DISSERTATIONES
MATHEMATICAE

(ROZPRAWY MATEMATYCZNE)

CCXXII

GRZEGORZ ANDRZEJCZAK
Some characteristic invariants
of foliated bundles

5.7133

CONTENTS

Introduction

The notion of a foliation was first introduced by Ch. Ehresmann and G. Reeb in 1944, in [9]. Since then numerous mathematicians have dealt with topics concerning foliations, concentrating in four: the intrinsic structure (holonomy groups, stability, etc.; cf. [32], [35]), the existence of foliations (in general or with some additional properties; cf. [39], [40]), the classification (cf. [14], [15]) and the theory of characteristic classes (and, more generally, invariants) of foliations. For more information and a bibliography, see [28].

The characteristic classes have been introduced as an important tool for classification of foliations. The first example of such a class is due to C. Godbillon and J. Vey ([12]). There have been many independently discovered approaches to characteristic invariants, among them those of Bernstein and Rosenfeld [2], [3], Bott and Haefliger [4], [5], [16], Godbillon and Vey [11], [41] and Kamber and Tondeur [20]–[24]. The last mentioned seems to be the most clearly geometrical. In papers [20]–[24] F. Kamber and Ph. Tondeur construct characteristic classes and invariants of foliated bundles. The concept of foliated bundles has been introduced in [18] and [19].

The present paper also concentrates on characteristic invariants of foliated bundles. We construct a new family of characteristic invariants which are elements of some abstract homology groups determined by the projected foliation of a foliated bundle. In Chapter 0 we prove those abstract homology groups are in fact cohomology groups of the base manifold with coefficients in some sheaves. We also give some necessary preliminaries.

In Chapter 1 we examine the category of foliated bundles with a reduction of structure group (shortly, RF-bundles). This is the category on which the characteristic invariants are defined. By definition, an RF-bundle is a family $(P(M, G), E, H, \xi)$, where $P(M, G)$ is a smooth principal bundle, E is a flat partial connection in the bundle, H is a closed subgroup of G and ξ is a homotopy class of sections $M \to P/H$; ξ can be regarded as a class of H-reductions of the structure group.

In Chapter 2 (cf. Theorem 2.3) we construct a characteristic diagram in the category of RF-bundles. It is a contravariant functor (cf. Theorem 2.9)

with assigns to any RF-bundle $\big(P(M, G), E, H, \xi\big)$ a commuting diagram of linear mappings, of the form

$$
\begin{array}{ccccccc}
\ker i_{G,H}^{*} & \hookrightarrow & I(G) & \to & I(H) & \to & \operatorname{coker} i_{G,H}^{*} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A & \to & B & \to & H^{\mathrm{ev}}(M) & \to & C,
\end{array}
$$

where $I(G)$ and $I(H)$ are the algebras of invariant polynomials on $\mathfrak{g}$ and $\mathfrak{h}$ respectively, $i_{G,H}^{*}: I(G) \to I(H)$ is the restriction map, A, B and C depend on the foliation of M induced by E only, the rows are exact and the columns $I(G) \to B$ and $I(H) \to H^{\mathrm{ev}}(M)$ are algebra homomorphisms ('primary characteristic homomorphisms' of the RF-bundle). The first of these homomorphisms is characteristic for the foliated bundle $\big(P(M, G), E\big)$ (cf. Definition 2.1), whereas the second is the Chern–Weil homomorphism for each of the H-reductions of $P(M, G)$ determined by elements of ξ. 'Primary' ('secondary') characteristic invariants are the elements of B and $H^{\mathrm{ev}}(M)$ (respectively, of A and C) contained in the images of the columns. The secondary characteristic homomorphisms $\ker i_{G,H}^{*} \to A$ and $\operatorname{coker} i_{G,H}^{*} \to C$ appear as a result of comparing the two independent structures attached to the principal bundle $P(M, G)$ – those structures are reflected by the primary characteristic homomorphisms. Consequently, the secondary characteristic invariants vanish if there is a section $\sigma \in \xi$ such that E is reducible through σ.

In the rest of the chapter, we follow [25] and [37] and give the local description of our characteristic homomorphism $I(G) \to B$. We also deform the flat partial connection and study the behaviour of the characteristic invariants.

In Chapter 3 we apply the above constructions to foliations and to framed foliations. We take advantage of the fact that the principal normal bundle of a foliation carries a canonical structure of a foliated bundle. Moreover, this bundle admits a canonical homotopy class of reductions of structure group to the orthogonal group (to the unit subgroup of the linear group if the foliation is framed). We define 'Pontrjagin classes' and 'secondary Pontrjagin classes' of foliations and study their properties (P. Molino [30] has constructed some Pontrjagin classes for foliations admitting a transversally projectable connection; they are different from the ones considered in the present paper). Secondary Pontrjagin classes prove to be obstructions to the existence of a bundle-like metric. We give a geometrical description (cf. Theorems 3.11 and 3.12) of foliations whose first Pontrjagin class, or first secondary Pontrjagin class, vanishes. We also notice that the classical characteristic classes (the exotic classes) of any foliation can be lifted, by using the Pontrjagin classes and the secondary Pontrjagin classes, to the homology groups of some ideals of differential forms vanishing on the foliation. Similarly the holonomy ring on a leaf (cf. [13], [38]) can be

determined if the Pontrjagin classes and the secondary Pontrjagin classes of the foliation are given.

Starting from the classical definition of concordance between two foliations of the same codimension (cf. [28]), we introduce the notion of concordance over a common subfoliation (cf. Definition 3.19) and find some invariants of this relation (cf. Theorem 3.21).

In Chapter 4 we study pairs composed of a foliation and its subfoliation. We introduce the concept of a Bott partial connection related to such a pair (cf. Definition 4.2) and use it to obtain 'relative Pontrjagin classes' and 'relative secondary Pontrjagin classes'. We prove (cf. Theorem 4.5) some formulae similar to the well-known multiplicative formula $p(\xi) = p(\eta) \cdot p(\xi/\eta)$ in $H^*(M)$, where $\eta \subset \xi$ are vector bundles over a manifold M and p is the total Pontrjagin class (here, on the category of vector bundles).

Finally, given a pair of foliations (F', F), $F \subset F'$, we find some conditions necessary for the existence of a foliation F'' such that $F = F' \cap F''$, and some weaker conditions necessary for the existence of two foliations $\bar{F}$ and F'' such that $F = \bar{F} \cap F''$ and $\bar{F}$ is concordant over F with the foliation F' (cf. Theorem 4.13). The nature of those conditions is the same as that of the Bott vanishing theorem (cf. [4]).

0. Preliminaries

0.1. Foliations. All the differentiable manifolds, mappings, vector bundles, the principal bundles and foliations considered in the present paper are assumed to be smooth (i.e. C^∞). We follow [26] in denoting classical objects related to a manifold.

DEFINITION 0.1.1. A *foliation F of codimension q of a manifold M* is a partition $\{L_\alpha\}_{\alpha \in A}$ of M into connected sets L_α (leaves) with the following property:

For every point in M there is a neighbourhood U and a chart $x = (x^1, \ldots, x^n)$: $U \to R^n$, $n = \dim M$, such that for each leaf L_α the components of $L_\alpha \cap U$ are described by the equations $x^1 = \text{const}, \ldots, x^q = \text{const}$.

Such a chart will be called an *integrating chart for F*.

Each leaf carries the canonical structure of an imbedded submanifold of M. The set of all vectors tangent to the leaves of F determines a subbundle E_F of the tangent bundle TM of M. It is referred to as the *tangent bundle of F*. The quotient $Q_F = TM/E_F$ is a vector bundle of dimension q, called the *normal bundle of F*. Its dual Q_F^* can be identified with a subbundle of the

cotangent bundle $T^* M$ of M. Namely, elements of Q_F^* will be regarded as covectors annihilating the appropriate fibres of E_F. We shall denote by $I_F \subset A^*(M)$ the ideal of differential forms on M generated by the set $\Gamma(Q_F^*)$ of all global sections of Q_F^*.

DEFINITION 0.1.2. A subbundle E of the tangent bundle TM of a manifold M is *involutive* if for any open $U \subset M$ and any two local sections $X, Y \in \Gamma(U, E)$ of E over U

$$[X, Y] \in \Gamma(U, E).$$

The following question arises naturally:

Given a subbundle E of TM (or, equivalently, Q^* of $T^* M$), does there exist a foliation F of M such that $E = E_F$ (respectively, $Q^* = Q_F^*$)?

The answer is given by the following classical result:

THEOREM OF FROBENIUS 0.1.3. *On a manifold M, the assignments $F \mapsto E_F$, $F \mapsto Q_F^*$ yield a 1-1 correspondence between*

(i) *foliations F of M,*

(ii) *involutive subbundles E of TM, and*

(iii) *subbundles Q^* of $T^* M$ such that the ideal generated in $A^*(M)$ by the set $\Gamma(Q^*)$ of all global sections of Q^* is closed under the exterior derivative.*

DEFINITION 0.1.4. A mapping $f \colon M' \to M$, M' and M being any manifolds, is *transverse to a foliation F of M* if

(0.1.1) $f_{*p}(T_p M') + E_F \cap T_{f(p)} M = T_{f(p)} M$ for $p \in M'$.

PROPOSITION 0.1.5. *If $f \colon M' \to M$ is a mapping transverse to a foliation F of M, then there is a unique foliation $f^* F$ of M' such that the union*

$$\bigcup_{p \in M'} (f_{*p})^{-1} (E_F \cap T_{f(p)} M) \subset TM'$$

coincides with the tangent bundle of $f^* F$, Furthermore, $\operatorname{codim} f^* F = \operatorname{codim} F$ and $f^*(I_F) \subset I_{f^* F}$, if $f^* \colon A^*(M) \to A^*(M')$ is the co-induced homomorphism.

Let F be a foliation of a manifold M. The sheaf of germs of (smooth) sections of Q_F^* generates in the sheaf $GA^*(M)$ of germs of differential forms on M the ideal GI_F of germs of differential forms vanishing on F. Obviously, $I_F = \Gamma(GI_F)$. We put

$$I_F^k = \Gamma(GI_F^k) \quad \text{for} \quad k = 1, 2, \ldots,$$

where GI_F^k means the kth power of the sheaf of algebras GI_F. For completeness, we set $I_F^0 = A^*(M)$. We shall write $GI_F^{k,h}$ and $I_F^{k,h}$ to denote the intersections $GI_F^k \cap GA^h(M)$ and $I_F^k \cap A^h(M)$, respectively, for $k, h = 1, 2, \ldots$, $k \leqslant h$. Observe that

$$I_F^{k,h} = \Gamma(GI_F^{k,h})$$

and

$$A^h(M)/I_F^{k,h} = \Gamma(GA^h(M)/GI_F^{k,h}), \quad \text{for all} \quad k \text{ and } h.$$

By Theorem 0.1.3, the exterior derivative d gives rise to differentials

$$d: GI_F^k \to GI_F^k$$

and

$$\bar{d}: GA^*(M)/GI_F^k \to GA^*(M)/GI_F^k,$$

for $k = 0, 1, 2, \ldots$

LEMMA 0.1.6. *Let F be the foliation of J^n, $J = (-1, 1) \subset \mathbf{R}$, composed of the sets $\{a\} \times J^{n-q}$, $a \in J^q$. For any positive integer k, the following two sequences*

(i) $I_F^{k,k} \xrightarrow{d} I_F^{k,k+1} \to \ldots \to I_F^{k,h-1} \xrightarrow{d} I_F^{k,h} \xrightarrow{d} I_F^{k,h+1} \to \ldots,$ *and*

(ii) $A^{k-1} \xrightarrow{d} A^k/I_F^{k,k} \xrightarrow{\bar{d}} A^{k+1}/I_F^{k,k+1} \xrightarrow{\bar{d}} \ldots$

$$\ldots \to A^{h-1}/I_F^{k,h-1} \xrightarrow{\bar{d}} A^h/I_F^{k,h} \xrightarrow{\bar{d}} A^{h+1}/I_F^{k,h+1} \to \ldots,$$

where $A^* = A^*(J^n)$ and the homomorphisms are induced from the exterior derivative, are exact.

Proof. In the natural coordinates $(x^1, \ldots, x^n)$ on J^n, I_F is generated by the forms $dx^1, \ldots, dx^q$. Any differential form φ on J^n is uniquely expressible as

$$(0.1.2) \qquad\qquad \varphi = \sum_K \varphi_K \wedge dx^K,$$

where the summation index K runs over all subsets of the set $\{1, \ldots, q\}$, the forms φ_K do not involve dx^j with $j \leqslant q$ and

$$dx^K = dx^{k^1} \wedge \ldots \wedge dx^{k^i}$$

if $K = \{k^1, \ldots, k^i\}$ and $k^1 < \ldots < k^i$.

For any form φ on J^n which does not involve dx^j, $j \leqslant q$, we define its truncated derivative $d'\varphi$ by

$$d'\varphi = (d\varphi)_\emptyset = \sum_{i > q} dx^i \wedge L_{\frac{\partial}{\partial x^i}} \varphi,$$

so that $d\varphi - d'\varphi \in I_F$. Here L stands for the Lie derivative.

As in the classical proof of the Poincaré lemma (cf. [8]), we consider a homotopy $g: \mathbf{R} \times J^n \to J^n$, $(s, x) \mapsto (1-s)x$. For any form $\varphi \in A^h(J^n)$, the induced form $g^*\varphi \in A^h(\mathbf{R} \times J^n)$ is uniquely expressible as $g^*\varphi = ds \wedge \alpha + \beta$, where α and β do not involve ds. We put

$$\chi(\varphi) = \int_{[0,1]} \alpha,$$

which means that we integrate the coefficients of α with respect to the variable s. Then, for $h \geqslant 1$, $d\varphi = 0$ implies $\varphi = -d\chi(\varphi)$.

If we consider another homotopy $g: R \times J^n \to J^n$, of the shape $(s, (x', x'')) \mapsto (x', (1-s)x'')$, where we identify J^n with $J^q \times J^{n-q}$, then the corresponding mapping χ' (constructed as χ above) satisfies the following condition:

if $d'\varphi$, $\deg \varphi \geqslant 1$, is defined and $d'\varphi = 0$, then $\varphi = -d'\chi'(\varphi)$.

(i') If $\varphi \in I_F^{k,h}$, $h > k \geqslant 1$, then expression (0.1.2) reduces to

$$\varphi = \sum_{i \geqslant k} \sum_{\text{card } K = i} \varphi_K \wedge dx^K,$$

where card K means the number of elements of K. Since

$$d\varphi - \sum_{\text{card } K = k} d'\varphi_K \wedge dx^K \in I_F^{k+1, h+1},$$

we have $d'\varphi_K = 0$ for all K such that card $K = k$ provided $d\varphi = 0$.

Assume that $\varphi \in I_F^{k,h}$ is a closed form. Consequently

$$\varphi_K = -d'\chi'(\varphi_K) \quad \text{for} \quad \text{card } K = k,$$

and

$$\bar{\varphi} := \varphi + d\big(\sum_{\text{card } K = k} \chi'(\varphi_K) \wedge dx^K \big) \in I_F^{k+1, h}.$$

It is easy to check that $\chi(\bar{\varphi}) \in I_F^{k, h-1}$ if $\bar{\varphi} \in I_F^{k+1, h}$. Thus

$$\varphi = -d\big(\chi(\bar{\varphi}) + \sum_{\text{card } K = k} \chi'(\varphi_K) \wedge dx^K \big)$$

and so sequence (i) is exact.

(ii') Consider an arbitrary form $\varphi \in A^h(J^n)$, $h \geqslant k \geqslant 1$, and assume that $d\varphi \in I_F^{k, h+1}$. We shall construct by induction some forms $\psi_1, \psi_2, \ldots$ $\ldots, \psi_k \in A^{h-1}(J^n)$ such that

(0.1.3) $\varphi + d(\psi_1 + \ldots + \psi_i) \in I_F^{i, h}$ for all $i \leqslant k$.

For $i = 1$, the only term outside I_F in (0.1.2), is $\varphi_\emptyset$. As $d\varphi - d'\varphi_\emptyset \in I_F$, we have $d'\varphi_\emptyset \in I_F$ and therefore $d'\varphi_\emptyset = 0$. We set $\psi_1 = \chi'(\varphi_\emptyset)$.

Suppose that we have constructed forms $\psi_1, \ldots, \psi_i$, $1 \leqslant i < k$, such that (0.1.3) holds. Then

$$\bar{\varphi} := \varphi + d(\psi_1 + \ldots + \psi_i) = \sum_{\text{card } K \geqslant i} \bar{\varphi}_K \wedge dx^K$$

and $d'\bar{\varphi}_K = 0$ for card $K = i$, as $d\bar{\varphi} = d\varphi \in I_F^k$ and

$$d\bar{\varphi} - \sum_{\text{card } K = i} d'\bar{\varphi}_K \wedge dx^K \in I_F^{i+1}.$$

Since $\deg \bar{\varphi}_K = h - \operatorname{card} K \geqslant 1$ if $\operatorname{card} K = i$, we have $\bar{\varphi}_K = -d'\chi'(\bar{\varphi}_K)$, and we may take

$$\psi_{i+1} = \sum_{\operatorname{card} K = i} \chi'(\bar{\varphi}_K) \wedge dx^K.$$

Putting $i = k$ in (0.1.3), we get $\varphi + d(\psi_1 + \ldots + \psi_k) \in I_F^{k,h}$. This proves that sequence (ii) is exact.

The next proposition follows directly from Lemma 0.1.6.

PROPOSITION 0.1.7. *Let F be a foliation of a manifold M and k a positive integer.*

(i) *The homology groups $H^{k+h}(I_F^k)$, $h = 0, 1, \ldots$, of the differential subalgebra (I_F^k, d) of $(A^*(M), d)$ are canonically isomorphic to the cohomology groups of M with coefficients in the sheaf $GI_F^{k,k} \cap d^{-1}(0)$ of closed germs from $GI_F^{k,k}$:*

$$H^{k+h}(I_F^k) \cong H^h\big(M; \; GI_F^{k,k} \cap d^{-1}(0)\big) \quad \textit{for } h = 0, 1, \ldots$$

(ii) *The homology groups $H^{k+h}\big(A^*(M)/I_F^k\big)$, $h = 0, 1, \ldots$, of the differential quotient algebra $\big(A^*(M)/I_F^k, \bar{d}\big)$, $\bar{d}$ being induced from the exterior derivative d, are canonically isomorphic to the cohomology groups of M with values in the sheaf $d^{-1} GI_F^{k,k} \subset GA^{k-1}(M)$:*

$$H^{k+h}\big(A^*(M)/I_F^k\big) \cong H^{h+1}(M; \; d^{-1} GI_F^{k,k}) \quad \textit{for} \quad h = 0, 1, \ldots$$

Proof. By Lemma 0.1.6, the sequences of sheaves

$$0 \to GI_F^{k,k} \cap d^{-1}(0) \hookrightarrow GI_F^{k,k} \xrightarrow{d} GI_F^{k,k+1} \xrightarrow{d} \ldots,$$

and

$$0 \to d^{-1} GI_F^{k,k} \hookrightarrow GA^{k-1} \xrightarrow{\bar{d}} GA^k/GI_F^{k,k} \xrightarrow{\bar{d}} GA^{k+1}/GI_F^{k,k+1} \xrightarrow{\bar{d}} \ldots,$$

where $GA^* = GA^*(M)$, are resolutions of $GI_F^{k,k} \cap d^{-1}(0)$ and of $d^{-1} GI_F^{k,k}$, respectively. This ends the proof.

There is a simple criterion to decide whether a form $\varphi \in A^h(M)$ belongs to the subspace $I_F^{k,h}$ or not.

PROPOSITION 0.1.8. *A differential form $\varphi \in A^h(M)$ belongs to $I_F^{k,h}$ if and only if*

(0.1.4) $\varphi(v_1, \ldots, v_h) = 0$

whenever at least $h - k + 1$ of the vectors $v_1, \ldots, v_h \in T_p M$ are tangent to F, for all $p \in M$.

Proof. Since the proposition is of local character, we may restrict ourselves, by using an integrating chart, to the elementary foliation F of J^n, as in Lemma 0.1.6. In the natural coordinates $(x^1, \ldots, x^n)$, the ideals I_F^k, $k \leqslant q$, are generated by all dx^K with $K \subset \{1, \ldots, q\}$ and $\operatorname{card} K = k$. Thus

$\varphi \in A^h(J^n)$ belongs to I_F^k if and only if in the decomposition (0.1.2) all φ_K with card $K < k$ vanish. If this is the case, then φ obviously satisfies (0.1.4).

To prove the converse, assume that (0.1.4) holds. Let l denote the largest integer $\leqslant k$ (possibly 0) such that $\varphi \in I_F^l$. There is a multiindex $K_0 = \{k_1, \ldots, k_l\}$, $k_1 < \ldots < k_l$, such that $\varphi_{K_0} \neq 0$ ($K_0 = \emptyset$ if $l = 0$). This means that $\varphi_{K_0}(w_1, \ldots, w_{h-l}) \neq 0$ for some vectors $w_1, \ldots, w_{h-l} \in T_p J^n$, $p \in J^n$. Let $v_1, \ldots, v_{h-l}$ denote the projections of $w_1, \ldots, w_{h-l}$, respectively, onto the subspace

$$E_p := \mathrm{Lin}\left\{ \frac{\partial}{\partial x^{q+1}}\bigg|_p, \ldots, \frac{\partial}{\partial x^n}\bigg|_p \right\} \subset T_p J^n,$$

tangent to F, parallel to the decomposition

$$T_p J^n = \mathrm{Lin}\left\{ \frac{\partial}{\partial x^1}\bigg|_p, \ldots, \frac{\partial}{\partial x^q}\bigg|_p \right\} \oplus E_p.$$

Then, there is

$$\varphi\left(v_1, \ldots, v_{h-l}, \frac{\partial}{\partial x^{k_1}}\bigg|_p, \ldots, \frac{\partial}{\partial x^{k_l}}\bigg|_p \right) = \varphi_{K_0}(v_1, \ldots, v_{h-l})$$

$$= \varphi_{K_0}(w_1, \ldots, w_{h-l}) \neq 0,$$

which contradicts (0.1.4) unless $l = k$. This ends the proof.

0.2. Foliated bundles. Throughout the paper, a principal bundle with structure group G, total space P and base manifold M will be denoted by $P(M, G)$. To denote the projection we shall use the letter π.

A pair $(\bar{f}, f)$ of mappings $\bar{f}: P' \to P$ and $f: M' \to M$ is a principal bundle homomorphism from $P'(M', G')$ to $P(M, G)$, associated with a group homomorphism $\hat{f}: G' \to G$, if $\pi \circ \bar{f} = f \circ \pi'$ and $\bar{f}(u'g') = \bar{f}(u') \cdot \hat{f}(g')$ for $u' \in P'$ and $g' \in G'$ (cf. [26], p. 53).

DEFINITION 0.2.1. (cf. [25]). A *partial connection* in a principal bundle $P(M, G)$ is a subbundle $E \subset TP$ such that

(i) $E_u \cap G_u = \{0\}$ for every $u \in P$ where G_u is the tangent space to the fibre through u;

(ii) $E_{ug} = (R_g)_* E_u$ for every $u \in P$ and $g \in G$, R_g being the right action of g on P.

The vectors in E will be called *partially horizontal*.

DEFINITION 0.2.2 (cf. [25]). A connection Γ in a principal bundle $P(M, G)$ is *adapted to a partial connection* E in $P(M, G)$ if Γ contains E as subbundles of TP.

It follows from the definition that a partial connection E in $P(M, G)$ projects onto a subbundle $\pi_* E \subset TM$. E is uniquely determined by its projected subbundle $\pi_* E \subset TM$ and any of the adapted connections. If a

curve in M is tangent to $\pi_* E$, then its horizontal lifts do not depend on the choice of the adapted connection.

DEFINITION 0.2.3 (cf. [25]). A partial connection E in $P(M, G)$ is *flat* if the subbundle $E \subset TP$ is involutive.

DEFINITION 0.2.4 (cf. [25]). A *foliated bundle* is a pair $(P(M, G), E)$, where E is a flat partial connection in the bundle $P(M, G)$.

If a partial connection is flat, then its projected subbundle is involutive. By Theorem 0.1.3, a foliated bundle $(P(M, G), E)$ gives rise to a pair of foliations of P and M respectively.

DEFINITION 0.2.5. For a foliated bundle $(P(M, G), E)$, a *projected foliation* is a foliation F of M such that $E_F = \pi_* E$.

PROPOSITION 0.2.6 (cf. [25]). *Given a foliation F of a manifold M and a family $\mathscr{C}$ of connections in a principal bundle $P(M, G)$, there exists a unique subset E of TP such that $\pi_*(E \cap T_u P) = E_F \cap T_{\pi(u)} M$ for $u \in P$ and the connection forms of connections from $\mathscr{C}$ annihilate E, if and only if, for any two connections $\Gamma, \Gamma' \in \mathscr{C}$, their connection forms ω, ω' satisfy*

$$(0.2.1) \qquad\qquad (\omega - \omega')(v) = 0 \quad if \quad \pi_* v \in E_F.$$

E, if it exists, is a partial connection in $P(M, G)$ and all the connections in $\mathscr{C}$ are adapted to E. E is flat if and only if

$$(0.2.2) \qquad \Omega(v, w) = 0 \quad \text{for } v, w \in TP \text{ such that } \pi_* v, \pi_* w \in E_F,$$

Ω being the curvature form of some (equivalently, any) connection from $\mathscr{C}$.

Any foliation F of codimension q of a manifold M generates a canonical partial connection in the principal bundle $LQ_F^*(M, \mathrm{Gl}(q))$ of all linear frames in the fibres of Q_F^*.

DEFINITION 0.2.7. A *Bott connection* in the bundle $LQ_F^*(M, \mathrm{Gl}(q))$ is the unique partial connection such that F is its projected foliation and, for any integrating chart $x = (x^1, \ldots, x^n)$ on an open subset U of M, the partially horizontal vectors are tangent to the image of the mapping $U \ni p \mapsto (dx_p^1, \ldots, dx_p^q) \in LQ_F^*$.

As has been shown in [25], a connection in $LQ_F^*(M, \mathrm{Gl}(q))$ is adapted to the Bott connection if and only if the corresponding covariant derivative V in Q_F^* satisfies the condition

$$(0.2.3) \qquad V_v \varphi = \iota_v(d\varphi) \quad \text{for} \quad \varphi \in \Gamma(Q_F^*) \text{ and } v \in E_F,$$

ι being the interior product. Since the Bott connection is flat, the bundle $LQ_F^*(M, \mathrm{Gl}(q))$ admits a canonical foliated bundle structure.

The following proposition will frequently be referred to in Chapters 1 and 2.

PROPOSITION 0.2.8 (cf. [26], pp. 79–82). *Let $(\bar{f}, f)$: $P'(M', G') \to P(M, G)$ be a principal bundle homomorphism associated with a group homomorphism $\hat{f}$: $G' \to G$.*

(i) *Let Γ' be a connection in $P'(M', G')$, ω' the connection form and Ω' the curvature form of Γ'. If f is a diffeomorphism of M' onto M, then there is a unique connection Γ in $P(M, G)$ such that the horizontal subspaces of Γ' are mapped into horizontal subspaces of Γ by $\bar{f}$. If ω and Ω are, respectively, the connection form and the curvature form of Γ', then $\bar{f}^* \omega = \hat{f}_* \circ \omega'$ and $\bar{f}^* \Omega = \hat{f}_* \circ \Omega'$.*

(ii) *Let Γ be a connection in $P(M, G)$, ω the connection form and Ω the curvature form of Γ. If $\hat{f}$ maps G' isomorphically onto G, then there is a unique connection Γ' in $P'(M', G')$ such that the horizontal subspaces of Γ' are mapped into horizontal subspaces of Γ by $\bar{f}$. If ω' and Ω' are, respectively, the connection form and the curvature form of Γ', then $\bar{f}^* \omega = \hat{f}_* \circ \omega'$ and $\bar{f}^* \Omega = \hat{f}_* \circ \Omega'$.*

The symbol $\hat{f}_*$ above stands for the induced homomorphism of Lie algebras.

