow x\cap K(T)) = \lim(\downarrow x\cap K(T))$, there is a $k\in\downarrow x\cap K(T)$ with $k\in\operatorname{int}\uparrow y$; then $y\leqslant k\leqslant x$. Conversely, if $y\leqslant k\leqslant x$ with $k\in K(T)$, then $x\in\uparrow k\subseteq\operatorname{int} y$. ■

COROLLARY 3.9. *If $T\in\mathscr{D}$, then $s_T(x) = c_T^{-1}(\downarrow x) = \{y\in T\colon c_T(y)\leqslant x\}$.*

Proof. Immediate from 3.8. ■

PROPOSITION 3.10. *Let $x\in L$ with $L\in\mathscr{CL}$. Then $s_L(x)\in K(PL)$ if and only if $x\in K(L)$.*

Proof. If $s_L(x)\in K(PL)$, then $s_L(x) = \downarrow y$ for some $y\in L$; hence $x = \sup s_L(x) = y$. By 3.7.3 we have $x\in\operatorname{int}\uparrow x$, i.e. $x\in K(L)$. Conversely,

let $x \in K(L)$; then $x = \sup s_L(x)$ implies $x \in s_L(x)$, since x is a compact element and $s_L(x)$ a sup-semilattice. Hence $s_L(x) = \downarrow x$. ■

COROLLARY 3.11. *If $T \in \mathscr{D}$ is a dual complete semilattice, then s_T: $T \to PT$ is a $\mathscr{D}$-morphism* (i.e. compact morphism: see 2.1–2.4)) *with $s_T|K(T) = i_T|K(T)$ and with $K(s_T) = c_T$: $T \to K(T)$* (upon identifying T with $K(PT)$ under i_T'). *Thus s_T and c_T^{op} are dual morphisms.*

Conversely, if $L \in \mathscr{CL}$ is a compact Lawson semilattice and if s_L is a morphism of compact semilattices, then $L \in \mathscr{D}$ (i.e. L is dual complete).

Proof. For the first part, we identify T and $\mathscr{F}(K(T)^{\mathrm{op}})$ under $t \to \downarrow t \cap K(T)$; then, by ([8], II-2.4, p. 35), the morphism $\mathscr{F}(c_T)$: $T \to \mathscr{F}(T^{\mathrm{op}}) = PT$ is given by $\mathscr{F}(c_T)(t) = c_T^{-1}(\downarrow t \cap K(T)) = c_T^{-1}(\downarrow t) = s_T(t)$, in view of 3.9. This proves $c_T = K(s_T)$ (by duality). But 3.10 shows that s_T is compact.

For the second part, let us assume that $L \in \mathscr{CL}$ and $s_L \in \mathscr{CL}$. Then $L \in \mathscr{L}$ since s_L is injective. The "dual" $K(s_L)$: $L \to K(L)$ is defined by $K(s_L)(t) = \min\{t' \in L\colon t \in s_L(t')\} \in K(L)$; by 3.8, $t \in s_L(t')$ is equivalent to the existence of a $k \in K(L)$ with $t \leqslant k \leqslant t'$. Hence $K(s_L)(t) = \min_{K(L)}(\uparrow t \cap K(T))$, and this makes $K(s_L)$ a right adjoint of j_L: $K(L) \to L$. Thus, $L \in \mathscr{D}$ by 2.7. iv. ■

c. The functor P: $\mathscr{CL} \to \mathscr{D}$. We are now in a position to prove an abstract characterization of the functor P:

THEOREM 3.12. *The functor P: $\mathscr{CL} \to \mathscr{D}$ is a (right) reflector, i.e. a right adjoint to the inclusion functor $\mathscr{D} \to \mathscr{CL}$. The front adjunction is given by s_T: $T \to PT$, the back adjunction by r_L: $PL \to L$.*

Proof. Let F: $T \to L$ be a $\mathscr{CL}$-morphism with $T \in \mathscr{D}$. Define f': $T \to PL$ by $f' = (Pf)s_T$. Since $s_T \in \mathscr{D}$ by 3.10 and $Pf \in \mathscr{D}$ by 3.3, we have $f' \in \mathscr{D}$. Moreover, $r_L f' = r_L(Pf)s_T = f r_T s_T = f$ by 3.3 and 3.6.i. Suppose that f'': $T \to PL$ is a $\mathscr{D}$-morphism with $r_L f'' = f$. Since f'' is compact, for any $k \in K(T)$ there is an $x \in L$ with $f''(k) = \downarrow x$; then $x = \sup f''(k) = r_L f''(k) = f(k)$. On the other hand, $f'(k) = (Pf)s_T(k) = (Pf)(\downarrow k)$ by 3.10, and $(Pf)(\downarrow k) = \downarrow f(\downarrow k) = \downarrow f(k)$. Thus, for all $k \in K(T)$ we have $f''(k) = \downarrow f(k) = f'(k)$. Since $K(T)$ is dense in T, we have $f'' = f'$. This shows that for each $\mathscr{CL}$-morphism f: $T \to L$ there is a unique $\mathscr{D}$-morphism f': $T \to PL$ with $r_L f' = f$; hence P is a right adjoint for the inclusion function $\mathscr{D} \to \mathscr{CL}$ with r_T as back adjunction. The front adjunction is $(1_T)' = (P1_T)s_T = s_T$. ■

The functor P is obviously faithful, since $Pf_1 = Pf_2$ implies $f_1 = f_2$ by 3.3 because of the surjectivity of r_L. But P is neither full (see 2.30) nor representative (see 2.13 or 2.14). However:

PROPOSITION 3.13. *The corestriction P: $\mathscr{CL} \to \mathscr{DL}$* (3.4) *is an equivalence of categories.*

Proof. This corestriction is faithful, full and representative by the definition of $\mathscr{DL}$. The inverse functor is the functor which associates with a $\mathscr{DL}$-object T the semilattice $K(T)$ with its associated Lawson topology according to 2.23iii. See also 2.29. ■

The coextension $P\colon \mathscr{CL}\to\mathscr{X}$ is not right adjoint to the inclusion functor $\mathscr{X}\to\mathscr{CL}$. This follows immediately from the following observation in view of the fact, that right adjoints preserve limits, hence, in particular, products:

PROPOSITION 3.14. *The coextension $P\colon \mathscr{CL}\to\mathscr{X}$ preserves finite products and fails to preserve infinite products.*

Proof. By duality we have a natural isomorphism between $P(\Pi_{\mathscr{CL}}T_a)$ and $\Pi_{\mathscr{X}}PT_a$ iff we have a natural isomorphism between the duals ([8], I-4.3, p. 21). By 3.2 we have $P(\Pi_{\mathscr{CL}}T_a)^{\wedge} \cong (\Pi_{\mathscr{CL}}T_a)^{\mathrm{op}} = \Pi_{\mathscr{S}}T_a^{\mathrm{op}}$ and $(PT_a)^{\wedge} = T_a^{\mathrm{op}}$. By ([8], I-43, p. 21) we know $(\Pi_{\mathscr{X}}PT_a)^{\wedge} \cong \Pi_{\mathscr{S}}(PT_a) \cong \Pi_{\mathscr{S}}T_a^{\mathrm{op}}$. For infinite families $\Pi_{\mathscr{S}}T_a^{\mathrm{op}} \neq \Pi_{\mathscr{S}}T_a^{\mathrm{op}}$, for finite families $\Pi_{\mathscr{S}}T_a^{\mathrm{op}} = \Pi_{\mathscr{S}}T_a^{\mathrm{op}}$ ([8], I-17, p. 8). ■

As an application we derive a result which we first proved in [8].

