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On the metamathematics of impredicative set theory

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CONTENTS

Introduction

This paper contains several results from the metamathematics of impredicative set theory M. This theory was in fact introduced by several authors. For instance as early as in 1948 Mostowski investigates it (without mentioning) in [11] and later Kelley uses it in his book [4] (see appendix to [4]). The first regular course in it can be found in Morse's book [8] who relates it to Knaster and Tarski: We should state here that the success of Kelley's book proliferated this set theory among topologists and people dealing with category theory. On the other hand, Mostowski made first regular use of the theory M in his book [10]. These facts urged several people to study metamathematics of M as well as relation of M to ZF and VNB set theories. We should note that not all the results of this paper are new. For instance we know that at least 4 people have proved relative consistency of M with $V = L$. They are (in alphabetic order): Jensen, Mostowski, Solovay, Tharp. The author of the present paper got to know about it more or less by "folk gossip". It should be mentioned, however, that Tharp's result (kindly communicated to us by U. Felgner) could be much stronger than ours since he does not use impredicative scheme of choice in the proof, i.e. he gets relative consistency of $M + V = L$ with respect to the theory M_0 (i.e. theory without the scheme of choice). We, however, do not know his proof and our proof strongly relies upon the scheme of choice.

Another reason to be interested in the theory M is that it strongly resembles the second order arithmetics (A_2). The results called *Theorem A*, i.e. the equiconsistency of M and $ZFC + T$ is quite similar to the equiconsistency of A_2 and ZFC. Further, they are both second order theories for respectively ZFC and the first order arithmetics. The method of trees, used in Section 3 owes much to ideas of Gandy in the case of the second order arithmetics. The similar situation occurs with Section 4 (on minimal model for M).

We would like to mention people who have had infinite patience to listen to how the ideas of the paper were growing and helped to give the paper its ultimate form. These are: Professor Mostowski, who suggested the problem, Dr. Zbierski from Warsaw who has read the paper several times, Professor van Dalen and Dr. Barendregt from Utrecht (where the main bulk of the paper was written). Apart from them Dr. Boffa from

Bruxelles discussed several times the paper with the author. Obviously their ideas are only in good parts of the paper.

The paper is organized as follows: First in Section 0 theory M is introduced and some facts about it and ZF set theory are proved, this part contains also the main tools to be used in the next paragraph. Section 1 contains the proof of sóme reflection principle for M and the proof of the fact that this reflection principle is at least as strong as the comprehension scheme. In Section 2 the technique of trees is introduced. It also contains the proof of the following Theorem A: *The consistency of M is equivalent to the consistency of $ZFC+T$, where ZFC is ZF set theory without power set axiom but with the scheme of choice and T is an axiom stating the existence of the family — roughly speaking — of from R_α for α inaccessible.* Section 3 contains the proof of the relative consistency of $M+V=L$. This is Theorem B. Section 4 contains the proof of existence of the minimal standard On — model for M. This is metatheorem C. The strong form of the axiom of constructibility is introduced. Section 5 contains a number of (easy) independence results for theories M and M_0 and Section 6 contains a modification of Lévy's hierarchy of formulas for set theory for the case of the theory M. Existence of partial truth predicates is proved exactly as in ZF case.

Throughout the paper standard set — theoretical terminology is used.

0. Set theory M

By set theory M we mean the theory built in the first order language α, with one two-place predicate ϵ (and equality). The theory is based on 7 axioms and 2 schemas.

DEFINITION 0.1. $Z(X) \leftrightarrow (EY)(X \epsilon Y)$.

The variables restricted to the predicate $Z(\cdot)$ are denoted by small Latin letters, general variables, ranging over all objects are denoted by capital Latin letters.

AXIOM I. $(X, Y)[(Z)(Z \epsilon X \leftrightarrow Z \epsilon Y) \rightarrow X = Y]$.

AXIOM II. $(x, y)(Ez)(T)(T \epsilon z \leftrightarrow T = x \vee T = y)$.

AXIOM III. $(x)(Ey)(Z)\big(Z \epsilon y \leftrightarrow (ET)(Z \epsilon T \ \& \ T \epsilon x)\big)$.

AXIOM IV. $(x)(Ey)(Z)\big(Z \epsilon y \leftrightarrow (t)(t \epsilon Z \rightarrow t \epsilon x)\big)$.

AXIOM V. $(Ex)\Big[(Ey)(y \epsilon x) \ \& \ (y)\big(y \epsilon x \rightarrow (Ez)\big(z \epsilon x \ \& \ (t)(t \epsilon z \leftrightarrow t \epsilon y \vee t = y)\big)\big)\Big]$.

AXIOM VI. $(X)\Big[(Ey)(y \epsilon X) \rightarrow (Ey)\big(y \epsilon X \ \& \ (z)(z \epsilon y \rightarrow z \notin X)\big)\Big]$.

Before we introduce Axioms VII, VIII and IX we need some definitions:

Definition 0.2. (a) The (Unique) object considered in Axiom II is called *pair of x and y* and denoted by $\{x, y\}$.

(b) The (Unique) object considered in Axiom III is called *union of x* and denoted by Ux.

(c) The (Unique) object considered in Axiom IV is called *powerset of x* and denoted by $\mathscr{P}x$.

(d) $\langle x, y\rangle \overset{\text{df}}{=} \{\{x\}, \{x, y\}\}$, $\langle x_1 \ldots x_n\rangle \overset{\text{df}}{=} \langle x_1, \langle x_2 \ldots x_n\rangle\rangle$.

(e) $\mathrm{Rel}(X) \leftrightarrow (y)\big(y \in X \to (Ez, t)\,(y = \langle z, t\rangle)\big)$.

(f) $\mathrm{Func}(X) \leftrightarrow \mathrm{Rel}(X)\ \&\ (y, z, t)\,(\langle y, z\rangle \in X\ \&\ \langle y, t\rangle \in X \to z = t)$.

(g) $X'y = z \leftrightarrow \mathrm{Func}(X)\ \&\ \langle y, z\rangle \in X$.

(h) $X''Y = Z \leftrightarrow \mathrm{Func}(X)\ \&\ (r)\big(r \in Y \to (Et)\,(t \in Z\ \&\ \langle r, t\rangle \in X)\big)\ \&\ (r)\big(r \in Z \to (Es)\,(s \in Y\ \&\ \langle s, r\rangle \in X)\big)$.

Axiom VII. $(X, y)\,[\mathrm{Func}(X) \to (Ez)\,(X''y = z)]$.

Axiom VIII$_\Phi$. $(EX)\,(x_{i_0} \ldots x_{i_n})\,(\langle x_{i_0} \ldots x_{i_n}\rangle \in X \leftrightarrow \Phi)$, we assume that $\mathrm{Fr}\,\Phi \subseteq \{i_0 \ldots i_n\}$, where $\mathrm{Fr}\,\Phi$ is the set of indices of free variables in Φ and X is not free in Φ.

The theory based on Axioms I–VIII is called M_0 and is sufficient when axiom of choice is added to do all practical mathematics. However (as we will see later), for our mathematical purposes we need a stronger theory M which includes additionally one more schema IX.

Definition 0.3. (a) The (Unique) object considered in Axiom VIII is denoted by $\{\langle x_{i_0} \ldots x_{i_n}\rangle;\ \Phi\}$.

(b) $\mathscr{D}(X) = \{y\colon (Ex)\,\langle y, x\rangle \in X\}$.

(c) $\mathscr{R}(X) = \{y\colon (Ex)\,\langle x, y\rangle \in X\}$.

(d) If X is a relation, then $X^{(y)} = \{t\colon \langle y, t\rangle \in X\}$.

Axiom IX. $(x)\,(EY)\,\Phi(x, Y) \to (EY)\,(x)\,\Phi(x, Y^{(x)})$.

Definition 0.4. (a) A formula Φ is called *predicative* if is logically equivalent to the formula, where all quantifiers range over sets, i.e. are relativized to the predicate $Z(\cdot)$.

(b) The theory $M_0^{\#}$ is the part of theory M_0 arising by restriction of schema VIII$_\Phi$ to predicative formulas.