LEMMA 0.2.9. *Let $(\bar{f}, f)$: $P'(M', \mathrm{Gl}(q)) \to P(M, \mathrm{Gl}(q))$ be a homomorphism of principal bundles, q being a positive integer. Suppose that Γ' and Γ are connections in $P'(M', \mathrm{Gl}(q))$ and $P(M, \mathrm{Gl}(q))$ respectively, such that the horizontal subspaces of Γ' are mapped into horizontal subspaces of Γ by $\bar{f}$. Let Q' and Q be the vector bundles associated with the above principal bundles, corresponding to the standard action of $\mathrm{Gl}(q)$ on $\mathbf{R}^q$, and let g: $Q' \to Q$ be the homomorphism induced from $\bar{f}$. If V' and V are the covariant derivatives in Q' and Q, determined by Γ' and Γ respectively, then*

$$g(V'_v \sigma') = V_{f_* v} \sigma,$$

for all sections σ' of Q' and σ of Q, such that $\sigma \circ f = g \circ \sigma'$ on a neighbourhood of the origin of $v \in TM'$.

Proof. Let v be any vector in TM' and assume that $\sigma \circ f = g \circ \sigma'$ on a neighbourhood of the origin of v. v is of the form $\dot{p}_0$ for some curve $t \mapsto p_t$ in M'. Let $t \mapsto u_t$ be its horizontal lift in P'. By the hypotheses, the curve $t \mapsto \bar{f}(u_t)$ is a horizontal lift of the curve $t \mapsto f(p_t)$. We have $g(u(\xi)) = \bar{f}(u)(\xi)$ for all $u \in P'$ and $\xi \in \mathbf{R}^q$, and so

$$g\left[\frac{1}{t} \left(u_0 u_t^{-1} (\sigma'(p_t)) - \sigma'(p_0) \right) \right] = \frac{1}{t} \left(\bar{f}(u_0) \bar{f}(u_t)^{-1} g \circ \sigma'(p_t) - g \circ \sigma'(p_0) \right)$$

$$= \frac{1}{t} \left(\bar{f}(u_0) \bar{f}(u_t)^{-1} \sigma \circ f(p_t) - \sigma \circ f(p_0) \right)$$

for all $t \neq 0$. If t tends to 0, then the above equality gives

$$g(V'_v \sigma') = V_{f_* v} \sigma,$$

which concludes the proof.

Remark 0.2.10. If a mapping $f: M' \to M$ is transverse to a foliation F of M, then f gives rise to a canonical homomorphism $\hat{f}: Q^*_{f^*F} \to Q^*_F$ which covers f and is an isomorphism on the fibres. Namely, the mapping $\alpha: f^*Q^*_F \to Q^*_{f^*F}$ of the shape

$$f^*Q^*_F = M' \times_{(f,\pi)} Q^*_F \ni (p, \omega) \mapsto \omega \circ f_{*p} \in Q^*_{f^*F}$$

is a vector bundle monomorphism (by (0.1.1), if ω annihilates $f_{*p}(T_p M')$ and $E_F \cap T_{f(p)} M$, then $\omega = 0$) and so, as $\operatorname{codim} f^*F = \operatorname{codim} F$, it is an isomorphism. $\hat{f}$ is the superposition of α^{-1} with the projection into Q^*_F. Consequently, $\hat{f}$ determines a canonical principal bundle homomorphism $(\bar{f}, f): LQ^*_{f^*F}(M', \operatorname{Gl}(q)) \to LQ^*_F(M, \operatorname{Gl}(q))$, $q = \operatorname{codim} F$. As we could have expected, this homomorphism transfers the Bott connection from one bundle into another.

PROPOSITION 0.2.11. *Let* $f: M' \to M$, F, $\hat{f}$ *and* $\bar{f}$ *be as in Remark 0.2.10. For any connection* Γ *in* $LQ^*_F(M, \operatorname{Gl}(q))$ *adapted to the Bott connection, the only connection* Γ' *in* $LQ^*_{f^*F}(M', \operatorname{Gl}(q))$ *whose horizontal subspaces are mapped into horizontal subspaces of* Γ *by* $\bar{f}_*$ *(cf. Proposition 0.2.8) is also adapted to the Bott connection.*

Proof. By Lemma 0.2.9, the covariant derivatives ∇' in $Q^*_{f^*F}$ and ∇ in Q^*_F, determined by Γ' and Γ respectively, are related in the sense that $\hat{f}(\nabla'_v f^*\varphi) = \nabla_{f_{*v}}\varphi$ and, equivalently, $\nabla'_v f^*\varphi = \alpha(p, \nabla_{f_{*v}}\varphi) = (\nabla_{f_{*v}}\varphi) \circ f_{*p}$, for $v \in TM$, $p = \pi(v)$ and $\varphi \in \Gamma(Q^*_F)$. Thus, for $v \in E_{f^*F}$,

$$(0.2.4) \qquad \nabla'_v f^*\varphi = (\iota_{f_{*v}} d\varphi) \circ f_{*p} = \iota_v d(f^*\varphi).$$

The identity $f^*\varphi = \alpha \circ (\operatorname{id}_{M'}, \varphi \circ f)$ for all $\varphi \in \Gamma(Q^*_F)$, together with the fact that α is an isomorphism, implies what follows:

(0.2.5)   if $(\varphi^1, \dots, \varphi^q)$ is a section of $LQ^*_F(M, \operatorname{Gl}(q))$ over an open subset U of M, then $(f^*\varphi^1, \dots, f^*\varphi^q)$ is a section of $LQ^*_{f^*F}(M', \operatorname{Gl}(q))$ over $f^{-1}(U)$.

Consequently, any section ψ of $Q^*_{f^*F}$ is locally expressible as a linear combination $\sum \alpha_i f^*\varphi^i$. By (0.2.4), $\nabla'_v\psi = \iota_v d\psi$ for all $v \in E_{f^*F}$, which ends the proof.

0.3. Chern–Weil homomorphism. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $I^k(G)$ denote the real vector space of all symmetric G-invariant k-linear mappings $a: \mathfrak{g} \times \dots \times \mathfrak{g} \to R$, where G acts on $\mathfrak{g}$ by the adjoint representation. Any mapping $a \in I^k(G)$ is uniquely determined by the values $a(t): = a(t, \dots, t)$ for $t \in \mathfrak{g}$. The direct sum

$$I(G) = \sum_{k=0}^{\infty} I^k(G), \quad I^0(G) = R,$$

is the algebra of G-invariant polynomials on $\mathfrak{g}$. The multiplication is given by the formula

$$(ab)(t_1, \ldots, t_{k+l}) = \frac{1}{(k+l)!} \sum_\sigma a(t_{\sigma(1)}, \ldots, t_{\sigma(k)}) b(t_{\sigma(k+1)}, \ldots, t_{\sigma(k+l)})$$

for $a \in I^k(G)$ and $b \in I^l(G)$, where the summation is taken over all permutations σ of the set $\{1, \ldots, k+l\}$.

Any group homomorphism $f: G' \to G$ determines an algebra homomorphism $f^*: I(G) \to I(G')$ such that $f^*(a) = a \circ (f_* \times \ldots \times f_*)$ for $a \in I(G)$, $f_*: \mathfrak{g}' \to \mathfrak{g}$ being the induced homomorphism of Lie algebras. If H is a closed subgroup of a Lie group G, then we shall use the symbol $i_{G,H}$ to denote the inclusion map. For any positive integer q, the following facts are well known:

(i) $I(\mathrm{Gl}(q)) = R[p_1, \ldots, p_q]$, where $p_i \in I^i(\mathrm{Gl}(q))$, $i = 1, \ldots, q$, are defined by the identity

$$\det(A + \lambda I_q) = \sum_{i=0}^q p_i(A) \lambda^{q-i} \quad \text{for} \quad A \in \mathfrak{gl}(q) \text{ and } \lambda \in R.$$

Here I_q is the unit matrix and $p_0 = 1$. One sets $p_i = 0$ for $i > q$.

(ii) $I(\mathrm{O}(q)) = R[\bar{p}_2, \bar{p}_4, \ldots, \bar{p}_{2[q/2]}]$, where $\bar{p}_{2i} = i^*_{\mathrm{Gl}(q),\mathrm{O}(q)}(p_{2i})$ for $i = 1, \ldots, [q/2]$. Moreover, for i odd, $i^*_{\mathrm{Gl}(q),\mathrm{O}(q)}(p_i) = 0$.

(iii) $i^*_{\mathrm{O}(q),\mathrm{SO}(q)}: I(\mathrm{O}(q)) \xrightarrow{\cong} I(\mathrm{SO}(q))$ for q odd.

(iv) $I(\mathrm{SO}(q)) = R[\hat{p}_2, \hat{p}_4, \ldots, \hat{p}_{q-2}, e]$, where $e \in I^{q/2}(\mathrm{SO}(q))$ is the Pfaffian (cf. [29], p. 309), and

$$i^*_{\mathrm{O}(q),\mathrm{SO}(q)}(\bar{p}_{2i}) = \hat{p}_{2i} \quad \text{for} \quad i = 1, \ldots, q/2 - 1,$$

$$i^*_{\mathrm{O}(q),\mathrm{SO}(q)}(\bar{p}_q) = e^2,$$

if q is even.

Consequently, $\ker i^*_{\mathrm{Gl}(q),\mathrm{O}(q)} = \ker i^*_{\mathrm{Gl}(q),\mathrm{SO}(q)}$ is the ideal generated by $p_1, p_3, p_5, \ldots$, $\operatorname{coker} i^*_{\mathrm{Gl}(q),\mathrm{SO}(q)} = 0$ and

$$\operatorname{coker} i^*_{\mathrm{Gl}(q),\mathrm{SO}(q)} = \begin{cases} 0 & \text{for } q \text{ odd,} \\ R[e]/(e^2) & \text{if } q \text{ is even.} \end{cases}$$

Given a polynomial $a \in I^k(G)$ and $\mathfrak{g}$-valued differential forms $\varphi_1, \ldots, \varphi_k$ on a manifold P, G being any Lie group and $\mathfrak{g}$ its Lie algebra, one defines a form $a(\varphi_1, \ldots, \varphi_k)$ on P so that

$$(0.3.1) \qquad a(\varphi_1, \ldots, \varphi_k) = \sum_{j_1, \ldots, j_k = 1}^r a(E_{j_1}, \ldots, E_{j_k}) \varphi_1^{j_1} \wedge \ldots \wedge \varphi_k^{j_k}$$

if

$$\varphi_i = \sum_{j=1}^r \varphi_i^j \otimes E_j, \quad i = 1, \ldots, k,$$

for any basis $E_1, \ldots, E_r$ of $\mathfrak{g}$, k being any positive integer. If φ is a $\mathfrak{g}$-valued form of even degree, then

$$(ab)(\varphi, \ldots, \varphi) = a(\varphi, \ldots, \varphi) \wedge b(\varphi, \ldots, \varphi) \quad \text{for} \quad a, b \in I(G).$$

Consider any principal bundle $P(M, G)$. We know that if $\mathfrak{g}$-valued forms $\varphi_1, \ldots, \varphi_k$ on P are G-equivariant and $a \in I^k(G)$, k being any positive integer, then there is a unique form $a[\varphi_1, \ldots, \varphi_k]$ on M such that $a(\varphi_1, \ldots, \varphi_k) = \pi^* a[\varphi_1, \ldots, \varphi_k]$. If $\tau: M \supset U \to P$ is a local section of the bundle, then

(0.3.2) $a[\varphi_1, \ldots, \varphi_k] | U = a(\tau^* \varphi_1, \ldots, \tau^* \varphi_k).$

PROPOSITION 0.3.1 (cf. [42]). *Let Ω be the curvature form of a connection in $P(M, G)$.*

(i) *For each $a \in I^k(G)$, $k = 1, 2, \ldots$, the $2k$-form $a[\Omega] := a[\Omega, \ldots, \Omega]$ is closed.*

(ii) *The de Rham cohomology class $w(a) \in H^{2k}_{deR}(M)$ of the form $a[\Omega]$ does not depend on the choice of the connection.*

(iii) *$w: I(G) \to H^{ev}(M) = \oplus H^{2k}_{deR}(M)$ is an algebra homomorphism (the Chern–Weil homomorphism associated with $P(M, G)$).*

For the proof see [6], [7].

We recall a geometrical construction, due to R. Bott [4] and H. Shulman [37], which is fundamental for the existence of the Chern–Weil homomorphism. Consider a family $\Gamma^0, \Gamma^1, \ldots, \Gamma^h$ of connections in a principal bundle $P(M, G)$. Let $\omega^0, \omega^1, \ldots, \omega^h$, respectively, be their connection forms. For any $t = (t_1, \ldots, t_h) \in R^h$, the form

(0.3.3) $\omega_t := \left(1 - \sum_i t_i\right)\omega^0 + \sum_i t_i \omega^i$

is a connection form on $P(M, G)$ and therefore the collection of all ω_t, $t \in R^h$, determines a $\mathfrak{g}$-valued 1-form $\omega^{0,1,\ldots,h}$ on $R^h \times M$, which is the connection form of some connection $\Gamma^{0,1,\ldots,h}$ in the bundle $(R^h \times P)(R^h \times M, G)$ induced from $P(M, G)$ by the projection $R^h \times M \to M$.

Let T_i denote the vector field on $R^h \times M$ tangent to all the lines $\tau \mapsto (t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_h; p)$ for $i = 1, \ldots, h$. For any $\beta \in A^{h+s}(R^h \times M)$, $s = 0, 1, \ldots,$

(0.3.4) $\iota_{T_h} \ldots \iota_{T_1} \beta \in A^s(R^h \times M),$

where ι denotes the interior product. In fact form (0.3.4) is only defined on $R^h \times M$ whereas its values are covectors tangent to M. One defines an operator $\int^h: A^{h+s}(R^h \times M) \to A^s(M)$, $s = 0, 1, \ldots,$ by putting

(0.3.5) $\int^h \beta = \int_{\Delta^h} \iota_{T_h} \ldots \iota_{T_1} \beta \quad \text{for} \quad \beta \in A^*(R^h \times M),$

where the integration is taken over the standard h-simplex

$$\Delta^h = \{(t_1, \ldots, t_h) \in \mathbf{R}^h; \, t_1, \ldots, t_h \geqslant 0, \sum_i t_i \leqslant 1\}.$$

We have

(0.3.6) $$\int^h \circ d + (-1)^{h+1} d \circ \int^h = \sum_{i=0}^{h} (-1)^i \int^{h-1} \circ (\varepsilon_i \times \mathrm{id}_M)^*,$$

where $\varepsilon_0, \ldots, \varepsilon_h \colon \mathbf{R}^{h-1} \to \mathbf{R}^h$ are of the form

$$\varepsilon_0(s_1, \ldots, s_{h-1}) = \left(1 - \sum_j s_j, s_1, \ldots, s_{h-1}\right)$$

and

$$\varepsilon_i(s_1, \ldots, s_{h-1}) = (s_1, \ldots, s_{i-1}, 0, s_i, \ldots, s_{h-1}) \quad \text{for} \quad i = 1, 2, \ldots$$

Turning back to the family of connections in $P(M, G)$ which we have been considering we define a linear mapping

$$\lambda(\Gamma^0, \Gamma^1, \ldots, \Gamma^h) \colon I(G) \to A^*(M)$$

as follows. For the case $h = 0$, let $\lambda(\Gamma^0)(a) = a[\Omega^0]$ for each $a \in I(G)$, where Ω^0 is the curvature form of Γ^0. For the general case, set

$$\lambda(\Gamma^0, \ldots, \Gamma^h)(a) = (-1)^{\binom{h}{2}} \int^h \lambda(\Gamma^{0,1,\ldots,h})(a).$$

On using (0.3.5), we see that for each $a \in I^k(G)$, $k = 0, 1, \ldots,$

(0.3.7) $\quad \lambda(\Gamma^0, \ldots, \Gamma^h)(a)$

$$= \frac{k!}{(k-h)!} \int_{\Delta^h} a[\omega^1 - \omega^0, \ldots, \omega^h - \omega^0, \Omega_t, \ldots, \Omega_t], \quad h \leqslant k,$$

where Ω_t is the curvature form for ω_t as in (0.3.3), and that

(0.3.8) $$\lambda(\Gamma^0, \ldots, \Gamma^h)(a) = 0 \quad \text{if} \quad h > k.$$

Moreover, $d \circ \lambda(\Gamma^0) = 0$ and, by (0.3.6),

(0.3.9) $\quad d \circ \lambda(\Gamma^0, \ldots, \Gamma^h) = \sum_{i=0}^{h} (-1)^i \lambda(\Gamma^0, \ldots, \hat{\Gamma}^i, \ldots, \Gamma^h) \quad \text{for } h \geqslant 1,$

where the symbol "$\hat{\ }$" means that the term is omitted.

PROPOSITION 0.3.2. *Let $(\bar{f}, f) \colon P'(M', G') \to P(M, G)$ be a homomorphism of principal bundles, associated with a group homomorphism $\hat{f} \colon G' \to G$. Let $\Gamma^{0'}, \ldots, \Gamma^{h'}$ and $\Gamma^0, \ldots, \Gamma^h$ be two families of connections in $P'(M', G')$ and $P(M, G)$, respectively. If their connection forms $\omega^{0'}, \ldots, \omega^{h'}$ and $\omega^0, \ldots, \omega^h$, respectively, satisfy $\bar{f}^* \omega^i = \hat{f}_* \circ \omega^{i'}$ for $i = 0, 1, \ldots, h$, where $\hat{f}_* \colon \mathfrak{g}' \to \mathfrak{g}$ is the induced homomorphism of Lie algebras, then*

$$f^* \circ \lambda(\Gamma^0, \ldots, \Gamma^h) = \lambda(\Gamma^{0'}, \ldots, \Gamma^{h'}) \circ \hat{f}^*.$$

Proof. The hypotheses imply that for any $t = (t_1, \ldots, t_h) \in R^h$, the connection forms $\omega_t' = (1 - \sum t_i)\omega^{0'} + \sum t_i \omega^{i'}$ and $\omega_t = (1 - \sum t_i)\omega^0 + \sum t_i \omega^i$ are related in the same way as $\omega^{i'}$ and ω^i:

$$\bar{f}^* \omega_t = \hat{f}_* \circ \omega_t'.$$

By [26], p. 81, this yields $\bar{f}^* \Omega_t = \hat{f}_* \circ \Omega_t'$ for the curvature forms. Consequently

$$\bar{f}^* a(\omega^1 - \omega^0, \ldots, \omega^h - \omega^0, \Omega_t, \ldots, \Omega_t)$$
$$= a(\bar{f}^* \omega^1 - \bar{f}^* \omega^0, \ldots, \bar{f}^* \omega^h - \bar{f}^* \omega^0, \bar{f}^* \Omega_t, \ldots, \bar{f}^* \Omega_t)$$
$$= a(\hat{f}_* \circ (\omega^{1'} - \omega^{0'}), \ldots, \hat{f}_* \circ (\omega^{h'} - \omega^{0'}), \hat{f}_* \circ \Omega_t', \ldots, \hat{f}_* \circ \Omega_t')$$
$$= (\hat{f}^* a)(\omega^{1'} - \omega^{0'}, \ldots, \omega^{h'} - \omega^{0'}, \Omega_t', \ldots, \Omega_t'),$$

which gives

$$f^* a[\omega^1 - \omega^0, \ldots, \omega^h - \omega^0, \Omega_t, \ldots, \Omega_t]$$
$$= (\hat{f}^* a)[\omega^{1'} - \omega^{0'}, \ldots, \omega^{h'} - \omega^{0'}, \Omega_t', \ldots, \Omega_t']$$

for any $a \in I^k(G)$, $k \geqslant h$, and any $t \in R^h$. By (0.3.7), this ends the proof.

The following lemma will be necessary in Chapter 2.

LEMMA 0.3.3. *If* $\Gamma^0, \ldots, \Gamma^h$ *are connections in* $P(M, G)$ *adapted to a flat partial connection* E, *then*

$$\lambda(\Gamma^0, \ldots, \Gamma^h)(a) \in I_F^{k, 2k-h} \quad for \quad a \in I^k(G),$$

$h, k = 0, 1, \ldots, h \leqslant k$. *Here* F *stands for the projected foliation.*

Proof. If $a \in I^k(G)$, then $\lambda(\Gamma^0, \ldots, \Gamma^h)(a) \in A^{2k-h}(M)$. By Proposition 0.1.8, we only have to prove that

$$\lambda(\Gamma^0, \ldots, \Gamma^h)(a)(v_1, \ldots, v_{2k-h}) = 0$$

if $v_1, \ldots, v_{2k-h} \in T_p M$, $p \in M$, and $v_1, \ldots, v_{k-h+1}$ are tangent to F. We shall show that $a[\omega^1 - \omega^0, \ldots, \omega^h - \omega^0, \Omega_t, \ldots, \Omega_t](v_1, \ldots, v_{2k-h}) = 0$ for all $t \in R^h$. Indeed, choosing $w_1, \ldots, w_{2k-h} \in T_u P$ such that $v_i = \pi_*(w_i)$ for all i, and $w_1, \ldots, w_{k-h+1} \in E$, u being an element of the fibre $\pi^{-1}(p) \subset P$, we get

$$a[\omega^1 - \omega^0, \ldots, \omega^h - \omega^0, \Omega_t, \ldots, \Omega_t](v_1, \ldots, v_{2k-h})$$
$$= a(\omega^1 - \omega^0, \ldots, \omega^h - \omega^0, \Omega_t, \ldots, \Omega_t)(w_1, \ldots, w_{2k-h}),$$

which vanishes by (0.3.1) and Proposition 0.2.6, since ω_t given by (0.3.3) is the connection form of some connection adapted to E for all $t \in R^h$ (observe that Ω_t appears in the above formula $k - h$ times only).

We have observed that the mapping $\lambda(\Gamma)$: $I(G) \to A^*(M)$, Γ being any connection in the bundle $P(M, G)$, is an algebra homomorphism. On the other hand, the linear mapping $\lambda(\Gamma_0, \Gamma_1)$: $I(G) \to A^*(M)$ cannot be a homomorphism of algebras (excluding trivial cases) for any two connections Γ_0

and Γ_1 in $P(M, G)$. The following proposition gives a "derivational rule" for $\lambda(\Gamma_0, \Gamma_1)(ab)$, $a, b \in I(G)$.

PROPOSITION 0.3.4. *Consider any two principal bundles $P(M, G)$ and $P'(M, G')$ over the same base. Let $\Gamma_0, \Gamma_1, \Gamma'_0$ and Γ'_1 be connections in $P(M, G)$ and $P'(M, G')$ respectively, and let $\omega_0, \omega_1, \omega'_0$ and ω'_1 be their connection forms. For t real, we put $\omega_t = t\omega_1 + (1-t)\omega_0$ and $\omega'_t = t\omega'_1 + (1-t)\omega'_0$. Let Ω_t and Ω'_t be the curvature forms determined by ω_t and ω'_t, respectively. For any $a \in I^k(G)$ and $b \in I^l(G)$, k, l any positive integers, there exists a differential form $\varphi \in A^{k+l-2}(M)$ such that*

$$\int_{0 \leqslant t \leqslant 1} (a[\Omega_t] \wedge lb[\omega'_1 - \omega'_0; \Omega'_t] + ka[\omega_1 - \omega_0; \Omega_t] \wedge b[\Omega'_t])$$

$$= \lambda(\Gamma_1)(a) \wedge \lambda(\Gamma'_0, \Gamma'_1)(b) + \lambda(\Gamma_0, \Gamma_1)(a) \wedge \lambda(\Gamma'_0)(b) + d\varphi.$$

Here, $a[\omega_1 - \omega_0; \Omega_t] = a[\omega_1 - \omega_0, \Omega_t, \ldots, \Omega_t]$ and, similarly, $b[\omega'_1 - \omega'_0; \Omega'_t] = b[\omega'_1 - \omega'_0, \Omega'_t, \ldots, \Omega'_t]$.

If the principal bundles coincide and $\Gamma_i = \Gamma'_i$ for $i = 0, 1$, then

(0.3.10) $\lambda(\Gamma_0, \Gamma_1)(ab)$

$$= \lambda(\Gamma_1)(a) \wedge \lambda(\Gamma_0, \Gamma_1)(b) + \lambda(\Gamma_0, \Gamma_1)(a) \wedge \lambda(\Gamma_0)(b) + d\varphi.$$

Proof. Integrating by parts, we get

$$\int_{0 \leqslant t \leqslant 1} (a[\Omega_t] \wedge lb[\omega'_1 - \omega'_0; \Omega'_t])$$

$$= a[\Omega_1] \wedge \int_{0 \leqslant t \leqslant 1} lb[\omega'_1 - \omega'_0; \Omega'_t] - \int_{0 \leqslant t \leqslant 1} \left(\frac{\partial}{\partial t} a[\Omega_t] \wedge \int_{0 \leqslant s \leqslant t} lb[\omega'_1 - \omega'_0; \Omega'_s] \right)$$

$$= \lambda(\Gamma_1)(a) \wedge \lambda(\Gamma'_0, \Gamma'_1)(b) -$$

$$- \int_{0 \leqslant t \leqslant 1} \left(d(ka[\omega_1 - \omega_0; \Omega_t]) \wedge \int_{0 \leqslant s \leqslant t} lb[\omega'_1 - \omega'_0; \Omega'_s] \right)$$

$$= \lambda(\Gamma_1)(a) \wedge \lambda(\Gamma'_0, \Gamma'_1)(b) - d \iint_{0 \leqslant s \leqslant t \leqslant 1} (ka[\omega_1 - \omega_0; \Omega_t] \wedge lb[\omega'_1 - \omega'_0; \Omega'_s]) -$$

$$- \int_{0 \leqslant t \leqslant 1} \left(ka[\omega_1 - \omega_0; \Omega_t] \wedge \int_{0 \leqslant s \leqslant t} d(lb[\omega'_1 - \omega'_0; \Omega'_s]) \right)$$

$$= \lambda(\Gamma_1)(a) \wedge \lambda(\Gamma'_0, \Gamma'_1)(b) + d\varphi + \lambda(\Gamma_0, \Gamma_1)(a) \wedge \lambda(\Gamma'_0)(b) -$$

$$- \int_{0 \leqslant t \leqslant 1} (ka[\omega_1 - \omega_0; \Omega_t] \wedge b[\Omega_t]),$$

where

$$\varphi = - \iint_{0 \leqslant s \leqslant t \leqslant 1} (ka[\omega_1 - \omega_0; \Omega_t] \wedge lb[\omega'_1 - \omega'_0; \Omega'_s]).$$

In the above reasoning, we have taken advantage of the identities

$$d(ka\,[\omega_1-\omega_0;\Omega_t]) = \frac{\partial}{\partial t}\,a\,[\Omega_t] \quad \text{and} \quad d(lb\,[\omega_1'-\omega_0';\Omega_s']) = \frac{\partial}{\partial s}\,b\,[\Omega_s']$$

(cf. [43], p. 113).

If, in particular, the principal bundles coincide and $\Gamma_i = \Gamma_i'$ for $i = 0, 1$, then (0.3.10) follows from the identity

$$a\,[\Omega_t] \wedge lb\,[\omega_1-\omega_0;\Omega_t] + ka\,[\omega_1-\omega_0;\Omega_t] \wedge b\,[\Omega_t]$$
$$= (k+l)(ab)\,[\omega_1-\omega_0;\Omega_t].$$

To show this identity, we consider the principal bundle $(R \times P)(R \times M, G)$ and the G-equivariant form Ω on $R \times P$ such that $\Omega = dt \wedge (\omega_1 - \omega_0) + \Omega_t$ at points of the set $\{t\} \times P$, for $t \in R$, dt being the differential of the projection onto R. Since elements of $I(G)$ are multilinear, we get

$$(ab)\,[\Omega] = (ab)\,[\Omega_t] + dt \wedge (k+l)(ab)\,[\omega_1-\omega_0;\Omega_t]$$

and, on the other hand,

$$(ab)\,[\Omega] = a\,[\Omega] \wedge b\,[\Omega]$$
$$= (a\,[\Omega_t] + dt \wedge ka\,[\omega_1-\omega_0;\Omega_t]) \wedge (b\,[\Omega_t] + dt \wedge lb\,[\omega_1-\omega_0;\Omega_t])$$

on $\{t\} \times M$, for $t \in R$. Comparing the terms which involve dt, we obtain the required identity and conclude the proof.

1. Category of RF-bundles

If $P(M, G)$ is a principal bundle and H a closed subgroup of G, then there is a canonical 1-1 correspondence between global sections $\sigma \colon M \to P/H$ of the fibre bundle associated with $P(M, G)$ with standard fibre G/H, and subbundles $Q(M, H)$ of $P(M, G)$ given by the assignment $\sigma \mapsto Q = \{u \in P; \sigma(\pi(u)) = uH\}$ (cf. [26], p. 58). Such a Q will be denoted by P_σ.