PROPOSITION 3.15. *Let $I = [0,1]\in\mathscr{CL}$ be the unit interval under min. Then PI is isomorphic to $\tilde{I} = (I\times\{0,1\})\setminus\{(0,0)\}$ with the lexicographic order under $\varphi\colon \tilde{I}\to PI$, $\varphi(r,0) = [0,r[$, $\varphi(r,1) = [0,r]$, and the diagram*

(i)

$$\begin{tikzcd} \tilde{I} \arrow[rr, "\varphi"] \arrow[dr, "Pr_I"'] & & PI \arrow[dl, "r_I"] \\ & I & \end{tikzcd}$$

commutes. If $T\in\mathscr{D}$ is a dual complete semilattice, then every morphism $f\colon T\to I$ factors uniquely in the following fashion

(ii)

$$\begin{tikzcd} T \arrow[rr, "f'"] \arrow[dr, "f"'] & & \tilde{I} \arrow[dl, "Pr_I"] \\ & I & \end{tikzcd}$$

$f'\in\mathscr{D}$. If $Q_0\subseteq I$ is the set of all dyadic rationals $m/2^n \geqslant 0$, set $C = \{(r,s)\in I\times\times\{0,1\} \mid r\notin Q_0 \Rightarrow s = 0\}$. Then C is isomorphic to the Cantor set under min [9], and $j\colon \tilde{I}\to C$, the left adjoint of the inclusion $C\to\tilde{I}$ is a $\mathscr{X}$-morphism such that $Pr_I = cj$ where $c\colon C\to I$ is the "Cantor morphism", i.e. the projection onto I. Hence every $f\colon T\to I$ factors indeed through c.

Proof. Exercise. ■

d. PL as a projective object. Certain lifting properties are most conveniently expressed in terms of projectivity in a category. It turns out that the concept of a relative projective is often more useful. We make these concepts precise in the following definition:

DEFINITION 3.16. (i) Suppose that $\mathscr{A}$ is a category and $\mathscr{E}$ a class of epics in $\mathscr{A}$. An object P of $\mathscr{A}$ is called an *$\mathscr{E}$-projective* iff for every $e \in \mathscr{E}$, $e\colon A \to B$ and every $p\colon P \to B$ there is a $q\colon P \to A$ with $eq = p$. If $\mathscr{E}$ is the class of *all* epics, then p is called a *projective*.

(ii) A morphism $p\colon P \to B$ is an *$\mathscr{E}$-projective cover of B*, iff P is an $\mathscr{E}$-projective and p is co-essential (i.e. p is epic and for any morphism $g\colon Y \to P$ the epimorphy of pg implies that of g).

Co-essentiality is a sort of minimality condition: Only very recently was it discovered that every epimorphism in $\mathscr{CL}$ is surjective [20]. We will comment on this fact in Section 5 (see 5.6 ff.). With this information we can immediately give the concept of co-essentiality in $\mathscr{CL}$ the following concrete meaning:

PROPOSITION 3.17. *A morphism $h\colon S \to T$ in $\mathscr{CL}$ is co-essential iff it is surjective and $h(S') = T$ implies $S' = S$ for each closed subsemilattice S' of S.* ■

NOTATION 3.18. In $\mathscr{CL}$ we denote with $\mathscr{E}$ the class of all surjective $\mathscr{CL}$-morphisms which are, in addition, lattice morphisms (i.e. preserve finite sups). The members of $\mathscr{E}$ are also called *$\mathscr{E}$-epics*.

It is now our objective to prove that PL is always an $\mathscr{E}$-projective.

PROPOSITION 3.19. *Let $e\colon S \to T$ be an $\mathscr{E}$-epic. Then there is a unique $\mathscr{D}$-morphism $f\colon PT \to PS$ such that:*

(i) $K(f)i'_S = i'_T e$ (see 3.2), *i.e. f is the dual of e^{op}.*

(ii) $er_S f = r_T$.

(iii) $(Pe)f = 1$.

$$\begin{tikzcd}
PS \arrow[rr, "Pe"] \arrow[dd, "r_S"'] & & PT \arrow[dd, "r_T"] \\
& PT \arrow[ul, "f"] \arrow[ur, "1"'] \arrow[dr, "r_T"'] & \\
S \arrow[rr, "e"'] & & T
\end{tikzcd}$$

Proof. The existence of f such that (i) holds follows from duality. Then 2.32 applies and shows (ii). Since $f(J) = e^{-1}(J)$, we have $(Pe)f(J) = e(e^{-1}(J)) = J$, since e is surjective. This establishes (iii). ■

Now we have

THEOREM 3.20. *Let L be a compact Lawson semilattice. Then r_L: $PL \to L$ is an $\mathscr{E}$-epic. Moreover, if e: $S \to L$ is any $\mathscr{E}$-epic, then there is a morphism $\bar{e}$: $PL \to S$ of compact Lawson semilattices such that $e\bar{e} = r_L$.*

$$\begin{tikzcd} & PL \arrow[dl, "\bar{e}"'] \arrow[d, "r_L"] \\ S \arrow[r, "e"'] & L \end{tikzcd}$$

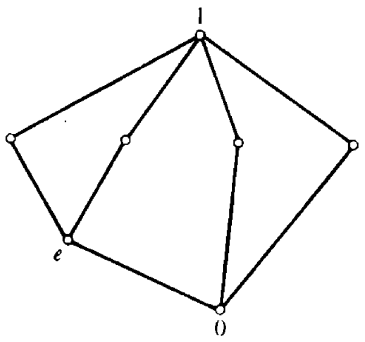
Proof. By 3.2, r_L preserves sups. Hence it is an $\mathscr{E}$-epic. Define f: $PL \to PS$ according to 3.19 and set $\bar{e} = r_S f$. ■

THEOREM 3.21. *For any compact Lawson semilattice L, the $\mathscr{D}$-object PL is an $\mathscr{E}$-projective.*

Proof. Let e: $S \to T$ be an $\mathscr{E}$-epic and p: $PL \to T$ a $\mathscr{CL}$-morphism. By 3.20, there is a morphism $\bar{e}$: $PT \to S$ with $e\bar{e} = r_T$. Since $PL \in \mathscr{D}$, by 3.12, there is a $\mathscr{D}$-morphism p': $PL \to PT$ such that $p = r_T p'$. If we set $q = \bar{e}p'$: $PL \to S$, then $eq = e\bar{e}p' = r_T p' = p$. The condition of 3.16. i is thus satisfied. ■

One might deplore the fact that the lifting property in Theorem 3.20 is established only for $\mathscr{E}$-morphisms. The following remarks show that this restriction is really necessary:

EXAMPLE 3.22. Let S be the semilattice indicated in the following diagram:



Let L be the quotient semilattice obtained from S by collapsing e and 0, and let e: $S \to L$ be the quotient map. One notes that e is a co-essential $\mathscr{CL}$-morphism and that r_L: $PL \to L$ is an isomorphism. If we had a morphism $\bar{e}$: $PL \to S$ with $e\bar{e} = r_L$, then the relation $e(\bar{e}r_L^{-1}) = 1_L$ would show that e is a retraction. But a co-essential retraction is always an isomorphism, thus e would have to be bijective, which it is not. Thus the lifting property of 3.20 fails. ■