Lemma 0.5 (Bernays–Gödel). *The theory $M_0^{\#}$ is finitely axiomatisable. A finite, equivalent axiomatics can be given as a variant of Gödels system Σ.*

Proof of this fact is known as Metatheorem 1 in Gödel's monograph [3].

Metalemma 0.6. *In the theory M_0 (and thence in M) it is provable that the set theory of Zermelo–Fraenkel (ZF) has a model. Moreover, one can prove in M_0 the existence of a natural model for ZF set theory (i.e. of a model of the form R_a).*

LEMMA 0.7. *If R_α is a model for ZF set theory, then α is a cardinal number with limit index.*

Proof. If α is not a cardinal, then $\overline{\overline{\alpha}} = \beta$ for some $\beta \in \alpha$. Then $\beta \in R_{\beta+1} \subseteq R_\alpha$. It is clear, in view of the fact, that R_α is a model for the power set axiom that α is a limit ordinal so $R_{\beta+4} \subseteq R_\alpha$ and therefore we have in R_α a well-ordering (i.e. relation being well-ordering) of type α. Since R_α is a model for ZF $\alpha \in R_\alpha$, contradiction. Thus $\alpha = \omega_\beta$ for some $\beta \subseteq \alpha$. We prove that $\beta = \alpha$. Because if $\beta \in \alpha$, then $\beta \in R_\alpha$ and since the function defined by the equation $f(\gamma) = \omega_\gamma$ is absolute with respect to R_α for α such that R_α is a model for ZF therefore if $\beta \in R_\alpha$, then $(\omega_\beta)^{R_\alpha} \in R_\alpha$ and so $\omega_\beta \in R_\alpha$ or $\alpha \in R_\alpha$ which is contradiction.

COROLLARY 0.8. *Similarly; if R_α is a model for ZF, then $\alpha = \omega_{\omega_\alpha}$ etc. In general; if $\Phi(x, y)$ is a formula which defines a function from On into On which is continuous and increasing and absolute with respect to R_α such that R_α is a model for ZF, then if α is such that R_α is a model for ZF, then $\Phi(\alpha, \alpha)$.*

LEMMA 0.9 ($ZF+AC$). *If α is strongly inaccessible, then $R_{\alpha+1}$ is a model for M (and hence for M_0). The predicate $Z(\cdot)$ of this model is exactly R_α.*

The proof of this fact is generally known; One should only mention that without the axiom of choice we are able to prove only that $R_{\alpha+1}$ is a model for M_0.

LEMMA 0.10 (Shepherdson [12]). *If R_α is a model for $M_0^{\#}$, then $\alpha = \delta+1$, where δ is strongly inacessible and thence R_α is a model for M_0.*

Proof. Clearly δ is strongly limit cardinal, i.e. $\beta < \delta \to 2^\beta < \delta$. So it is sufficient to prove that δ is regular. Assume now that δ is not regular then there is $\beta \in \delta$ and a function $f \in^\beta \delta$ such that:

$$\delta = \lim_{\gamma \in \beta} f_\gamma,$$

Now $f \in R_{\delta+1}$ (because each of its elements has rank smaller then δ) so as β is a set in $R_{\delta+1}$ so is δ which is contradition.

Therefore we have in M_0 the following curious phenomenon.

COROLLARY 0.11. *In M_0 the existence of natural model for ZF is provable, consistency of ZF is equivalent to the consistency of $M_0^{\#}$ but the existence of a natural model for M_0 is not provable in M. In fact it is quite often that R_α is a model for ZF and R is not a model for M_0.*

The reason for this is that there is deep difference between substitution schema of ZF and substitution axiom of M_0.

LEMMA 0.12. *If α is weakly inccessible, then $\mathscr{P}(L_\alpha) \cap L_{\alpha(+)L}$ is a model for M.*

Proof. In L the generalized continuum hypothesis holds and therefore α is not only weakly inaccessible in L but also strongly inaccessible

(One notes that α is weakly inaccessible in L because V is an end extension of L and "α being weakly inaccessible" is Π_1 formula.)

But in L, for inaccessible α, $L_\alpha = R_\alpha$ thus $L_\alpha = R_\alpha \cap L$. Now $(R_{\alpha+1})^L = (\mathscr{P}(R_\alpha^L))^L = (\mathscr{P}(L_\alpha))^L = \mathscr{P}(L_\alpha) \cap L = \mathscr{P}(L_\alpha) \cap L_{\alpha(+)L}$.

Now conclusion easily follows from absoluteness of satisfaction in transitive classes and 0.9.

METALEMMA 0.13. *In M_0 it is provable that there are ordinals α such that R_α is a model for ZF. But one cannot prove (even in M) that there is a regular ordinal with this property.*

Proof. If it were provable there is regular ordinal α such that R_α is a model for ZF, then in view of 0.7 we would be able to prove in M the existence of weakly inaccessible cardinal and in particular in view of 0.12 that M has a model which is in contradition with second Gödel theorem (Feferman [1], § 8).

We know that ZF is interpretable in M by predicate $Z(\cdot)$ (i.e. axioms of ZF, relativised to $Z(\cdot)$ are provable in M). We can ask what sentences should be added to ZF to get new theory ZF_1 such that consistency of ZF_1 is equivalent to consistency of M. We can take as ZF_1 the set of these sentences Φ of the language of ZF that $\Phi^{(Z)}$ is provable in M. (It is easy to see that set of Gödel numbers of formulas of ZF_1 is recursively enumerable). However, our definition of ZF_1 is artificial and one would like to know some natural axiomatics for ZF_1 extending ZF. In this subject several attempts were made by the author yet without success. We give here one with negative solution due to M. Boffa.

LEMMA 0.14. *There are accessible numbers α such that $R_\alpha \prec V$ and $L_\alpha \prec L$ and R_α is union of elementary sequence R_{α_ξ}, L_α is union of elementary sequence L_{α_ξ} of type α.*

Proof. The function $\varphi(\xi) = \alpha_\xi$ "ξ-th index of elementary submodel of V of form R_β" is continuous and increasing and hence has a critical point, the same for L_α's.

COROLLARY 0.15 (Boffa). *There is a standard model N for ZF such that (*) $N = \bigcup_{\xi \in On \cap N} A_\xi$ where $\{A_\xi\}$ is a continuous tower of elementary submodels of N and N is not extendable to the model N' for M with preservation of sets, i.e. with $Z^{(N')} = N$.*

Hence $ZF = ZF + \varphi_1$ is not equivalent to ZF_1 where

$$\varphi_1\colon\ (\alpha)\,(E\varphi)\,[\vec{\varphi} = \alpha \ \&\ (\beta)_\alpha R_{\varphi_\beta} \prec \bigcup_{\gamma \in \alpha} R_{\varphi_\gamma}].$$

Moreover, much stronger condition on N that of type (*) is not sufficient for the fact that $\mathrm{Th}(N) \supset ZF_1$.

Proof. Take first critical number of the function φ_ξ considered in proof 0.14 and use absoluteness of R_α with respect to models of form R_β.

We prove the following strengthening

LEMMA 0.16. *If Φ is a sentence of the language of set theory such that in M, it is provable $\Phi^{(Z)}$, then there is a standard model for $ZF+\Phi$ not extendable to a model for M_0 with the same sets.*

Proof. If $M \vdash \Phi^{(Z)}$, then there is the smallest α such that R_α is a model for ZF and Φ. We claim that this is the model. Indeed assume that there is $X \subseteq R_\alpha+1$ such $R_\alpha \cup X$ is a model for M, then as $M \vdash \Phi^{(Z)}$ and as a reflection principle is provable in M we have $\beta \in \alpha$ such that $R_\alpha \cup X \models (R_\beta \models ZF+\Phi)$ thence $R_\beta \models ZF+\Phi$, contradition.

U. Felgner informed us in Oberwolfach in april 1971 that he was able to prove (analogically as he had proved in $M_0^{\#}$ case) the following

Fact 0.17. If $M_0+E \vdash \Phi^{(Z)}$, where Φ is a sentence of set theory, then $M_0+AC \vdash \Phi^{(Z)}$ (where AC is a local form of the axiom of choice and E is the global form of it).