DEFINITION 1.1. Sections $\sigma_0, \sigma_1 \colon M \to P/H$ are *homotopic* if there exists a (smooth) mapping $\tau \colon M \times R \to P/H$ such that $\tau(\cdot, t) \colon M \to P/H$ is a section for any real t, $\tau(\cdot, 0) = \sigma_0$, $\tau(\cdot, 1) = \sigma_1$ and $\tau(p, t)$ is independent of t for (p, t) in some neighbourhood of $M \times \{0, 1\}$.

Observe that the principal bundles $P_{\sigma_0}(M, H)$ and $P_{\sigma_1}(M, H)$ are then isomorphic.

The homotopy between sections $M \to P/H$ is obviously an equivalence relation. The equivalence classes will be called *homotopy classes* of *H-reductions* of $P(M, G)$ (of the structure group of $P(M, G)$).

DEFINITION 1.2. A *foliated bundle with a reduction of structure group*, shortly an *RF-bundle*, is a collection $(P(M, G), E, H, \xi)$ where $(P(M, G), E)$ is a foliated bundle, H is a closed subgroup of G and ξ is a homotopy class of *H-reductions* of the bundle $P(M, G)$.

DEFINITION 1.3. For any two RF-bundles $B' = (P'(M', G'), E', H', \xi')$ and $B = (P(M, G), E, H, \xi)$, a triple $(\bar{f}, f, \hat{f})$, with $\bar{f}: P' \to P$, $f: M' \to M$ and $\hat{f}: G' \to G$, is an *RF-morphism of B' into B*, to be denoted as $(\bar{f}, f, \hat{f}): B' \to B$, if
    (i) $\hat{f}$ is a group homomorphism and $\hat{f}(H') \subset H$,
    (ii) the pair $(\bar{f}, f)$ is a principal bundle homomorphism of $P'(M', G')$ into $P(M, G)$, associated with $\hat{f}$,
    (iii) $\bar{f}_*(E_{u'}) \subset E_{\bar{f}(u')}$ for $u' \in P'$, and
    (iv) for any sections $\sigma' \in \xi'$ and $\sigma \in \xi$, the mappings

$$(\mathrm{id}_{M'}, \sigma \circ f), (\mathrm{id}_{M'}, \bar{f}/\circ\sigma'): \ M' \to M' \times_{(f, \pi)} P/H$$

are homotopic sections of the associated bundle $f^*P/H \to M'$, $f^*P = M' \times_{(f, \pi)} P$ being the total space of the principal bundle $(f^*P)(M', G)$ induced from $P(M, G)$ by f (cf. [26], p. 60). Here $\bar{f}/$ stands for the quotient mapping $P'/H' \to P/H$ induced from $\bar{f}$, and the symbol $\times_{(f, \pi)}$ denotes the Whitney product.
    Condition (iv) can be re-formulated as follows:
    (iv') for any sections $\sigma' \in \xi'$, and $\sigma \in \xi$, there exists a mapping $\tau: M' \times \boldsymbol{R} \to P/H$ such that $\bar{\pi} \circ \tau = f \circ \mathrm{pr}_{M'}$, $\tau(\cdot, 0) = \sigma \circ f$, $\tau(\cdot; 1) = \bar{f}/\circ\sigma'$, and τ is independent of t on a neighbourhood of $M' \times \{0, 1\}$. Here $\bar{\pi}$ and $\mathrm{pr}_{M'}$ stand for the projections $P/H \to M$ and $M' \times \boldsymbol{R} \to M'$, respectively.

DEFINITION 1.4. For any two RF-morphisms

$$(\bar{f}, f, \hat{f}): \ B' \to B \qquad \text{and} \qquad (\bar{g}, g, \hat{g}): \ B'' \to B',$$

their *superposition* $(\bar{f}, f, \hat{f}) \circ (\bar{g}, g, \hat{g}): B'' \to B$ is defined as the triple $(\bar{f} \circ \bar{g}, f \circ g, \hat{f} \circ \hat{g})$.
    Observe that the triple of the superpositions satisfies the conditions of Definition 1.3. Indeed, if $B'' = (P''(M'', G''), E'', H'', \xi'')$, $B' = (P'(M', G'), E', H', \xi')$ and $B = (P(M, G), E, H, \xi)$, and the homotopy classes ξ'', ξ' and ξ are represented by σ'', σ' and σ, respectively, then the sections $(\mathrm{id}_{M''}, \sigma \circ (f \circ g))$, $(\mathrm{id}_{M''}, (\bar{f} \circ \bar{g})/\circ\sigma''): M'' \to M'' \times_{(f \, g, \bar{\pi})} P/H$ are both homotopic to $(\mathrm{id}_{M''}, \bar{f}/\circ\sigma' \circ g)$ and so they are homotopic, which proves (iv). Conditions (i)–(iii) are obvious.

DEFINITION 1.5. A *category RFB of foliated bundles with a reduction of structure group* is a category whose objects are RF-bundles, morphisms are

determined by RF-morphisms and the superposition map is as in Definition 1.4.

A foliated bundle $(P(M, G), E)$ can be identified with the RF-bundle $(P(M, G), E, G, \{id\})$, id: $M \xrightarrow{\approx} P/G$ being the canonical diffeomorphism.

DEFINITION 1.6. An RF-bundle $B = (P(M, G), E, H, \xi)$ is *reducible* if there are a foliated bundle $(P(M, H), E)$, with $\pi_* E = \pi_* E$, and an RF-morphism $(\bar{f}, id_M, i_{G,H})$: $(P(M, H), E, H, \{id\}) \dashrightarrow B$, $i_{G,H}$ being the inclusion $H \subset G$.

In other words, the reducibility of $(P(M, G), E, H, \xi)$ means that the partially horizontal subspaces are tangent to $P_\sigma \subset P$ for some $\sigma \in \xi$ (cf. [26], p. 84).

We shall now try to elucidate the meaning of Definition 1.3.

PROPOSITION 1.7. (i) *Let* $B = (P(M, G), E, H, \xi)$ *be any RF-bundle,* F' *a foliation of a manifold* M' *and* f: $M' \to M$ *a mapping satisfying* $f_*(E_{F'}) \subset \pi_* E$. *Then there exists a unique RF-bundle* $(f, F')^* B = (\bar{P}(M', G), \bar{E}, H, \bar{\xi})$ *such that* $\bar{P}(M', G) = (f^* P)(M', G)$ *is the bundle induced from* $P(M, G)$ *by* f, $\pi_* \bar{E}$ $= E_{F'}$ *and* $(\bar{g}, f, id_G)$: $(f, F')^* B \to B$, *for* $\bar{g}$: $\bar{P} = M' \times_{(f, \pi)} P \to P$ *the projection into the second factor.*

(ii) *If, moreover,* $P'(M', G)$ *is a principal bundle over* M' *and a pair* $(\bar{f}, f)$ *is a principal bundle homomorphism of* $P'(M', G)$ *into* $P(M, G)$, *then there is a unique RF-bundle* $B' = (P'(M', G), E', H, \xi')$ *such that* $\pi_* E' = E_{F'}$ *and* $(\bar{f}, f, id_G)$: $B' \to B$.

Proof. We recall the formula for the tangent space of a Whitney product. If M_1, M_2 and M are manifolds and f_i: $M_i \to M$, $i = 1, 2$, are mappings transverse to each other, then the projections of the Whitney product $N = M_1 \times_{(f_1, f_2)} M_2 = \{(p, q) \in M_1 \times M_2; f_1(p) = f_2(q)\}$ into M_1 and M_2 give rise to an isomorphism

(1.1) $T_{(p,q)} N \xrightarrow{\approx} T_p M_1 \times_{(f_{1*p}, f_{2*q})} T_q M_2$, $p \in M_1$, $q \in M_2$,

(cf. [36]).

If ω is a connection form on $P(M, G)$, then, by Proposition 0.2.8, $\bar{\omega} = \bar{g}^* \omega$ is a connection form on $\bar{P}(M', G)$. The corresponding curvature forms Ω and $\bar{\Omega}$ satisfy $\bar{\Omega} = \bar{g}^* \Omega$. By Proposition 0.2.6, if ω and ω' are the connection forms of connections adapted to E, then

$$(\bar{\omega} - \bar{\omega}')(v) = (\omega - \omega')(\bar{g}_* v) = 0 \quad \text{and} \quad \bar{\Omega}(v, w) = \Omega(\bar{g}_* v, \bar{g}_* w) = 0,$$

for $v, w \in T\bar{P}$ such that $\bar{\pi}_* v, \bar{\pi}_* w \in E_{F'}$, since then $\pi_* \bar{g}_* v, \pi_* \bar{g}_* w \in \pi_* E$. Here $\bar{\pi}$ stands for the projection $\bar{P} \to M'$. Thus, also by Proposition 0.2.6, there is a unique subset $\bar{E}$ of $T\bar{P}$ such that

$$\bar{\pi}_*(\bar{E} \cap T_{(p,u)} \bar{P}) = E_{F'} \cap T_p M' \quad \text{for all } (p, u) \in \bar{P}$$

and all the connection forms $\bar{\omega} = \bar{g}^* \omega$ with ω adapted to E annihilate $\bar{E}$, and this subset is a flat partial connection in $\bar{P}(M', G)$. On the other hand, each $\bar{g}^* \omega$ annihilates $(\bar{g}_*)^{-1}(E)$ and, as we shall see, $(\bar{\pi}_*)^{-1}(E_{F'}) \cap (\bar{g}_*)^{-1}(E)$ projects onto $E_{F'}$. Indeed, let (p, u) be any point of $\bar{P}$ and $v \in E_{F'} \cap T_p M'$ any vector tangent to F' at p. We have $f_{*p} v \in \pi_* E \cap T_{f(p)} M$ and so $f_{*p} v = \pi_{*u} w$ for some $w \in E_u$. By (1.1), the pair (v, w) forms a vector in $T_{(p,u)} \bar{P}$, such that $\bar{g}_*(v, w) = w \in E$. Thus we get

$$v = \bar{\pi}_{*(p,u)}(v, w) \quad \text{and} \quad (v, w) \in (\bar{g}_{*(p,u)})^{-1}(E_u),$$

which proves that

$$\bar{\pi}_{*(p,u)}\left((\bar{\pi}_*)^{-1}(E_{F'}) \cap (\bar{g}_*)^{-1}(E) \cap T_{(p,u)} \bar{P}\right) = E_{F'} \cap T_p M'.$$

Consequently, $(\bar{\pi}_*)^{-1}(E_{F'}) \cap (\bar{g}_*)^{-1}(E)$ and $\bar{E}$ must coincide. In particular, $\bar{g}_*(\bar{E}) \subset E$. We see that $\bar{E}$ is the desired partial connection. If a flat partial connection in the bundle $\bar{P}(M', G)$ projects onto $E_{F'}$ and is mapped into E by $\bar{g}$, then all the connection forms $\bar{g}^* \omega$ with ω adapted to E annihilate it and so the flat partial connection must be equal to $\bar{E}$. To end the proof of (i) let us observe that all the sections

$$(\mathrm{id}_{M'}, \sigma \circ f) \colon M' \to M' \times_{(f, \pi)} P/H = \bar{P}/H \quad \text{for} \quad \sigma \in \xi$$

are in the same homotopy class $\bar{\xi}$, which is uniquely determined by any of them.

If, moreover, $(\bar{f}, f)$ is principal bundle homomorphism of $P'(M', G)$ into $P(M, G)$, then the canonical isomorphism

$$((\pi', \bar{f}), \mathrm{id}_{M'}) \colon P'(M', G) \xrightarrow{\cong} \bar{P}(M', G), \quad \text{where } (\pi', \bar{f})(u) = (\pi'(u), \bar{f}(u)) \in \bar{P}$$

for $u \in P'$, transfers the RF-bundle structure from $\bar{P}(M', G)$ to $P'(M', G)$. This proves (ii) and ends the proof of the proposition.

PROPOSITION 1.8. *Let* $B' = (P'(M', G'), E', H', \xi')$ *be any RF-bundle and let* $\bar{P}(M', G)$ *be a principal bundle over* M'. *For any principal bundle homomorphism* $(\bar{h}, \mathrm{id}_{M'}) \colon P'(M', G') \to \bar{P}(M', G)$ *associated with a group homomorphism* $\hat{f} \colon G' \to G$, *and any closed subgroup* H *of* G *containing* $\hat{f}(H')$, *there is a unique RF-bundle* $\bar{B} = (\bar{P}(M', G), \bar{E}, H, \bar{\xi})$ *such that* $\pi_* E = \bar{\pi}_* \bar{E}$ *and* $(\bar{h}, \mathrm{id}_{M'}, \hat{f}) \colon B' \to \bar{B}$.

Proof. By Proposition 0.2.8, any connection Γ' in $P'(M', G')$ determines a unique connection $\bar{\Gamma}$ in $\bar{P}(M', G)$ such that $\bar{h}$ maps the horizontal subspaces of Γ' into horizontal subspaces of $\bar{\Gamma}$. If ω' and $\bar{\omega}$ are the connection forms of Γ' and $\bar{\Gamma}$ respectively, then $\bar{h}^* \bar{\omega} = \hat{f}_* \circ \omega'$, where $\hat{f}_* \colon \mathfrak{g}' \to \mathfrak{g}$ is the induced homomorphism of Lie algebras. Similarly, $\bar{h}^* \bar{\Omega} = \hat{f}_* \circ \Omega'$ for the curvature forms. Thus the family of all connections in $P'(M', G')$ adapted to E' determines some family $\mathscr{C}$ of connections in $\bar{P}(M', G)$. We check the hypotheses of Proposition 0.2.6.

Let ω and ω' be the connection forms of any two connections adapted to E. We shall denote the corresponding connection forms on $\bar{P}$ by $\bar{\omega}$ and $\bar{\omega}'$ respectively. Consider arbitrary vectors $v, w \in T_{\bar{h}(u)} \bar{P}$ such that $\bar{\pi}_* v, \bar{\pi}_* w \in \pi'_* E$. We have $\bar{\pi}_* v = \pi'_* v'$ and $\bar{\pi}_* w = \pi'_* w'$ for some $v', w' \in E'_u$. Here u stands for any point of P'. The vectors $v - \bar{h}_* v'$ and $w - \bar{h}_* w'$ are vertical and therefore we get

$$(\bar{\omega} - \bar{\omega}')(v) = (\bar{\omega} - \bar{\omega}')(\bar{h}_* v') = \hat{f}_*(\omega(v') - \omega'(v')) = 0$$

and

$$\bar{\Omega}(v, w) = \bar{\Omega}(\bar{h}_* v', \bar{h}_* w') = \hat{f}_*(\Omega(v', w')) = 0,$$

where Ω and $\bar{\Omega}$ are the curvature forms corresponding to ω and $\bar{\omega}$, respectively. Thus we have verified conditions (0.2.1), (0.2.2) over the image of the mapping $\bar{h}$, which meets every fibre of the bundle $\bar{P}(M', G)$. Since the forms $\bar{\omega} - \bar{\omega}'$ and $\bar{\Omega}$ are G-equivariant, the hypotheses of Proposition 0.2.6 are satisfied and thus there exists a unique flat partial connection $\bar{E}$ in $\bar{P}(M', G)$ which projects onto $\pi'_* E'$ and is such that any connection in $\mathscr{C}$ is adapted to $\bar{E}$. We shall show that $\bar{h}$ maps E' onto $\bar{E}$. For any $v \in E'_u$, $u \in P'$, we have $\bar{\pi}_* \bar{h}_* v \in \pi'_* E'$ and $\bar{\omega}(\bar{h}_* v) = \hat{f}_*(\omega(v)) = 0$ if $\bar{\omega} \in \mathscr{C}$ corresponds to ω, and so $\bar{h}_* v \in \bar{E}_{\bar{h}(u)}$. If $w \in \bar{E}_{\bar{h}(u)}$, then $\bar{\pi}_* w = \pi'_* w'$ for some $w' \in E'_u$, and $w = \bar{h}_* w$ as $\bar{\pi}_*(w - \bar{h}_* w') = \bar{\omega}(w - \bar{h}_* w') = 0$ for any $\bar{\omega} \in \mathscr{C}$. Consequently, $\bar{h}_*(E'_u) = \bar{E}_{\bar{h}(u)}$ for all $u \in P'$.

Any partial connection in $\bar{P}(M', G)$ which agrees with $\bar{E}$ on the image of $\bar{h}$ must coincide with $\bar{E}$ since $\bar{h}(P')$ meets every fibre of the bundle $\bar{P}(M', G)$.

Finally, let us observe that the sections $\bar{h}/\circ \sigma'$: $M' \to \bar{P}/H$, $\sigma' \in \xi'$, are in the same homotopy class $\bar{\xi}$, which is uniquely determined by any of them. This ends the proof.

PROPOSITION 1.9. *For any two RF-bundles* $B' = (P'(M', G'), E', H', \xi')$ *and* $B = (P(M, G), E, H, \xi)$, *any RF-morphism* $(\bar{f}, f, \hat{f})$ *of* B' *into* B *factors through*

$$((\pi', \bar{f}), \mathrm{id}_{M'}, \hat{f}): B' \to (f, F')^* B \quad and \quad (\bar{g}, f, \mathrm{id}_G): (f, F')^* B \to B,$$

where $(f, F')^* B$ *and* $\bar{g}$ *are as in Proposition 1.7 (i) and* F' *is the projected foliation of* M'.

Proof. We have $(f, F')^* B = (\bar{P}(M', G), \bar{E}, H, \bar{\xi})$, where $\bar{P} = f^* P = M' \times_{(f, \pi)} P$, $\bar{E} = (\bar{\pi}_*)^{-1}(\pi'_* E') \cap (\bar{g}_*)^{-1}(E)$ and $(\mathrm{id}_{M'}, \sigma \circ f) \in \bar{\xi}$ for $\sigma \in \xi$. Thus all we need to show is $(\pi', \bar{f})/\circ \sigma' \in \bar{\xi}$ for $\sigma' \in \xi'$, and

$$(\pi', \bar{f})_*(E') \subset (\bar{\pi}_*)^{-1}(\pi'_* E') \cap (\bar{g}_*)^{-1}(E).$$

Since $\bar{g} \circ (\pi', \bar{f}) = \bar{f}$ and $\bar{\pi} \circ (\pi', \bar{f}) = \pi'$, the above inclusion is equivalent to $\bar{f}_*(E') \subset E$, the latter being obviously satisfied as the triple $(\bar{f}, f, \hat{f})$ is an RF-morphism. By the same argument, $(\pi', \bar{f})/\circ \sigma' = (\mathrm{id}_{M'}, \bar{f}/\circ \sigma') \in \bar{\xi}$.

2. Characteristic diagram for RF-bundles

In this chapter we shall construct a contravariant functor from the category RFB of RF-bundles to a category of commuting diagrams

$$A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow A_4$$
$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$
$$B_1 \rightarrow B_2 \rightarrow B_3 \rightarrow B_4$$

with exact rows. Namely, with any RF-bundle $(P(M,G), E, H, \xi)$ we associate (in the way described below) a commuting diagram

(2.1)
$$
\begin{array}{ccccccc}
\ker i^*_{G,H} & \lhook\joinrel\longrightarrow & I(G) & \xrightarrow{\dot{i}_{G,H}} & I(H) & \longrightarrow & \operatorname{coker} i^*_{G,H} \\
\downarrow{\scriptstyle w'_2} & & \downarrow{\scriptstyle w'_1} & & \downarrow{\scriptstyle w''_1} & & \downarrow{\scriptstyle w''_2} \\
\bigoplus_i H^{2i-1}(A^*(M)/I^i_F) & \xrightarrow{d_*} & \bigoplus_i H^{2i}(I^i_F) & \xrightarrow{a_*} & H^{ev}(M) & \xrightarrow{\beta_*} & \bigoplus_i H^{2i}(A^*(M)/I^i_F)
\end{array}
,
$$

where $i^*_{G,H}$ is the restriction map, the lower row is a direct sum of the appropriate parts of the long exact homology sequences associated with

(2.2) $0 \rightarrow I^i_F \overset{\alpha}{\hookrightarrow} A^*(M) \overset{\beta}{\rightarrow} A^*(M)/I^i_F \rightarrow 0$ for $i = 0, 1, \ldots,$

and F is the projected foliation of M. Moreover, w''_1 is the Chern–Weil homomorphism of any subbundle $P_\sigma(M, H)$ of $P(M, G)$, $\sigma \in \xi$. Since homotopic sections $M \rightarrow P/H$ give rise to isomorphic subbundles, w''_1 is independent of the choice of σ.

DEFINITION 2.1. The *generalized Chern–Weil homomorphism*

$$w_E: I(G) \rightarrow \bigoplus_{i \geqslant 0} H^{2i}(I^i_F)$$

of a foliated bundle $(P(M, G), E)$ is a mapping represented on the cochain level by $\lambda(\Gamma)$ for any connection Γ adapted to E:

$$w_E(a) = [\lambda(\Gamma)(a)]_k \in H^{2k}(I^k_F) \quad \text{for} \quad a \in I^k(G), \; k = 0, 1, \ldots$$

Here F stands for the projected foliation and the indexed braces $[\;]_k$ denote the homology class in a homology group of I^k_F for $k = 0, 1, \ldots$

By Lemma 0.3.3, $\lambda(\Gamma)(a) \in I^k_F$ if $a \in I^k(G)$ and Γ is adapted to E. By (0.3.9), if $a \in I^k(G)$ and Γ^0 and Γ^1 are any two connections in $P(M, G)$ adapted to E, then

$$\lambda(\Gamma^1)(a) - \lambda(\Gamma^0)(a) = d\lambda(\Gamma^0, \Gamma^1)(a),$$

and $\lambda(\Gamma^0, \Gamma^1)(a) \in I^k_F$, as stated in Lemma 0.3.3. This proves the correctness of Definition 2.1.

Observe that the direct sum $\bigoplus H^{2i}(I_F^i)$ carries a canonical structure of a graded commutative ring. The multiplication structure is standard:

$$[\varphi]_k \cdot [\psi]_l = [\varphi \wedge \psi]_{k+l}.$$

Obviously, $\varphi \wedge \psi \in I_F^{k+l}$ if $\varphi \in I_F^k$ and $\psi \in I_F^l$. This structure makes w_E a graded algebra homomorphism.

DEFINITION 2.2. For any RF-bundle $B = (P(M, G), E, H, \xi)$ its *left (right)* *'primary' characteristic homomorphism* $w_1' = w_1'(B)$ (respectively, $w_1'' = w_1''(B)$) coincides with the generalized Chern–Weil homomorphism of $(P(M, G), E)$ (the Chern–Weil homomorphism of any of the subbundles $P_\sigma(M, H), \sigma \in \xi$). The values of w_1' and of w_1'' are *primary characteristic invariants of B.*

Comparing the primary characteristic homomorphisms of an RF-bundle, we get the following

THEOREM 2.3. *Let* $B = (P(M, G), E, H, \xi)$ *be an RF-bundle. There is a unique commuting diagram* (2.1) *such that* w_1' *and* w_1'' *are the primary characteristic homomorphisms of B and for any pair* (Γ_σ, Γ) *of connections in the principal bundle* $P(M, G)$ *such that* Γ *is adapted to E and* Γ_σ *is reducible to* $P_\sigma(M, H), \sigma$ *being any element of* ξ,

$$w_2'(a) = [\lambda(\Gamma_\sigma, \Gamma)(a) + I_F^k]$$

for $a \in \ker i_{G,H}^* \cap I^k(G)$, $k = 1, 2, \ldots$ *Here F stands for the projected foliation of M, and the braces denote the homology class.*

Proof. To start with, let us observe that the diagram

$$
\begin{array}{ccc}
I(G) & \xrightarrow{i_{G,H}^*} & I(H) \\
\downarrow{\scriptstyle w_1'} & & \downarrow{\scriptstyle w_1''} \\
\oplus H^{2i}(I_F^i) & \xrightarrow{\alpha_*} & H^{ev}(M)
\end{array},
$$

where the mapping α_* is induced from sequence (2.2), commutes. Indeed, it follows from Definition 2.1 that $\alpha_* \circ w_1'$ is the Chern–Weil homomorphism of $P(M, G)$. The commutativity is a consequence of naturality of the Chern–Weil homomorphism.

We divide the proof of the theorem into two lemmas.

LEMMA 2.4. *There exists a unique linear mapping*

$$w_2'': \operatorname{coker} i_{G,H}^* \to \bigoplus_{i \geq 1} H^{2i}(A^*(M)/I_F^i)$$

such that the diagram

$$
\begin{array}{ccc}
I(H) & \to & \operatorname{coker} i_{G,H}^* \\
\downarrow{\scriptstyle w_1''} & & \downarrow{\scriptstyle w_2''} \\
H^{ev}(M) & \xrightarrow{\beta_*} & \oplus H^{2i}(A^*(M)/I_F^i)
\end{array},
$$

where the upper arrow is the projection and β_* is induced from the sequence (2.2), commutes.·

Proof. We have $\beta_* \circ w_1'' \circ i_{G,H}^* = \beta_* \circ \alpha_* \circ w_1' = 0$, and so the superposition $\beta_* \circ w_1''$ annihilates $\operatorname{im} i_{G,H}^*$. Thus it factorizes uniquely through the projection $I(H) \to I(H)/\operatorname{im} i_{G,H}^* = \operatorname{coker} i_{G,H}^*$.

LEMMA 2.5. *Let Γ and Γ_σ be connections in $P(M, G)$ such that Γ is adapted to the partial connection E and Γ_σ is reducible to $P_\sigma(M, H)$, σ being an element of ξ. Then the linear mapping $\lambda(\Gamma_\sigma, \Gamma) \colon I(G) \to A^*(M)$ gives rise to a homomorphism*

$$w_2' \colon \ker i_{G,H}^* \to \bigoplus_{i \geqslant 1} H^{2i-1}\left(A^*(M)/I_F^i\right),$$

$$w_2'(a) = [\lambda(\Gamma_\sigma, \Gamma)(a) + I_F^i] \quad \textit{for} \quad a \in \ker i_{G,H}^* \cap I^i(G),$$

which depends on the given RF-bundle only. Moreover, the diagram

$$
\begin{array}{ccc}
\ker i_{G,H}^* & \lhook\joinrel\longrightarrow & I(G) \\
\Big\downarrow{\scriptstyle w_2'} & & \Big\downarrow{\scriptstyle w_1'} \\
\bigoplus H^{2i-1}\left(A^*(M)/I_F^i\right) & \overset{d_*}{\to} & \bigoplus H^{2i}(I_F^i),
\end{array}
$$

where the mapping d_* is induced from the exterior derivative, commutes.

Proof. If $\hat{\Gamma}_\sigma$ is the restriction of Γ_σ to $P_\sigma(M, H)$, and $\hat{\omega}_\sigma$ and ω_σ are the connection forms of $\hat{\Gamma}_\sigma$ and Γ_σ respectively, then $i_\sigma^* \omega_\sigma = i_{G,H_*} \circ \hat{\omega}_\sigma$, where $i_\sigma \colon P_\sigma \hookrightarrow P$ is the inclusion (cf. Proposition 0.2.8). By Proposition 0.3.2,

$$\lambda(\Gamma_\sigma) = \lambda(\hat{\Gamma}_\sigma) \circ i_{G,H}^*,$$

which proves that

$$d\lambda(\Gamma_\sigma, \Gamma)(a) = \lambda(\Gamma)(a) - \lambda(\Gamma_\sigma)(a) = \lambda(\Gamma)(a)$$

if $a \in \ker i_{G,H}^*$. By Lemma 0.3.3, $\lambda(\Gamma_\sigma, \Gamma)(a) + I_F^k$ is a cocycle if $a \in \ker i_{G,H}^* \cap \cap I^k(G)$, and so w_2' is well-defined. Moreover, there is

$$d_* \circ w_2'(a) = [d\lambda(\Gamma_\sigma, \Gamma)(a)]_k = [\lambda(\Gamma)(a)]_k = w_1'(a)$$

for $a \in \ker i_{G,H}^* \cap I^k(G)$, $k = 1, 2, \ldots$

(i) Independence of the choice of the adapted connection Γ. Let Γ' be a connection in $P(M, G)$ adapted to E and different from Γ. By (0.3.9)

$$\lambda(\Gamma_\sigma, \Gamma')(a) - \lambda(\Gamma_\sigma, \Gamma)(a) = \lambda(\Gamma, \Gamma')(a) - d\lambda(\Gamma_\sigma, \Gamma, \Gamma')(a)$$

for all a. Thus we get

$$[\lambda(\Gamma_\sigma, \Gamma)(a) + I_F^k] = [\lambda(\Gamma_\sigma, \Gamma')(a) + I_F^k]$$

since $\lambda(\Gamma, \Gamma')(a) \in I_F^k$ if $a \in I^k(G)$ (cf. Lemma 0.3.3).