The situation illustrated in Example 3.22 can be described accurately:

PROPOSITION 3.23. *Let e: $S \to L$ be a co-essential $\mathscr{CL}$-morphism such that there is an $\bar{e}$: $PL \to S$ with $e\bar{e} = r_L$, then e is an $\mathscr{E}$-epic (and preserves, in fact, arbitrary sups).*

Proof. Since r_L is surjective, e is surjective. Let d: $S \to PL$ be the right adjoint of $\bar{e}$ (see 1.9 and 1.26). Then d preserves sups by 1.7, and thus $r_L d$ preserves sups by 3.2. By 1.12, we have $\bar{e}d = 1$, hence $e = e\bar{e}d = r_T d$ and thus e preserves arbitrary sups. ■

We point out that a lifting of some sort will always exist:

PROPOSITION 3.24. *Let f: $S \to L$ be a surjective $\mathscr{CL}$-morphism. Then there exists a function f^*: $PL \to S$ with $ff^* = r_L$ with the following properties:*

(i) *f^* preserves arbitrary sups,*

(ii) *f^* is continuous from below and upper semicontinuous* (see 1.14, 1.16).

Proof. Let d: $PL \to PS$ be the right adjoint of Pf and set $f^* = r_S d$. Then 2.31 applies to show $ff^* = r_L$. Since r_S and d preserve sups (3.2, 1.7), we have that (i), and (ii) follows from 1.20 and the continuity of r_S. ■

Finally, we remark, somewhat outside the scope of the context of PL, that for any compact Lawson semilattice L there is always a projective P in the category $\mathscr{CL}$ and a surjective morphism p: $P \to L$; such a morphism will have the lifting property for all f: $S \to L$ in $\mathscr{CL}$. Indeed, in the category of compact spaces there is a projective compact space (i.e. an extremally disconnected compact space) S and a surjective continuous function ξ: $X \to L$ (example: let X be the Stone–Čech compactification of the underlying (discrete) set L_d of L and ξ the map induced by $L_d \to L$). Now let $F(X)$ be the free compact Lawson semilattice generated by X, namely, the $\cap$-semilattice of all compact non-empty subsets of X. If φ: $X \to T$ is any continuous function into a $\mathscr{CL}$-semilattice T, then there is a unique morphism φ': $F(X) \to T$ with $\varphi'(\{x\}) = \varphi(x)$, $x \in X$; indeed $\varphi'(Y) = \inf \varphi(Y)$ satisfies the requirement. Now take $p = \xi'$: $F(X) \to L$ and suppose that f: $S \to L$ is a $\mathscr{CL}$-morphism. Since X is a projective compact space, there is a continuous function ψ: $X \to S$ with $f\psi = \xi$. Set $\bar{f} = \psi'$, then $f\bar{f} = f\psi' = (f\psi)' = \xi' = p$.

It should be noted, however, that $F(X)$ is vast (even by comparison with the already large PL) and that it is not particularly functorial in L (although, by choosing for X some canonical projective cover in the category of compact spaces, it could be made functorial).

Unfortunately, r_L: $PL \to L$, despite its agreeable universal properties which we have exhibited, is not an $\mathscr{E}$-projective cover in general. It is practically trivial, to derive, starting from a surjective $\mathscr{CL}$-morphism, a co-essential $\mathscr{CL}$-morphism:

LEMMA 3.25. *Let f: $S \to T$ be a surjective $\mathscr{CL}$-morphism. Let $\mathscr{X}$ be the family of all compact semilattices $S' \subseteq S$ with $f(S') = T$. Then $(\mathscr{X}, \supseteq)$ is inductive, and if S_0 is a minimal element of $\mathscr{X}$, then $e = f|S_0$: $S_0 \to T$ is a co-essential morphism.*

Proof. Immediate from Zorn's Lemma and 3.17. ■

Thus, using the axiom of choice via 3.25 we find subsemilattices $Q \subseteq PL$ such that $r_L|Q\colon Q \to L$ is co-essential. We will show in the next chapter, that Q is in fact uniquely determined. Under suitable circumstances we will be able to give an accurate description of Q, but in general it seems to remain somewhat mysterious.

4. On the fine structure of PL

a. The construction of $A(L)$. The first definition is connected with 3.7.

DEFINITION 4.1. Let $L \in \mathscr{C}$ be a compact semilattice. We write $x \ll y$ if and only if $y \in \operatorname{int} \uparrow x$ (or if and only if $x \in (\downarrow y)_0 = s_L(x)$ should $L \in \mathscr{CL}$ according to 3.7). We set $A(L) = \{x \in L\colon x \in (\operatorname{int} \uparrow x)^-\}$. Thus $x \in A(L)$ if and only if $x = \lim x_n$ for some net (x_n) such that $x \ll x_n$. We let $B(L)$ denote the supsubsemilattice of L generated by $A(L)$. ■

It is immediate that $\ll$ is a transitive relation and that $x \ll y$ implies that $x \leqslant y$, whereas the converse may fail. The relation $x \ll x$ is equivalent to $x \in K(L)$ (see [8], II-3.3). The element 0 is always contained in $A(L)$. More generally $K(L) \subseteq A(L)$.

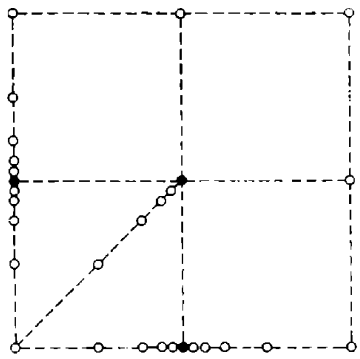
EXAMPLE 4.2. Let $r_n \in]0, 1/2[$ be a sequence converging monotonically to 0 and let L be the compact subsemilattice of $[0, 1]^2$ composed of the following sets:

(i) $\{(a, b)\colon a, b \in \{0, 1/2, 1\}\}$

(ii) $\{(1/2 - r_n, 1/2 - r_n)\colon n = 1, 2, \ldots\}$

(iii) $\{(1/2 \pm r_n, 0)\colon n = 1, 2, \ldots\}$

(iv) $\{(0, 1/2 \pm r_n)\colon n = 1, 2, \ldots\}$



We note that $L \in \mathscr{Z}$ and $A(L) = L \setminus \{(1/2, 1/2)\}$. Thus $A(L)$ contains the elements $(1/2, 0)$ and $(0, 1/2)$, but not their sup $(1/2, 1/2)$. Also $A(L)$ contains $(1/2, 1)$ and $(1, 1/2)$, but not their inf $(1/2, 1/2)$. Finally, the ascending sequence $(1/2 - r_n, 1/2 - r_n) \in A(L)$ has no sup in $A(L)$. ■

The closure properties of $A(L)$ are, therefore, somewhat meager. In any case we have:

PROPOSITION 4.3. (i) *Let L be a compact semilattice. Then $A(L)$ is closed under infs of downward directed sets.*

(ii) *Let L be a compact semilattice satisfying*

(UC) *for each $(a, b) \in L \times L$ the restriction of the function $(x, y) \to x \vee y$: $L \times L \to L$ to $\uparrow(a, b)$ is continuous at (a, b).*

Then $A(L)$ is a sup-semilattice, i.e. $B(L) = A(L)$. Note that condition (UC) *is satisfied if L is a topological lattice.*

Proof. (i) Let $D \subseteq A(L)$ be downwards directed, and let $a = \inf_L D$. Then $\bigcup \{\operatorname{int} \uparrow d\colon d \in D\} \subseteq \operatorname{int} \uparrow a$; if U is an open neighborhood of a in L then there is a $d \in D$ with $d \in U$, since $d = \lim D$ (see [8], II-1.1, p. 29). Since $d \in A(L)$, we have $U \cap \operatorname{int} \uparrow d \neq \emptyset$. It follows that $U \cap \operatorname{int} \uparrow a \neq \emptyset$. Hence $a \in A(L)$.