Now we give a number of technical lemmas and definitions which will be used in sequel.

DEFINITION 0.18. (a) $X\eta Y \leftrightarrow (Ex)(X = Y^{(x)})$.

(b) $X \simeq Y \leftrightarrow (Z)(Z\eta X \leftrightarrow Z\eta Y)$.

DEFINITION 0.19. $\mathrm{Bord}(X) \leftrightarrow \mathrm{Rel}(X)$ & $(y)\big(y \in \mathscr{D}(X) \to \langle y, y\rangle \in X\big)$ & & $(y, z)(\langle y, z\rangle \in X$ & $\langle z, y\rangle \in X \to y = z)$ & $(y, z, t)(\langle y, z\rangle \in X$ & $\langle z, t\rangle \in X \to$ $\to \langle y, t\rangle \in X)$ & $(y, z)(y \in \mathscr{D}X$ & $z \in \mathscr{D}X \to \langle y, z\rangle \in X \vee \langle y, z\rangle \in X)$ & $(Y)(Y \subseteq$ $\subseteq \mathscr{D}X$ & $Y \neq \emptyset \to (Et)_Y(u)_Y(\langle t, u\rangle \in X$ & $X \subseteq On^2)$.

The fact we restrict ourselves to well-orderings of ordinals does not affect our construction since we have the following fact:

If R is well-ordering, then there is a $Y \subseteq On^2$ and $f\colon \mathscr{D}R \xrightarrow[\text{onto}]{1-1} \mathscr{D}Y$ such that $\langle x, y\rangle \in R \leftrightarrow \langle fx, fy\rangle \in Y$.

The proof of this fact in M_0 follows from the following observations:

$$\mathscr{D}R = \bigcup_{\alpha \in On} \big(\mathscr{D}R \cap (R_{\alpha+1} - R_\alpha)\big)$$

and On can be splitted into On parts each similar to On.

DEFINITION 0.20. (a) $O_X(y) = \{z\colon \langle z, y\rangle \in X$ & $z \neq y\}$.

(b) $X \restriction y = X \cap \big(O_X(y)\big)^2$.

(c) $X_1 \sim X_2 \leftrightarrow \mathrm{Bord}\,X_1$ & $\mathrm{Bord}\,X_2$ & $(EZ)\big(\mathrm{Func}Z$ & $\mathscr{D}Z = \mathscr{D}X_1 \cup$ $\cup \mathscr{R}X_1$ & $\mathscr{R}Z = \mathscr{D}X_2 \cup \mathscr{R}X_2$ & $1\text{-}1(Z)$ & $(s, t)(\langle s, t\rangle \in X_1 \leftrightarrow \langle Z's, Z't\rangle \in$ $\in X_2)\big)$.

(d) $1\text{-}1(Y) \leftrightarrow \mathrm{Func}(Y)$ & $\mathrm{Func}(\{\langle z_1, z_2\rangle\colon \langle z_2, z_1\rangle \in Y\})$.

(e) $X_1 \leqslant X_2 \leftrightarrow \mathrm{Bord}\,X_1$ & $\mathrm{Bord}\,X_2$ & $[(Ey)(X_1 \sim X_2 \restriction y) \vee X_1 \sim$ $\sim X_2]$.

LEMMA 0.21. (a) $(X)(\mathrm{Bord}X \to X \leqslant X)$.

(b) $(X, Y, Z)[\mathrm{Bord}X \,\&\, \mathrm{Bord}Y \,\&\, \mathrm{Bord}Z \to (X \leqslant Y \,\&\, Y \leqslant Z \to X \leqslant Z)]$.

(c) $(X, Y)[\mathrm{Bord}X \,\&\, \mathrm{Bord}Y \,\&\, X \leqslant Y \,\&\, Y \leqslant X \to X \sim Y]$.

(d) $(X, Y)[\mathrm{Bord}X \,\&\, \mathrm{Bord}Y \to (X \leqslant Y \vee Y \leqslant X)]$.

The proof of the facts (a) and (b) is clear and the proof of (c) and (d) is analogous to corresponding ZF case. However, we will give the proof of 0.21 (d) since it gives rise to an unsolved problem.

Proof of 0.21. (d) Let X, Y be well-orderings; We form the class $U = \{\langle z, t\rangle : X \upharpoonright z \sim Y \upharpoonright t \,\&\, z \in \mathscr{D}X \cup \mathscr{R}X \,\&\, t \in \mathscr{D}Y \cup \mathscr{R}Y\}$. U is one-to-one function. Actually while proving 0.21 (c) we prove that similarity function is unique. Now we have three possible cases:

(a) $\mathscr{D}U = \mathscr{D}X \cup \mathscr{R}X$. (b) $\mathscr{R}U = \mathscr{D}Y \cup \mathscr{R}Y$. (c) $\mathscr{D}U \neq \mathscr{D}X \cup \mathscr{R}Y \,\&\, \&\, \mathscr{R}U \neq \mathscr{D}Y \cup \mathscr{R}Y$.

We prove that case (c) is impossible. Indeed then $\mathscr{D}X \cup \mathscr{R}Y - \mathscr{D}U \neq \emptyset \neq \mathscr{D}Y \cup \mathscr{R}Y - \mathscr{R}U$. Let a, b be responsibly first elements of the classes $(\mathscr{D}X \cup \mathscr{R}X) - \mathscr{D}U$ and $(\mathscr{D}Y \cup \mathscr{R}Y) - \mathscr{R}U$, then $\langle a, b\rangle \in U$ which is contradition.

LEMMA 0.22 (Gaifman). *If the theory* $M_0^{\#}$ *is enriched by assumptions* AC *and "There exists a measurable cardinal", then* 0.21 (d) *is provable.*

The proof can be derived from Gaifman [2].

PROBLEM 0.23 (Gaifman). *Is* 0.21 (d) *provable in* $M_0^{\#}+AC$? In other words: *Is* $M_0^{\#}+AC+\neg$ 0.21 (d) *consistent?*

DEFINITION 0.24. Let X be a class of pairs and assume that $(y)(Y\eta X \to \mathrm{Bord}(Y))$. Well-ordering T has property $\mathrm{Sup}X$ iff

(i) $(Y)(Y\eta X \to Y \leqslant T)$.

(ii) $(x)_{\mathscr{D}T}(EY)(Y\eta X \,\&\, T\upharpoonright x \leqslant Y)$.

THEOREM 0.25. *If* $\mathscr{D}X$ *is well-orderable, i.e. there is a relation which well-orders* $\mathscr{D}X$ *and* X *satisfies assumption of* 0.24, *then there is* Y *with property* $\mathrm{Sup}X$. Y *is unique up to isomorphism (i.e. relation* $\sim$).

Proof. We form class $M = \bigcup_{g \in \mathscr{D}X} \{g\} \times \big(\mathscr{D}(X^{(g)}) \cup \mathscr{R}(X^{(g)})\big)$. We introduce relation I in M by the following:

$$\langle g, h\rangle I \langle g_1, h_1\rangle \leftrightarrow X^{(g)}\upharpoonright h \sim X^{(g_1)}\upharpoonright h_1.$$

Its clear that I is equivalence. Now M is well-ordered by lexicographical product of given ordering of $\mathscr{D}X$ and all $X^{(g)}$. Therefore we can choose one element from each equivalence class. Call the selector W. We introduce relation $<$ in W by the following

$$\langle g, h\rangle < \langle g_1, h_1\rangle \leftrightarrow X^{(g)}\upharpoonright h \leqslant X^{(g_1)}\upharpoonright h_1.$$

It is easy to see that relation $<$ is not smaller than all $X^{(g)}$ for $g \in \mathscr{D}X$. Since we want $<$ to be longer than all of them we put some new object at the end of W.

The uniqueness follows directly from 0.21 (d) and 0.24.

PROBLEM 0.26. *Is* 0.25 *provable in* $M_0^{\#}+AC$?

We conjecture that it is not.

