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NUMERICAL SOLUTION OF A THREE-DIMENSIONAL INTEGRAL EQUATION

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I

Some of the problems of mathematical physics for different kind of physical fields have been solved in infinite domains with anomalous bodies. These problems can frequently be reduced to the solution of an integral equation.

The following problem is to be solved:

$$\Delta u+k^2 u = f, \quad k(M) = \begin{cases} k_0, & M \notin V, \\ k_T, & M \in V, \end{cases} \tag{1}$$

V is a finite domain. Here k_0 may be a constant, but if $k \equiv k_0$, the problem can be solved more easily than the original one.

Let us rewrite the equation (1) in the form

$$\Delta u+k_0^2 u = f-(k^2-k_0^2)u. \tag{2}$$

The Green's function $G(M, M_0)$ of this equation is the solution of the equation

$$\Delta G+k_0^2(M)G = \delta(R_{MM_0}), \tag{3}$$

where R_{MM_0} is the distance between points M and M_0. Then, the solution of the equation (2) can be written formally as

$$u(M) = u_0(M)+\int_V G(M, M_0)(k_0^2-k^2)u(M_0)\,dV_{M_0}, \tag{4}$$

where

$$u_0(M) = \int G(M, M_0)f(M_0)\,dV_{M_0}$$

is a known function. Now, if in (4) the points M belong to V, we get an integral equation in a finite domain.

II

If we consider the problem of propagation of electro-magnetic waves, then equation (1) is the system of differential equations in three-dimensional domain with discontinuous coefficients. This problem is used for mathematical modelling of electro-magnetic sounding. In this case the kernel of the integral equation $\hat{E}(M, M_0)$ is the set of vectors of the electric fields at the point M, induced by electric dipoles placed at the point M_0 and parallel to the coordinate axes; $\underline{u}$ and $\underline{u}_0$ are the vectors of electric field [1]. The matrix $\hat{E}(M, M_0)$ has non-integrable singularity as $M \to M_0$. At the same time, $\underline{u}$ and $\underline{u}_0$ change much more slowly. For resulting algorithms analytical separation of the singularity is very important. Besides, sparse grid can be used for $\underline{u}$, whereas for $\hat{E}$ we have to use the dense grid.

To obtain numerical solutions, we partition the domain V into elementary domains V_j, $j = 1, \ldots, K$.

These elementary domains are so small that $\underline{u}$ can be assumed to be constant within each of them: $\underline{u} = \underline{u}(M_j)$, $M_j \in V_j$. Then the equation (4) can be reduced to a system of algebraic equations

$$\underline{u}(M_j) + \sum_{m=1}^{K} \hat{\alpha}_{jm} \underline{u}(M_m) = \underline{u}_0(M_j), \qquad j = 1, \ldots, K, \tag{5}$$

$$\hat{\alpha}_{jm} = (k_0^2 - k_T^2) \int_{V_m} \hat{E}(M_j, M_0) \, dV_{M_0}. \tag{6}$$

III

Let the equation (1) be solved in a stratified domain

$$k_0 = \begin{cases} 0 & \text{if } z < 0, \\ k_1 & \text{if } 0 < z < h, \\ k_2 & \text{if } h < z. \end{cases}$$

In this case, an effective method for solving the equation (3) is Hankel's transformation with the Bessel's functions J_k of the first kind [2, 3]. The components of the matrix $\hat{E}$ can be expressed by the integrals

$$G = \int_0^\infty J_n(tr) v(t, z, z_0) t \, dt, \qquad n = 0, 1, \tag{7}$$

and their derivatives with respect to x and y, where $r = \sqrt{(x-x_0)^2 + (y-y_0)^2}$, $M = (x, y, z)$, $M_0 = (x_0, y_0, z_0)$. v can be discontinuous if $z = z_0$. It is seen that if $r \to 0$ and $z \to z_0$, G tends to infinity.