(ii) Let $a, b \in A(L)$. Then $(a, b) = \lim(a_n, b_n)$ with $a \ll a_n$, $b \ll b_n$. By hypothesis we have $a \vee b = \lim a_n \vee b_n$. The following lemma will show that $a \vee b \ll a_n \vee b_n$ and thus finish the proof. ■

LEMMA 4.4. *In any $L \in \mathscr{C}$ we have the following propositions:*

(i) *If $a \ll x$ and $b \ll y$ then $a \vee b \ll x \vee y$.*

(ii) *$\uparrow \operatorname{int} \uparrow u = \operatorname{int} \uparrow u$ for all u.*

Proof. First we note that (ii) $\Rightarrow$ (i): $a \ll x$ and $b \ll y$ means $x \in \operatorname{int} \uparrow a$ and $y \in \operatorname{int} \uparrow b$; but this implies $x \vee y \in \operatorname{int} \uparrow a \cap \operatorname{int} \uparrow b$ by (ii). Since obviously $\operatorname{int} \uparrow a \cap \operatorname{int} \uparrow b \subseteq \operatorname{int}(\uparrow a \cap \uparrow b) = \operatorname{int} \uparrow (a \vee b)$, we conclude (i). Secondly, we prove (ii): $w \in \operatorname{int} \uparrow u$ means $v \leqslant w$ for some $v \in \operatorname{int} \uparrow u$; by the continuity of $z \to vz$ there is an open neighborhood W of w with $vW \subseteq \operatorname{int} \uparrow u \subseteq \uparrow u$; thus $W \subseteq \uparrow u$ and therefore $W \subseteq \operatorname{int} \uparrow u$, showing $w \in \operatorname{int} \uparrow u$. ■

We should bear in mind that in the following discussion we can read $A(L) = B(L)$ if L is a topological lattice.

NOTATION 4.5. For brevity we will set $\mathrm{Id}(B(L)) = \mathscr{F}(B(L)^{\mathrm{op}})$; thus for $J \subseteq B(L)$ we have $J \in \mathrm{Id}(B(L))$ iff

(i) J is a sup-semilattice.

(ii) If $b \in B(L)$ and $b \leqslant j \in J$, then $b \in J$. We recall from [8], II-2 that $\mathrm{Id}(B(L))$ may be identified with the dual of the sup-semilattice $B(L)$. In particular, it has a unique $\mathscr{Z}$-topology. ■

LEMMA 4.6. *Let $L \in \mathscr{C}$. If $J \in PL$, then $J \cap B(L) \in \mathrm{Id}(B(L))$, and if $I \in \mathrm{Id}(B(L))$, then $\downarrow_L I = \{x \in L\colon$ There is an $a \in I$ with $x \leqslant a\}$ is in PL. In particular, the functions*

$$\lambda_L\colon PL \to \mathrm{Id}(B(L)), \quad \lambda_L(J) = J \cap B(L),$$

and

$$\varrho_L\colon \mathrm{Id}(B(L)) \to PL, \quad \varrho_L(I) = \downarrow_L I$$

are well defined.

Proof. (i) If $J \in PL$, let $a, b \in J \cap B(L)$, $c \in B(L)$. If $c \leqslant a$, then $c \in J$, hence $c \in J \cap B(L)$. Further, $a \vee b \in J \cap B(L)$. Thus, $J \cap B(L) \in \mathrm{Id}(B(L))$.

(ii) If $I \in \mathrm{Id}(B(L))$ and $x, y \in \downarrow_L I$, then there are $a, b \in I$ with $x \leqslant a$, $y \leqslant b$, whence $x \vee y \leqslant a \vee b \in I$. Thus $x \vee y \in \downarrow_L I$. Therefore, $\downarrow_L I \in PL$. ■

PROPOSITION 4.8. *For a compact semilattice $L \in \mathscr{C}$, the pair (λ_L, ϱ_L) is a Galois connection between PL and $\mathrm{Id}(B(L))$. We have $\lambda_L \varrho_L = 1$, whence the left adjoint λ_L is surjective, whereas the right adjoint ϱ_L is injective. The left adjoint λ_L: $PL \to \mathrm{Id}(B(L))$ is a $\mathscr{X}$-morphism relative to the canonical $\mathscr{X}$-topologies of PL and $\mathrm{Id}(B(L))$* (3.2 and 4.5).

Proof. First we observe that $J \supseteq \downarrow_L I$ is equivalent to $J \cap B(L) \supseteq I$ for $J \in PL$ and $I \in \mathrm{Id}(B(L))$; by 1.4.3 this shows that (λ_L, ϱ_L) is a Galois connection. From $(\downarrow_L I) \cap B(L) = I$ it follows that λ_L is surjective and ϱ_L injective. In order to show that a function between $\mathscr{CL}$-objects is a $\mathscr{CL}$-morphism, by 1.25 we must prove that it preserves arbitrary infs and sups of upward directed sets. The first preservation property is clear from 1.7; we must prove the second. Let $\{J_x\colon x \in X\}$ be an upward directed collection of elements in PL. Then $\sup_{PL}\{J_x\colon x \in X\} = \bigcup\{J_x\colon x \in X\}$, whence $\lambda_L(\sup_{PL} J_x) = (\bigcup J_x) \cap B(L) = \bigcup(J_x \cap B(L)) = \bigcup \lambda_L(J_x) = \sup_{\mathrm{Id}(B(L))} \lambda_L(J_x)$. ■

At this point we specialize to Lawson semilattices.

PROPOSITION 4.9. *Let L be a compact Lawson semilattice. Then the following statements hold*:

(a) $s_L(x) = (\downarrow x)_0 = \{y \in L\colon$ there is $a \in A(L)$ with $y \leqslant a \ll x\} = \{y \in L\colon$ there is $b \in B(L)$ with $y \leqslant b \ll x\} = \downarrow_L(\downarrow x)_0 \cap A(L) = \downarrow_L(\downarrow x)_0 \cap B(L) = \varrho_L \lambda_L s_L(x)$. *In particular*, $s_L(1) = L_0 = \downarrow_L(A(L))$.

(b) $x = \sup_L((\downarrow x)_0 \cap A(L)) = \sup_L(\downarrow x \cap A(L)) = \sup_L(\downarrow x \cap B(L))$ *for all* $x \in L$.

(c) $\sup_L \lambda_L(J) = \sup_L(J \cap A(L)) = \sup_L J = r_L(J)$ *for all* $J \in PL$.

(d) $\mathrm{im}\, s_L \subseteq \mathrm{im}\, \varrho_L$.

(e) $L = B(L)^- = A(L)^-$.