DEFINITION 0.27. The predicate $\Phi(\cdot)$ is called *codable* iff there is X such that $(Y)\big(Y\eta X \leftrightarrow \Phi(Y)\big)$.

DEFINITION 0.28. The class X is called *finite sequence of classes* if $\mathscr{D}X \subseteq \omega$ & $\overline{\overline{\mathscr{D}X}} < \omega$.

LEMMA 0.29. *If the predicate* $\Phi(\cdot)$ *is codable, then also the following predicate* $\Psi(\cdot)$ *is codable:*

$\Psi(X) \leftrightarrow$ "*X is a finite sequence of classes*" & $\Phi(X^{(i)})$ *for all* $i \in \mathscr{D}X$.

Proof. Let Y be a code for Φ. We form an auxilary class $U = \bigcup\limits_{\substack{s\subseteq\omega\\ s<\aleph_0}}{}^{s}\ \mathscr{D}Y$ and for every $w \in U$ the class $t_w = \{\langle i, x\rangle : \langle w(i), x\rangle \in X\}$.

Now we form the class: $R = \{\langle w, s\rangle : s \in t_w\}$. This is the code for Ψ.

1. Reflection principles in M

DEFINITION 1.1. Let X be a class of pairs and Φ a formula of set theory, $\mathrm{Fr}\,\Phi$ set of indices of free variables of Φ. We define the relation

$$X \models \Phi[\vec{x}]$$

between the class X, the formula (i.e. a natural number) Φ and finite sequence $\vec{x} \in {}^{\mathrm{Fr}\Phi}\mathscr{D}X$ as follows:

(a) If Φ is an atomic formula i.e. Φ: $v_i \in v_j$ or $v = v_j$, then $X \models \Phi[\vec{x}]$ iff respectively

$$X^{(x_i)} \in X^{(x_j)} \quad \text{or} \quad X^{(x_i)} = X^{(x_j)}.$$

(b) If Φ: $\neg\Psi$, then $X \models \Phi[\vec{x}]$ iff it is not true that $X \models \Psi[\vec{x}]$.

(c) If Φ: Φ_1 & Φ_2, then $X \models \Phi[\vec{x}]$ iff $X \models \Phi_1[\vec{x}\restriction \mathrm{Fr}\,\Phi_1]$ and $X \models$ $\models \Phi_2[\vec{x}\restriction \mathrm{Fr}\,\Phi_2]$.

(d) If Φ: $(Ev_i)\Phi_1$ and $\mathrm{Fr}\,\Phi = \mathrm{Fr}\,\Phi_1$, then $X \models \Phi[\vec{x}]$ iff $X \models \Phi_1[\vec{x}]$.

(e) If Φ: $(Ev_i)\Phi_1$ and $\mathrm{Fr}\,\Phi \neq \mathrm{Fr}\,\Phi_1$, then $X \models \Phi[\vec{x}]$ iff $(Ey)_{DX}$ $(X \models \Phi_1[\vec{x} \cup \{\langle i, y\rangle\}])$.

METALEMMA 1.2. *There is a formula* Stsf$(\cdot,\cdot,\cdot)$ *of three free variables such that*

$$\mathrm{Stsf}(X, i, \vec{x}) \leftrightarrow X \models \Phi[\vec{x}],$$

where $i = \ulcorner\Phi\urcorner$.

We write the formula Stsf as follows. "There exists a class which codes finite sequence of classes enumerated by Gödel numbers of subformulas of Φ and such that if: k is the Gödel number of an atomic formula $v_i \in v_j$, then $y^{(k)}$ is $E_{ij}(X)$, if k is the Gödel number of an atomic formula $v_i = v_j$, then $Y^{(k)}$ is $I_{ij}(X)$ if k is the Gödel number of a negation of formula with Gödel number l, then $Y^{(k)} = {}^{\mathrm{Fr}(\ulcorner k\urcorner)}\mathscr{D}X - Y^{(l)}$ if k is the Gödel number of a formula which is a conjunction of formulas with the Gödel numbers l_1 and l_2, then $Y^{(k)}$ is a class of functions $\vec{x} \in {}^{\mathrm{Fr}(\ulcorner k\urcorner)}\mathscr{D}X$ such that $\vec{x}\restriction \mathrm{Fr}(\ulcorner l_1\urcorner) \in Y^{(l_1)}$ and $x\restriction \mathrm{Fr}(\ulcorner l_2\urcorner) \in Y^{(l_2)}$ and if k is of the form $(Ev_i)\Phi_1$, then $Y^{(k)}$ is the class of sequences $\vec{x}$ such that there is $y \in \mathscr{D}X$ such that $\vec{x} \cup \{\langle i, y\rangle\} \in Y(\ulcorner\Phi_1\urcorner)$. Here

$$E_{ij}(X) = \{\{\langle i, x\rangle, \langle j, y\rangle\}:\ x^{(x)} \in X^{(y)}\},$$
$$I_{ij}(X) = \{\{\langle i, x\rangle, \langle j, y\rangle\}:\ X^{(x)} = X^{(y)}\}.$$

We remark that we imitate here in fact Mostowski's construction [11]. This is possible because the definition in the case of the existential quantifier turned out to be predicative

DEFINITION 1.3. $\mathrm{Rep}(X, x, Y) \leftrightarrow Y = X^{(x)}$.

Fact 1.4. $(Ex)\mathrm{Rep}(X, x, Y) \leftrightarrow Y\eta X$.

LEMMA 1.5. *If* $X \simeq Y$ (cf. 0.12 (b)) *and* Φ *is a formula and for every* $i \in \mathrm{Fr}\,\Phi$ *we have respectively* $\mathrm{Rep}(X, x_i, Y_i)$ *and* $\mathrm{Rep}(Y, x_i', Y_i)$, *then*

$$X \models \Phi[\vec{x}] \leftrightarrow Y \models \Phi[\vec{x}'].$$

Proof. For atomic formulas it is obvious, the same for boolean connectives and dummy quantifiers. We consider now the case Φ: $(Ev_i)\Phi_1$ and $i \in \mathrm{Fr}\,\Phi_1$. Then by definition: $X \models \Phi[\vec{x}] \leftrightarrow (Ey)_{\mathscr{D}X} X \models \Phi_1[\vec{x} \cup \{\langle i, y\rangle\}$. By 1.3 and 1.4 $\mathrm{Rep}(X, y, X^{(y)})$ so $X^{(y)}\eta X$ and therefore $X^{(y)}\eta Y$ so there is $y_1 \in \mathscr{D}Y$ such that $\mathrm{Rep}(Y, y_1, X^{(y)})$. By induction hypothesis $Y \models \Phi_1[\vec{x}' \cup \{\langle i, y_1\rangle\}]$ so

$$(Ey)_{\mathscr{D}Y} Y \models \Phi_1[\vec{x}' \cup \{\langle i, y\rangle\}] \quad \text{or} \quad Y \models \Phi[\vec{x}'].$$

Clearly the opposite implication can be proved by the same method.

DEFINITION 1.6. $\mathrm{Inc}(X, Y) \leftrightarrow (Z)(Z\eta X \rightarrow Z\eta Y)$.

DEFINITION 1.7 (Scheme). (a) The predicate $U(\cdot, \cdot)$ is called a *sequence* iff

$$(X, Y)[U(X, Y) \rightarrow \mathrm{Bord}(Y)].$$

(b) The sequence $U(\cdot, \cdot)$ is called *function like on the first coordinate* iff

$$U(X_1, T) \,\&\, U(X_2, T) \rightarrow X_1 \simeq X_2.$$

(c) The sequence $U(\cdot, \cdot)$ is called *function like on the second coordinate* iff

$$U(X, T_1) \,\&\, U(X, T_2) \rightarrow T_1 \sim T_2.$$

(d) The sequence $U(\cdot, \cdot)$ is called *monotonic* iff

$$U(X, T) \,\&\, U(X_1, T_1) \,\&\, T \leqslant T_1 \rightarrow \mathrm{Inc}(X, X_1).$$

(e) The sequence $U(\cdot, \cdot)$ is called *continuous* iff from the fact that T is a limit well-ordering it follows:

(i) $U(X, T) \,\&\, U(X_1, T_1) \,\&\, T \leqslant T_1 \rightarrow \mathrm{Inc}(X, X_1)$,

(ii) $U(X, T) \,\&\, Y\eta X \rightarrow (EX_1, T_1)\big(T_1 < T \,\&\, U(X_1, T_1) \,\&\, Y\eta X_1\big)$.