(ii) Independence of the choice of the reducible connection Γ_σ. Let Γ_σ' be

a connection in $P(M, G)$ reducible to a connection $\hat{\Gamma}'_\sigma$ in $P_\sigma(M, H)$ and different from Γ_σ. By Proposition 0.3.2, $\lambda(\Gamma_\sigma, \Gamma'_\sigma) = \lambda(\hat{\Gamma}_\sigma, \hat{\Gamma}'_\sigma) \circ i^*_{G,H}$ and so

$$\lambda(\Gamma'_\sigma, \Gamma)(a) - \lambda(\Gamma_\sigma, \Gamma)(a) = d\lambda(\Gamma_\sigma, \Gamma'_\sigma, \Gamma)(a) - \lambda(\Gamma_\sigma, \Gamma'_\sigma)(a)$$

$$= d\lambda(\Gamma_\sigma, \Gamma'_\sigma, \Gamma)(a)$$

for $a \in \ker i^*_{G,H}$ (cf. (0.3.9)). This proves

$$[\lambda(\Gamma_\sigma, \Gamma)(a) + I^k_F] = [\lambda(\Gamma'_\sigma, \Gamma)(a) + I^k_F] \quad \text{if} \quad a \in I^k(G).$$

(iii) Independence of the choice of the reducing section $\sigma \in \xi$. Let τ: $M \times R \to P/H$ be a homotopy between $\sigma = \tau(\cdot, 0)$ and $\sigma' = \tau(\cdot, 1), \sigma' \in \xi$. τ determines a section

$$(\tau, \mathrm{pr}_R): M \times R \to (P \times R)/H,$$

with $(P \times R)(M \times R, G)$ being the principal bundle induced from $P(M, G)$ by the projection $M \times R \to M$. Let Γ_σ be a connection in the bundle $(P \times R)$ $(M \times R, G)$ reducible to the subbundle $(P \times R)_{(\tau, \mathrm{pr}_R)}(M \times R, H)$. Let i_t and $\bar{i}_t$, $t \in R$, be the injections $M \ni p \mapsto (p, t) \in M \times R$ and $P \ni u \mapsto (u, t) \in P \times R$, respectively. By Proposition 0.2.8, the principal bundle homomorphism $(\bar{i}_t, i_t)$ pulls Γ_τ back to some connection Γ^t_τ in $P(M, G)$, reducible to $P_{\tau(\cdot, t)}(M, H)$, for all $t \in R$. Moreover, if ω_τ and ω^t_τ are the connection forms of Γ_τ and Γ^t_τ respectively, then $\omega^t_\tau = \bar{i}^*_t \omega_\tau$ for $t \in R$. We extend Γ^t_τ trivially to some connection Γ'_τ in $(P \times R)(M \times R, G)$ such that its connection form ω'_τ satisfies $\omega'_\tau = \mathrm{pr}^*_P \omega^1_\tau$, pr_P being the projection $P \times R \to P$. Γ'_τ is reducible to $(P_{\sigma'} \times R)(M \times R, H)$.

Since Γ_τ and Γ'_τ are both reducible to principal bundles with structure group H,

$$d\lambda(\Gamma_\tau, \Gamma'_\tau)(a) = \lambda(\Gamma'_\tau)(a) - \lambda(\Gamma_\tau)(a) = 0$$

for $a \in \ker i^*_{G,H}$. Furthermore, by Proposition 0.3.2, we get

$$i^*_0 \circ \lambda(\Gamma_\tau, \Gamma'_\tau) = \lambda(\Gamma^0_\tau, \Gamma^1_\tau), \quad \text{whereas} \quad i^*_1 \circ \lambda(\Gamma_\tau, \Gamma'_\tau) = \lambda(\Gamma^1_\tau, \Gamma^1_\tau) = 0$$

(observe that $\omega^1_\tau = i^*_t \omega'_\tau$ for all $t \in R$). In the de Rham cohomology, the induced homomorphisms $i^*_0, i^*_1: H^*(M \times R) \to H^*(M)$ coincide. Thus, for every $a \in \ker i^*_{G,H}$, $\lambda(\Gamma^0_\tau, \Gamma^1_\tau)(a)$ is an exact form. If, moreover, Γ is an adapted connection, then the form

$$\lambda(\Gamma^1_\tau, \Gamma)(a) - \lambda(\Gamma^0_\tau, \Gamma)(a) = -\lambda(\Gamma^0_\tau, \Gamma^1_\tau)(a) + d\lambda(\Gamma^0_\tau, \Gamma^1_\tau, \Gamma)$$

is also exact, which proves

$$[\lambda(\Gamma^0_\tau, \Gamma)(a) + I^k_F] = [\lambda(\Gamma^1_\tau, \Gamma)(a) + I^k_F],$$

for $a \in \ker i^*_{G,H} \cap I^k(G)$, $k = 1, 2, \ldots$ This concludes the proof of the lemma and of Theorem 2.3 as well.

DEFINITION 2.6. For any RF-bundle $B = (P(M, G), E, H, \xi)$ its *left* (*right*) '*secondary*' *characteristic homomorphism* $w'_2 = w'_2(B)$ (respectively, w''_2

$= w_2''(B))$ is the linear mapping constructed as in Lemma 2.5 (Lemma 2.4). The values of w_2' and w_2'' are 'secondary' characteristic invariants of B.

DEFINITION 2.7. For any RF-bundle $B = \big(P(M, G), E, H, \xi\big)$ its characteristic diagram is the diagram given by Theorem 2.3.

PROPOSITION 2.8. Let $B = \big(P(M, G), E, H, \xi\big)$ be any RF-bundle. If $w_1' = w_1'(B)$ and $w_2' = w_2'(B)$ are the left characteristic homomorphisms of B, then

$$w_2'(ab) = w_2'(a) \cdot w_1'(b) \quad \text{for } a \in \ker i_{G,H}^* \text{ and } b \in I(G),$$

where the multiplication is as follows:

(2.3)
$$H^{2i-1}\big(A^*(M)/I_F^i\big) \otimes H^{2j}(I_F^j) \to H^{2(i+j)-1}\big(A^*(M)/I_F^{i+j}\big),$$
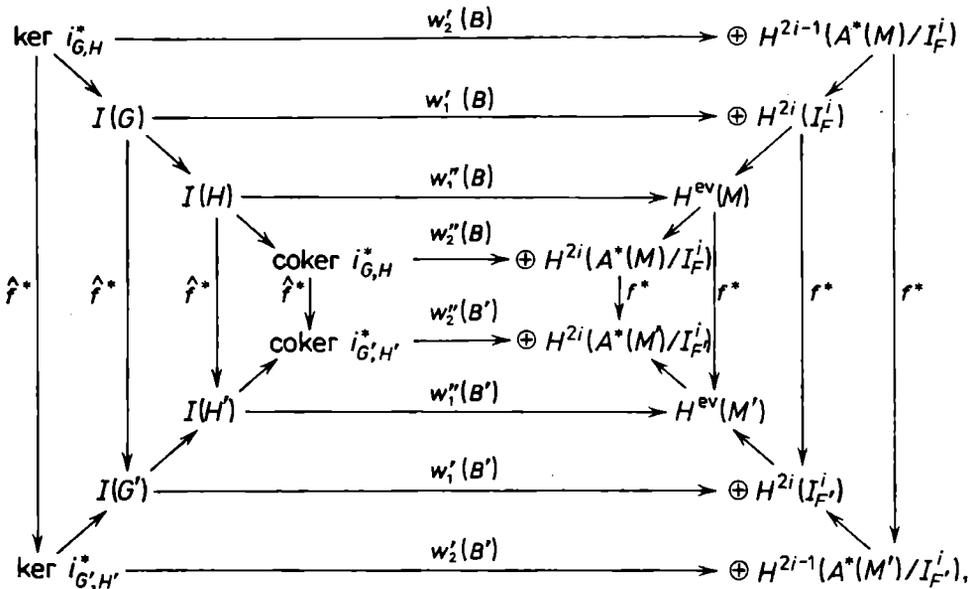$$[\varphi + I_F^i] \otimes [\psi]_j \mapsto [\varphi \wedge \psi + I_F^{i+j}],$$

for $i = 1, 2, \ldots$ and $j = 0, 1, \ldots$

Proof. Let Γ and Γ_σ be as in Theorem 2.3. By (0.3.10), $\lambda(\Gamma_\sigma, \Gamma)(ab) = \lambda(\Gamma_\sigma, \Gamma)(a) \wedge \lambda(\Gamma)(b) + \varphi$, where φ is an exact form on M, if $a \in \ker i_{G,H}^*$ and $b \in I(G)$. Consequently, we get

$$[\lambda(\Gamma_\sigma, \Gamma)(ab) + I_F^{i+j}] = [\lambda(\Gamma_\sigma, \Gamma)(a) \wedge \lambda(\Gamma)(b) + I_F^{i+j}]$$
$$= [\lambda(\Gamma_\sigma, \Gamma)(a) + I_F^i] \cdot [\lambda(\Gamma)(b)]_j$$

if $a \in I^i(G)$ and $b \in I^j(G)$. This ends the proof.

THEOREM 2.9. Let $(\bar{f}, f, \hat{f}): B' \to B$ be an RF-morphism of an RF-bundle $B' = \big(P'(M', G'), E', H', \xi'\big)$ into another RF-bundle $B = \big(P(M, G), E, H, \xi\big)$. Then, the diagram,

commutes. *Here the left "wall" is induced from* $\hat{f}^*: I(G) \to I(G')$ *and* $(\hat{f} \mid H')^*: I(H) \to I(H')$, *and the right "wall" is a direct sum of mappings induced, on the homology level, from the homomorphisms*

$$
\begin{array}{ccccccc}
0 \to & I_F^k & \hookrightarrow & A^*(M) & \to & A^*(M)/I_F^k & \to 0 \\
& \downarrow f^\circ \mid I_F^k & & \downarrow f^\circ & & \downarrow & \\
0 \to & I_{F'}^k & \hookrightarrow & A^*(M') & \to & A^*(M')/I_{F'}^k & \to 0
\end{array} \quad ,
$$

$k = 0, 1, \ldots$, *between short exact sequences of differential algebras, F and F' being the projected foliations of M and M' respectively. For the sake of clarity, all the mappings which form the "walls" are denoted by the same symbol* $-\hat{f}^*$ *and* f^* *respectively.*

Proof. What we need to prove is

$$f^* \circ w_2'(B) = w_2'(B') \circ \hat{f}^*,$$
$$f^* \circ w_1'(B) = w_1'(B') \circ \hat{f}^*,$$
$$f^* \circ w_1''(B) = w_1''(B') \circ \hat{f}^*,$$
$$f^* \circ w_2''(B) = w_2''(B') \circ \hat{f}^*.$$

We shall show that the above four mappings coincide with the appropriate characteristic homomorphisms of the RF-bundle $(f, F')^* B = (\bar{P}(M', G), \bar{E}, H, \bar{\xi})$ (cf. Proposition 1.7). Assume the notation of Proposition 1.9.

(i) Let Γ and Γ_σ be connections in $P(M, G)$ such that Γ is adapted to E and Γ_σ reduces to a connection $\hat{\Gamma}_\sigma$ in $P_\sigma(M, H)$, for some $\sigma \in \xi$. Let ω, ω_σ and $\hat{\omega}_\sigma$ be their respective connection forms. It follows from the proof of Proposition 1.7 that $\bar{\omega} := \bar{g}^* \omega$ is the connection form of some connection $\bar{\Gamma}$, adapted to $\bar{E}$, in the induced bundle $\bar{P}(M', G)$. By Proposition 1.7, $\bar{\sigma} := (\text{id}_{M'}, \sigma \circ f)$ is an element of $\bar{\xi}$. As $\bar{g}$ maps the total space $\bar{P}_{\bar{\sigma}}$ of the subbundle determined by $\bar{\sigma}$ into P_σ,

$$
\begin{aligned}
\bar{P}_{\bar{\sigma}} &= \{(p, u) \in M' \times P; \, f(p) = \pi(u) \text{ and } \bar{\sigma}(p) = (p, u)H\} \\
&= \{(p, u) \in M' \times P; \, f(p) = \pi(u) \text{ and } \sigma(f(p)) = uH\} \\
&= \bar{g}^{-1}(P_\sigma),
\end{aligned}
$$

the forms $\omega_{\bar{\sigma}} = \bar{g}^* \omega_\sigma$ and $\hat{\omega}_{\bar{\sigma}} = (\bar{g} \mid \bar{P}_{\bar{\sigma}})^* \hat{\omega}_\sigma$ are the connection forms of some connections $\Gamma_{\bar{\sigma}}$ and $\hat{\Gamma}_{\bar{\sigma}}$ in $\bar{P}(M', G)$ and $\bar{P}_{\bar{\sigma}}(M', H)$, respectively (cf. Proposition 0.2.8). Obviously, $\Gamma_{\bar{\sigma}}$ is an extension of $\hat{\Gamma}_{\bar{\sigma}}$. By Proposition 0.3.2, we get

$$\lambda(\Gamma_{\bar{\sigma}}, \bar{\Gamma}) = f^* \circ \lambda(\Gamma_\sigma, \Gamma),$$
$$\lambda(\bar{\Gamma}) = f^* \circ \lambda(\Gamma),$$

and

$$\lambda(\hat{\Gamma}_{\bar{\sigma}}) = f^* \circ \lambda(\hat{\Gamma}_\sigma).$$

Consequently

$$w_2'\big((f, F')^* B\big)(a) = [\lambda(\Gamma_{\bar{\sigma}}, \bar{\Gamma})(a) + I_{F'}^k] = [f^* \lambda(\Gamma_\sigma, \Gamma)(a) + I_{F'}^k]$$
$$= f^* [\lambda(\Gamma_\sigma, \Gamma)(a) + I_F^k] = f^* w_2'(B)(a)$$

for $a \in \ker i_{G,H}^* \cap I^k(G)$, $k = 1, 2, \ldots$,

$$w_1'\big((f, F')^* B\big)(a) = [\lambda(\bar{\Gamma})(a)]_k = [f^* \lambda(\Gamma)(a)]_k$$
$$= f^* [\lambda(\Gamma)(a)]_k = f^* w_1'(B)(a)$$

for $a \in I^k(G)$, $k = 0, 1, \ldots$,

$$w_1''\big((f, F')^* B\big)(b) = [\lambda(\hat{\Gamma}_{\bar{\sigma}})(b)] = [f^* \lambda(\hat{\Gamma}_\sigma)(b)] = f^* w_1''(B)(b)$$

for $b \in I(H)$, and

$$w_2''\big((f, F')^* B\big)(b + \operatorname{im} i_{G,H}^*) = [\lambda(\hat{\Gamma}_{\bar{\sigma}})(b) + I_{F'}^k]$$
$$= [f^* \lambda(\hat{\Gamma}_\sigma)(b) + I_{F'}^k]$$
$$= f^* [\lambda(\hat{\Gamma}_\sigma)(b) + I_F^k]$$
$$= f^* w_2''(B)(b + \operatorname{im} i_{G,H}^*)$$

for $b \in I^k(H)$, $k = 1, 2, \ldots$

(ii) For $\sigma' \in \xi'$, let Γ' and $\Gamma_{\sigma'}$ be connections in $P'(M', G')$ such that Γ' is adapted to E' and $\Gamma_{\sigma'}$ reduces to a connection $\hat{\Gamma}_{\sigma'}$ in $P'_{\sigma'}(M', H')$. Let ω', $\omega_{\sigma'}$ and $\hat{\omega}_{\sigma'}$ be their respective connection forms. Set

$$\bar{\sigma} = (\operatorname{id}_{M'}, \bar{f} / \circ \sigma'): \; M' \to f^* P/H,$$

so that $\bar{\sigma} \in \bar{\xi}$, by Proposition 1.9. The principal bundle homomorphism $((\pi', \bar{f}), \operatorname{id}_{M'}): P'(M', G') \to \bar{P}(M', G)$ carries $P'_{\sigma'}(M', H')$ into $\bar{P}_{\bar{\sigma}}(M', H)$. Namely, if $u \in P'_{\sigma'}$, then $(\pi'(u), \bar{f})(u) = (\pi'(u), \bar{f}(u)) \in \bar{P}_{\bar{\sigma}}$ since $(\bar{f} / \circ \sigma')(\pi'(u)) = \bar{f}/(uH') = \bar{f}(u) H$ and therefore $\bar{\sigma}(\pi'(u)) = (\pi'(u), \bar{f}(u)) H$. Thus we can transfer the connections Γ', $\Gamma_{\sigma'}$ and $\hat{\Gamma}_{\sigma'}$ from $P'(M', G')$ and $P'_{\sigma'}(M', H')$ to $\bar{P}(M', G)$ and $\bar{P}_{\bar{\sigma}}(M', H)$ respectively. Let $\bar{\Gamma}$, $\Gamma_{\bar{\sigma}}$ and $\hat{\Gamma}_{\bar{\sigma}}$ be the respective transferred connections and let $\bar{\omega}$, $\omega_{\bar{\sigma}}$ and $\hat{\omega}_{\bar{\sigma}}$ denote their connection forms.

By Propositions 1.8 and 1.9, $\bar{\Gamma}$ is adapted to $\bar{E}$. Obviously, $\Gamma_{\bar{\sigma}}$ is an extension of $\Gamma_{\bar{\sigma}}$. The connection forms are related as follows:

$$(\pi', \bar{f})^* \, \bar{\omega} = \hat{f}_* \circ \omega',$$

$$(\pi', \bar{f})^* \, \omega_{\bar{\sigma}} = \hat{f}_* \circ \hat{\omega}_{\sigma'},$$

and

$$((\pi', \bar{f}) \mid P'_{\sigma'})^* \, \hat{\omega}_{\bar{\sigma}} = \hat{f}_* \circ \hat{\omega}_{\sigma'}.$$

By Proposition 0.3.2, we get

$$\lambda(\Gamma_{\bar{\sigma}}, \bar{\Gamma}) = \lambda(\Gamma_{\sigma'}, \Gamma') \circ \hat{f}^*,$$

$$\lambda(\bar{\Gamma}) = \lambda(\Gamma') \circ \hat{f}^*$$

and

$$\lambda(\hat{\Gamma}_{\bar{\sigma}}) = \lambda(\hat{\Gamma}_{\sigma'}) \circ (\hat{f} \mid H')^*.$$

Consequently

$$w_2'\big((f, F')^* B\big)(a) = [\lambda(\Gamma_{\bar{\sigma}}, \bar{\Gamma})(a) + I_{F'}^k]$$

$$= [\lambda(\Gamma_{\sigma'}, \Gamma') \hat{f}^*(a) + I_{F'}^k] = w_2'(B') \hat{f}^*(a)$$

for $a \in \ker i_{G,H}^* \cap I^k(G)$, $k = 1, 2, \ldots$,

$$w_1'\big((f, F')^* B\big)(a) = [\lambda(\bar{\Gamma})(a)]_k = [\lambda(\Gamma') \hat{f}^*(a)]_k = w_1'(B') \hat{f}^*(a)$$

for $a \in I^k(G)$, $k = 0, 1, \ldots$,

$$w_1''\big((f, F')^* B\big)(b) = [\lambda(\hat{\Gamma}_{\bar{\sigma}})(b)] = [\lambda(\hat{\Gamma}_{\sigma'})(\hat{f} \mid H')^*(b)] = w_1''(B')(\hat{f} \mid H')^*(b)$$

for $b \in I(H)$, and

$$w_2''\big((f, F')^* B\big)(b + \operatorname{im} i_{G,H}^*) = [\lambda(\hat{\Gamma}_{\bar{\sigma}})(b) + I_{F'}^k]$$

$$= [\lambda(\hat{\Gamma}_{\sigma'})(\hat{f} \mid H')^*(b) + I_{F'}]$$

$$= w_2''(B')\big((\hat{f} \mid H')^*(b) + \operatorname{im} i_{G',H'}^*\big)$$

for $b \in I^k(H)$, $k = 1, 2, \ldots$ This ends the proof of the theorem.

COROLLARY 2.10. *If* $B = \big(P(M, G), E, H, \xi\big)$ *is a reducible RF-bundle* (cf. Definition 1.6), *then its secondary characteristic homomorphisms vanish*:

$$w_2'(B) = 0 \quad and \quad w_2''(B) = 0.$$

Proof. The statement is obvious since $\ker i_{H,H}^* = 0 = \operatorname{coker} i_{H,H}^*$.

The secondary characteristic invariants are therefore obstructions to the reducibility of RF-bundles.

Our constructions are based on the existence of global connections in a principal bundle. In the complex analytic or algebraic case no global complex analytic or algebraic connection need exist. On the other hand,

however, such connections are always constructible locally. This is one of the motivations for a local construction of the primary characteristic homomorphisms. Here the word "local" means that the construction arises from a family of local connections in a principal bundle. Such a construction for the Chern–Weil homomorphism (= the right primary characteristic homomorphism) is well-known (cf. [8], [37]). The resulting homomorphism takes its values in the Čech–de Rham cohomology of the base space. We shall define in a similar way some Čech–de Rham-type cohomology groups which will be canonically isomorphic to $H^{i+k}(I_F^i)$, $i, k = 0, 1, \ldots$, for a given foliation F.

Let F be a foliation of a manifold M, k any nonnegative integer and $\mathscr{U} = \{U_\alpha\}_{\alpha \in A}$ an open covering of M. We define a double chain complex $\check{C}_{\mathscr{U}}^{*,*}(GI_F^k)$ where

$$\check{C}_{\mathscr{U}}^{p,q}(GI_F^k) = \prod_{(\alpha_0, \ldots, \alpha_p)} \Gamma(U_{\alpha_0} \cap \ldots \cap U_{\alpha_p}, GI_F^{k,k+q})$$

(cf. Section 0.1), the product being over all ordered $(p+1)$-tuples $(\alpha_0, \ldots, \alpha_p) \in A^{p+1}$ such that $U_{\alpha_0} \cap \ldots \cap U_{\alpha_p} \neq \emptyset$, and the differentials

$$d': \check{C}_{\mathscr{U}}^{p,q}(GI_F^k) \to \check{C}_{\mathscr{U}}^{p+1,q}(GI_F^k),$$

$$d'': \check{C}_{\mathscr{U}}^{p,q}(GI_F^k) \to \check{C}_{\mathscr{U}}^{p,q+1}(GI_F^k), \quad p, q = 0, 1, \ldots,$$

are as follows:

$$(d'\varphi)_{(\alpha_0, \ldots, \alpha_{p+1})} = \sum_{i=0}^{p+1} (-1)^i \varphi_{(\alpha_0, \ldots, \hat{\alpha}_i, \ldots, \alpha_{p+1})} | U_{\alpha_0} \cap \ldots \cap U_{\alpha_{p+1}}$$

and

$$(d''\varphi)_{(\alpha_0, \ldots, \alpha_p)} = (-1)^p d\varphi_{(\alpha_0, \ldots, \alpha_p)}.$$

Notice that there is a canonical injection

$$I_F^{k,k+q} \hookrightarrow \prod_{\alpha \in A} \Gamma(U_\alpha, GI_F^{k,k+q}) = \check{C}_{\mathscr{U}}^{0,q}(GI_F^k)$$

for $q = 0, 1, \ldots$ By standard methods, we find the sequence

$$0 \to I_F^{k,k+q} \hookrightarrow \check{C}_{\mathscr{U}}^{0,q}(GI_F^k) \xrightarrow{d'} \check{C}_{\mathscr{U}}^{1,q}(GI_F^k) \xrightarrow{d'} \ldots,$$

exact for each q. Consequently, the chain mapping

$$e_{\mathscr{U}}: I_F^k \hookrightarrow \check{C}_{\mathscr{U}}^{0,*}(GI_F^k) \hookrightarrow \check{C}_{\mathscr{U}}^{*}(GI_F^k),$$

the last term being the total complex of $C_{\mathscr{U}}^{*,*}(GI_F^k)$, induces an isomorphism in homology (cf. [8], p. 14).

If a covering $\mathscr{V} = \{V_\beta\}_{\beta \in B}$ of M is finer than $\mathscr{U}$ and $\mu: B \to A$ is any function such that $V_\beta \subset U_{\mu(\beta)}$ for $\beta \in B$, then the formula

$$(\mu^*\varphi)_{(\beta_0, \ldots, \beta_p)} = \varphi_{(\mu(\beta_0), \ldots, \mu(\beta_p))} | V_{\beta_0} \cap \ldots \cap V_{\beta_p}$$

for $\varphi \in \check{C}_{\mathcal{U}}^{p,q}(GI_F^k)$ defines a chain homomorphism

$$\mu^*: \check{C}_{\mathcal{U}}^{*,*}(GI_F^k) \to \check{C}_{\mathcal{V}}^{*,*}(GI_F^k)$$

which satisfies $\mu^* \circ e_{\mathcal{U}} = e_{\mathcal{V}}$, and is such that the mapping

$$t_{\mathcal{V}}^{\mathcal{U}}: H\check{C}_{\mathcal{U}}^{*}(GI_F^k) \to H\check{C}_{\mathcal{V}}^{*}(GI_F^k)$$

induced from μ^* in the homology of the total complexes does not depend on the choice of μ. Moreover, if $\mathcal{W}$ is any covering of M finer than $\mathcal{V}$, then $t_{\mathcal{W}}^{\mathcal{U}} = t_{\mathcal{W}}^{\mathcal{V}} \circ t_{\mathcal{V}}^{\mathcal{U}}$. Obviously, $t_{\mathcal{U}}^{\mathcal{U}} = \mathrm{id}$. Thus the family of all the homology groups $H\check{C}_{\mathcal{U}}^{*}(GI_F^k)$, $\mathcal{U}$ being any open covering of M, together with the mappings $t_{\mathcal{V}}^{\mathcal{U}}$ for $\mathcal{V}$ finer than $\mathcal{U}$, form a direct system. Its limit $\check{H}_{\mathrm{deR}}^{*}(M; GI_F^k)$ is canonically isomorphic to $H^{*+k}(I_F^k)$ (cf. Proposition 0.1.7) and the isomorphism is established by the family of all the homomorphisms

$$e_{\mathcal{U}}^{*}: H^{*+k}(I_F^k) \xrightarrow{\cong} H\check{C}_{\mathcal{U}}^{*}(GI_F^k),$$

$\mathcal{U}$ being any open covering of M.

THEOREM 2.11. *Let* $(P(M, G), E)$ *be any foliated bundle and let* $\mathcal{U}$ $= \{U_\alpha\}_{\alpha \in A}$ *be an open covering of* M. *If* $\Gamma = \{\Gamma_\alpha\}_{\alpha \in A}$ *is a family of connections in the restricted bundles* $P(M, G)|U_\alpha$, $\alpha \in A$, *respectively, such that each* Γ_α *is adapted to the partial connection* $E|U_\alpha$, *then for any* $a \in I^k(G)$, $k = 0, 1, \ldots$, *the chain*

$$\lambda(\Gamma)(a) = \sum_{p=0}^{k} (-1)^{\binom{p}{2}} \lambda^p(\Gamma)(a) \in \bigoplus_{p=0}^{k} \check{C}_{\mathcal{U}}^{p,k-p}(GI_F^k) = \check{C}_{\mathcal{U}}^{k}(GI_F^k),$$

where

$$\lambda^p(\Gamma)(a)_{(\alpha_0,\ldots,\alpha_p)} = \lambda(\Gamma_{\alpha_0}, \ldots, \Gamma_{\alpha_p})(a),$$

is a cycle and its homology class in $H\check{C}_{\mathcal{U}}^{*}(GI_F^k)$ *coincides with* $e_{\mathcal{U}}^{*}(w_E(a))$.

Thus any family of local connections adapted to E suffices for determining w_E with values in $\bigoplus \check{H}_{\mathrm{deR}}^{k}(M; GI_F^k)$.