Proof (a) By 3.7 we have $y \in (\downarrow x)_0$ if and only if $y \ll x$ (4.1). Let $A = \uparrow y \setminus \mathrm{int} \uparrow y$. Since $L \in \mathscr{CL}$, for each $a \in A$ there is $b \in L$ such that $a \notin \uparrow b$ and $b \ll x$. Then $y \vee b \ll x$ by 4.4. Thus $a \in A \cap L \setminus \uparrow(y \vee b)$ which is an open subset of A. In this fashion we secure an open cover of A. Then from the compactness of A we obtain a finite subcover. This is equivalent to the existence of a set $\{b_1, \ldots, b_m\}$ such that $y \leqslant b_i \ll x$ for $i = 1, \ldots, m$ and $A \subseteq \bigcup(L \setminus b_i)$. This then implies that $y \leqslant \sup\{b_1, \ldots, b_m\} \ll x$ with $\sup\{b_1, \ldots, b_m\} \in \mathrm{int} \uparrow y$. Denote $\sup\{b_1, \ldots, b_m\}$ by c_1. Should c_1 not be in $A(L)$ we may repeat the process above to secure $c_2 < c_1$ such that $y \ll c_2 \ll c_1 \ll x$. By this method we either find an element c_i with the desired property or we obtain a decreasing net $c_1 \gg c_2 \gg \ldots$. Then $\bigcup \uparrow c_i$

is an open filter on L containing x and from 4.3.i, $\inf \bigcup \uparrow c_i \geqslant y$ and belongs to $A(L)$. Conversely, suppose that there is $a \in A(L)$ such that $y \leqslant a \ll x$. Then *a fortiori* $y \leqslant b \ll x$ for some $b \in B(L)$, and if this condition holds, then $x \in \operatorname{int} \uparrow b \subseteq \operatorname{int} \uparrow y$. Hence $y \in s_L(x)$. This establishes the first three equalities. The others are immediate from the definitions. If $a \in A(L)$, then $a \ll 1$, whence $L_0 = \downarrow_L(A(L))$.

(b) By 2.19 we have $x = \sup(\downarrow x)_0$, hence (a) proves the first equality. The second and third are trivial from

$$(\downarrow x)_0 \cap A(L) \subseteq (\downarrow x) \cap A(L) \subseteq \downarrow x \cap B(L) \subseteq \downarrow x.$$

(c) If $J \subseteq PL$, let $x = \sup_L J$. Then $(\downarrow x)_0 \subseteq J \subseteq \downarrow x$ by 2.19, whence (c) is a consequence of (b).

(d) is immediate from (a).

(e) Let $x \in L$ and suppose that U is an open convex neighborhood of x. Then there is $u \in U$ such that $u \ll x$ since L is a Lawson semilattice. Then from 4.9 there is $a \in A(L)$ such that $u \leqslant a \ll x$. Since x has a basis of open convex neighborhoods, we have $L = A(L)^-$ and $L = B(L)^-$. ■

Now we sharpen 4.9. (d). First some notation:

NOTATION 4.10. We say that $A(L)$ is *cofinal in* $J \in PL$ if for any $x \in J$ there is an $a \in J \cap A(L)$ with $x \leqslant a$. We let $P_A L$ be the set of all $J \in PL$ in which $A(L)$ is cofinal.

Note. If $A(L) = B(L)$ (as, e.g., in the situation of 4.3), then $P_A L = \operatorname{im} \varrho_L$.

PROPOSITION 4.11. *If L is a compact Lawson semilattice, then* $s_L(L) \subseteq P_A L \subseteq s_L(L)^-$. In other words, $(P_A L)^- = s_L(L)^-$. *If* $A(L) = B(L)$, *then* $(\operatorname{im} \varrho_L)^- = (\operatorname{im} s_L)^-$.

Proof. By 4.9.a we have $s_L(L) \subseteq P_A L$. Now take a $J \in P_A(L)$; we have to prove $J \in s_L(L)^-$. A basic neighborhood of J is $W(x, F) = \{J' \in PL: x \in J',\ J' \cap F = \emptyset\}$ for some $x \in J$ and some finite set F with $F \cap J = \emptyset$. Since $x \in J \in P_A L$, there is an $a \in J \cap A(L)$ with $x \leqslant a$. Now we have $a \in (\operatorname{int} \uparrow a)^-$ on one hand and $a \notin F$ on the other; since $\uparrow F$ is closed, the open neighborhood $PL \setminus \uparrow F$ of a must intersect $\operatorname{int} \uparrow a$. Hence there is a $y \in \operatorname{int} \uparrow a$ with $y \notin \uparrow F$. By 3.7 we have $a \in (\downarrow y)_0$, and since $\downarrow y \cap F = \emptyset$, *a fortiori* $(\downarrow y)_0 \cap \cap F = \emptyset$. Hence $s_L(y) = (\uparrow y)_0 \in W(a, F) \subseteq W(x, f)$. Thus $W(x, F) \cap \operatorname{im} s_L \neq \emptyset$, as we had to show. ■

One recalls that in 3.10 we showed that $i_L(K(L)) = K(PL) \cap \operatorname{im} s_L$. Now we show more:

PROPOSITION 4.12. *Let L be a compact Lawson semilattice. Then*

(0) $i_L(K(L)) = K(PL) \cap \operatorname{im} s_L$; $K(PL) = \operatorname{im} i_L$,

(i) $i_L(A(L)) = K(PL) \cap P_A L$,

(ii) $i_L(B(L)) = K(PL) \cap \operatorname{im} \varrho_L = \varrho_L(K(\mathrm{Id}(B(L))))$.

Proof. As we noted, (0) was proved in 3.10. Proof of (i): Let $J \in i_L(A(L))$, then $J = \downarrow a$ for some $a \in A(L)$; but then, trivially, $A(L)$ is cofinal in J, whence $J \in K(PL) \cap P_A L$. Conversely, let $J \in K(PL) \cap P_A L$. From $J \in K(PL)$ we deduce $J = \downarrow x$ for some $x \in L$ (3.2). Since $J \in P_A L$, there must be an $a \in A(L) \cap J$ with $x \leqslant a$; but then $x = a$ and $J = i_L(a) \in A(L)$. The proof of the first equality in (ii) is similar in view of the fact that $J \in \operatorname{im} \varrho_L$ iff $\downarrow_L(J \cap B(L)) = J$. In order to prove the second equality we recall, that for $I \in \mathrm{Id}(B(L))$ we have $I \in K(\mathrm{Id}(B(L)))$ iff $I = \downarrow_{B(L)} b$ for some $b \in B(L)$ (see [8], II-3.3, p. 38). But this is the case iff $\varrho_L(I) = \downarrow_L(I)$ equals $\downarrow_L b$ for some $b \in B(L)$, and this is equivalent to $\varrho_L(I) \in i_L(B(L))$. Since ϱ_L is injective by 4.8, this proves the second equality. ■

The following proposition summarizes diagrammatically our present situation.

PROPOSITION 4.13. *Let L be a compact Lawson semilattice. Define r'_L: $\mathrm{Id}(B(L)) \to L$ by $r'_L(I) = \sup_L I$. Then r'_L is a $\mathscr{CL}$-morphism and there is a commutative diagram of $\mathscr{CL}$-objects*

$$\begin{tikzcd}
 & \mathrm{Id}(B(L)) \arrow[d, "\varrho_L"'] \arrow[dr, "1"] & \\
 & PL \arrow[r, two heads, "\lambda_L"] \arrow[dr, "r_L"] & \mathrm{Id}(B(L)) \arrow[d, "r'_L"] \\
L \arrow[ur, "i_L"] \arrow[r, "1"'] & L \arrow[u, "s_L"] \arrow[r, "1"'] & L
\end{tikzcd}$$

in which all surjective maps are $\mathscr{CL}$-morphisms.