Convention 1.8. Iff $U(\cdot, \cdot)$ is function like on both coordinates we shall write $X = U_T$ instead of $U(X, T)$.

(f) If $U(\cdot, \cdot)$ is a sequence, then

$$V_U(X) \leftrightarrow (ET)(X\eta U_T).$$

THEOREM 1.9 (Scheme) (Reflection principle for M). *Let $U(\cdot, \cdot)$ be a sequence function like on both coordinates monotonous, continuous and such that $(T)(EX)U(X, T)$.*

Let $V_U(\cdot)$ be a corresponding predicate (1.7.(f)). *Then* $(T)(ET_1)\{T < T_1 \,\&\, (Y_{i_0} \ldots Y_{i_k})[\mathrm{Rep}(U_{T_1}, x_{i_0}, Y_{i_0}) \,\&\, \ldots \,\&\, \mathrm{Rep}(U_{T_1}, x_{i_k}, Y_{i_k}) \rightarrow U_{T_1} \models$ $\models \Phi[\vec{x}] \leftrightarrow \Phi^{V_u}(Y_{i_0}, \ldots, Y_{i_k})$, *where* $\mathrm{Fr}\,\Phi = \{i_0 \ldots i_k\}$, $\vec{x} \in {}^{\mathrm{Fr}\Phi}\mathscr{D}U_{T_1}$, *such that* $\vec{x}(i_m) = x_{i_m}$.

The proof of 1.9 we will get similarly as in the ZF-case by finding certain critical points for certain mappings. Namely for the formula Φ we are going to construct the predicate $R_\Phi(\cdot, \cdot)$ with both arguments being the well-orderings. Second argument T_1 of R_Φ will show well-ordering T' such that the values of Skolem functions for Φ and for arguments from U_T (where T is first argument of R_Φ) can be found in $U_{T'}$.

First we prove the following:

LEMMA 1.10 (Scheme). *Under assumptions* 1.9 *there exists a predicate: R_Φ of the language of the set theory such that* $R_\Phi(T_1, T_2) \rightarrow \mathrm{Bord}(T_1) \,\&\, \mathrm{Bord}(T_2)$.

R is function like on the second coordinate and has the following properties:

$$R_\Phi(T_1, T_3) \,\&\, R_\Phi(T_2, T_3) \to T_1 \sim T_2$$

and it is continuous and increasing. And if T_1 is the critical point of R_Φ, then the thesis of 1.9 *is true.*

Proof. For an atomic Φ we take

$$R_\Phi(T_1, T_2) \leftrightarrow \mathrm{Bord}(T_1) \,\&\, \mathrm{Bord}(T_2) \,\&\, T_1 \sim T_2.$$

For Φ: $\neg \Psi$

$$R_\Phi(T_1, T_2) \leftrightarrow R_\Psi(T_1, T_2).$$

For Φ: $\Phi_1 \,\&\, \Phi_2$

$$R_\Phi(T_1, T_2) \leftrightarrow (ET_3)\big(R_{\Phi_1}(T_1, T_3) \,\&\, R_{\Phi_2}(T_3, T_2)\big).$$

For Φ: $(Ev_i)\Phi_1$ if $i \notin \mathrm{Fr}\,\Phi_1$, then:

$$R_\Phi(T_1, T_2) \leftrightarrow R_{\Phi_1}(T_1, T_2).$$

So we assume now that $i \in \mathrm{Fr}\,\Phi_1$.

For every well-ordering T and the sequence $\vec{x} \in^{\mathrm{Fr}\Phi} \mathscr{D} U_T$ let $T_{\vec{x}}$ denote the smallest well-ordering T_0 such that if there is Z such that $V_U(Z)$ and such that $\Phi^{V_U}(U_T^{(x_{i_0})}, \ldots, U_T^{(x_{i_k})}, Z)$, then Z can be found in U_{T_0} or T if there is no such T_0. Since (by Lemma 0.22) we have the class coding all finite sequences of η-elements U_T so by IX we have the class W of well-orderings coresponding to finite sequences of η-elements of the class U_T and by 0.25 the supremum of W. One should remark that the use of Axiom IX is crucial in this point.

Now the operation $F(\cdot, \cdot)$ describing the mapping is not necessarily continuous and increasing. We will prove, however, that every operation is majorized by another operation $G(\cdot, \cdot)$ which is already continuous and increasing (it will be corollary to 1.11). Now we have $F(T_1, T_2) \,\&\, G(T_1, T_3) \to T_2 \leqslant T_3$. Since T_2 shows us the bound for the occurrence of values of the Skolem function for Φ so does T_3 since $U(\cdot, \cdot)$ is monotonic.

Now

$$R_\Phi(T_1, T_2) \leftrightarrow (ET_3)\big(R_{\Phi_1}(T_1, T_3) \,\&\, G(T_3, T_2)\big).$$

It is clear that $R_\Phi(\cdot, \cdot)$ satisfies conditions of lemma.

LEMMA 1.11. *If $R(T_1, T_2)$ is a predicate with properties mentioned in* 1.10, *then* $(T)\,(ET_1)\big(T \leqslant T_1 \,\&\, R(T_1, T_1)\big)$.

Proof. Similarly as in the ZF case we show that $R(T, T_1) \to T \leqslant T_1$. Now using Axiom IX we form a class X such that $\mathscr{D}X = \omega$ and $(n)R(X^{(n)}, X^{(n+1)})$. Now $T_1 = \mathrm{Sup}\,X$ is an appropriate well-ordering.

The proof of 1.9 is now an easy corollary to 1.10 and 1.11. However, we need to fill the gap in the proof of 1.10 and so we prove

THEOREM 1.12 (On inductive definitions). *Let* $\approxeq_1$ *and* $\approxeq_2$ *be equivalences definable in M. Let* $\Phi_1(\cdot,\cdot)$, $\Phi_2(\cdot,\cdot)$ *be two formulas satisfying respectively*

$$(X)(EY)(Z)[\Phi_1(X,Z)\leftrightarrow Y\approxeq_1 Z],\qquad (X)(EY)(Z)[\Phi_2(X,Z)\leftrightarrow Y\approxeq_2 Z].$$

Then there is a formula $\Psi(\cdot,\cdot)$ *such that* Ψ *is a sequence and*

(a) $\Psi(T,X)\ \&\ \Psi(T,Y)\ \&\neg\ \mathrm{Lim}(T)\to X\approxeq_1 Y$,

(b) $\Psi(T,X)\ \&\ \Psi(T,Y)\ \&\ \mathrm{Lim}(T)\to X\approxeq_2 Y$,

(c) $\Psi(T,X)\ \&\ \Psi(T^{+1},Y)\to\Phi_1(X,Y)$,

(d) $\Psi(T,X)\ \&\ \mathrm{Lim}(T)\to(EU)\big(\Phi_2(U,X)\ \&\ (Y)\,[(Ex)\Psi(T\restriction x,Y)\leftrightarrow$ $\leftrightarrow Y\eta U]\big)$.

This is proved, as a usual theorem in *ZF* case; note that Axiom IX has to be used in the proof.

Now we are able to prove

LEMMA 1.1.3. *If* $F(\cdot,\cdot)$ *is a predicate such that* $F(T_1,T_2)\to\mathrm{Bord}(T_1)\ \&$ $\mathrm{Bord}(T_2)$, *then there is a continuous and increasing G which majorizes F.*

Proof. We define G inductively as follows: Assume T is a limit; we have already $G(T\restriction x, S_{T\restriction x})$ for $x\in\mathscr{D}T$ then by IX we form class W such that $G(T\restriction x, W^{(x)})$ and by 0.25 we form supremum H of W.