Proof of the theorem. First observe that $\lambda(\Gamma)(a)$ is a cycle for every $a \in I^k(G)$, $k = 0, 1, \ldots$ We have

$$\left(d'(-1)^{\binom{p-1}{2}} \lambda^{p-1}(\Gamma)(a)\right)_{(\alpha_0,\ldots,\alpha_p)}$$

$$= (-1)^{\binom{p-1}{2}} \sum_{i=0}^{p} (-1)^i \lambda(\Gamma_{\alpha_0}, \ldots, \hat{\Gamma}_{\alpha_i}, \ldots, \Gamma_{\alpha_p})(a)$$

$$= (-1)^{\binom{p-1}{2}} d\lambda(\Gamma_{\alpha_0}, \ldots, \Gamma_{\alpha_p})(a)$$

$$= -\left(d''(-1)^{\binom{p}{2}} \lambda^p(\Gamma)(a)\right)_{(\alpha_0,\ldots,\alpha_p)}$$

for $p = 1, \ldots, k$,

$$\left(d'' \lambda^0(\Gamma)(a)\right)_\alpha = d\lambda(\Gamma_\alpha)(a) = 0,$$

and

$$\left(d'\,\lambda^k(\Gamma)(a)\right)_{(\alpha_0,\ldots,\alpha_{k+1})} = d\lambda(\Gamma_{\alpha_0},\ldots,\Gamma_{\alpha_{k+1}})(a) = 0,$$

by (0.3.8). Consequently, we get

$$(d'+d'')\lambda(\Gamma)(a) = \sum_{p=1}^{k} d'(-1)^{\binom{p-1}{2}}\lambda^{p-1}(\Gamma)(a) + d''(-1)^{\binom{p}{2}}\lambda^p(\Gamma)(a) = 0.$$

Let Γ be a global connection adapted to E. We set

$$\lambda(\Gamma,\Gamma)(a) := \sum_{p=0}^{k-1} (-1)^{\binom{p+1}{2}}\lambda^p(\Gamma,\Gamma)(a) \in \bigoplus_{p=0}^{k-1} \check{C}_{\mathscr{H}}^{p,k-1-p}(GI_F^k) = \check{C}_{\mathscr{H}}^{k-1}(GI_F^k),$$

where

$$\lambda^p(\Gamma,\Gamma)(a)_{(\alpha_0,\ldots,\alpha_p)} = \lambda(\Gamma,\Gamma_{\alpha_0},\ldots,\Gamma_{\alpha_p})(a),$$

for $a\in I^k(G)$. We have

$$\left(d'(-1)^{\binom{p}{2}}\lambda^{p-1}(\Gamma,\Gamma)(a)\right)_{(\alpha_0,\ldots,\alpha_p)}$$

$$= (-1)^{\binom{p}{2}}\sum_{i=0}^{p}(-1)^i\lambda(\Gamma,\Gamma_{\alpha_0},\ldots,\hat{\Gamma}_{\alpha_i},\ldots,\Gamma_{\alpha_p})(a)$$

$$= (-1)^{\binom{p}{2}}\left(\lambda(\Gamma_{\alpha_0},\ldots,\Gamma_{\alpha_p})(a) - d\lambda(\Gamma,\Gamma_{\alpha_0},\ldots,\Gamma_{\alpha_p})(a)\right)$$

$$= \left((-1)^{\binom{p}{2}}\lambda^p(\Gamma)(a) - d''(-1)^{\binom{p+1}{2}}\lambda^p(\Gamma,\Gamma)(a)\right)_{(\alpha_0,\ldots,\alpha_p)}$$

for $p=1,\ldots,k-1$,

$$d'(-1)^{\binom{k}{2}}\lambda^{k-1}(\Gamma,\Gamma)(a) = (-1)^{\binom{k}{2}}\lambda^k(\Gamma)(a),$$

and

$$\left(d''\lambda^0(\Gamma,\Gamma)(a)\right)_{\alpha} = d\lambda(\Gamma,\Gamma_{\alpha})(a)$$

$$= \lambda(\Gamma_{\alpha})(a) - \lambda(\Gamma)(a)\,|\,U_{\alpha}$$

$$= \left(\lambda^0(\Gamma)(a) - e_{\mathscr{H}}\lambda(\Gamma)(a)\right)_{\alpha}.$$

Consequently, we obtain

$$(d'+d'')\lambda(\Gamma,\Gamma)(a) = \lambda(\Gamma)(a) - e_{\mathscr{H}}\lambda(\Gamma)(a) \quad \text{for} \quad a\in I^k(G),$$

which proves that the homology classes of $\lambda(\Gamma)(a)$ and $e_{\mathscr{H}}\lambda(\Gamma)(a)$ coincide. This concludes the proof.

Remark 2.12. It can be shown that each foliated bundle $(P(M,G),E)$ admits a family $\{\psi_\alpha\}_{\alpha\in A}$ of local trivializations $\psi_\alpha\colon \pi^{-1}(U_\alpha)\to U_\alpha\times G$ such that the G-coordinate of ψ_α is constant on the components of $L\cap\pi^{-1}(U)$ for any leaf L of the foliation of P determined by E. Then the connections Γ_α in $P(M,G)\,|\,U_\alpha$, $\alpha\in A$, which arise from the trivial flat connections in the bundles $(U_\alpha\times G)(U_\alpha,G)$, are all adapted to E. Applying Theorem 2.11 to the family $\{\Gamma_\alpha\}_{\alpha\in A}$, we are able to describe w_E in terms of the transition functions.

Remark 2.13. If G is a Lie group with a finite number of components and K is its maximal compact subgroup, then any principal bundle with structure group G admits a unique homotopy class of K-reductions (cf. [26], p. 58, Th. 5.7). Thus a foliated bundle $(P(M, G), E)$ gives rise to a unique RF-bundle $(P(M, G), E, K, \xi)$, and ξ consists of all sections $M \to P/K$. Since all the maximal compact subgroups of G are conjugate, the secondary characteristic invariants do not depend on the choice of K. Indeed, for any $g \in G$, the right multiplication $R_g \colon P \to P$, $u \mapsto ug$, and the adjoint automorphism $\mathrm{ad}(g^{-1})$ of G, $x \mapsto g^{-1} xg$, determine an RF-isomorphism

$$(R_g, \mathrm{id}_M, \mathrm{ad}(g^{-1})) \colon (P(M, G), E, K, \xi) \to (P(M, G), E, K', \xi'),$$

where $K' = g^{-1} Kg$ and ξ' is the class of all sections $M \to P/K'$.

Remark 2.14. If $(P(M, G), E)$ is a foliated bundle and Γ is a connection in $P(M, G)$ adapted to E, then the homomorphism $\lambda(\Gamma) \colon I(G) \to A^*(M)$ preserves the filtration:

$$\lambda(\Gamma) \mathscr{F}^{2p} I(G) \subset I_F^p \quad \text{for} \quad \mathscr{F}^{2p} I(G) = \bigoplus_{k \geqslant p} I^k(G), \quad p = 0, 1, \ldots$$

Here F stands for the projected foliation. Thus $\lambda(\Gamma)$ induces a map of the spectral sequences associated with the corresponding filtered complexes. The induced mappings are

$$\lambda^i \colon I^k(G) \to A^{i,k}/B^{i,k},$$

where $A^{i,k} = \mathrm{im}\big(H^{2k}(I_F^k/I_F^{k+1}) \to H^{2k}(I_F^k/I_F^{k+1})\big)$ and

$$B^{i,k} = \mathrm{im}\big(\partial \colon H^{2k-1}(I_F^{k+1-i}/I_F^k) \to H^{2k}(I_F^k/I_F^{k+1})\big)$$

for $i, k = 1, 2, \ldots$; $\lambda^i(a)$ is the image of $w_E(a)$ under the superposition $H^{2k}(I_F^k) \to A^{i,k} \to A^{i,k}/B^{i,k}$ for $a \in I^k(G)$. Thus those of the derived characteristic classes (for the definition, see [23]) which are determined by $\lambda(\Gamma)$ can be described in terms of the primary characteristic invariants.

For any RF-bundle its secondary and left primary characteristic homomorphisms take values in some homology groups which depend, in general, on the projected foliation. If one deforms the flat partial connection, leaving the projected foliation fixed, then the following question arises: what is the behaviour of the corresponding characteristic invariants?

Theorem 2.15. Let $B_t = (P(M, G), E_t, H, \xi)$, $t \in R$, be a family of RF-bundles such that all the projected foliations coincide with a given foliation F of M. Assume that, for any $u \in P$ and any vector $v \in E_F \cap T_{\pi(u)} M$, the partially horizontal lifts $\bar{v}_t \in E_t \cap T_u P$, $t \in R$, form a smooth curve in $T_u P$. Then

(i) the right characteristic homomorphisms $w_1''(B_t)$ and $w_2''(B_t)$ do not depend on t;

(ii) the superpositions

$$I^k(G) \xrightarrow{\;w_1'(B_t)\;} H^{2k}(I_F^k) \xrightarrow{\;\alpha_{k-1}^k\;} H^{2k}(I_F^{k-1})$$

and

$$\ker i_{G,H}^* \cap I^k(G) \xrightarrow{\;w_2'(B_t)\;} H^{2k-1}\big(A^*(M)/I_F^k\big) \xrightarrow{\;\beta_{k-1}^k\;} H^{2k-1}\big(A^*(M)/I_F^{k-1}\big),$$

where the mappings α_{k-1}^k and β_{k-1}^k are induced from the inclusion $I_F^k \subset I_F^{k-1}$, for $k = 1, 2, \ldots$, are independent of t.

Proof. Statement (i) is evident since $w_2''(B_t)$ is determined by $w_1''(B_t)$, which depends on ξ only.

It follows from the hypotheses that E_t, $t \in R$, can be joined to give a (smooth) subbundle E of $T(P \times R)$. Indeed, if $X_1, \ldots, X_q$ form a local basis of E_F over some open subset U of M, then their partially horizontal lifts $\bar{X}_1(t), \ldots, \bar{X}_q(t)$ form a basis of $E_t | \pi^{-1}(U)$ for $t \in R$. Since the lifts depend smoothly on t, they determine a basis of $E | \pi^{-1}(U) \times R$. Clearly, E is a flat partial connection in $(P \times R)(M \times R, G)$. Indeed, if Γ is a connection adapted to E, and ω and Ω are its connection form and its curvature form, respectively, then the canonical injection $(\bar{i}_t, i_t): P(M, G) \to (P \times R)(M \times R, G)$ pulls Γ back to a connection Γ_t in $P(M, G)$, for $t \in R$. Here $\bar{i}_t: P \to P \times R$ and $i_t: M \to M \times R$ put t at the second place. It follows from the fact that $\bar{i}_t^* \omega$ and $\bar{i}_t^* \Omega$ are, respectively, the connection form and the curvature form of Γ_t that Γ_t is adapted to E_t, and that

$$\Omega(v, w) = 0 \quad \text{if} \quad v, w \in E_{(u,t)}, \quad \text{for} \quad (u, t) \in P \times R.$$

That is why E is flat.

We extend the H-reduction structure (common for all B_t) trivially onto the bundle $(P \times R)(M \times R, G)$. Let $B = \big((P \times R)(M \times R, G), E, H, \eta\big)$ be the RF-bundle which embraces the above two structures. For every $t \in R$, we have $(\bar{i}_t, i_t, \mathrm{id}_G): B_t \to B$ and so, by Theorem 2.9,

$$i_t^* \circ w_1'(B) = w_1'(B_t)$$

and

$$i_t^* \circ w_2'(B) = w_2'(B_t).$$

To end the proof it suffices to show that the superpositions

$$\alpha_{k-1}^k \circ i_t^*: H^{2k}(I_{F \times R}^k) \to H^{2k}(I_F^{k-1})$$

and

$$\beta_{k-1}^k \circ i_t^*: H^{2k-1}\big(A^*(M \times R)/I_{F \times R}^k\big) \to H^{2k-1}\big(A^*(M)/I_F^{k-1}\big)$$

are independent of $t \in R$ for $k = 1, 2, \ldots$ Here $F \times R$ stands for the projected foliation of E — it consists of the sets $L \times \{t\} \subset M \times R$, where L is any leaf of F and t ranges over R. Observe that the ideal $I_{F \times R}$ is generated in $A^*(M \times R)$ by $\mathrm{pr}_M^* I_F$ and $dt = d\,\mathrm{pr}_R$, where $\mathrm{pr}_M: M \times R \to M$ and $\mathrm{pr}_R: M \times R \to R$ are the projections.

Any differential form $\varphi \in A^h(M \times R)$, $h = 0, 1, \ldots$, is uniquely expressible as

$$(2.4) \qquad\qquad \varphi = dt \wedge \psi + \bar\varphi,$$

where ψ and $\bar\varphi$ satisfy $\iota_T \psi = \iota_T \bar\varphi = 0$, T being the unit vector field on $M \times R$ tangent to all the lines $\{p\} \times R$, $p \in M$. Applying ι_T to both sides of (2.4), we see that $\psi = \iota_T \varphi$. Here ι denotes the interior product. $\bar\varphi = \varphi - dt \wedge \iota_T \varphi$ will be referred to as the free part of φ. For φ such that $\iota_T \varphi = 0$, we shall denote the free part of $d\varphi$ by $d'\varphi$ (cf. the proof of Lemma 0.1.6). Observe that $di_t^* \varphi = i_t^* d\varphi = i_t^* d'\varphi$ for $t \in R$.

Let $\varphi \in I_{F \times R}^{k,h}$ be a closed form, $k \geqslant 1$. By (2.4), we get

$$0 = -dt \wedge d'(\iota_T \varphi) + dt \wedge \iota_T d\bar\varphi + d'\bar\varphi.$$

This implies $d'(\iota_T \varphi) = \iota_T d\bar\varphi$, and so, for any $s \in R$,

$$(2.5) \qquad\qquad i_s^* d'(\iota_T \varphi) = i_s^* \iota_T d\bar\varphi.$$

The left-hand side of (2.5) is equal to $di_s^*(\iota_T \varphi)$ whereas the right-hand side satisfies

$$(i_s^* \iota_T d\bar\varphi)(X_1, \ldots, X_h)(p) = (d\bar\varphi)(T, Y_1, \ldots, Y_h)(p, s)$$

$$= \frac{\partial}{\partial t}\bigg|_s \bar\varphi(Y_1, \ldots, Y_h)(p, t)$$

$$= \frac{\partial}{\partial t}\bigg|_s (i_t^* \bar\varphi)(X_1, \ldots, X_h)(p)$$

$$= \frac{\partial}{\partial t}\bigg|_s (i_t^* \varphi)(X_1, \ldots, X_h)(p)$$

for $p \in M$ and any vector fields $X_1, \ldots, X_h$ on M, $Y_1, \ldots, Y_h$ being their extensions onto $M \times R$ such that $Y_{i(p,t)} = i_{t*} X_{ip}$ for all $t \in R$ and all i. Integrating both sides of (2.5) with respect to s, we get

$$(2.6) \quad i_{t_1}^* \varphi - i_{t_0}^* \varphi = \int_{t_0 \leqslant s \leqslant t_1} di_s^*(\iota_T \varphi) = d \int_{t_0 \leqslant s \leqslant t_1} i_s^*(\iota_T \varphi) \quad \text{for} \quad t_0 \leqslant t_1.$$

By Proposition 0.1.8, since

$$i_s^*(\iota_T \varphi)(v_1, \ldots, v_{h-1}) = \varphi(T_{(p,s)}, i_{s*}v_1, \ldots, i_{s*}v_{h-1}) = 0$$

if at least $h - 1 - (k-1) + 1 = h - k + 1$ of the vectors $v_1, \ldots, v_{h-1} \in T_p M$, $p \in M$, are tangent to F, we conclude that $i_s^*(\iota_T \varphi) \in I_F^{k-1,h-1}$. Thus, for any $t_0, t_1 \in R$, the difference (2.6) is a boundary in I_F^{k-1}, and consequently the mapping $\alpha_{k-1}^k \circ i_t^*$ is independent of t.

Now, let $\varphi \in A^{2k-1}(M \times R)$ satisfy $d\varphi \in I_{F \times R}^k$, $k \geqslant 1$. By (2.4)

$$d\varphi = dt \wedge \left(-d'(\iota_T \varphi) + \iota_t d\bar\varphi\right) + d'\bar\varphi.$$

As above, $d\varphi \in I^k_{F \times R}$ implies $i^*_s (\iota_T d\varphi) \in I^{k-1}_F$, which means that

$$i^*_s \left(-d'(\iota_T \varphi) + \iota_T d\bar{\varphi} \right) \in I^{k-1}_F,$$

and so

$$-di^*_s (\iota_T \varphi) + \frac{\partial}{\partial t}\bigg|_s (i^*_t \varphi) \in I^{k-1}_F$$

for $s \in R$. Consequently,

$$i^*_{t_1} \varphi - i^*_{t_0} \varphi = \int_{t_0 \leqslant s \leqslant t_1} \left(\frac{\partial}{\partial t}\bigg|_s (i^*_t \varphi) - di^*_s (\iota_T \varphi) \right) + d \int_{t_0 \leqslant s \leqslant t_1} i^*_s (\iota_T \varphi)$$

and

$$i^*_{t_1} \varphi - i^*_{t_0} \varphi \equiv d \int_{t_0 \leqslant s \leqslant t_1} i^*_s (\iota_T \varphi) \quad (\text{mod } I^{k-1}_F) \quad \text{for} \quad t_0 \leqslant t_1.$$

Finally, the homology class of $i^*_t \varphi + I^{k-1}_F$ does not depend on t. This concludes the proof of (ii) and of the theorem as well.

3. Pontrjagin classes of foliations

For any foliation F of codimension q of a manifold M, the dual normal principal bundle $LQ^*_F(M, \text{Gl}(q))$ of all linear frames in Q^*_F admits a canonical structure of a foliated bundle. The canonical flat partial connection in $LQ^*_F(M, \text{Gl}(q))$ is the Bott connection given by Definition 0.2.7. As the orthogonal group $O(q)$ is a maximal compact subgroup of the linear group $\text{Gl}(q)$, there is a unique homotopy class of $O(q)$-reductions of $LQ^*_F(M, \text{Gl}(q))$. Clearly, any $O(q)$-reduction corresponds to a fibre metric in Q^*_F and the correspondence is 1-1. If the foliation F is transversely orientable (which means that Q^*_F is an orientable vector bundle), then, on choosing an orientation form $\varepsilon \in \Gamma(\Lambda^q Q^*_F) \subset A^q(M)$, we fix a homotopy class of $SO(q)$-reductions. In any case, the bundle $LQ^*_F(M, G(q))$ is given an RF-bundle structure. The RF-bundles obtained in that way will be denoted by $B(F)$ and $B(F, \varepsilon)$ respectively.

DEFINITION 3.1. Let F be a foliation of codimension q of a manifold M. The *Pontrjagin classes of F* are the elements

$$p_i(F) = w'_1(B(F))(p_i) \in H^{2i}(I^i_F), \quad i = 0, 1, \ldots$$

The sum

$$p(F) = 1 + p_1(F) + \ldots + p_q(F) \in \bigoplus_i H^{2i}(I^i_F)$$

is a *total Pontrjagin of F*.

The secondary characteristic invariants

$$s_{2j+1}(F) := w_2'(B(F))(p_{2j+1}) \in H^{4j+1}(A^*(M)/I_F^{2j+1}), \quad j = 0, 1, \ldots,$$

of $B(F)$ are *secondary Pontrjagin classes of F*.

By Proposition 2.8 and the fact that $w_1'(B(F))$ is an algebra homomorphism, all the characteristic invariants of F (i.e. of $B(F)$) can be described in terms of the Pontrjagin classes and the secondary Pontrjagin classes of F.

DEFINITION 3.2. If the codimension q of F is even, F is a transversely orientable foliation and an orientation form ε is given, then the only nontrivial right characteristic invariant of $B(F, \varepsilon)$,

$$e(F, \varepsilon) := w_2''(B(F, \varepsilon))(\bar{e}) \in H^q(A^*(M)/I_F^{q/2}),$$

where $\bar{e} \in \operatorname{coker} i^*_{\text{Gl}(q), \text{SO}(q)}$ is the coset containing the Pfaffian e, is an *Euler class of the pair* (F, ε).

Obviously, $e(F, -\varepsilon) = -e(F, \varepsilon)$.

PROPOSITION 3.3. *If a smooth mapping* $f: M' \to M$ *is transverse to a foliation F of M, then there is*

$$p(f^*F) = f^*p(F)$$

and

$$s_{2j+1}(f^*F) = f^*s_{2j+1}(F) \quad for \quad j = 0, 1, \ldots$$

If F is transversely orientable and ε is an orientation form, then

$$e(f^*F, f^*\varepsilon) = f^*e(F, \varepsilon).$$

Proof. It is a straightforward consequence of Theorem 2.9 and the fact that, by Proposition 0.2.11, the triple $(\bar{f}, f, \operatorname{id}_{\text{Gl}(q)})$ is an RF-morphism of $B(f^*F)$ into $B(F)$ (of $B(f^*F, f^*\varepsilon)$ into $B(F, \varepsilon)$, in the oriented case), $\bar{f}$ being as in Remark 0.2.10, and $q = \operatorname{codim} F = \operatorname{codim} f^*F$.

PROPOSITION 3.4. *If a foliation F of a manifold M admits a bundle-like metric* (for the definition, see [33]), *then the secondary Pontrjagin classes of F vanish. If moreover, F is transversally orientable and its codimension is even, then also each of the Euler classes of F vanishes.*

Proof. A. Morgan ([31]) has proved that F admits a bundle-like metric if and only if there exists a Riemannian metric g on M such that its Levi–Civita connection Γ is adapted to the Bott connection E in $LQ_F^*(M, \text{Gl}(q))$. Since Γ is reducible to the subbundle of orthonormal frames, so is E. Thus the RF-bundle $B(F)$ ($B(F, \varepsilon)$ if ε is an orientation form) is reducible in the sense of Definition 1.6, and so we are able to apply Corollary 2.10 to it.

For any vector bundle Q over a manifold M, we shall identify local sections σ of the principal bundle $LQ(M, \text{Gl}(q))$ of linear frames in Q, q being the fibre dimension of Q, with sequences $(\sigma^1, \ldots, \sigma^q)$ of local sections of

Q such that $\sigma(p) = (p; \sigma^1(p), \ldots, \sigma^q(p))$ for all p, and call them local trivializations of Q.

LEMMA 3.5. *Let $LQ(M, \mathrm{Gl}(q))$ be the principal bundle of linear frames in a q-dimensional vector bundle Q over a manifold M. Let $\sigma = (\sigma^1, \ldots, \sigma^q)$ be a trivialization of Q over an open subset U of M. For any connection Γ in $LQ(M, \mathrm{Gl}(q))$ with the connection form ω, the corresponding covariant derivative ∇ in Q satisfies*

$$\nabla_v \sigma^i = \sum_j \sigma^*(\omega_i^j)(v) \cdot \sigma^j(p) \quad for \quad i = 1, \ldots, q \text{ and } v \in T_p M, \; p \in U.$$

Here $\omega = \sum \omega_j^i \otimes E_i^j$, where $E_i^j \in \mathfrak{gl}(q)$ is a matrix such that the entry at the i-th column and the j-th row is 1 and other entries are all zero for $i, j = 1, \ldots, q$.

The lemma is a straightforward consequence of [27], Chap. V.

Under the above assumption, we shall use the notation

$$\vartheta_j^i = \vartheta_{\sigma j}^i := \sigma^*(\omega_i^j) \quad for \quad i, j = 1, \ldots, q,$$

so

$$\nabla_v \sigma^i = \sum_j \vartheta_j^i(v)\sigma^j(p) \quad for \quad v \in TM, \; p = \pi(v), \text{ and all } i.$$

We shall denote the form

$$\sigma^* \omega = \sum_{i,j} \vartheta_j^i \otimes E_j^i$$

by $\vartheta = \vartheta_\sigma$ and refer to it as the coefficient form of ∇ (with respect to the local trivialization σ). For the curvature form $\Omega = \sum \Omega_i^j \otimes E_j^i$ of Γ, the forms

$$R_j^i = R_{\sigma j}^i := \sigma^* \Omega_i^j, \quad i, j = 1, \ldots, q,$$

satisfy

$$R_j^i = d\vartheta_j^i - \sum_k \vartheta_k^i \wedge \vartheta_j^k$$

for all i and j. We shall denote the form

$$\sigma^* \Omega = \sum_{i,j} R_j^i \otimes E_j^i$$

by $R = R_\sigma$ and refer to it as the curvature form of ∇ (with respect to σ). We shall use the matrix notation, $\vartheta = [\vartheta_j^i; i, j \leqslant q]$ and $R = [R_j^i; i, j \leqslant q]$.

Applying (0.3.2) to (0.3.7), we get $\lambda(\Gamma)(a)|U = a(R)$ for $a \in I(\mathrm{Gl}(q))$ and, if $\bar{\Gamma}$ is another connection in $LQ(M, \mathrm{Gl}(q))$ with the corresponding coefficient form $\bar{\vartheta}$ on U,

$$\lambda(\Gamma, \bar{\Gamma})(a)|U = \int_{0 \leqslant t \leqslant 1} ha(\bar{\vartheta} - \vartheta, R_t, \ldots, R_t) \quad for \quad a \in I^h(\mathrm{Gl}(q)), \; h = 0, 1, \ldots$$

Here $R_t = [R^i_{tj}; i, j \leqslant q]$, $R^i_{tj} = d\vartheta^i_{tj} - \sum \vartheta^i_{tk} \wedge \vartheta^k_{tj}$ and $\vartheta^i_{tj} = (1-t)\vartheta^i_j + t\bar\vartheta^i_j$ for $i, j = 1, \ldots, q$ and $t \in [0, 1]$. The above formulae are our motivation for writing $\lambda(V)$ and $\lambda(V, \bar V)$ instead of $\lambda(\Gamma)$ and $\lambda(\Gamma, \bar\Gamma)$ respectively, $\bar V$ being the covariant derivative in Q determined by $\bar\Gamma$, in the situation described above. Thus

$$(3.1) \qquad\qquad \lambda(V)(a) | U = a(R)$$

and

$$(3.2) \qquad\qquad \lambda(V, \bar V)(a) | U = \int\limits_{0 \leqslant t \leqslant 1} ha(\bar\vartheta - \vartheta, R_t, \ldots, R_t)$$

for $a \in I^h(\text{Gl}(q))$, $h = 1, 2, \ldots$

DEFINITION 3.6. Let Q be a vector bundle over a manifold M and $LQ(M, \text{Gl}(q))$ the frame bundle associated with Q, q being the fibre dimension of Q, and let E be a flat partial connection in the principal bundle $LQ(M, \text{Gl}(q))$. A covariant derivative V in Q is *adapted to* E if the connection determined in the frame bundle by V is adapted to E.

Sometimes a foliation F of codimension q of a manifold M is described by a family $\{f_U; U \in \mathscr{U}\}$ of submersions $f_U: M \supset U \to \mathbb{R}^q$ locally constant on the leaves of F, $\mathscr{U}$ being an open covering of M. For each $U \in \mathscr{U}$, the differentials df^i_U, $i = 1, \ldots, q$, of the coordinates of f_U form a local trivialization of Q^*_F.

LEMMA 3.7. *A covariant derivative V in Q^*_F, F being a foliation of codimension q of a manifold M, is adapted to the Bott connection in $LQ^*_F(M, \text{Gl}(q))$ if and only if, for any local trivialization $(\psi^1, \ldots, \psi^q)$ of Q^*_F composed of closed 1-forms on an open subset U of M, the coefficient form $\vartheta = [\vartheta^i_j; i, j \leqslant q]$ of V satisfies $\vartheta^i_j \in \Gamma(U, GI_F)$ for all i and j.*

Proof. If V is adapted to the Bott connection, then (0.2.3) holds: $V_v \varphi = \iota_v(d\varphi)$ for $v \in E_F$ and for all sections φ of Q^*_F. This implies $V_v \psi^i = 0$, and so $\vartheta^i_j(v) = 0$, for $v \in E_F | U$ and $i, j = 1, \ldots, q$. Consequently, all ϑ^i_j are local sections of GI_F.