So, the integrals $\hat{\alpha}_{jm}$ (6) are 4-fold singular integrals. For separation of singularity we have taken into consideration physical arguments.

If the space is homogeneous (for example, $k_0 \equiv k_2$ if the body V is in the second layer $z > h$), the components of the Green's function $\hat{E}^0$ can be expressed in the explicit form

$$E^0_{ln} = A\delta_{ln} + \frac{1}{k_2^2}\frac{\partial^2 A}{\partial x_l \partial x_n}, \qquad l, n = 1, 2, 3,$$

(8) $$A = \frac{1}{4\pi R} e^{ik_2 R}, \qquad R = \sqrt{r^2 + (z - z_0)^2},$$

$$x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad \mathrm{Re}(ik_2) < 0.$$

It is clear that in the neighbourhood of M, $E^0_{ln} = O(R^{-3})$, and so it is a nonintegrable singularity.

To integrate $\hat{E}^0$ for $j = m$, we use potential theory [4]; the corresponding integral becomes

$$\hat{\alpha}^0_{jj} = (k_2^2 - k_T^2)(q_1 \hat{I} + \hat{Q}_1),$$

(9) $$q_1 = \lim_{\varrho \to 0} \int_{V_\varrho} E^0_{ll}(M_j, M_0)\, dV_{M_0} = -\frac{1}{3k_2^2},$$

$$\hat{Q}_1 = \lim_{\varrho \to 0} \int_{V_j \cap V_\varrho} \hat{E}^0_{jj}(M_j, M_0)\, dV_{M_0},$$

where $\hat{I}$ is the unit matrix and V_ϱ is the ball having center at M_j and radius ϱ.

For some symmetric integration domains, for example for a ball or a cube, the integral $\hat{Q}_1$ can be rewritten in the form

(10) $$\hat{Q}_1 = q_2 \cdot \hat{I}, \qquad q_2 = \frac{2}{3} \int_{V_j} A\, dV_{M_0}$$

in which an integrable singularity only occurs. Further, for the ball of radius R_0 we have the explicit formula

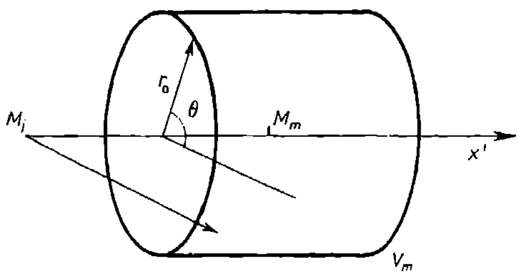
(11) $$q_2 = \frac{2}{3k_2^2}\big((1 - ik_2 R_0)\exp(ik_2 R_0) - 1\big).$$

IV

Since the number of coefficients of the system (5) increases as $9K^3$, it is very important to develop a quick but sufficiently precise algorithm for 4-fold integrals $\hat{\alpha}_{jm}$ (6). The separation of $\hat{E}^0$ is very useful because the corresponding integrals $\hat{\alpha}^0_{jm}$ are much greater than the remainder parts $\hat{\alpha}^1_{jm}$. Since $\hat{E}^0$ has explicit form (8), the $\hat{\alpha}^0_{jm}$ will be only 3-fold integrals. On the other hand, the integrals $\hat{\alpha}^1_{jm}$ do not contain any more singularity and so they can be calculated with lower relative precision.

Another method for speeding up the calculations is the variation of the integration domain. Numerical experiments show that if a compact domain V_j is replaced by another compact one with equal volume (for example, the cube by the ball), this gives rise to a minor difference in the results obtained.

Naturally, division of the body V into elementary cubes can be easier than into balls. If we use a ball for the integral q_2, we obtain exactly formula (11), and at the same time the 3-fold integral (10) for a cube can be integrated analitically with respect to one variable only. If the volumes of the cube and the ball are equal, the relative difference between the two q_2's for different values of parameters does not exceed 2%.