If T is not a limit we proceed as follows: T has the last element x_0. Let $G(T\restriction x_0, H)$ and $F(T,H_1)$. I put $G(T,H_2)\leftrightarrow H_2\sim\max(H+1,H_1)$, where $H+1$ rougly speaking arises form H by putting the first element of H at the end.

It is clear that G is continuous and increasing.

DEFINITION 1.14. (a) Condition A_U is the following formula: $(X)\,V_U(X)$.

(b) When we say that condition A is true we mean: for some particular predicate U, condition A_U is true.

Condition A is called sometimes a *split principle* or an *approximation principle* since it asserts that the *superuniverse* of all classes can be approximated from below by definable small (i.e. codable) superclasses.

THEOREM 1.15 (Scheme). *Let* $\vartheta=\langle|\vartheta|,\varepsilon\rangle$ *be a standard model for M satisfying condition A. In particular let* A_U *be true in* ϑ.

Then ϑ *has property of reflection; i.e. if T is a well-ordering in* ϑ, *then there is* T_1 *in* ϑ *such that* $T\leqslant T_1$ *and*

$$\vartheta\models\mathrm{Rep}(U_{T_1},x_{i_m},Y_{i_m}),\qquad 0\leqslant m\leqslant k,$$

$$U_{T_1}\models\Phi[\vec{x}]\leftrightarrow\vartheta\models\Phi[Y_{i_0},\dots,Y_{i_k}].$$

Note that the meaning of the sign $\models$ on both sides of the equivalence is different.

Proof. By 1.9.

At the moment even the consistency of A seems doubtful but later; in Section 4 we will prove consistency of A

THEOREM 1.16. *If the existence of inaccessible cardinal is consistent with ZF, then condition A cannot be proved in M.*

Proof. It is clear by Easton theorem that we can assume that $2^\theta = \theta^{++}$ for inaccessible θ and so the model $R_{\theta+1}$ would be of power θ^{++}.

However, it is clear that for any $X \in R_{\theta+1}$

$$\overline{\overline{\{A: A\eta X\}}} \leqslant \overline{\overline{\mathscr{D}X}} \leqslant \theta .$$

From the other hand for each well-ordering

$$T \subseteq \theta^2 = (On_{R_{\theta+1}})^2, \quad \overline{T} < \theta^+.$$

Thence if condition A is satisfied in $R_{\theta+1}$, then $\overline{\overline{R_{\theta+1}}} \leqslant \theta \cdot \theta^+ = \theta^+$.

Thence in the model where $\overline{\overline{R_{\theta+1}}} = \theta^{++}$ condition A cannot be satisfied.

DEFINITION 1.17. Let S be a scheme:

$$(X)(EY)[X\eta Y \,\&\, \mathrm{Inc}(V, Y) \,\&\, (Y_{i_0}, \ldots, Y_{i_k})]\big(\mathrm{Rep}(Y, x_{i_0}, Y_{i_0}) \,\&\, \ldots \\ \&\, \mathrm{Rep}(Y, x_{i_k}, Y_{i_k}) \to Y \models \Phi[\vec{x}] \leftrightarrow \Phi(Y_{i_0}, \ldots, Y_{i_k})\big)$$

(where V is the universe of all sets).

THEOREM 1.18. $M + A \vdash S$.

Proof. We take as Y, U_T such that $X\eta U_T$ and $R_\Phi(T, T)$, where R is an appropriate predicate (cf. 1.10).

THEOREM 1.19. *Let* $M_0' = M_0^\# + S + \Delta_1$ *comprehention; then* $M_0' \vdash M_0$.

Proof. We need to prove the scheme of class existence;

$$A_\Phi\colon\ (EY)(u)\big(u \in Y \leftrightarrow Z(u) \,\&\, \Phi\big).$$

One of the conditions in S is that for every set $x\eta Y$. Thence the notion of the set is absolute, if $Y \models (EZ)(X \in Z)$, then X is really a set and conversly. We take as Y the class coding all parameters in Φ and reflecting Φ, i.e. $Y \models \Phi \leftrightarrow \Phi$. Now by the Δ_1 class existence scheme

$$\{x\colon \Phi\} = \{x\colon Y \models \Phi\} \quad \text{exists}.$$

Remark 1.20. It is easy to see that the proof of our reflection principle can be carried out without much trouble to the second order arithmetics with an axiom of choice or to any other arithmetics of finite order (always assuming an appropriate scheme of choice).

2. The trees

DEFINITION 2.1. The class F is called a *well founded tree* (*w.f.t.*) iff

(i) F is a function, $F \subseteq On^2$.

(ii) For every $\emptyset \neq Y \subseteq \mathscr{D}F$ there is $y \in Y$ such that $y \notin F * Y$.

(iii) There is an element MAX_F such that for every $x \in \mathscr{D}F$ there is a natural number $n = n(x) \geqslant 1$ such that

$$F^{(n)}(x) = \underbrace{F \circ \ldots \circ F}_{n}(x) = \mathrm{MAX}_F.$$

Fact 2.2. MAX_F is unique. Also $n(x)$ is unique (for each x).

Proof. Assume that $a_1 \neq a_2$ are both maximal in sense 2.1 (iii) then there is $k \geqslant 1$ such that $F^{(k)} a_1 = a_1$. The orbit of a_1 is then a set which is non-empty and contradicts 2.1 (ii). Similarly we prove the uniqueness of $n(x)$.

DEFINITION 2.3. If F is w.f.t, then R_F is defined as follows:

$$\langle x, y\rangle \in R_F \leftrightarrow (Ek)_{\omega-\{0\}}(F^{(k)}x = y).$$

LEMMA 2.4. (i) R_F *is a partial well-ordering, i.e.* (a) *for every non-empty* $Y \subseteq \mathscr{D}R_F$ *there is* $y \in Y$ *minimal in* Y; (b) R_F *is antisymmetric.* (ii) MAX_F *is the biggest element of* R_F. (iii) F *determines uniquely* R_F *and conversly.*

If R *is a partial well-ordering with the biggest element and with the property that* $\langle x, y\rangle \in R$ *implies that there is* k *such that there is a sequence* $\boldsymbol{x}$ *of the length* $k+1$ *such that* $\boldsymbol{x}(0) = x$, $\boldsymbol{x}(k) = y$ *and* $\boldsymbol{x}(l+1)$ *is an immediate successor* (*in* R) *of* $\boldsymbol{x}(l)$, *then there exists exactly one function* F *such that* $R = R_F$.

Proof. (i) follows directly from 2.1 (ii) and the fact that F is a function. (ii) Follows directly from 2.1 (iii). (iii) Let $\mathrm{Succ}_R(x, y)$ mean that y is the successor of x in R we define $F(x) = y \leftrightarrow \mathrm{Succ}_R(x, y)$. Now the biggest element of R becomes MAX_F (by existence of path). The minimality condition implies 2.1 (ii) and the fact that R is a partial well-ordering implies that F is a function. Clearly $R_F = R$.

DEFINITION 2.5. If R is a relation, then $O_R(x) = \{y\colon \langle y, x\rangle \in R \;\&\; y \neq x\}$.

LEMMA 2.6. *If* F *is w.f.t., then there is no sequence* $\boldsymbol{x} \in {}^{\omega}\mathscr{D}F$ *such that* $(n)_\omega\big(F\boldsymbol{x}(n+1) = \boldsymbol{x}(n)\big)$.

Proof. The existence of such a sequence contradicts 2.1 (ii). However, since our trees are built of ordinals, i.e. elements of well-ordered class we can prove also that under 2.1 (i) and 2.1 (iii), 2.6 implies 2.1 (ii).

LEMMA 2.7. *If* F *is w.f.t. and* R_F *is a corresponding partial well-ordering, then, for every* $x \in \mathscr{D}F \cup \mathscr{R}F$, *we have*: $\big(O_{R_F}(x) \cup \{x\}\big)^Z \cap R_F$ *is a partial well-ordering satisfying conditions from* 2.4 (iii). *The biggest element of it is* x.