To prove the converse, let us fix an arbitrary point $p \in M$. Assume that $\psi^1, \ldots, \psi^q$, $d\psi^i = 0$ for all i, form a basis of sections of Q^*_F over some neighbourhood U of p and that the corresponding coefficient form ϑ of V satisfies $\vartheta(v) = 0$ for $v \in E_F | U$. This means that $V_v \psi^i = 0$ for $v \in E_F | U$, $i = 1, \ldots, q$. If φ is a section of Q^*_F, then there are functions α_i, $i = 1, \ldots, q$, on U such that $\varphi | U = \sum \alpha_i \cdot \psi^i$, and so we get

$$V_v \varphi = V_v \sum_i \alpha_i \psi^i = \sum_i v(\alpha_i) \psi^i_p = \iota_v \sum_i d\alpha_i \wedge \psi^i = \iota_v(d\varphi)$$

for $v \in E_F \cap T_p M$. Since p is arbitrary, $V_v \varphi = \iota_v d\varphi$ for all $v \in E_F$. This concludes the proof.

We shall now examine the first Pontrjagin class $p_1(F) \in H^2(I_F)$ and the first secondary Pontrjagin class $s_1(F) \in H^1\big(A^*(M)/I_F\big)$.

Each foliation F of a manifold M determines a canonical 2-fold covering M_F of M. M_F is the unit 0-dimensional sphere bundle in $\Lambda^q Q_F^*$, $q = \operatorname{codim} F$, which means that

$$M_F = \Lambda^q Q_F^* / R^+,$$

where R^+ acts on $\Lambda^q Q_F^*$ by multiplication. Let $\pi_F \colon M_F \to M$ be the projection. We shall denote the canonical nontrivial involution in the fibres $\pi_F^{-1}(p)$, $p \in M$, by "$-$": if $\bar{p} \in \pi_F^{-1}(p)$, then $\pi_F^{-1}(p) = \{\bar{p}, -\bar{p}\}$.

Obviously, if F is transversely orientable, then M_F is a disjoint union of two copies of M. The induced foliation $\pi_F^* F$ of M_F is always transversely orientable.

DEFINITION 3.8. For any foliation F of a manifold M, a global nowhere zero section ω of the induced bundle $\pi_F^* \Lambda^q Q_F^*$, $q = \operatorname{codim} F$, is a 2-*folded orientation form of* F if $\omega(-\bar{p}) = -\omega(\bar{p})$ for $\bar{p} \in M_F$.

Any fibre metric g in $\Lambda^q Q_F^*$ determines a unique 2-folded orientation form ω of F such that for each unit covector $u \in \Lambda^q Q_F^*$, $\omega(uR^+) = u$. The correspondence $g \mapsto \omega$ is 1-1 and onto.

LEMMA 3.9. *For any 2-folded orientation form* ω *of a foliation* F *of a manifold* M *there is a 1-form* η *on* M *such that*

$$d\omega = \pi_F^* \eta \wedge \omega.$$

Proof. Let ω be a 2-folded orientation form of F. We fix a family $\{x_U; U \in \mathcal{U}\}$ of integrating charts $x_U = (x_U^1, \ldots, x_U^n) \colon U \to R^n$ for F, $n = \dim M$, $\mathcal{U}$ being an open covering of M. The forms $dx_U^1 \wedge \ldots \wedge dx_U^q$ are nowhere zero local sections of $\Lambda^q Q_F^*$ and, thus there are nonvanishing functions $f_U \colon \pi_F^{-1}(U) \to R$ such that

$$\omega \,|\, \pi_F^{-1}(U) = f_U \, \pi_F^*(dx_U^1 \wedge \ldots \wedge dx_U^q),$$

for $U \in \mathcal{U}$. Consequently,

$$d\omega \,|\, \pi_F^{-1}(U) = d\log|f_U| \wedge \omega|\pi_F^{-1}(U) \quad \text{for} \quad U \in \mathcal{U}.$$

Let $\{\varphi_U; U \in \mathcal{U}\}$ be a partition of unity subordinate to the covering $\mathcal{U}$. Since each f_U satisfies $f_U(-\bar{p}) = -f_U(\bar{p})$ for $\bar{p} \in M_F$, we have $|f_U| = \pi_F^* h_U$ for some function $h_U \colon U \to R^+$, $U \in \mathcal{U}$, and we get

$$d\omega = \pi_F^* \Big(\sum_{U \in \mathcal{U}} \varphi_U \, d\log h_U \Big) \wedge \omega.$$

This ends the proof.

LEMMA 3.10. *If ω is a 2-folded orientation form of a foliation F of a manifold M, and η is a 1-form on M such that $d\omega = \pi_F^* \eta \wedge \omega$, then*

(3.3) $$s_1(F) = [\eta + I_F] \in H_*^1\big(A^*(M)/I_F\big)$$

and therefore

$$p_1(F) = [d\eta]_1 \in H^2(I_F).$$

Proof. First observe that the class $[\eta + I_F]$ does not depend on the choice of ω and η. Indeed, if any other 1-form η' satisfies $d\omega = \pi_F^* \eta' \wedge \omega$, then $\pi_F^* (\eta - \eta') \wedge \omega = 0$ proves that $\eta - \eta' \in I_F$. On the other hand, for any two 2-folded orientation forms ω and ω' of F there is a function $f: M \to \boldsymbol{R} \setminus \{0\}$ such that $\omega' = \pi_F^* f \cdot \omega$. If $d\omega = \pi_F^* \eta \wedge \omega$, then

$$d\omega' = \pi_F^* df \wedge \omega + \pi_F^* f \cdot d\omega = \pi_F^* (d \log|f| + \eta) \wedge \omega.$$

Obviously, the homology classes $[\eta + I_F]$ and $[d \log|f| + \eta + I_F]$ in $H^1\big(A^*(M)/I_F\big)$ coincide.

Now, let g be a Riemannian metric on M and ω_g the corresponding 2-folded orientation form of F composed of all unit covectors in $\Lambda^q Q_F^*$. If $x = (x^1, \ldots, x^n): M \supset U \to \boldsymbol{R}^n$, $n = \dim M$, is any integrating chart for F, then

$$\omega_g(\varepsilon dx_p^1 \wedge \ldots \wedge dx_p^q \, \boldsymbol{R}^+) = \varepsilon(\det [g^{ij}(p); i,j \leqslant q])^{-1/2} dx_p^1 \wedge \ldots \wedge dx_p^q$$

for $p \in U$, $\varepsilon = \pm 1$. Here $g^{ij} = g(dx^i, dx^j)$ for $i, j \leqslant q$. Differentiating the above equality, we get

(3.4) $$d\omega_g|\pi_F^{-1}(U) = \pi_F^*(-\tfrac{1}{2}d \log \det [g^{ij}; i,j \leqslant q]) \wedge \omega_g|\pi_F^{-1}(U).$$

Let V^g be the covariant derivative in Q_F^* given by the orthogonal projection of the Levi–Civita covariant derivative $\bar{V}$ in T^*M onto $Q_F^* \subset T^*M$,

$$\bar{V}_v \varphi - V_v^g \varphi \in (Q_F^*)^\perp \quad \text{for} \quad v \in TM \text{ and } \varphi \in \Gamma(Q_F^*).$$

Since

$$\bar{V}_{\partial/\partial x^a} dx^b = -\Gamma_{ac}^b dx^c,$$

where Γ_{ac}^b, $a, b, c = 1, \ldots, n$, are the Christoffel symbols, the coefficient form $\bar{\vartheta} = [\bar{\vartheta}_c^b; b, c \leqslant n]$ of $\bar{V}$ satisfies $\bar{\vartheta}_c^b = -\Gamma_{ac}^b dx^a$ for all b and c. Here and below we use the classical Einstein summation convension. We assume that the indices a, b, c, ... run from 1 to n, whereas the indices i, j, k, ... run from 1 to q.

It follows from the definition that

$$\vartheta_{gj}^i = \bar{g}_{jk} g^{ak} \bar{\vartheta}_a^i, \quad i,j = 1, \ldots, q,$$

for the coefficient form $\vartheta_g = [\vartheta_{gj}^i; i,j \leqslant q]$ of V^g. Here

$$[\bar{g}_{ij}; i,j \leqslant q] := [g^{ij}; i,j \leqslant q]^{-1}.$$

Consequently,

$$(3.5) \qquad \vartheta^i_{gi} = -\tfrac{1}{2}\bar{g}_{ij}\,g^{aj}\,g^{ib}\,(g_{ab|c}+g_{bc|a}-g_{ac|b})\,dx^c$$

$$= -\tfrac{1}{2}\bar{g}_{ij}\,g^{aj}\,g^{ib}\,g_{ab|c}\,dx^c$$

$$= \tfrac{1}{2}\bar{g}_{ij}\,g^{aj}\,g_{ab}\,g^{ib}_{|c}\,dx^c$$

$$= \tfrac{1}{2}\bar{g}_{ij}\,dg^{ij}$$

$$= \tfrac{1}{2}d\log\det\,[g^{ij};i,j\leqslant q].$$

According to Lemma 3.7, if V is a covariant derivative adapted to the Bott connection in $LQ^*_F(M,\mathrm{Gl}(q))$ and $\vartheta = [\vartheta^i_j;i,j\leqslant q]$ is its coefficient form with respect to the section $(dx^1,\ldots,dx^q)\colon U\to LQ^*_F$, then (3.4) and (3.5) give

$$d\omega_g|\pi_F^{-1}(U) = \pi_F^*(\vartheta^i_i - \vartheta^i_{gi}) \wedge \omega_g|\pi_F^{-1}(U).$$

This means that $d\omega_g = \pi_F^*\lambda(V^g,V)(p_1)\wedge\omega_g$ on $\pi_F^{-1}(U)$ and so everywhere on M_F since U is the domain of an arbitrary integrating chart. Notice that $p_1 = \mathrm{trace}\colon \mathrm{gl}(q)\to \boldsymbol{R}$.

By definition, $s_1(F) = [\lambda(V^g,V)(p_1)+I_F]$. Consequently, (3.3) holds, as was to be shown.

THEOREM 3.11. *Let F be a foliation of a manifold M. The following conditions are equivalent*:

(i) $s_1(F) = 0$;

(ii) *F admits a closed 2-folded orientation form*;

(iii) *for any Riemannian metric g on M, there exists a positive function α on M such that the distribution (subbundle) orthogonal to E_F is minimal (i.e. its mean curvature tensor vanishes) with respect to the metric αg*;

(iv) *there is a Riemannian metric g on M such that the distribution orthogonal to E_F is minimal.*

If F is transversely orientable, then (ii) *means that F is defined by a closed nonvanishing q-form on M.*

Proof. (i) $\Rightarrow$ (iii) Let x,ω_g,V^g,V and the corresponding coefficient forms ϑ_g and ϑ be as in the proof of Lemma 3.10. According to (i), there is a real function f on M such that $d\lambda(V^g,V)(p_1)-df\in I_F$, and so

$$\vartheta^i_i - \vartheta^i_{gi} - df\,|\,U \in \Gamma(U,GI_F).$$

By Lemma 3.7, the above relation reduces to

$$\tfrac{1}{2}d\log\,[\det g^{ij};i,j\leqslant q]+df\,|\,U \in \Gamma(U,GI_F),$$

which immediately gives

$$d\big((\det\,[\mathbf{g}^{ij};i,j\leqslant q])^{1/2}\,e^f\big)\in\Gamma(U,GI_F).$$

Thus the Riemannian metric αg, $\alpha = e^{-2f/q}$, satisfies

$$d\left(\det\left[(\alpha g)^{ij}; i,j \leqslant q\right]\right)^{-1/2} \wedge dx^1 \wedge \ldots \wedge dx^q = 0.$$

This means the 2-folded orientation form $\omega_{\alpha g}$ determined by the metric αg is closed over U. Since U is the domain of an arbitrary integrating chart,

$$d\omega_{\alpha g} = 0.$$

By [1], this is equivalent to the minimality (with respect to αg) of the orthogonal complement of E_F in TM.

(iii) $\Rightarrow$ (iv) It is evident.

(iv) $\Rightarrow$ (ii) It is a straightforward consequence of [1].

(ii) $\Rightarrow$ (i) It follows from (3.3).

The first secondary Pontrjagin class $s_1(F)$ appears in [38], where the equivalence (i) $\Leftrightarrow$ (ii) is also pointed out.

THEOREM 3.12. *Let F be a foliation of a manifold M. The following conditions are equivalent*:

(i) $p_1(F) = 0$;

(ii) *for any 2-folded orientation form ω of F, there is a closed 1-form η on M such that*

$$d\omega = \pi_F^* \eta \wedge \omega;$$

(iii) *there is a family $\{f_U; U \in \mathcal{U}\}$ of submersions $f_U = (f_U^1, \ldots, f_U^q): U \to \mathbf{R}^q$ locally constant on the leaves of F, $q = \operatorname{codim} F$, such that the "transition functions" k_{UV}, $U, V \in \mathcal{U}$, defined by*

$$(3.6) \qquad df_U^1 \wedge \ldots \wedge df_U^q = k_{UV}\, df_V^1 \wedge \ldots \wedge df_V^q \qquad on \qquad U \cap V,$$

are locally constant, $\mathcal{U}$ being an open covering of M.

By [10], a foliation F of codimension 1 satisfies the condition (ii) if and only if it is transversely affine (cf. [10] for the definition).

Remark 3.13. Condition (ii) of Theorem 3.11 can be expressed as follows: there is such a family $\{f_U; U \in \mathcal{U}\}$ of submersions $f_U: U \to \mathbf{R}^q$ locally constant on the leaves of F that the "transition functions" k_{UV}, $U, V \in \mathcal{U}$, defined by (3.6), take the values ± 1 only.

Proof of Theorem 3.12. (i) $\Rightarrow$ (ii) Let ω be a 2-folded orientation form of F and assume that there is $d\omega = \pi_F^* \eta \wedge \omega$, $\eta \in A^1(M)$. By Lemma 3.10, if $p_1(F) = 0$, then $d\eta = d\varphi$ for some $\varphi \in I_F^{1,1}$. Thus $d\omega = \pi_F^*(\eta - \varphi) \wedge \omega$ and $d(\eta - \varphi) = 0$.

(ii) $\Rightarrow$ (iii) Let ω be a 2-folded orientation form of F, and $\{x_U; U \in \mathcal{U}\}$ a family of integrating charts $x_U = (x_U^1, \ldots, x_U^n): U \to \mathbf{R}^n$ for F, $n = \dim M$, $\mathcal{U}$ being an open covering of M. We assume $x_U(U) = (-1, 1)^n$ for all $U \in \mathcal{U}$.

For each $U \in \mathcal{U}$, there is a real function α_U on U such that

$$\omega(\varepsilon dx_{Up}^1 \wedge \ldots \wedge dx_{Up}^q \mathbf{R}^+) = \varepsilon \alpha_U(p)\, dx_{Up}^1 \wedge \ldots \wedge dx_{Up}^q$$

for $p \in U$, $\varepsilon = \pm 1$. Thus, $d\omega^* = \pi_F^* \eta \wedge \omega$ implies

$$\eta | U - d\log|\alpha_U| \in \Gamma(U, GI_F) \qquad for \qquad U \in \mathcal{U}.$$

If, moreover, $d\eta = 0$, then there are functions $\beta_U: U \to R$ such that $\eta|U = d\beta_U$ and $\beta_U - \beta_V$ is locally constant on $U \cap V$, for $U, V \in \mathscr{U}$.

For $U \in \mathscr{U}$, we shall modify the submersion $(x_U^1, \ldots, x_U^q): U \to R^q$ so as to obtain another one, $f_U = (f_U^1, \ldots, f_U^q)$, such that

(3.7) $df_U^1 \wedge \ldots \wedge df_U^q = \alpha_U e^{-\beta_U} dx_U^1 \wedge \ldots \wedge dx_U^q.$

If all f_U, $U \in \mathscr{U}$, satisfy (3.7), then

$$e^{\beta_U(p)} df_{U_p}^1 \wedge \ldots \wedge df_{U_p}^q = \omega(dx_{U_p}^1 \wedge \ldots \wedge dx_{U_p}^q, R^+)$$

$$= \omega\big(\varepsilon(p) dx_{V_p}^1 \wedge \ldots \wedge dx_{V_p}^q, R^+\big)$$

$$= \varepsilon(p) e^{\beta_V(p)} df_{V_p}^1 \wedge \ldots \wedge df_{V_p}^q$$

for $p \in U \cap V$, $U, V \in \mathscr{U}$. Here $\varepsilon = \operatorname{sign} \det[\partial x_U^i/\partial x_V^j : i, j \leqslant q]$. Thus (3.6) holds for $k_{UV} = \varepsilon e^{\beta_V - \beta_U}$ which is locally constant.

To get (3.7), let us observe that

$$d(\alpha_U e^{-\beta_U}) = \alpha_U e^{-\beta_U}(d \log|\alpha_U| - \eta|U) \in \operatorname{Lin}\{dx_U^1, \ldots, dx_U^q\}$$

and therefore the form

$$\varphi_U = \alpha_U e^{-\beta_U} dx_U^1 + \sum_{i=2}^{q} \left(t \mapsto \int_0^{t_1} \big((\alpha_U e^{-\beta_U}) \circ x_U^{-1}\big)_i(\tau, t_2, \ldots, t_n) d\tau\right) \circ x_U dx_U^i$$

is closed for $U \in \mathscr{U}$. Here $t = (t_1, \ldots, t_n)$ ranges over $x_U(U) = (-1, 1)^n$. Since elements of $\mathscr{U}$ are contractible, there exist functions $f_U^1: U \to R$ such that $df_U^1 = \varphi_U$, $U \in \mathscr{U}$. We set $f_U^i = x_U^i$ for $i = 2, \ldots, q$, $U \in \mathscr{U}$.

(iii) $\Rightarrow$ (i) Let $\{f_U; U \in \mathscr{U}\}$ be a family of submersions such as in (iii), and assume that ω is any 2-folded orientation form of F. Let $\alpha_U: U \to R$ be a function defined by

$$\omega(\varepsilon df_{U_p}^1 \wedge \ldots \wedge df_{U_p}^q, R^+) = \varepsilon \alpha_U(p) df_{U_p}^1 \wedge \ldots \wedge df_{U_p}^q$$

for $p \in U$ and $\varepsilon = \pm 1$, $U \in \mathscr{U}$. We have $\alpha_U(p) = \pm \alpha_V(p) k_{UV}(p)$ for $p \in U \cap V$, and so $|\alpha_U| = |\alpha_V| |k_{UV}|$ on $U \cap V$, $U, V \in \mathscr{U}$. Since the functions k_{UV} are locally constant, there is a global 1-form η on M such that $\eta|U = d \log|\alpha_U|$ for all U. Furthermore,

$$d\omega = \pi_F^* \eta \wedge \omega.$$

By Lemma 3.10, $p_1(F) = 0$ as $d\eta = 0$. This ends the proof.

If the dual normal bundle Q_F^* of a foliation F of a manifold M is trivial, then any homotopy class t of trivializations of Q_F^* together with the Bott connection E gives rise to a well-defined RF-bundle $B(F, t) = \big(LQ_F^*(M, \operatorname{Gl}(q)), E, \{I_q\}, t\big)$, $q = \operatorname{codim} F$. Here $I_q \in \operatorname{Gl}(q)$ stands for the unit matrix. This gives us an opportunity to define and study (left) characteristic invariants on the classical category of framed foliations.

DEFINITION 3.14. A *framed foliation* is a pair (F, t) composed of a foliation F and a homotopy class t of trivializations of Q_F^*.

DEFINITION 3.15. For any framed foliation (F, t) of a manifold M, its *secondary Pontrjagin classes* are the left secondary characteristic invariants

$$s_i(F, t) := w_2'\big(B(F, t)\big)(p_i) \in H^{2i-1}\big(A^*(M)/I_F^i\big), \quad i = 1, 2, \ldots,$$

of $B(F, t)$.

By Proposition 2.8, any other secondary characteristic invariant of $B(F, t)$ (and therefore of the framed foliation) can be described in terms of $s_i(F, t)$ and $p_j(F)$, $i, j = 1, 2, \ldots$

PROPOSITION 3.16. *If (F, t) is a framed foliation of a manifold M, and M' is another manifold and a mapping $f: M' \to M$ is transverse to F, then there is a canonical homotopy class $f^* t$ of trivializations of $Q_{f^*F}^*$ such that*

$$s_i(f^*F, f^* t) = f^* s_i(F, t) \quad for \quad i = 1, 2, \ldots$$

Proof. According to (0.2.5), if $\psi = (\psi^1, \ldots, \psi^q) \in t$, then $f^* \psi := (f^* \psi^1, \ldots, f^* \psi^q)$ is a trivialization of $Q_{f^*F}^*$. The homotopy class of $f^* \psi$ is independent of the choice of ψ — we shall denote it by $f^* t$. It follows from Lemma 0.2.9 that, if ψ is parallel with respect to a connection Γ in $LQ_F^*(M, \mathrm{Gl}(q))$, then $f^* \psi$ is parallel with respect to the connection induced from Γ by the canonical principal bundle homomorphism $(\bar{f}, f)$: $LQ_{f^*F}^*(M', \mathrm{Gl}(q)) \to LQ_F^*(M, \mathrm{Gl}(q))$ (cf. Remark 0.2.10). Consequently, $(\bar{f}, f, \mathrm{id}_{\mathrm{Gl}(q)})$: $B(f^*F, f^* t) \to B(F, t)$, and the proposition follows from Theorem 2.9.

Remark 3.17. It follows from the theorem of Frobenius that, for any trivialization $\psi = (\psi^1, \ldots, \psi^q)$ of Q_F^*, there are 1-forms ϑ_j^i, $i, j \leqslant q$, such that

$$d\psi^i = \vartheta_j^i \wedge \psi^j \quad for \quad i = 1, \ldots, q.$$

As before, we use the Einstein summation convention. Observe that the covariant derivative V in Q_F^* such that $\vartheta = [\vartheta_j^i; i, j \leqslant q]$ is its coefficient form (with respect to ψ) is adapted to the Bott connection. Indeed,

$$V_v \alpha_i \psi^i = v(\alpha_i) \psi_p^i + \alpha_i(p) \vartheta_j^i(v) \psi_p^j = \iota_v d(\alpha_i \psi^i)$$

for any real functions $\alpha_i \in \mathscr{F}(M)$, $i = 1, \ldots, q$, and all $v \in E_F$, $p = \pi(v)$ (cf. (0.2.3)). On the other hand, there is a flat covariant derivative in Q_F^* such that ψ is parallel, and so its coefficient form (with respect to ψ) is 0. Thus, by (3.2),

$$w_2'\big(B(F, t)\big)(a) = \Big[\int_{0 \leqslant \tau \leqslant 1} a(\vartheta, R_\tau, \ldots, R_\tau) + I_F^k \Big]$$

for $a \in I^k(\mathrm{Gl}(q))$, $k = 1, 2, \ldots$, where $R_\tau = [R_{\tau j}^i; i, j \leqslant q]$, $R_{\tau j}^i = \tau d\vartheta_j^i - \tau^2 \vartheta_k^i \wedge \vartheta_j^k$ for $\tau \in R$, and t is the homotopy class of ψ.

It follows from the above that the Pontrjagin classes and the secondary Pontrjagin classes of a framed foliation realize what has been postulated in

[34] — they are invariants of the foliation, described in terms of the coefficient forms ϑ_j^i, $i, j \leqslant q$. For some purposes, they seem to be the most elementary invariants.

PROPOSITION 3.18. *Let (F, t) be a framed foliation of a manifold M. The odd secondary Pontrjagin classes of (F, t) do not depend on t. More precisely,*

$$(3.8) \qquad s_{2k+1}(F, t) = s_{2k+1}(F) \qquad for \qquad k = 0, 1, \ldots$$

Proof. Let $\psi \in t$, $\psi = (\psi^1, \ldots, \psi^q)$, be a trivialization of Q_F^*. There is a unique fibre metric g in Q_F^* such that $(\psi_p^1, \ldots, \psi_p^q)$ is an orthonormal frame for every $p \in M$. In other words, the section $\psi \colon M \to LQ_F^*$ composed with the projection $LQ_F^* \to LQ_F^*/O(q)$ gives rise to a section $M \to LQ_F^*/O(q)$ of the associated quotient bundle. Thus the identity mapping determines an RF-morphism $B(F, t) \to B(F)$, and (3.8) follows directly from Theorem 2.9.

DEFINITION 3.19. If F and F_i', $i = 0, 1$, are foliations of a manifold M such that $Q_{F_0}^*$, $Q_{F_1}^* \subset Q_F^*$ and $\operatorname{codim} F_0' = \operatorname{codim} F_1'$, then F_0' and F_1' are *concordant over* F if there exists a foliation F' of $M \times \mathbf{R}$ such that

(i) the injections $i_s \colon M \to M \times \{s\} \hookrightarrow M \times \mathbf{R}$ are transverse to F' and $F_s' = i_s^* F'$ for $s = 0, 1$, and

(ii) one has $Q_{F'}^* \subset Q_{F \times \mathbf{R}}^*$, $F \times \mathbf{R} = \{L \times \{s\}; (L, s) \in F \times \mathbf{R}\}$ being a foliation of $M \times \mathbf{R}$.

Framed foliations (F_i', t_i), $i = 0, 1$, are *concordant over* F if there is a framed foliation (F', t) of $M \times \mathbf{R}$ such that F' satisfies (ii) and

(i') the injections i_s are transverse to F' and $(F_s', t_s) = (i_s^* F', i_s^* t)$ for $s = 0, 1$.

If F is a trivial pointwise foliation, then F_0' and F_1' are concordant over F if and only if they are concordant in the classical sense ([28]).

If F' and F are foliations of a manifold M such that $Q_{F'}^* \subset Q_F^*$, then Proposition 1.7(ii) applied to $B(F')$ ($B(F', t)$ if t is a homotopy class of trivializations of $Q_{F'}^*$), to the identity mapping $LQ_{F'}^*(M, \operatorname{Gl}(q')) \to LQ_{F'}^*(M, \operatorname{Gl}(q'))$, $q' = \operatorname{codim} F'$, and to the foliation F of M, give us a well-defined RF-bundle with F as the projected foliation. We shall denote that RF-bundle by $B(F')|F$ (respectively, by $B(F', t)|F$).

PROPOSITION 3.20. *If F' ((F', t)) and F are foliations of a manifold M such that $Q_{F'}^* \subset Q_F^*$, then*

$$w_1'\big(B(F')|F\big) = i_F^{F'} \circ w_1'\big(B(F')\big)$$

and

$$w_2'\big(B(F')|F\big) = i_F^{F'} \circ w_2'\big(B(F')\big)$$

(respectively, $w_2'\big(B(F', t)|F\big) = i_F^{F'} \circ w_2'\big(B(F', t)\big)$), where $i_F^{F'}$ stands for the suitable one from among the homomorphisms $\bigoplus H^{2i}(I_{F'}^i) \to \bigoplus H^{2i}(I_F^i)$ and $\bigoplus H^{2i-1}\big(A^*(M)/I_{F'}^i\big) \to \bigoplus H^{2i-1}\big(A^*(M)/I_F^i\big)$, induced from the inclusion $I_{F'} \hookrightarrow I_F$.

Proof. It is a straightforward consequence of Theorem 2.9.

THEOREM 3.21. (i) *If foliations F_0' and F_1' of a manifold M are concordant over a foliation F of M, then the superpositions*

(a) $$I^i\big(\mathrm{Gl}(q')\big) \xrightarrow{w_1(B(F_s'))} H^{2i}(I_{F_s'}^i) \xrightarrow{i_F^{F_s'}} H^{2i}(I_F^i) \xrightarrow{\alpha_{i-1}^i} H^{2i}(I_F^{i-1}),$$

and

(b) $$I^i\big(\mathrm{Gl}(q')\big) \cap \ker i_{\mathrm{Gl}(q'),O(q')}^{*} \xrightarrow{w_2(B(F_s'))} H^{2i-1}\big(A^*(M)/I_{F_s'}^i\big) \xrightarrow{i_F^{F_s'}}$$
$$\xrightarrow{i_F^{F_s'}} H^{2i-1}\big(A^*(M)/I_F^i\big) \xrightarrow{\beta_{i-1}^i} H^{2i-1}\big(A^*(M)/I_F^{i-1}\big),$$

where $q' = \mathrm{codim}\, F_s'$, are independent of $s = 0, 1$ for $i = 1, 2, \ldots$ Here α_{i-1}^i and β_{i-1}^i are the homomorphisms induced from the inclusion $I_F^i \hookrightarrow I_F^{i-1}$.