Proof. It remains to verify that (i) $r_L = r'_L \lambda_L$ and that (ii) $r'_L \in \mathscr{CL}$. But since r_L is surjective, (i) $\Rightarrow$ (ii) by Lemma 2.25. Thus we prove (i): Let $J \in PL$. Then $r_L(J) = \sup_L J = \sup_L \lambda_L(J)$ by 4.9.c.; but $\sup_L \lambda_L(J) = r'_L \lambda_L(J)$. ■

Note that in r'_L: $\mathrm{Id}(B(L)) \to L$ we have found a "more economical" representation of L as a quotient of a $\mathscr{Z}$-object than was given by r_L: $PL \to L$; the trouble with r'_L, however, is that its universal properties (if any) are not clear; recall that in 3.12, 3.20 and 3.21 we obtained good insight into the universal properties of r_L.

b. On the geometric structure of PL.

THEOREM 4.14. *For any compact Lawson semilattice L the set $\bar{P}_A L$, defined by*

$$\bar{P}_A L = s_L(L)^- = (P_A L)^-,$$

is the unique smallest closed subset $C \subseteq PL$ with $r_L(C) = L$.

Proof: Let C be a closed subset of PL with $r_L(C) = L$; by 4.11 it suffices to show $s_L(L) \subseteq C$. Thus let $x \in L$; we must show that $(\downarrow x)_0 = s_L(x) \in C$. Let $y \in (\downarrow x)_0$. Since $r_L(C) = L$, there is a $J_y \in C$ with $\sup J_y = r_L(J_y) = y$. Hence $(\downarrow y)_0 \subseteq J_y$ by 2.19; and $y \in (\downarrow x)_0$ implies $J_y \subseteq \downarrow y \subseteq (\downarrow x)_0$. Thus

(i) $s_L(y) \subseteq J_y \subseteq s_L(x)$.

If y ranges through the upward directed set $(\downarrow x)_0$, then $s_L(x) = \lim s_L(y)$ by 3.6.iii. Hence, from (i) above we obtain by squeezing

(ii) $s_L(x) = \lim J_y$.

Since C is closed and $J_y \in C$, we conclude $s_L(x) \in C$. ■

DEFINITION 4.15. If $L \in \mathscr{CL}$ we define QL to be the smallest multiplicatively closed compact subset containing $s_L(L)$, i.e.

$$QL = \Big(\bigcup_{n=1}^{\infty} (\bar{P}_A L)^n\Big)^- .$$

We recall that it is our convention that a semilattice is a commutative idempotent monoid.

LEMMA 4.17. *QL is a semilattice.*

Proof. Recall $L_0 = s_L(1)$; by definition of QL we have $L_0 \in QL$. Then $L_0 \cdot QL$ is a multiplicatively closed compact subset of PL containing $L_0 \cdot s_L(L) = s_L(L)$. Hence $L_0 \cdot QL = QL$, i.e. L_0 is the identity of QL. ■

PROPOSITION 4.18. *If L is a compact Lawson semilattice then QL is the unique smallest multiplicatively closed compact subset C of PL with $r_L(C) = L$.*

Proof. Immediate consequence of Theorem 4.14. ■

DEFINITION 4.19. We denote by q_L: $QL \to L$ the restriction $r_L|QL$, and with μ_L: $QL \to \mathrm{Id}(B(L))$ the restriction $\lambda_L|QL$.

We note that q_L is surjective by 4.18, and μ_L is surjective at least if $A(L) = B(L)$ by 4.11.

PROPOSITION 4.20. *There is a commutative diagram of $\mathscr{CL}$-objects* (see p. 48) *in which all surjective maps and μ_L are $\mathscr{CL}$-morphisms and in which the dotted portion of the diagram applies if $A(L) = B(L)$.* ■

We have now yet another representation of L as a quotient of a $\mathscr{L}$-object (namely, QL), this one being probably less economical than r'_L. On the other hand, we can formulate the following universal property:

$$\begin{tikzcd}
\mathrm{Id}(B(L)) \arrow[d, dashed, "q_L"'] \arrow[dr, dashed, "1"] & & \\
QL \arrow[r, "\mu_L"] \arrow[dr, "r_L"] & \mathrm{Id}(B(L)) \arrow[d, "r'_L"] & \\
L \arrow[u, "s_L"] \arrow[r, "1"'] & L &
\end{tikzcd}$$

PROPOSITION 4.21. *Let L be a compact Lawson semilattice. Then*

(i) q_L: $QL \to L$ *is co-essential* (see 3.17), *and*

(ii) *for any $\mathscr{E}$-epic e: $S \to L$* (see 3.18) *there is a morphism $\bar{e}$: $QL \to S$ with $e\bar{e} = q_L$.*

(iii) *If $L \in \mathscr{D}$, then q_L: $QL \to L$ is an isomorphism; in particular, every $\mathscr{E}$-epic $S \to L$ is a retraction.*

Proof. (i) follows, via 3.17, from 4.18.(ii): By 3.20 there is an e': $PL \to S$ with $ee' = r_L$; set $\bar{e} = e'|QL$. (iii): By 3.6 and 3.11, s_L is a $\mathscr{CL}$-injection, hence $s_L(L)$ is a compact sub-semilattice of PL, hence equals QL by 4.15. ∎

The reader will note that we do not assert that q_L itself is an $\mathscr{E}$-epic; indeed this would be the case if QL were a sublattice of PL for which there does not seem to be any reason. Furthermore, there is no evidence to believe that QL is an $\mathscr{E}$-projective, contrary to the case of PL (3.21). Finally, again in contrast with PL, the assignment $L \to QL$ albeit somewhat canonically constructed, is in no obvious fashion functorial.

COROLLARY 4.22. *If $A(L) = B(L)$, then r'_L: $\mathrm{Id}(B(L)) \to L$ is co-essential.*

Proof. By an earlier remark, μ_L: $QL \to \mathrm{Id}(B(L))$ is surjective in this case. If C is a closed subsemilattice of $\mathrm{Id}(B(L))$ with $r'_L(C) = L$, then $\mu_L^{-1}(C)$ is a closed subsemilattice of QL which maps onto L by q_L, hence is QL by 4.21.i. Thus $C = \mathrm{Id}(B(L))$ since μ_L is surjective. ∎

The whole situation is unsatisfactory insofar as the character of QL remains mysterious. There is one special situation in which we can completely clarify the situation; for this purpose we have to postulate properties for $A(L)$ which are not generally satisfied (4.2) and which go considerably beyond what was discussed in 4.3.