Proof. Call this new formed relation $R_F(x)$; we have $R_F(x) \subseteq R_F$ and $\mathscr{D}R_F(x) = O_{R_F}(x)$. Now clearly $R_F(x)$ is a partial well-ordering since both conditions are hereditary when one takes a subclass. Since $x \in \mathscr{D}R_F(x) \cup \mathscr{R}R_F(x)$ thence x is the biggest element of $R_F(x)$. The finite sequence for x, y such that $\langle x, y\rangle \in R_F(x)$ is the same as in R_F since $\mathscr{D}R_F(x) = O_{R_F}(x)$.

DEFINITION 2.8. F_x is w.f.t. determined by $R_F(x)$.

Fact 2.9. (a) $\mathrm{MAX}_{F_x} = x$.

(b) $F = F_{\mathrm{MAX}_F}$.

(c) $(x)_{\mathscr{D}F\cup\mathscr{R}F}(F_x \subseteq F)$.

(d) $(x)_{\mathscr{D}F\cup\mathscr{R}F}(y, z)_{\mathscr{D}F_x}\big(F_x(y) = z \leftrightarrow F(y) = z\big)$.

DEFINITION 2.10. If F is w.f.t., then $\mathrm{Rank}(F, x, T)$ is defined to have the following properties:

(a) If x is minimal in R_F, then $\mathrm{Rank}(F, x, \emptyset)$.

(b) If x is not minimal, then assume that $\mathrm{Rank}(F, y, Y^{(y)})$ for $y \in F^{(-1)}*\{x\}$. Then $\mathrm{Rank}(F, x, \mathrm{Sup}\, X)$.

THEOREM 2.11. (i) $(x)_{\mathscr{D}F\cup\mathscr{R}F}(ET)\mathrm{Rank}(F, x, T)$.

(ii) $(F, x, T_1, T_2)\big(\mathrm{Rank}(F, x, T_1)\ \&\ \mathrm{Rank}(F, x, T_2) \to T_1 \sim T_2\big)$.

(iii) (F, x, y, T_1, T_2) $\mathrm{Rank}\big((F, x, T_1)\ \&\ \mathrm{Rank}(F, x, T_2)\ \&\ xR_Fy \to$ $\to T_1 < T_2\big)$.

Proof. (i) Assume not; Take x minimal in the class of these elements which have no rank; Then

$$(y)_{F^{-1}*\{x\}}(ET)\mathrm{Rank}(F, y, T).$$

By Axiom IX we can form class X such that

$$(y)_{F^{-1}*\{x\}}\big(\mathrm{Rank}(F, y, X^{(y)})\big).$$

Since $\mathscr{D}F \subseteq On$ so it is well-orderable and we can form a supremum of X. So $\mathrm{Rank}(F, x, \mathrm{Sup}\, X)$ contradiction.

(ii) Follows from the lemma on the uniqueness of the supremum.

(iii) Follows from the fact that $X^{(t)} < \mathrm{Sup}\, X$ by an easy introduction on length of the "path" joining x and y.

DEFINITION 2.12. $\mathrm{Rank}(F, T) \leftrightarrow \mathrm{Rank}(F, \mathrm{MAX}_F, T)$.

LEMMA 2.13. (i) $(F)\,(ET)\mathrm{Rank}(F, T)$.

(ii) $(F, T_1, T_2)\,\big[\mathrm{Rank}(F, T_1) \to \big(\mathrm{Rank}(F, T_2) \leftrightarrow T_1 \sim T_2\big)\big]$.

(iii) *If* F_1, F_2 *are isomorphic, then* $\mathrm{Rank}(F_1, T) \leftrightarrow \mathrm{Rank}(F_2, T)$.

Proof. (i) follows directly from 2.11 (i). (ii) follows directly from 2.11 (a) and (iii). (iii) It is enough to prove that if u is an isomorphism of F_1 and F_2, then

$$\mathrm{Rank}(F_1, x, T) \leftrightarrow \mathrm{Rank}(F_2, u_x, T).$$

DEFINITION 2.14. The w.f.t. F is called *reduced w.f.t.* iff

$$(x, y, z)_{\mathscr{D}F \cup \mathscr{R}F}[y, z \in F^{-1} * \{x\} \ \& \ y \neq z \rightarrow \sim(F_y \sim F_z)]$$

(i.e. F_y is not isomorphic to F_z).

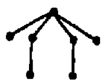
DEFINITION 2.15 (Reduction procedure). Assume that for every $y \in F^{-1} * \{x\}$ the tree G_y (called *reduct of* F_y) is given. We form the tree G_x as follows; First we form a new relation G'_x by the equation

$$\langle y, x\rangle \in G'_x \leftrightarrow (z)\,[z \in y \ \& \ z \in F^{-1} * \{x\} \rightarrow \sim (G_y \sim G_z)].$$

Now

$$G_x = \Big(\bigcup_{y \in G'^{-1}_x * \{x\}} G_y \cup G'_x\Big).$$

The rather complicated definition needs an explanation; We simply cut out subtrees which are too rich; e.g. the tree

 will be reduced to the tree  the tree  will be reduced first to one  and then to 

Further explanation will be clear when we prove 2.20.

LEMMA 2.16. *The tree G_x obtained in* 2.13 *is reduced.*

Proof. It is clear that it is enough to consider $y, z \in G^{-1} * \{x\}$. But then it is just clear from the definition of G_x.

DEFINITION 2.17. The tree G_x is called a *reduct of* F_x. In case when $x = \mathrm{MAX}_F$ we call G_x a reduct of F.

METALEMMA 2.18. (a) *There is a formula* $\mathrm{Red}(F, x, y)$ *describing*:

Y is a reduct of F_x.

(b) *There is formula* $\mathrm{Red}(F, G)$ *describing*:

G is a reduct of F.

Proof. (a) Follows from the principle of inductive definitions. (b) Follows from (a).

The formula $\mathrm{Red}(F, x, Y)$ has the following property:

$$\mathrm{Red}(F, x, Y) \ \& \ \mathrm{Red}(F, x, Y_1) \rightarrow Y = Y_1$$

and therefore the formula $\mathrm{Red}(F, Y)$ has a property

$$\mathrm{Red}(F, Y) \ \& \ \mathrm{Red}(F, Y_1) \rightarrow Y = Y_1.$$

Remark 2.19. Every w.f.t. can be extended to a well-ordering; i.e.

$$(X)\big(X \text{ w.f.t.}\big) \rightarrow (ET)\big(\mathrm{Bord}(T) \ \& \ X \subseteq T\big).$$

Proof. We describe T as follows: Let the level of the degree S denotes the class of these objects in $\mathcal{D}x \cup \mathcal{D}x$ which have in X rank S. Then the relation $x \leqslant^* y \leftrightarrow$ [The degree of the level of x is smaller then the degree of the level of y] or [x, y belong to the same level and $x \in y$] is a well-ordering which extends X (by 2.11 (iii)).

LEMMA 2.20. (i) *If F is reduced w.f.t., then the unique automorphism of F is the identity.*

(ii) *If F, G are reduced w.f.t., then if $F \sim G$, then there is the unique isomorphism of F and G.*

Proof. (i) Assume not; let φ be non-trivial automorphism of F and G. Then there is x minimal in the class of all z such that $\varphi \upharpoonright F_z$ is non-trivial. Then there are $z_1, z_2 \in F^{-1} * \{x\}$ such that $\varphi * F_{z_1} = F_{z_2}, z_1 \neq z_2$ and φ is trivial on F_{z_1} and F_{z_2}. Then $F_{z_1} \sim F_{z_2}$.

(ii) If φ, ψ are different isomorphisms, then $\varphi^{-1} \circ \psi$ is a non-trivial automorphism of F.

LEMMA 2.21. *The w.f. reduced trees are exactly w.f.t. without non-trivial automorphisms.*

Proof. By 2.20 (i) we only need to prove that if F has no non-trivial automorphisms, then F is irreducible.

Assume that F is not reduced. Then there are x, y, z

$$y, z \in F^{-1} * \{x\} \quad \text{and} \quad F_y \sim F_z.$$

Now the mapping induced by a similarity map of F_y and F_z and the transposition of y and z is a non-trivial automorphism of F.