(ii) *If t_s is a homotopy class of trivializations of $Q_{F_s}^*$, $s = 0, 1$, and the framed foliations (F_0', t_0) and (F_1', t_1) are concordant over F, then the superposition*

(b′) $$I^i\big(\mathrm{Gl}(q')\big) \xrightarrow{w_2(B(F_s', t_s))} H^{2i-1}\big(A^*(M)/I_{F_s'}^i\big) \xrightarrow{i_F^{F_s'}} H^{2i-1}\big(A^*(M)/I_F^i\big) \xrightarrow{\beta_{i-1}^i}$$
$$\xrightarrow{\beta_{i-1}^i} H^{2i-1}\big(A^*(M)/I_F^{i-1}\big)$$

is also independent of $s = 0, 1$.

Proof. Let F' be a foliation of $M \times R$ realizing the concordance of F_0' and F_1' over F. Given a connection in the bundle $LQ_{F'}^*(M \times R, \mathrm{Gl}(q'))$, a parallel displacement along the lines $\{p\} \times R$, $p \in M$, give rise to a diffeomorphism $\alpha: LQ_{F_0'}^* \times R \xrightarrow{\cong} LQ_{F'}^*$ since $LQ_{F_0'}^*$ is canonically diffeomorphic to $i_0^* LQ_{F'}^*$ (cf. Remark 0.2.10). Consequently, we obtain a principal bundle isomorphism

$$(\alpha, \mathrm{id}_{M \times R}): (LQ_{F_0'}^* \times R)(M \times R, \mathrm{Gl}(q')) \xrightarrow{\cong} LQ_{F'}^*(M \times R, \mathrm{Gl}(q')).$$

Since $Q_F^* \subset Q_{F \times R}^*$, we have $E_{F \times R} \subset E_{F'} = \pi_* E$, where E is the Bott connection in $LQ_{F'}^*(M \times R, \mathrm{Gl}(q))$. Thus we may apply Proposition 1.7(ii) to the principal bundle homomorphism

$$(\alpha \circ \bar{i}_s, i_s): LQ_{F_0'}^*(M, \mathrm{Gl}(q')) \to LQ_{F'}^*(M \times R, \mathrm{Gl}(q')),$$

to the foliation F of M, and to the RF-bundle $\big(LQ_{F'}^*(M \times R, \mathrm{Gl}(q')), E, H, \xi\big)$, where $H = O(q')$ and ξ is the class of all $O(q')$-reductions (if F_i' is provided with a homotopy class t_i of trivializations of $Q_{F_i'}^*$, $i = 0, 1$, then H is the unit subgroup and $\xi = t$ is a homotopy class of trivializations of $Q_{F'}^*$ such that $t_0 = i_0^* t$ and $t_1 = i_1^* t$) for $s \in R$. As a result, we get a 1-parameter family of RF-bundles,

$$B_s = \big(LQ_{F_0'}^*(M, \mathrm{Gl}(q')), E_s, H, \xi_s\big), \quad s \in R.$$

Observe that ζ_s is independent of s. Indeed, if $\sigma\colon M\times R\to LQ^*_{F'}/H$ represents ζ, then ζ_s is represented by

$$\sigma_s\colon M\to LQ^*_{F'_0}/H, \qquad \sigma_s=\left(\pi,(\alpha\circ\bar i_s)/\right)^{-1}\circ(\mathrm{id}_M,\sigma\circ i_s),$$

where

$$(\mathrm{id}_M,\sigma\circ i_s)\colon M\to M\times_{(i_s,\pi)}LQ^*_{F'}/H$$

and $\left(\pi,(\alpha\circ\bar i_s)/\right)$ is the bundle diffeomorphism

$$LQ^*_{F'_0}/H\ni uH\mapsto\left(\pi(u),\alpha(u,s)\,H\right)\in M\times_{(i_s,\pi)}LQ_{F'}/H \qquad \text{for}\quad s\in R.$$

Consequently, all the sections σ_s, $s\in R$, are homotopic. As the partial connections E_s, $s\in R$, can be put together to obtain a partial connection in $(LQ^*_{F'_0}\times R)(M\times R,\mathrm{Gl}(q'))$, the family of all B_s, $s\in R$, satisfies the hypotheses of Theorem 2.15 and so the superpositions

(a) $$I^i(\mathrm{Gl}(q'))\xrightarrow{\ w'_1(B_s)\ }H^{2i}(I_F^i)\xrightarrow{\ \alpha^i_{i-1}\ }H^{2i}(I_F^{i-1}),$$

and

(b) $$I^i(\mathrm{Gl}(q'))\cap$$

$$\cap\ker i^*_{\mathrm{Gl}(q'),H}\xrightarrow{\ w'_2(B_s)\ }H^{2i-1}\left(A^*(M)/I_F^i\right)\xrightarrow{\ \beta^i_{i-1}\ }H^{2i-1}\left(A^*(M)/I_F^{i-1}\right),$$

$i=1,2,\ldots$, are independent of $s\in R$. It now suffices to observe that B_s is isomorphic to $B(F'_s)|F$ or to $B(F'_s,t_s)|F)$ for $s=0,1$. In conclusion, the theorem follows from Proposition 3.20.

Remark 3.22. The Bott characteristic homomorphism ([4]) for a foliation F of a manifold M maps the homology groups of some graded differential algebra WO_q, $q=\mathrm{codim}\,F$, (W_q for a framed foliation) into $H^*_{\mathrm{deR}}(M)$. One has

$$WO_q=R[c_1,\ldots,c_q]_q\otimes\Lambda(u_1,u_3,\ldots,u_{2[(q+1)/2]-1}),$$

where $\deg c_i=2i$, $\deg u_j=2j-1$, $dc_i=0$ and $du_j=c_j$ for j odd and all i. Here $R[c_1,\ldots,c_q]_q$ is the truncated algebra of polynomials modulo elements of degree greater than $2q$. Similarly,

$$W_q=R[c_1,\ldots,c_q]_q\otimes\Lambda(u_1,u_2,\ldots,u_q),$$

where $\deg c_i=2i$, $\deg u_i=2i-1$, $dc_i=0$ and $du_i=c_i$, for all i.

The exotic characteristic classes of the foliation (i.e. those which are not Pontrjagin classes of Q^*_F) are the values of the Bott characteristic homomorphism on the subspace of $H(WO_q)$ spanned by the monomials

$$c_I u_J=c_{i_1}\ldots c_{i_k}\otimes u_{j_1}\wedge\ldots\wedge u_{j_h},$$

where $I=(i_1,\ldots,i_k)$, $i_1\leqslant\ldots\leqslant i_k$, and $J=(j_1,\ldots,j_h)$, $j_1<\ldots<j_h$, are sequences such that

$$\sum_{s=1}^{k}i_s+j_1>q,$$

k and l being any positive integers. Obviously, all the Bott characteristic classes of a framed foliation are exotic.

Taking advantage of the products

$$\mu^{bd}: \; H^a(I_F^b) \otimes H^c(A^*(M)/I_F^d) \to H^{a+c}(I_F^b),$$

$$[\varphi]_b \otimes [\psi + I_F^d] \mapsto [\varphi \wedge \psi]_b,$$

defined for $b+d > q$, we can associate any monomial $c_I u_J$ with some class in $H^*(I_F^{|I|})$, $|I| = i_1 + \ldots + i_k$, in the way described below. We associate $c_I u_J \in WO_q$ with

$$(p_I s_J)(F) := \mu^{|I|,j_h}(\cdot, s_{j_h}(F)) \circ \ldots \circ \mu^{|I|,j_1}(\cdot, s_{j_1}(F))(p_I(F)),$$

where $p_I(F) = p_{i_1}(F) \ldots p_{i_k}(F)$. Similarly, for a framed foliation (F, t), we associate $c_I u_J \in W_q$ with

$$(p_I s_J)(F, t) := \mu^{|I|,j_h}(\cdot, s_{j_h}(F, t)) \circ \ldots \circ \mu^{|I|,j_1}(\cdot, s_{j_1}(F, t))(p_I(F)).$$

It follows from the construction of the Bott characteristic homomorphism for F and for (F, t) that the exotic characteristic class corresponding to $c_I u_J$ coincides with the image of $(p_I s_J)(F)$ and that of $(p_I s_J)(F, t)$, respectively, under the mapping $H^*(I_F^{|I|}) \to H^*_{deR}(M)$ induced from the inclusion $I_F^{|I|} \hookrightarrow A^*(M)$ for all admissible I and J. Roughly speaking, the exotic characteristic classes of F can be lifted, by using the Pontrjagin classes and the secondary Pontrjagin classes, from $H^*_{deR}(M)$ to $H^*(I_F^b)$ with b depending on the particular exotic class. Let us note here that the Pontrjagin classes of F are lifts (up to a real factor) of the Pontrjagin classes of Q_F^*.

Remark 3.23. If L is a leaf of a foliation F of a manifold M, then the inclusion $i_L: L \hookrightarrow M$ induces a homomorphism $i_L^*: H^a(A^*(M)/I_F^b) \to H^a_{deR}(L)$ for $a, b = 0, 1, \ldots$ Consequently, $i_L^* s_{2i+1}(F) \in H^{4i+1}_{deR}(L)$, $i = 0, 1, \ldots$ $(i_L^* s_j(F, t) \in H^{2j-1}_{deR}(L)$, $j = 1, 2, \ldots$, if F is framed by t) are well-defined cohomology classes. They generate the holonomy ring of the foliation F on the leaf L (cf. [13], [38]).

4. Relative Pontrjagin classes of foliations

For any two foliations F' and F of a manifold M, the conditions $E_F \subset E_{F'}$ and $Q_{F'}^* \subset Q_F^*$ are equivalent to each other and stipulate that the leaves of F' should be foliated by leaves of F.

Definition 4.1. For any two foliations F' and F of the same manifold, F is a *subfoliation of F'* if the leaves of F' are foliated by leaves of F. If this is the case, the pair (F', F) is called a *flag foliations*.

The notion of the Bott connection in $LQ_F^*(M, \mathrm{Gl}(q))$, for a foliation F of codimension q of a manifold M, can be generalized to the case of a flag of foliations.

Let (F', F) be an arbitrary but fixed flag of foliations of a manifold M. Consider the family of all the covariant derivatives V in the vector bundle

$$Q_{F/F'}^* := Q_F^*/Q_{F'}^*,$$

such that

(4.1) $V_v \tilde{\varphi} = [\iota_v(d\varphi)]^{\tilde{}}$ for $\varphi \in \Gamma(Q_F^*)$ and $v \in E_F$.

Here we use the symbol "$\tilde{}$" (tilde) to denote the projection of Q_F^* onto $Q_{F/F'}^*$. If $\varphi \in \Gamma(Q_{F'}^*)$, then we have $\iota_w(d\varphi) \in Q_F^*$ for all $w \in E_{F'}$, and so $[\iota_v(d\varphi)]^{\tilde{}} = 0$ for $v \in E_F \subset E_{F'}$. Moreover, the right-hand side of (4.1) behaves as covariant derivatives do, for it is equal to the coset of $\bar{V}_v \varphi$, $\bar{V}$ being any covariant derivative in Q_F^* adapted to the Bott connection.

By Lemma 3.5, the family of the corresponding connections in the frame bundle $LQ_{F/F'}^*(M, \mathrm{Gl}(q''))$ of $Q_{F/F'}^*$, $q'' = \mathrm{codim}\, F - \mathrm{codim}\, F'$, satisfies the hypotheses of Proposition 0.2.6. Thus there is a unique flat partial connection to which all the covariant derivatives are adapted and which projects onto E_F.

DEFINITION 4.2. For any flag (F', F) of foliations of a manifold M, the *Bott connection in the principal bundle* $LQ_{F/F'}^*(M, \mathrm{Gl}(q''))$, $q'' = \mathrm{codim}\, F - -\mathrm{codim}\, F'$, is the unique flat partial connection with F as the projected foliation and such that all the covariant derivatives adapted to it satisfy (4.1).

Providing $LQ_{F/F'}^*(M, \mathrm{Gl}(q''))$ with the canonical structure of all $O(q'')$-reductions (cf. Remark 2.13) together with the Bott connection we get an RF-bundle depending canonically on the flag (F', F). This RF-bundle will be denoted by $B(F/F')$.

Similarly, if $Q_{F/F'}^*$ is a trivial vector bundle, then the symbol $B(F/F', t)$ will denote the RF-bundle composed of the principal bundle, the Bott connection and of some homotopy class t of trivializations of $Q_{F/F'}^*$.

DEFINITION 4.3. For any flag (F', F) of foliations of a manifold M, the *relative Pontrjagin classes* $p_k(F/F')$, $k = 0, 1, \ldots$, and the *relative secondary Pontrjagin classes* $s_{2i+1}(F/F')$, $i = 0, 1, \ldots$, of F modulo F' are the left characteristic invariants,

$$p_k(F/F') = w_1'\big(B(F/F')\big)(p_k) \in H^{2k}(I_F^k),$$

and

$$s_{2i+1}(F/F') = w_2'\big(B(F/F')\big)(p_{2i+1}) \in H^{4i+1}\big(A^*(M)/I_F^{2i+1}\big),$$

$i, k = 0, 1, \ldots,$ of $B(F/F')$. The *total relative Pontrjagin class of F modulo F'* is the sum

$$p(F/F') := \sum_{k=0}^{q''} p_k(F/F') \in \bigoplus_k H^{2k}(I_F^k),$$

where $q'' = \operatorname{codim} F - \operatorname{codim} F'$.

If, moreover, the vector bundle $Q_{F/F'}^*$ is trivial and t is a homotopy class of trivializations of $Q_{F/F'}^*$, then the *relative secondary Pontrjagin classes of F modulo F' with respect to t* are defined and equal to the left secondary invariants

$$s_k(F/F', t) := w_2'\big(B(F/F', t)\big)(p_k) \in H^{2k-1}\big(A^*(M)/I_F^k\big), \quad k = 1, 2, \ldots,$$

of $B(F/F', t)$.

Obviously, as in Proposition 3.18, $s_{2i+1}(F/F', t) = s_{2i+1}(F/F')$ for all i.

The nature of the relative characteristic classes can be clarified as follows.

THEOREM 4.4. *Let (F', F) be a flag of foliations of a manifold M. For any leaf L of F', the characteristic invariants of the foliation $F|L$ of L, determined by leaves of F, coincide with the restrictions of the relative characteristic invariants of F modulo F' to L. Precisely, if $i_L\colon L \hookrightarrow M$ is the inclusion, then $i_L^*\colon I_F \to I_{F|L}$ induces homomorphisms*

$$i_L^*\colon \bigoplus_k H^{2k}(I_F^k) \to \bigoplus_k H^{2k}(I_{F|L}^k)$$

and

$$i_L^*\colon H^{2i-1}\big(A^*(M)/I_F^i\big) \to H^{2i-1}\big(A^*(L)/I_{F|L}^i\big), \quad i = 1, 2, \ldots,$$

and we have

$$i_L^* p(F/F') = p(F|L)$$

and

$$i_L^* s_{2j+1}(F/F') = s_{2j+1}(F|L), \quad j = 0, 1, \ldots$$

If, moreover, $Q_{F/F'}^*$ is a trivial vector bundle and t is a homotopy class of trivializations of this bundle, then t determines canonically a homotopy class $t|L$ of trivializations of $Q_{F|L}^*$ and

$$i_L^* s_k(F/F', t) = s_k(F|L, t|L), \quad k = 1, 2, \ldots$$

Proof. Observe that the vector bundle $Q_{F|L}^*$ is canonically isomorphic to the bundle $i_L^* Q_{F/F'}^*$ induced from $Q_{F/F'}^*$ by i_L. The isomorphism is established by the mapping α_L,

$$i_L^* Q_{F/F'}^* \ni (p, \tilde{\omega}) \mapsto \omega \circ i_{L*p} \in Q_{F|L}^*.$$

Here, as before, $\tilde{\omega}$ means the coset of $\omega \in Q_F^*$ in $Q_{F/F'}^* = Q_F^*/Q_{F'}^*$. Indeed, α_L is a monomorphism since $\omega \circ i_{L*p} = 0$ implies $\omega \in Q_{F'}$ and so $\tilde{\omega} = 0$. Since both

vector bundles are of the same dimension, this homomorphism is really an isomorphism. The inverse isomorphism gives rise to a bundle mapping $\hat{i}_L\colon Q^*_{F|L} \to Q^*_{F/F'}$ which covers i_L and is an isomorphism on the fibres. We use i_L to get a principal bundle homomorphism $(\hat{i}_L, i_L)\colon LQ^*_{F|L}(L, \mathrm{Gl}(q''))$ $\to LQ^*_{F/F'}(M, \mathrm{Gl}(q''))$, $q'' = \mathrm{codim}\, F - \mathrm{codim}\, F' = \mathrm{codim}\, F|L$. If Γ is a connection in $LQ_{F/F'}(M, \mathrm{Gl}(q''))$ adapted to the Bott connection and ∇ is the corresponding covariant derivative in $Q^*_{F/F'}$, then the connection Γ_L in $LQ^*_{F|L}(L, \mathrm{Gl}(q''))$, induced from Γ by $(\hat{i}_L, i_L)$, determines in $Q^*_{F|L}$ a covariant derivative ∇^L such that

$$\nabla^L_v \varphi = \alpha_L\big(\pi(v),\, \nabla_{i_{L*}v} \tilde{\psi}\big) \quad \text{for} \quad \varphi \in \Gamma(Q^*_{F|L}) \quad \text{and} \quad v \in TL,$$

$\tilde{\psi}$ being any section of $Q^*_{F/F'}$ satisfying $\varphi = i^*_L \psi$ (and, therefore, $\tilde{\psi} \circ i_L = \hat{i}_L \circ \varphi$) on a neighbourhood of the origin $\pi(v)$ of v (cf. Lemma 0.2.9). Consequently, if Γ is adapted to the Bott connection, then

$$\nabla^L_v \varphi = \alpha_L\big(\pi(v),\, (\iota_{i_{L*}v}\, d\psi)^{\sim}\big) = (\iota_{i_{L*}v}\, d\psi) \circ i_{L*\pi(v)} = \iota_v\, d(i^*_L \psi) = \iota_v\, d\varphi$$

for $v \in E_{F|L}$, and so Γ_L is also adapted to the Bott connection. Therefore, $(\hat{i}_L, i_L)$ maps the Bott connection in $LQ^*_{F|L}(L, \mathrm{Gl}(q''))$ to the Bott connection in $LQ^*_{F/F'}(M', \mathrm{Gl}(q''))$, and the theorem follows from Proposition 1.7(ii) and Theorem 2.9.

THEOREM 4.5. (i) *For any flag* (F', F) *of foliations of a manifold* M,

$$p(F) = i^{F'}_F\big(p(F')\big)\, p(F/F')$$

in the ring $\bigoplus H^{2k}(I^k_F)$, *and*

$$s_{2k+1}(F) = \sum_{i+j=k} i^{F'}_F\big(s_{2i+1}(F')\big)\, p_{2j}(F/F') + s_{2j+1}(F/F')\, i^{F'}_F\big(p_{2i}(F')\big)$$

for $k = 0, 1, \ldots$, *the multiplication in the formula for the secondary classes being as in* (2.3). *As before,* $i^{F'}_F$ *stands for the suitable mapping out of those induced from the inclusion* $I_{F'} \hookrightarrow I_F$.

(ii) *If* F' *is provided with a homotopy class* t' *of trivializations of* $Q^*_{F'}$, *then*

$$s_{2k+1}(F) = \sum_{i=1}^{2k+1} i^{F'}_F\big(s_i(F', t')\big)\, p_{2k+1-i}(F/F') + s_{2k+1}(F/F')$$

for $k = 0, 1, \ldots$

(iii) *If the flag* (F', F) *is provided with a homotopy class* t'' *of trivializations of* $Q^*_{F/F'}$, *then*

$$s_{2k+1}(F) = i^{F'}_F\big(s_{2k+1}(F')\big) + \sum_{i=0}^{2k} s_{2k+1-i}(F/F', t'')\, i^{F'}_F\big(p_i(F')\big)$$

for $k = 0, 1, \ldots$

(iv) *If* F, F' *and* (F', F) *are provided with homotopy classes* t, t' *and* t'' *of*

trivializations of Q_F^*, $Q_{F'}^*$ and $Q_{F/F'}^*$, respectively, and t, t' and t'' are related in the sense that there is a trivialization $\varphi = (\varphi^1, \ldots, \varphi^q)$ in t such that $\varphi' := (\varphi^1, \ldots, \varphi^{q'}) \in t'$ and $\varphi'' := (\tilde{\varphi}^{q'+1}, \ldots, \tilde{\varphi}^q) \in t''$, $q = \text{codim}\, F$ and $q' = \text{codim}\, F'$, then

$$s_k(F, t) = \sum_{i=1}^{k} i_F^{F'}\big(s_i(F', t')\big) p_{k-i}(F/F') + s_k(F/F', t'')$$

$$= i_F^{F'}\big(s_k(F', t')\big) + \sum_{i=0}^{k-1} s_{k-i}(F/F', t'')\, i_F^{F'}\big(p_i(F')\big)$$

for $k = 1, 2, \ldots$

Proof. Fix an arbitrary Riemannian metric g on M. Let $\bar{V}$ be any covariant derivative in Q_F^*. We shall denote by H the orthogonal complement of E_F in $E_{F'}$, by $E_{F'}^\perp$ the complement of $E_{F'}$ in TM, and by C the complement of $Q_{F'}^*$ in Q_F^*.

Starting from $\bar{V}$ we define another covariant derivative V in Q_F by putting

(4.2)
$$V_v \psi = \begin{cases} \iota_v d\psi & \text{for} \quad v \in E_F \\ \iota_v d\psi^- + \bar{V}_v \psi^\perp & \text{for} \quad v \in H, \\ \bar{V}_v \psi & \text{for} \quad v \in E_{F'}^\perp, \end{cases}$$

for each $\psi \in \Gamma(Q_F^*)$. Here $\psi^- \in \Gamma(Q_{F'}^*)$ and $\psi^\perp \in \Gamma(C)$ are the components of ψ.

Let us fix an arbitrary point p of M and consider any local trivialization $\varphi = (\varphi^1, \ldots, \varphi^q)$ of Q_F^* over a neighbourhood U of p, such that φ^i, $i = 1, \ldots, q'$, are local sections of $Q_{F'}^*$. We assume that the indices $i, j, \ldots$ range over the set $\{1, \ldots, q'\}$, and the indices $r, s, \ldots$, over the set $\{q'+1, \ldots, q\}$.

If $\vartheta = [\vartheta_b^a; a, b \leqslant q]$ is the coefficient form of V with respect to φ, then we have $\vartheta_r^i = 0$ for all i and r. Consequently, the curvature form $R = [R_b^a; a, b \leqslant q]$ of V satisfies

(4.3) $R_j^i = d\vartheta_j^i - \vartheta_k^i \wedge \vartheta_j^k$, $R_s^r = d\vartheta_s^r - \vartheta_t^r \wedge \vartheta_s^t$,   and $R_r^i = 0$,

for all i, j, r, and s.

We see that V restricted to sections of $Q_{F'}^*$ determines a covariant derivative V' in $Q_{F'}^*$ such that $\vartheta' := [\vartheta_j^i; i, j \leqslant q']$ is its coefficient form (with respect to $\varphi' := (\varphi^1, \ldots, \varphi^q)$) and its curvature form is $R' := [R_j^i; i, j \leqslant q']$. On the other hand, the formula

(4.4) $V_v'' \tilde{\psi} = (V_v \psi^\perp)^\sim$ for $\psi \in \Gamma(Q_F^*)$ and $v \in TM$,

defines a covariant derivative V'' in $Q_{F/F'}^*$. As before, $\psi^\perp \in \Gamma(C)$ denotes the component of ψ. We have

$$V_v'' \tilde{\varphi}^r = (V_v \varphi^r)^\sim = \big(\vartheta_i^r(v)\, \varphi_p^i + \vartheta_s^r(v)\, \varphi_p^s\big)^\sim = \vartheta_s^r(v)\, \tilde{\varphi}_p^s$$

for $v \in T_p M$, $p \in U$, and all r, which proves that $\vartheta'' := [\vartheta_s^r; q' < r, s \leqslant q]$ is the coefficient form of V'' (with respect to $\varphi'' := (\tilde{\varphi}^{q'+1}, \dots, \tilde{\varphi}^q)$). By (4.3), the curvature form R'' of V'' is equal to $[R_s^r; q' < r, s \leqslant q]$. Consequently, the coefficient form ϑ and the curvature form R of V can be written in the matrix form as follows:

$$(4.5) \qquad \vartheta = \left[\begin{array}{c|c} \vartheta' & 0 \\ \hline \dots & \vartheta'' \end{array} \right] \quad \text{and} \quad R = \left[\begin{array}{c|c} R' & 0 \\ \hline \dots & R'' \end{array} \right].$$

(i) By (4.5), in the algebra $A^*(U)$, we have

$$\det(I_q + R) = \det(I_{q'} + R') \wedge \det(I_{q''} + R'')$$

and therefore

$$\sum_{k=0}^{q} p_k(R) = \sum_{i=0}^{q'} p_i(R') \wedge \sum_{j=0}^{q''} p_j(R''),$$

where $q'' = q - q'$, and I_q, $I_{q'}$ and $I_{q''}$ are the unit matrices in $\mathrm{Gl}(q)$, $\mathrm{Gl}(q')$ and $\mathrm{Gl}(q'')$ respectively, considered as constant 0-forms on U. The above formula gives

$$(4.6) \qquad \sum_{k=0}^{q} \lambda(V)(p_k) = \sum_{i=0}^{q'} \lambda(V')(p_i) \wedge \sum_{j=0}^{q''} \lambda(V'')(p_j)$$

on U (cf. (3.1)) and so at p. Since $p \in M$ is taken arbitrarily, (4.6) holds everywhere on M.

According to (4.2) and (4.4), each of the covariant derivatives V, V', and V'', is adapted to the appropriate Bott connection, and so the forms $\lambda(V)(p_k)$, $\lambda(V')(p_i)$, and $\lambda(V'')(p_j)$, determine the Pontrjagin classes $p_k(F)$, $p_i(F')$ and $p_j(F/F')$, for $i, j, k = 0, 1, \dots$ By (4.6),

$$\lambda(V)(p_k) = \sum_{i+j=k} \lambda(V')(p_i) \wedge \lambda(V'')(p_j)$$

which proves

$$p_k(F) = \sum_{i+j=k} i_F^{F'}(p_i(F')) \, p_j(F/F'),$$

for $k = 0, 1, \dots$

To prove the formula for the secondary Pontrjagin classes let us take any covariant derivatives V'^g and $\bar{V}^g$ in Q_F^* and in its orthogonal complement C in Q_F^*, respectively, metric with respect to g. Using them, we construct another metric covariant derivative V^g in Q_F^*,

$$(4.7) \qquad \begin{array}{lll} V_v^g \psi = V_v'^g \psi & \text{for} & \psi \in \Gamma(Q_F^*) \text{ and } v \in TM, \quad \text{and} \\ V_v^g \psi = \bar{V}_v^g \psi & \text{for} & \psi \in \Gamma(C) \text{ and } v \in TM. \end{array}$$

The coefficient form ϑ_g of V^g with respect to φ can be written as

$$\vartheta_g = \left[\begin{array}{c|c} \vartheta'_g & 0 \\ \hline \cdots & \vartheta''_g \end{array}\right]$$

where ϑ'_g is the coefficient form of V'^g with respect to φ' and ϑ''_g can be regarded as the coefficient form of the covariant derivative V''^g in $Q^*_{F/F'}$ defined by

$$V''^g_v \tilde{\psi} = (\bar{V}^g_v \psi)^{\sim} \quad \text{for} \quad \psi \in \Gamma(C) \text{ and } v \in TM.$$

Obviously, V''^g is a metric covariant derivative.