THEOREM 4.23. *Let L be a compact Lawson semilattice such that $A(L)$ is closed under finite sups and arbitrary infs. Then μ_L: $QL \to \mathrm{Id}(A(L))$ is an isomorphism (i.e. QL has the dual $A(L)^{\mathrm{op}}$) and q_L: $QL \to L$ is an $\mathscr{E}$-epic.*

Proof. First we recall that $A(L) = B(L)$ since $A(L)$ is a sup-semilattice. We claim that ϱ_L: $\mathrm{Id}(A(L)) \to PL$ preserves arbitrary infs: Suppose that $\{I_x: x \in X\}$ is an arbitrary collection of elements in $\mathrm{Id}(A(L))$. Then $\bigcap I_x \subseteq \bigcap \downarrow_L I_x$, whence $\downarrow_L(\bigcap I_x) \subseteq \bigcap \downarrow_L I_x$. Now let $s \in \bigcap \downarrow_L I_x$. Then for each $x \in X$ there is an $a_x \in I_x$ with $s \leqslant a_x$. Let $a = \inf_L\{a_x: x \in X\}$. By hypothesis, $a \in A(L)$. Further, $s \leqslant a \leqslant a_x \in I_x$ for all x, hence $a \in I_x$ for all x. Thus $s \in \downarrow_L(\bigcap I_x)$. Therefore $\downarrow_L(\bigcap I_x) = \bigcap \downarrow_L I_x$, i.e. $\varrho_L(\inf_{\mathrm{Id}(A(L))} I_x) = \inf_{PL} \varrho_L(I_x)$. Recall that $\varrho_L(A(L)) = L_0$; the corestriction ϱ_L: $\mathrm{Id}(A(L)) \to L_0$. Now PL is a monoid morphism preserving arbitrary infs and arbitrary sups by 4.8 and 1.7; hence, by 1.26, is a $\mathscr{CL}$-morphism. Hence $(\mathrm{im}\, s_L)^- = (\mathrm{im}\, \varrho_L)^- = \mathrm{im}\, \varrho_L$ (see 4.11) is a compact semilattice, hence equals QL by definition 4.15. Then, by 4.13 and 4.19, ϱ_L is an isomorphism. Since ϱ_L preserves sups, QL is a sub-lattice of PL; then $q_L = r_L|QL$ preserves sups by 3.2 and is, therefore, an $\mathscr{E}$-epic (3.18). ■

This final application of Galois connections in the context of topological semilattices concludes the theoretical part of the paper, but we add a few concrete examples and some applications in order to illustrate what we have done.

5. Examples, applications

EXAMPLE 5.1. (i) Let $L_1 = (I \times [0, 1/2]) \cup (\{1\} \times I) \subseteq I \times I$, $L_2 = I$ and let $f\colon L_1 \to L_2$ be the projection onto the first factor. Take $J = L_1 \in PL_1$; then $J_0 = J \setminus \{(1, 1)\}$ and $f(J_0) = I$, whereas $f(J)_0 = I \setminus \{1\}$. Thus we do not in general have $(Pf)(J_0) = \big((Pf)(J)\big)_0$, whence $L \to QL$ is not functorial without grave restrictions on the morphisms.

(ii) Let $L = [0, 1/2]^2 \cup [1/2, 1]^2 \subseteq I^2$ and $x = (1/2, 1)$, $y = (1, 1/2)$. Then $s_L(x) s_L(y) = [0, 1/2]^2$, whereas $s_L(xy) = [0, 1/2[^2$.

(iii) A $\mathscr{Z}$-object T is called *well-ordered* if $T = K(T)$. Finite products and closed subsemilattices of well-ordered semilattices are well-ordered. If T is well-ordered, then $T \in \mathscr{D}$, $A(T) = T$ and r_T is an isomorphism. The theory degenerates in this case.

(iv) Let $L = (I \times \{0, 1\}) \setminus \{(0, 0)\}$ with the lexicographic order. Then $L = PI$ by 3.15. We may identify PL with $(I \times \{0, 1/2, 1\}) \setminus \{(0, 0), (0, 1/2)\}$, $K(PL)$ with $PL \setminus (]0, 1] \times \{0\})$, and $QL = s_L(L)$ with $PL \setminus \setminus (]0, 1] \times \{1/2\})$. The function r_L identifies $(r, 0)$ and $(r, 1/2)$.

(v) Let X be a set. Consider $L = 2^X$. Then $(r_x)_{x \in X} \in A(2^X)$ iff $r_x = 0$ for all but a finite number of the x. Hence $A(2^X) = K(2^X) = ({}^X2)^{\mathrm{op}}$. The hypotheses of 4.23 are satisfied. Thus $QL = ({}^X2)^{\wedge} = 2^X$; in fact, $q_L\colon QL \to L$ is an isomorphism. This shows e.g. that every $\mathscr{E}$-epic $S \to L$ is a retract by 4.21.iii.

(vi) Let T be a compact chain. Then $A(T) = T \setminus K_{\mathrm{co}}(T)$, where $K_{\mathrm{co}}(T)$ is the set of all points which are isolated from above (see [8], II-3,12, p. 42). The hypotheses of Theorem 4.23 are satisfied. Hence $QT \cong \mathrm{Id}\big(A(T)\big)$, and QT is a compact chain with dual $A(T)^{\mathrm{op}}$. This chain is non-metric, non-separable if $A(T)$ is uncountable ([8], III-3.1, p. 93 and [9], 1.6). In particular, if C is the standard Cantor chain, then QC is a non-metric, non-separable chain. While C can be injected into 2^N, QC cannot be injected into $Q(2^N) \cong 2^N$ (see (v) above). Note that any injection $C \to 2^N$ induces an injection $PC \to P(2^N)$. Once again this shows that $L \to QL$ is not functorial in a practical fashion. ■

We would like to expand Example 5.1.v a bit more systematically:

PROPOSITION 5.2. *Let $\{L_j\colon j\in J\}$ be a family of compact semilattices. Then $A(\Pi L_j) = \{(x_j)_{j\in J}$: there is a finite set $F\subseteq J$ such that $x_j = 0$ for $j\in J\setminus F$ and $x\in A(L_j)$ for $j\in F\}$. If $A(L_j)$ is closed under arbitrary infs and finite sups for all j, then this is true for $A(\Pi L_j)$, and $Q(\Pi L_j) = \Pi Q L_j$.*

Proof. The calculation of $A(\Pi L_j)$ is straightforward. Since every non-void subset X of $A(\Pi L_j)$ contains an element $x = (x_j)_{j\in J}$ with $x_j = 0$ for j outside some finite set $F\subseteq J$, then $a = (a_j)_{j\in J} = \inf X$ is such that $a_j = 0$ for $j\in J\setminus F$ and $a_j\in A(L_j)$ for $j\in F$. Hence $a\in A(\Pi L_j)$. Thus 4.23 applies and shows $Q(L_j)^{\wedge} = A(L_j)^{\mathrm{op}}$ and $Q(\Pi L_j)^{\wedge} = A(\Pi L_j)^{\mathrm{op}} = \Pi A(L_j)^{\mathrm{op}} = \Pi Q(L_j)^{\wedge}$. The final assertion then follows by duality. ∎

Remark. In view of 3.14 one could say that with respect to product preservation, Q behaves better than P.

As sample applications of the theory of the functor P we derive some known results on Lawson semilattices:

APPLICATION 5.3. *A quotient of a Lawson semilattice is a Lawson semilattice.*

Proof. Let $f\colon L_1\to L_2$ be a surjective morphism of compact semilattices such that L_1 is Lawson. Then we have a commutative diagram

$$\begin{tikzcd} PL_1 \arrow[r, "Pf"] \arrow[d, "r_{L_1}"'] & PL_2 \arrow[d, "r_{L_2}"] \\ L_1 \arrow[r, "f"'] & L_2 \end{tikzcd}$$

in which all maps are surjective (2.31) and Pf, r_{L_1}, f are morphisms in $\mathscr{C}$ (i.e. are continuous) (3.2). Since Pf is surjective, graph $r_{L_2} = (\mathrm{graph}\, f r_{L_1})\circ\circ(\mathrm{graph}\, Pf)^{-1}$; hence Lemma 2.25 applies and shows that r_{L_2} is a $\mathscr{C}$-morphism, hence L_2 is a Lawson semilattice by 3.1. ∎

APPLICATION 5.4. *Let L be a compact semilattice and T be a chain in L satisfying the conditions:*

(i) *If $X\subseteq T$ then $\sup_L X\in T$.*

(ii) $K(T)\subseteq K(L)$.