$\{r, \theta, x'\}$

$r_0 = b\frac{2}{\sqrt{\pi}}$

Fig. 1

Now, let $j \neq m$. If the domain of integration for $\hat{\alpha}^0_{jm}$ (which is a 3-fold integral) is a cube, one can integrate analytically with respect to one variable. If the domain is a cylinder with axis along $\overrightarrow{M_j M_m}$ (see Fig. 1) then the integral can be integrated in two variables. The relative difference will not be more than 3%, but for most of the values of the parameters the difference is smaller than 1%.

V

After separation of $\hat{E}^0$ from Green's function the components of the remainder part, i.e. $\hat{E}^1$, take the form of integrals,

$$G^1 = \int_0^\infty J_n(tr)\, v^1(t)\, e^{ik_2(z+z_0)}\, t\, dt$$

and their derivatives in x and y. With the aid of this formula, it can be shown that in the (4-fold) integrals $\hat{\alpha}^1_{jm}$ integration with respect to z_0 can be performed analytically. Therefore it is useful to choose V_j to be a cylinder with different cross-sections. For $j = m$ a circular cylinder can be chosen. Then the integrals $\hat{\alpha}^1_{jj}$ can be integrated analytically in all the three space variables.

The difference between the integrals $\hat{\alpha}^1_{jj}$ over a cylinder and a cube is not more than 5% for those parameters which can be of interest in practice.

If $j \neq m$, we use the following body (in cylindrical coordinates):

$$a-b \leqslant r \leqslant a+b, \quad -b/a \leqslant \theta \leqslant b/a, \quad -b \leqslant z_0' \leqslant b$$

(see Fig. 2), where $2b$ is edge of the cube and a is the distance between

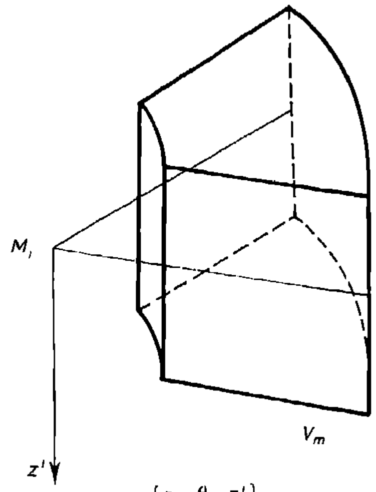


Fig. 2

centers of V_j and V_m [5]. The volume of this body is $8b^3$. Here $\hat{\alpha}^1_{jm}$ can be integrated analytically with respect to z_0 and θ, and often in r, as well.

The error which may arise from replacement of the cube by this body would be very small. So the choice is satisfactory.

Finally, to quicken the algorithms for calculation of the integrals $\hat{\alpha}^1_{jm}$ we must have economic methods for evaluation of infinite integrals with Bessel's functions. Now we would like to refer also to paper [6], in which one can find a method that can be effective for the problems discussed above.

References

[1] V. I. Dmitriev and R. H. Farzan, *Method for calculating anomalous electromagnetic field from local inhomogeneity*, in: Mathematical models of the electromagnetic sounding and mapping in geophysics, Coll. of papers, Eötvös Loránd University, Budapest 1980.

[2] В. И. Дмитриев, *Электромагнитные поля в неоднородных средах*, Изд. МГУ, Москва 1969.

[3] Gy. Molnárka and T. G. Pavlova, *Method for calculating the electromagnetic field of a source, buried into stratified medium*, Cited above Coll. of papers.

[4] А. Н. Тихонов и А. А. Самарский, *Уравнения математической физики*, Наука, Москва 1972.

[5] R. H. Farzan, *Propagation of electromagnetic waves in stratified media with local inhomogeneity*, in: Mathematical models in physics and chemistry and numerical methods of their realization, Teubner-Texte zur Math. 61 (1984).

[6] V. Bezvoda, R. Farzan, K. Segeth and G. Takó, *On numerical evaluation of integrals involving Bessel functions*, Apl. Mat. (Praha) 5 (1986).

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