For any real s, the form $\vartheta_s = s\vartheta + (1-s)\vartheta_g$ can be written as

$$\vartheta_s = \left[\begin{array}{c|c} \vartheta'_s & 0 \\ \hline \cdots & \vartheta''_s \end{array}\right],$$

where $\vartheta'_s = s\vartheta' + (1-s)\vartheta'_g$ and $\vartheta''_s = s\vartheta'' + (1-s)\vartheta''_g$ (cf. (4.5)). Consequently, the curvature form $R_s = [R^a_{sb}; a, b \leqslant q]$, $R^a_{sb} = d\vartheta^a_{sb} - \vartheta^a_{sc} \wedge \vartheta^c_{sb}$, can be written in a similar way,

$$R_s = \left[\begin{array}{c|c} R'_s & 0 \\ \hline \cdots & R''_s \end{array}\right],$$

where R'_s and R''_s correspond to ϑ'_s and ϑ''_s, respectively. We take advantage of the above matrix representations to obtain the identity

$$\det\left(I_q + R_s + ds \wedge (\vartheta - \vartheta_g)\right)$$
$$= \det\left(I_{q'} + R'_s + ds \wedge (\vartheta' - \vartheta'_g)\right) \wedge \det\left(I_{q''} + R''_s + ds \wedge (\vartheta'' - \vartheta''_g)\right)$$

over $U \times R$, ds being the differential of the projection onto R. Comparing the terms which involve ds, we get

$$kp_k(\vartheta - \vartheta_g, R_s, \ldots, R_s)$$
$$= \sum_{i+j=k} p_i(R'_s) \wedge jp_j(\vartheta'' - \vartheta''_g, R''_s, \ldots, R''_s) + ip_i(\vartheta' - \vartheta'_g, R'_s, \ldots, R'_s) \wedge p_j(R''_s)$$

for all k and all s. By (0.3.2), the above equality gives

(4.8) $kp_k[\omega - \omega_g, \Omega_s, \ldots, \Omega_s]$

$$= \sum_{i+j=k} p_i[\Omega'_s] \wedge jp_j[\omega'' - \omega''_g, \Omega''_s, \ldots, \Omega''_s] + ip_i[\omega' - \omega'_g, \Omega'_s, \ldots, \Omega'_s] \wedge p_j[\Omega''_s]$$

on U and so at p. Here ω, ω', ω'', ω_g, ω'_g and ω''_g are the connection forms of the connections defined by V, V', V'', V^g, V'^g and V''^g in the bundles $LQ^*_F(M, \mathrm{Gl}(q))$, $LQ^*_{F'}(M, \mathrm{Gl}(q'))$ and $LQ^*_{F/F'}(M, \mathrm{Gl}(q''))$, respectively, and Ω_s, Ω'_s and Ω''_s are the curvature forms of the connections whose connection forms are, respectively, $s\omega + (1-s)\omega_g$, $s\omega' + (1-s)\omega'_g$ and $s\omega'' + (1-s)\omega''_g$, for

$s \in R$. Since the point $p \in M$ is taken arbitrarily, (4.8) holds everywhere on M. By Proposition 0.3.4, integrating (4.8) (with k replaced by $2k+1$) with respect to the variable s over the unit segment, we get

$$\lambda(\nabla^g, \nabla)(p_{2k+1})$$

$$= \sum_{i+j=k} \lambda(\nabla')(p_{2i}) \wedge \lambda(\nabla''^g, \nabla'')(p_{2j+1}) + \lambda(\nabla'^g, \nabla')(p_{2i+1}) \wedge \lambda(\nabla'')(p_{2j}) + \chi_k,$$

where χ_k is an exact form on M for $k = 0, 1, \ldots$ This ends the proof of (i).

(ii) If t' is a homotopy class of trivializations of Q_F^*, represented, say, by $\varphi' = (\varphi^1, \ldots, \varphi^{q'})$, then we may assume that $g(\varphi^i, \varphi^j) = \delta^{ij}$, $i, j = 1, \ldots, q'$ (observe that, up till now, the metric g has been quite arbitrary). Now, if ∇'^g is such that $\nabla'^g \varphi^i = 0$ for all i, then the covariant derivative ∇^g, defined by (4.7), remains metric and induces a flat covariant derivative (namely, ∇'^g) in Q_F^*. According to Proposition 0.3.4, integrating (4.8) (with k replaced by $2k+1$), we get

$$\lambda(\nabla^g, \nabla)(p_{2k-1})$$

$$= \sum_{i=1}^{2k+1} \lambda(\nabla'^g, \nabla')(p_i) \wedge \lambda(\nabla'')(p_{2k+1-i}) + \lambda(\nabla''^g, \nabla'')(p_{2k+1}) + \mu_k,$$

where μ_k is an exact form on M for $k = 0, 1, \ldots$ This proves (ii).

(iii) If t'' is a homotopy class of trivializations of $Q_{F/F'}^*$ and $\varphi'' = (\tilde{\varphi}^1, \ldots, \tilde{\varphi}^{q''})$ is a representative of t'', $\varphi^1, \ldots, \varphi^{q''}$ being global sections of Q_F^*, then we may assume that $g(\varphi^u, \varphi^v) = \delta^{uv}$ for $u, v = 1, \ldots, q''$. Now, C coincides with the subbundle of Q_F^* spanned by $\varphi^1, \ldots, \varphi^{q''}$. We put $\bar{\nabla}^g \varphi^u = 0$ for all u and define ∇^g as in (4.7). The covariant derivative ∇''^g induced in $Q_{F/F'}^*$ is flat and so $\lambda(\nabla''^g)(p_j) = 0$, $j = 1, 2, \ldots$ Integrating (4.8) (with k replaced by $2k+1$), we get

$$\lambda(\nabla^g, \nabla)(p_{2k+1})$$

$$= \lambda(\nabla'^g, \nabla')(p_{2k+1}) + \sum_{i=0}^{2k} \lambda(\nabla)(p_i) \wedge \lambda(\nabla''^g, \nabla'')(p_{2k+1-i}) + \lambda_k,$$

where λ_k is an exact form on M. This proves (iii).

(iv) Let $\varphi = (\varphi^1, \ldots, \varphi^q)$ be as in the hypothesis. In order not to change the construction developed in the proof of (i) we assume that the metric g is chosen in such a way that φ is a field of orthonormal frames of Q_F^*. As ∇'^g and $\bar{\nabla}^g$ we take the flat covariant derivatives

$$\nabla'^g \varphi^i = 0 \quad \text{and} \quad \bar{\nabla}^g \varphi^r = 0 \quad \text{for} \quad i = 1, \ldots, q' \text{ and } r = q'+1, \ldots, q.$$

Thus ∇^g and ∇''^g are also flat and φ and φ'' are parallel global sections of $LQ_F^*(M, \mathrm{Gl}(q))$ and $LQ_{F/F'}^*(M, \mathrm{Gl}(q''))$ respectively. Consequently, the classes $s_k(F, t)$, $s_i(F', t')$ and $s_j(F/F', t'')$ can be obtained from the forms $\lambda(\nabla^g, \nabla)(p_k)$,

$\lambda(V'^g, V')(p_i)$ and $\lambda(V''^g, V'')(p_j)$, respectively, for $i, j, k = 1, 2, \ldots$ According to Proposition 0.3.4, integrating (4.8) with respect to the variable s over the unit segment, we get

$$\lambda(V^g, V)(p_k) = \sum_{i=1}^{k} \lambda(V'^g, V')(p_i) \wedge \lambda(V'')(p_{k-i}) + \lambda(V''^g, V'')(p_k) + \bar{\mu}_k$$

$$= \lambda(V'^g, V')(p_k) + \sum_{i=0}^{k-1} \lambda(V')(p_i) \wedge \lambda(V''^g, V'')(p_{k-i}) + \bar{\lambda}_k,$$

where $\bar{\mu}_k$ and $\bar{\lambda}_k$ are exact forms on M for $k = 1, 2, \ldots$ This proves (iv) and concludes the proof of the theorem.

Remark 4.6. It follows from Theorem 4.5 that all the relative characteristic invariants of F modulo F' can be described in terms of the characteristic invariants of F and of F'.

The total Pontrjagin class $p(F/F')$ can be computed from the formula

$$p(F/F') = p(F) i_F^{F'} (p(F'))^{-1}$$

(observe that $i_F^{F'} p(F') = 1 + i_F^{F'} p_1(F') + \ldots \in \bigoplus H^{2i}(I_F^i)$ is an invertible element). The formulae for the secondary Pontrjagin classes are far more complicated. For example,

$$s_1(F/F') = s_1(F) - i_F^{F'} s_1(F'),$$

$$s_3(F/F') = s_3(F) - s_1(F) i_F^{F'} p_2(F') - i_F^{F'} s_3(F') -$$
$$- i_F^{F'} s_1(F') \left[p_2(F) - p_1(F) i_F^{F'} p_1(F') + i_F^{F'} (p_1(F')^2 - 2p_2(F')) \right].$$

COROLLARY 4.7. *For a foliation F of a manifold M, if there exists a foliation F' of codimension q of M such that $p(F') = 1$ and all its leaves are foliated by leaves of F, then*

$$p_i(F) = 0 \quad for \quad \operatorname{codim} F - q < i \leqslant \operatorname{codim} F.$$

If, moreover, all the secondary Pontrjagin classes of F' vanish, then

$$s_{2k+1}(F) = 0 \quad for \quad \operatorname{codim} F - q < 2k+1 \leqslant \operatorname{codim} F.$$

On the other hand, if the quotient vector bundle $Q^*_{F/F'}$ is trivial, then

$$s_{2k+1}(F) = 0 \quad for \quad \max(q, \operatorname{codim} F - q) < 2k+1 \leqslant \operatorname{codim} F.$$

Proof. If F' exists, then Theorem 4.5(i) implies $p(F) = p(F/F')$ and so the highest Pontrjagin classes of F must vanish. If, moreover, $s_{2k+1}(F') = 0$ for all k, then also $s_{2k+1}(F) = s_{2k+1}(F/F') = 0$ for $2k+1 > \operatorname{codim} F - q$.

The condition $p(F') = 1$ together with the triviality of $Q^*_{F/F'}$ implies, by Theorem 4.5(iii),

$$s_{2k+1}(F) = s_{2k+1}(F/F') + i_F^{F'} s_{2k+1}(F').$$

For $2k+1 > \max(q, \operatorname{codim} F - q)$ both terms vanish. This ends the proof.

Corollary 4.7 gives us some conditions necessary, for example, for the existence of a closed nonvanishing global 1-form in I_F (the case $q = 1$ and $s_1(F') = 0$).

LEMMA 4.8. *Let (F', F) be a flag of foliations of a manifold M. Let $\{\psi_U; U \in \mathcal{U}\}$ be a family of local trivializations $\psi_U = (\psi_U^1, ..., \psi_U^q)$ of Q_F^* over U, such that $\psi_U^i \in \Gamma(U, Q_{F'}^*)$ for $i = 1, ..., q'$, $q' = \operatorname{codim} F'$, and all the forms $\psi_U^1, ..., \psi_U^q$ are closed for $U \in \mathcal{U}$, $\mathcal{U}$ being an open covering of M. For any two elements U, V of $\mathcal{U}$, there are real functions c_{UVb}^a on $U \cap V$, $a, b = 1, ..., q$, such that*

$$(4.9) \qquad \psi_U^a | U \cap V = \sum_{b=1}^q c_{UVb}^a \psi_V^b | U \cap V \quad for \quad a = 1, ..., q.$$

If for every $U, V \in \mathcal{U}$ we have $c_{UVs}^r = \operatorname{const.}$, $q' < r, s \leqslant q$, then $p(F/F') = 1$.

Proof. On applying the projection $Q_F^* \to Q_{F/F'}^*$ to (4.9), we get

$$\tilde{\psi}_U^r | U \cap V = \sum_{s=q'+1}^q c_{UVs}^r \tilde{\psi}_V^s | U \cap V \quad for \quad q' < r \leqslant q, U, V \in \mathcal{U}.$$

Since all the coefficients c_{UVs}^r are constant, the covariant derivative V'' in $Q_{F/F'}^*$, such that $V'' \tilde{\psi}_U^r = 0$ for all $r, U \in \mathcal{U}$, is well-defined. Obviously, it is adapted to the Bott connection. By (3.1), we have $\lambda(V'')(p_i) = 0$ for i positive, as all the coefficient forms of V'' are zero. This concludes the proof.

COROLLARY 4.9. *Let F be a foliation of codimension q of a manifold M. Let $0 = q_0 < q_1 < ... < q_{h-1} < q_h = q$ be an increasing sequence of integers, and h any positive integer. Assume that there is a family $\{\psi_U; U \in \mathcal{U}\}$, $\mathcal{U}$ being an open covering of M, of local trivializations $\psi_U = (\psi_U^1, ..., \psi_U^q): U \to LQ_F^*$ composed of closed forms, such that the transition functions c_{UVb}^a, $a, b = 1, ..., q$ (cf. (4.9)), can be written in the matrix form as follows:*

$$[c_{UVb}^a; a, b \leqslant q] = \begin{bmatrix} {}_1c_{UV} & 0 & 0 & ... & 0 \\ ... & {}_2c_{UV} & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & {}_hc_{UV} \end{bmatrix},$$

where ${}_rc_{UV} = [c_{UVb}^a; q_{r-1} < a, b \leqslant q_r]$ for $r = 1, ..., h$, and all the entries above the "diagonal" are zero:

$$c_{UVb}^a = 0 \quad if \quad a \leqslant q_r < b \quad for \ some \ r \leqslant h-1,$$

for all $U, V \in \mathcal{U}$. If all ${}_rc_{UV}$, $r = 1, ..., h$ and $U, V \in \mathcal{U}$, are constant, then $p(F) = 1$.

Proof. It follows from the hypotheses that there are foliations

$F_1, \ldots, F_{h-1}$ of codimension $q_1, \ldots, q_{h-1}$, respectively, such that $\psi_U^1, \ldots, \psi_U^{q_r}$ are sections of $Q_{F_r}^*$ for $U \in \mathcal{U}$ and $r = 1, \ldots, h-1$. By Theorem 4.5(i) and Lemma 4.8,

$$p(F) = i_F^{F_1}\big(p(F_1)\big) i_F^{F_2}\big(p(F_2/F_1)\big) \ldots i_F^{F_{h-1}}\big(p(F_{h-1}/F_{h-2})\big)\big(p(F/F_{h-1})\big) = 1.$$

Consider any three foliations F, F' and F'' of codimension, respectively, q, q' and q'' of a manifold M and assume them to satisfy $Q_F^* = Q_{F'}^* \oplus Q_{F''}^*$. In other words, the leaves of F' and the leaves of F'' are mutually transverse and intersect themselves along leaves of F. We shall write $F = F' \cap F''$ to denote such a situation. Observe that the vector bundles $Q_{F/F'}^*$ and $Q_{F''}^*$ are then canonically isomorphic.

PROPOSITION 4.10. *We have*

$$p(F/F') = i_F^{F''} p(F'')$$

and, for odd i,

$$s_i(F/F') = i_F^{F''} s_i(F'').$$

If t is a homotopy class of trivializations of $Q_{F''}^*$ and $\hat{t}$ is the corresponding homotopy class of trivializations of $Q_{F/F'}^*$, then

$$s_i(F/F', \hat{t}) = i_F^{F''} s_i(F'', t) \quad \text{for all } i.$$

Proof. If $\bar{V}$ and V are covariant derivatives in $Q_{F/F'}^*$ and $Q_{F''}^*$ respectively, corresponding to each other in the sense that $\bar{V}\tilde{\varphi} = (V\varphi)^{\sim}$ for all $\varphi \in \Gamma(Q_{F''}^*)$, then $\bar{V}$ satisfies (4.1) if V satisfies (0.2.3) (with F replaced by F''). This is a straightforward consequence of the inclusion $E_F \subset E_{F''}$. Equivalently, if V is adapted to the Bott connection, so is $\bar{V}$. Since $\lambda(\bar{V}) = \lambda(V)$, this implies the equality $p_k(F/F') = i_F^{F''} p_k(F'')$, $k = 0, 1, \ldots$

Evidently, the isomorphism $Q_{F''}^* = Q_{F/F'}^*$ transfers any metric (flat) covariant derivative V^1 from $Q_{F''}^*$ to $Q_{F/F'}^*$, and there is $\lambda(\bar{V}^1, \bar{V}) = \lambda(V^1, V)$, $\bar{V}^1$ being the corresponding metric (respectively, flat) covariant derivative in $Q_{F/F'}^*$. This proves the formulae for the secondary Pontrjagin classes.

COROLLARY 4.11. *Let F' and F'' be foliations of a manifold M, such that $Q_{F'}^* \cap Q_{F''}^* = 0$. Put $F = F' \cap F''$. We have*

$$p(F) = i_F^{F'}\big(p(F')\big) i_F^{F''}\big(p(F'')\big)$$

and

$$s_{2k+1}(F) = \sum_{i+j=k} i_F^{F'}\big(s_{2i+1}(F')\big) i_F^{F''}\big(p_{2j}(F'')\big) + i_F^{F''}\big(s_{2j+1}(F'')\big) i_F^{F'}\big(p_{2i}(F')\big)$$

for $k = 0, 1, \ldots$

If t' and t'' are homotopy classes of trivializations of $Q_{F'}^*$ and $Q_{F''}^*$

respectively, and t is the homotopy class of trivializations of $Q_F^* = Q_{F'}^* + Q_{F''}^*$ determined by t' and t'', then

$$s_k(F, t) = i_F^{F''}\left(s_k(F'', t'')\right) + \sum_{i=1}^{k} i_F^{F'}\left(s_i(F', t')\right) i_F^{F''}\left(p_{k-i}(F'')\right)$$

$$= \sum_{i=0}^{k-1} i_F^{F''}\left(s_{k-i}(F'', t'')\right) i_F^{F'}\left(p_i(F')\right) + i_F^{F'}\left(s_k(F', t')\right)$$

for $k = 1, 2, \ldots$

Proof. It is a straightforward consequence of Theorem 4.5 and Proposition 4.10.

Another consequence of Theorem 4.5 is the following

PROPOSITION 4.12. *If foliations F_0 and F_1 of a manifold M are concordant over a foliation F of M, then the superpositions*

$$I^i\left(Gl(q'')\right) \xrightarrow{w_1^{i}(B(F/F_s))} H^{2i}(I_F^i) \xrightarrow{\alpha_{i-1}^{i}} H^{2i}(I_F^{i-1}), \quad i = 1, 2, \ldots,$$

where $q'' = \operatorname{codim} F - \operatorname{codim} F_0$, do not depend on $s = 0, 1$.

Proof. As the superposed mappings are linear, it suffices to show the independence on a linear basis of $I\left(Gl(q'')\right)$.

By Theorem 4.5, $p(F/F_s) = p(F) i_F^{F_s}\left(p(F_s)\right)^{-1}$. Thus each class $p_i(F/F_s)$, and any monomial

$$p_I(F/F_s) = p_{i_1}(F/F_s) \cdots p_{i_r}(F/F_s), \quad I = (i_1, \ldots, i_r),$$

are linear combinations of classes of the shape $p_J(F) i_F^{F_s}\left(p_K(F_s)\right)$, where

$$p_J(F) = p_{j_1}(F) \ldots p_{j_m}(F) \quad \text{for} \quad J = (j_1, \ldots, j_m),$$

and

$$p_K(F_s) = p_{k_1}(F_s) \ldots p_{k_n}(F_s) \quad \text{for} \quad K = (k_1, \ldots, k_n),$$

I, J and K being any finite sequences of positive integers. Using the product

$$H^{2a}(I_F^a) \otimes H^{2b}(I_F^{b-1}) \ni [\varphi]_a \otimes [\psi]_{b-1} \mapsto [\varphi \wedge \psi]_{a+b-1} \in H^{2(a+b)}(I_F^{a+b-1}),$$

we get

$$\alpha_{|J|+|K|-1}^{|J|+|K|}\left(p_J(F) i_F^{F}\left(p_K(F_s)\right)\right) = p_J(F) \alpha_{|K|-1}^{|K|}\left(i_F^{F}\left(p_K(F_s)\right)\right)$$

$$= p_J(F) \alpha_{|K|-1}^{|K|}\left(i_F^{F} w_1'\left(B(F_s)\right)(p_{k_1} \cdots p_{k_n})\right),$$

which does not depend on s by Theorem 3.21. Consequently, for any sequence $I = (i_1, \ldots, i_r)$ of indices,

$$\alpha_{|I|-1}^{|I|}\left(w_1'\left(B(F/F_s)\right)(p_{i_1} \cdots p_{i_r})\right) = \alpha_{|I|-1}^{|I|}\left(p_I(F/F_s)\right)$$

is independent of $s = 0, 1$. This concludes the proof.

Suppose that we are given a flag (F', F) of foliations of a manifold M. G. Joubert, R. Moussu and D. Tischler ([17]) have constructed a number of conditions necessary for F to be an intersection $F_1 \cap F_2$ of two foliations of M. On the other hand, even if F is really an intersection of two foliations, the question "Does there exist a foliation F'' of M such that $F = F' \cap F''$?" remains open. The following theorem states that the answer is "no" unless some conditions are satisfied.

THEOREM 4.13 (generalized Bott vanishing theorem). *Let (F', F) be a flag of foliations of a manifold M.*

(i) *If there exists a foliation F'' of M such that $F = F' \cap F''$, then, for any sequence $(k_1, \ldots, k_r)$, $r = 2, 3, \ldots$, of positive integers, we have*

$$p_{k_1}(F/F') \ldots p_{k_r}(F/F') = 0 \quad if \quad k_1 + \ldots + k_r > \operatorname{codim} F - \operatorname{codim} F'.$$

(ii) *If F' is concordant over F with a foliation $\bar{F}$ such that $F = \bar{F} \cap F''$ for some foliation F'' of M, then, for any sequence $(k_1, \ldots, k_r)$, $r = 2, 3, \ldots$, of positive integers,*

$$\alpha_{k-1}^k \left(p_{k_1}(F/F') \ldots p_{k_r}(F/F') \right) = 0 \ if \ k = k_1 + \ldots + k_r > \operatorname{codim} F - \operatorname{codim} F'.$$

Proof. If there are foliations $\bar{F}$ and F'' of M such that $F = \bar{F} \cap F''$, then $p(F/\bar{F}) = i_F^{F''} p(F'')$, by Proposition 4.10. Consequently,

$$p_{k_1}(F/\bar{F}) \ldots p_{k_r}(F/\bar{F}) = i_F^{F''} \left(p_{k_1}(F'') \ldots p_{k_r}(F'') \right) = 0$$

for $k = k_1 + \ldots + k_r > \operatorname{codim} F - \operatorname{codim} \bar{F} = \operatorname{codim} F''$, as then $I_{F''}^k = 0$ and therefore

$$p_{k_1}(F'') \ldots p_{k_r}(F'') \in H^{2k}(I_{F''}^k) = 0.$$

For $\bar{F} = F'$, this proves (i). If $\bar{F}$ and F' are concordant over F, then

$$\alpha_{k-1}^k \left(p_{k_1}(F/F') \ldots p_{k_r}(F/F') \right) = \alpha_{k-1}^k \left(p_{k_1}(F/\bar{F}) \ldots p_{k_r}(F/\bar{F}) \right),$$

by Proposition 4.12. This proves (ii) and ends the proof of the theorem.

References

[1] G. Andrzejczak, *On some form determined by a distribution on a Riemannian Manifold,* Coll. Math. 41(1978), 79–83.

[2] I. N. Bernstein and B. I. Rosenfeld, *Characteristic classes of foliations,* Funct. Analysis and its Appl. 6(1972), 68–69.

[3] —, —, *Homogeneous spaces of infinite-dimensional Lie algebras and characteristic classes of foliations,* Russ. Math. Surveys 28(1973), 107–143.

[4] R. Bott, *Lectures on characteristic classes and foliations,* in *Lectures on Algebraic and Differential Topology,* Springer Lecture Notes, 279(1972), 1–94.

[5] R. Bott and A. Haefliger, *On characteristic classes of Γ-foliations,* Bull. Amer. Math. Soc. 78(1972), 1039–1044.

[6] H. Cartan, *La transgression dans un groupe de Lie et dans un espace fibré principal,* Colloque de Topologie, Bruxelles (1950), 57–71.

[7] S. S. Chern, *Differential geometry of fiber bundles,* Proc. Intern. Congress Math. (1950), Vol. II, 397–411.

[8] J. L. Dupont, *Curvature and Characteristic Classes,* Springer Lecture Notes, 640(1978).

[9] Ch. Ehresmann and G. Reeb, *Sur les champs d'éléments de contact de dimension p complètement intégrables,* C. R. Acad. Sci. Paris 218(1944), 955–957.

[10] E. Ferida and P. M. D. Furness, *Transversally affine foliations,* Glasgow Math. J. 17 (1976) 2, 106–111.

[11] C. Godbillon, *Cohomologies d'algèbres de Lie de champs de vecteurs,* Séminaire Bourbaki, 25e année (1971/72), no 421.

[12] C. Godbillon and J. Vey, *Un invariant des feuilletages de codimension un,* C. R. Acad. Sci. Paris 273(1971), 92–95.

[13] R. Goldman, *The holonomy ring on the leaves of foliated manifolds,* J. Diff. Geom. 11(1976), 411–449.

[14] A. Haefliger, *Homotopy and integrability, Manifolds,* Springer Lecture Notes, 197, Amsterdam 1970, 133–166.

[15] —, *Feuilletages sur les variétés ouvertes,* Topology 9(1970), 183–194.

[16] —, *Sur les classes caractéristiques des feuilletages,* Séminaire Bourbaki, 24e année (1971/72), no. 412.

[17] G. Joubert, R. Moussu and D. Tischler, *Sur les classes caractéristiques des feuilletages produits,* C. R. Acad. Sci. Paris 275(1972)3, 171–174.

[18] F. Kamber and Ph. Tondeur, *Invariant differential operators and cohomology of Lie algebra sheaves,* Differentialgeometrie im Grossen, Juli 1969, Berichte aus dem Math. Forschunginstitut Oberwolfach, Heft 4, Mannheim (1971), 177–230.

[19] —, —, *Invariant differential operators and cohomology of Lie algebra sheaves,* Mem. Amer. Math. Soc. 113, 1971.

[20] —, —, *Algèbres de Weil semi-simpliciales,* C. R. Acad. Sci. Paris 276(1973), 1177–1179.

[21] —, —, *Homomorphisme caractéristique d'un fibré principal feuilleté,* ibid., 1407–1410.

[22] —, —, *Classes caractéristiques dérivées d'un fibré principal feuilleté,* ibid., 1449–1452.

[23] —, —, *Characteristic invariants of foliated bundles*, Manuscripta Mathematica 11(1974), 51–89.

[24] —, —, *Semi-simplicial Weil algebras and characteristic classes for foliated bundles in Čech cohomology*, Proc. Symp. Pure Math. 27(1975), 283–294.

[25] —, —, *Foliated Bundles and Characteristic Classes*, Springer Lecture Notes, 493(1975).

[26] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vol. I, Interscience Publ., New York 1963.

[27] J. L. Koszul, *Lectures on fibre bundles and differential geometry*, Tata Inst. of Fund. Research, Bombay 1960.

[28] H. B. Lowson, *Foliations*, Bull. Amer. Math. Soc. 80(1974), 369–418.

[29] J. W. Milnor and J. Stasheff, *Characteristic Classes*, Annals of Math. Studies 76, Princeton Univ. Press, Princeton 1974.

[30] P. Molino, *Propriétés cohomologiques et propriétés topologiques des feuilletages a connexion transvèrse projetable*, Topology 12(1973), 317–325.

[31] A. Morgan, *Holonomy and metric properties of foliations in higher codimension*, Proc. Amer. Math. Soc. 58(1976), 255–261.

[32] G. Reeb, *Sur certaines propriétés topologiques des variétés feuilletées*, Actual. Sci. Ind. No. 1183, Hermann, Paris 1952.

[33] B. Reinhart, *Foliated manifolds with bundle-like metrics*, Ann. of Math. 69(1959), 119–132.

[34] —, —, *Holonomy invariants for framed foliations*, Geometrie Différentielle, Coll. Santiago de Compostela, Espagne 1972, Springer Lecture Notes, 392, 47–52.

[35] H. Rummler, *Kompakte Blätterungen durch Minimalflächen*, Habilitationschrift an der Univ. Freiburg.

[36] H. Serre, *Lie Algebras and Lie Groups* (in Russian), Izdat. "Mir", Moscow 1969.

[37] H. Shulman, *Characteristic classes of foliations*, Ph. D. Thesis, University of California, Berkeley, 1972.

[38] H. Shulman and D. Tischler, *Leaf invariants for foliations and the van East isomorphism*, J. Diff. Geom. 11(1976), 535–546.

[39] W. Thurston, *The theory of foliations of codimension greater then one*, Comment. Math. Helv. 49(1974), 214–231.

[40] —, *The existence of foliations*, Notices Amer. Math. Soc. 21(1974), A-407.

[41] J. Vey, *Quelques constructions relatives aux Γ-structures*, C. R. Acad. Sci. Paris 276(1973), 1151–1153.

[42] A. Weil, *Geometrie différentielle des espaces fibrés*, unpublished.

[43] R. O. Wells, *Differential Analysis on Complex Manifolds*, Prentice-Hall, Inc. Englewood Cliffs, N. J., 1973.