(iii) *$t_1 < t_2$ in T implies $t_1 \ll t_2$ (3.20).*

Let $d\colon T\to L$ be the inclusion map and let $g\colon L\to T$ be its left adjoint. Give T the order topology, relative to which T is compact by (i). *Then g is a morphism of compact semilattices such that $g|T = 1_T$. There is, in fact, a commutative diagram*

$$\begin{tikzcd}
PT \arrow[rr, "1"] \arrow[dd, "r_T"'] \arrow[dr, "Pd"] & & PT \arrow[dd, "r_T"] \\
& PL \arrow[ur, "Pg"'] \arrow[dd, "r_l"] & \\
T \arrow[rr, "1"] \arrow[dr, "d"'] & & T \\
& L \arrow[ur, "g"'] &
\end{tikzcd}$$

Proof. Since d preserves sups by (i), there is a left adjoint g (1.9). Let $t \in T$. If $t \in K(T)$, then $t \in K(L)$ by (ii); hence $d(\operatorname{int} \uparrow t) = [t, 1]_T \subseteq \uparrow_L t = \operatorname{int} \uparrow d(t)$. If $t \in T \setminus K(T)$, then $t \notin \operatorname{int} \uparrow_T t$ and $t' \in \operatorname{int} \uparrow_T t$ implies $t < t'$; hence $t \ll t'$ by (iii). So $t' \in \operatorname{int} \uparrow_L t$ by definition. Thus, for all $t \in T$ we have $d(\operatorname{int} \uparrow t) = \operatorname{int} \uparrow d(t)$. Hence g is continuous by 1.19. Since it preserves infs (1.7), g is a morphism of compact semilattices. The commutivity of the diagram follows from 2.31. ■

Conditions (i), (ii), and (iii) are also necessary for the existence of a continuous left adjoint $L \to T$.

We remark how this process yields enough totally ordered images of a Lawson semilattice to separate points: First it suffices to show that for each Lawson semilattice there is one totally ordered nondegenerate semilattice quotient (for if $x < y$ in L are to be separated, it suffices to separate identity and zero in Lx/Ly which is in $\mathscr{CL}$ by 5.3). If L is disconnected there is at least one $k \in K(L)$, $k \neq 0$, and 5.4 applies with $T = \{0, k\}$. Assume that L is connected. Note that

(i) *if $a \ll b$ in L with $a, b \in A(L)$* (4.1), *then there is a $c \in A(L)$ with $a \ll c \ll b$.*

(Use 1.17 and the definition of $\ll$ and $A(L)$). Define by induction, using (i), an order injection $d\colon Q_0 \to A(L)$ where Q_0 are the dyadic rationals in $I = [0, 1]$ with $q < R$ in Q_0 implying $d(q) \ll d(r)$. Extend d to I by $d(t) = \sup d(\downarrow t \cap Q_0)$. Set $T = d(I)$ and note $K(T) = 0$ (since $K(I) = 0$); moreover, 5.4 (i) and (iii) are clearly satisfied. We apply 5.4 and obtain Lawson's theorem:

COROLLARY 5.5. *The morphisms $L \to I$ on a Lawson semilattice $L \in \mathscr{CL}$ separate the points.* ■

We conclude by discussing, without the details of the proofs, for which we refer to [20], some applications to the question of epimorphisms in $\mathscr{CL}$. In 5.4 above we gave what amounted to a characterization of morphisms into a chain. This specializes readily to a characterization of morphisms $I^n \to I$ of the cube:

APPLICATION 5.6. *Let $d\colon I \to I^n$ be a function. Then the following statements are equivalent:*

(I) (1) *d preserves arbitrary sups.*

(2) $d(0) = 0$.

(3) *If $t < t'$ in I and $\mathrm{pr}_k d(t) = pr_k d(t')$, then $\mathrm{pr}_k d(t) = 0$ for $k \in \{1, \ldots, n\}$.*

(II) *d has a right adjoint $g\colon I^n \to I$ given by $g(x_1, \ldots, x_n) = \max\{t \in I\colon \mathrm{pr}_k d(t) \leqslant x_k,\ k = 1, \ldots, n\}$, and g is a $\mathscr{CL}$-morphism.* ■

EXAMPLE 5.7. Let $d_j\colon I \to I^n$ be given by $d_1(t) = (\frac{1}{2}t, \ldots, \frac{1}{2}t)$ and $d_2(t) = (t, \frac{1}{2}t, \ldots, \frac{1}{2}t)$, then the d_j satisfy condition (I) of 5.6. Their adjoints g_j have the following properties:

(a) *If $m = (\frac{1}{2}, \ldots, \frac{1}{2})$ is the mid-point of the cube, then $g_1(m) \neq g_2(m)$.*

(b) *If $G \subseteq I^n$ is the subsemilattice of all $(x_1, \ldots, x_n)$ with $x_1 = 1$ or $x_1 = 0$ or $x_2 = 0$ or $\ldots$ or $x_n = 0$, then $g_1|G = g_2|G$.* ■

Since, in an obvious way, each endomorphism of I^n is described in terms of morphisms $I^n \to I$, the characterization 5.6 gives a hold also on endomorphisms of I^n. For example, in [20] it is shown that there are retractions of I^n of arbitrarily small displacement which push the boundary inside (except for the identity):

APPLICATION 5.8. *For any $0 < \varepsilon \leqslant 1$ there exists a retraction of compact semilattices $F_\varepsilon\colon I^n \to I^n$ such that*

(a) $\|F_\varepsilon(x) - x\| \leqslant \varepsilon$ *for all $x \in I^n$ (with the maximum norm on I^n).*

(b) $F_\varepsilon(I^n) \setminus \{1\} \subseteq \operatorname{int} I^n$. ■

With 5.8 as a tool one can show the following existence theorem for endomorphisms:

APPLICATION 5.9. *Let $S \subseteq T$ be closed subsemilattices of the cube I^n and suppose that $t \in T \setminus S$. Then there exists an endomorphism $\varphi\colon I^n \to I^n$ such that*

(a) $\varphi(t) = m$ (the midpoint of I^n).

(b) $\varphi(S) \subseteq G$ (with G as in 5.7.(b)). ■

Using what was shown in 5.5 and taking 5.7 and 5.9 together, one recognizes rather directly, that the morphisms $L \to I$ separate even more strongly than indicated in 5.5:

THEOREM 5.10. *Let $S \subseteq T$ be compact Lawson semilattices. Then for each $t \in T \setminus S$ there is a pair of morphisms $f_j^t\colon T \to I$, $j = 1, 2$, with $f_1^t(t) \neq f_2^t(t)$ and $f_1^t|S = f_2^t|S$.* ■

If $S \neq T$ one can form the morphisms $f_j\colon T \to I^{T \setminus S}$ given by $f_j(x) = (f_j^t(x))_{t \in T \setminus S}$ and obtain

COROLLARY 5.11. *In the category $\mathscr{CL}$ every closed subsemilattice is an equalizer (i.e. the precise set where two morphisms agree).* ■

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