LEMMA 2.22. *If F is reduced, then F_x is reduced for every $x \in \mathcal{D}F \cup \mathcal{R}F$.*

Proof. Non-trivial automorphism of F_x can be extended to non-trivial automorphism of F.

DEFINITION 2.23. $AMAX_F = F^{-1} * \{MAX_F\}$; $AMAX_F$ is a class of almost maximal elements of F.

Here it should be said that since we consider the trees to imitate "the sets of rank bigger then *On*" therefore if $x \in AMAX_F$ we imagine F_x to "belong" to F.

DEFINITION 2.24. (i) $F_1 \mathrm{Eps} F_2 \leftrightarrow (Ey)_{AMAX_{F_2}} (EX_1, X_2)$ [$\mathrm{Red}(F_1, X_1)$ & $\mathrm{Red}(F_2, y, X_2)$ & $X_1 \sim X_2$];

(ii) $F_1 \mathrm{Eq} F_2 \leftrightarrow (EX_2, X_2)\big(\mathrm{Red}(F_1, X_1)$ & $\mathrm{Red}(F_2, X_2)$ & $X_1 \sim X_2\big)$;

(iii) $F_1 \mathrm{In} F_2 \leftrightarrow (Z)(Z \mathrm{Eps} F_1 \to Z \mathrm{Eps}\ F_2)$.

LEMMA 2.25. $\mathrm{Red}(F_1, F_2) \to \big(\mathrm{Rank}(F_1, T) \leftrightarrow \mathrm{Rank}(F_2, T)\big)$.

Proof. Assume that $\mathrm{Red}(F_1, F_2)$ we analize the construction of F_2 under the inductive hypothesis (i.e. for $T_1 < T$ the lemma is valid).

Then: if $\mathrm{Red}(X_y, Y_y)$, then $\mathrm{Rank}(F_1, y, T) \leftrightarrow \mathrm{Rank}(F_2, y, T)$. So we see that if in $F_1^{-1}*\{x\}$ there is an object with rank T, then we can find another object with the same rank in $F_2^{-1}*\{x\}$.

Therefore the coresponding suprema are equal. (We agree that it needs the proof that $X \simeq Y \to \mathrm{Sup}\,X \sim \mathrm{Sup}\,Y$ but it is quite simple.)

THEOREM 2.26. (i) *If* $X\,\mathrm{Eps}\,Y$ *and* $Y\,\mathrm{Eq}\,Z$, *then* $X\,\mathrm{Eps}\,Z$.

(ii) *If* $X\,\mathrm{Eq}\,Y$ *and* $Y\,\mathrm{Eps}\,Z$, *then* $X\,\mathrm{Eps}\,Z$.

Proof. (i) The first assumption implies that the reduct of X is similar to reduct of Y_x for some $x \in A\,\mathrm{MAX}_y$. The tree Y_x is not necessarily reduced but we can see that its reduct will be similar to a tree starting in an almost maximal vertex of the reduct of Y. So the same holds for some almost maximal vertex of the reduct of Z which just means that $X\,\mathrm{Eps}\,Z$.

(ii) Is even simpler.

LEMMA 2.27. *The relation* Eps *is well founded, i.e.*

$$(EY)\,(Y\,\mathrm{Eps}\,X) \to (EY)\,[Y\,\mathrm{Eps}\,X \,\&\, (Z)\,(Z\,\mathrm{Eps}\,Y \to \neg Z\,\mathrm{Eps}\,X)].$$

Proof. Take $y \in A\,\mathrm{MAX}_x$ with the smallest rank in X. Then we claim that X_y is the required Y. Indeed, from 2.25 we can derive that the trees equal in sense of Eq have the same rank and it is clear that the construction of a formula Eps was as follows:

If $X\,\mathrm{Eps}\,Y$, *then there is* $x \in A\,\mathrm{MAX}_y X\,\mathrm{Eq}\,Y_x$.

So we know that any tree which is an element (in the sense of the relation Eps) of Y has a rank smaller then Y so it cannot be an element (in the sense of relation the Eps) of X.

LEMMA 2.28. *The relation* Eq *has properties of equivalence.*

Proof. Obvious.

METATHEOREM 2.29. *Formal expressions corresponding to the axioms of extensionality and foundation are true among trees, i.e. The sentences* **Ext** *and* **Found** *arising from extensionality and foundation by substitution of* Eq *and* Eps *instead of* $=$ *and* $\in$ *and restricting quantifiers to trees are provable in* M.

Proof. (i) Foundation case; it follows from 2.27.

(ii) Extensionality case. Therefore let us assume $X\,\mathrm{In}\,Y$ and $Y\,\mathrm{In}\,X$.

We can assume that X and Y are reduced. We need to construct an isomorphism of X and Y. If $x \in A\,\mathrm{MAX}_X$, then there is exactly one $y \in A\,\mathrm{MAX}_Y$ such that $X_x\,\mathrm{Eq}\,Y_y$ this gives us the isomorphism of X and Y.

METATHEOREM 2.30. *Let* Φ *be a formula arising from* Φ *by a procedure described in* 2.29. *Let* $\{i_0, \ldots, i_k\} = \mathrm{Fr}\,\Phi$. *Then*

$$X_{i_0}\mathrm{Eq}\,Y_{i_0} \,\&\ldots\&\, X_{i_k}\mathrm{Eq}\,Y_{i_k} \to [\Phi(X_{i_0}\ldots X_{i_k}) \leftrightarrow \Phi(Y_{i_0}\ldots Y_{i_k})].$$

Proof. By easy induction.

METATHEOREM 2.31. **Pair, Union, Infinity** *are true among trees.*

Proof. It is clear how to construct the tree imitating ω. The union tree for X arises from X just by cutting out $A\mathrm{MAX}_X$ and joining $\bigcup_{x \in A\mathrm{MAX}_X} (A\mathrm{MAX}_{X_x})$ directly with MAX_X. The pair tree for X and Y is arising by the procedure known as "Hilbert Hotel". Since we will use it often in future we describe it now in this particular case. We split On into 3 disjoint parts: One: one point and two parts: each similar to On. In these big parts we copy X and Y (this needs no form of an the axiom of choice since X and Y are built up from ordinals and there is at most one order preserving mapping of $\mathscr{D}X \cup \mathscr{R}X$ onto the initial segment of On) now we have disjoint w.f.t. X_1 and Y_1 and we "glue" them together with the remaining one ordinal as the maximal element.

METATHEOREM 2.32. **Substitution schema of ZF** *is true among trees.*

Proof. Let F be a tree and Φ a predicate. For Y Eps$F$ we have a unique (up to Eq) Z such that $\Phi(Y, Z)$. But by 2.30 this is nothing more but $(x)_{A\mathrm{MAX}_F}(EZ)\Phi(F_x, Z)$.

Now by a corresponding occurrence of IX we have X such that $\Phi(F_x, X^{(x)})$.

Now we repeat the Hilbert Hotel procedure (cf. 2.31) using not two but On disjoint parts each similar to On. We glue them together into one tree. This is the tree containing all images.

LEMMA 2.33. *For every ordinal a there is a set W_a of trees such that if* $\mathrm{Rank}(X, T)$, $T < E \cap a^2$, *X reduced, then there is $Z \in W_a$ such that $X \sim Z$.*

Proof. We construct W_a by induction as follows; In limit points we just take unions. W_{a+1} is formed as follows.

For each $x \in \mathscr{P}(W_a)$ we form by the Hilbert Hotel procedure a tree F_x which is reduced and has a property: $Y\mathrm{Eps}F_x \leftrightarrow (EZ)_x(Y\mathrm{Eq}Z)$.

LEMMA 2.34. *Let* $\boldsymbol{V}$ *be a tree with a property*

$$(Y)\left[Y\mathrm{Eps}\boldsymbol{V} \leftrightarrow (Ea)_{On}\big(\mathrm{Rank}(Y, E \cap a^2)\big)\right].$$

(A tree like this can be formed using 2.33.*) Then if* Sin($\cdot$) *is a formula expressing strong inaccessibility, then* **Sin(V)**.

Proof. Only a powerset relativized to V needs the proof which is equally clear by 2.33.

METATHEOREM 2.35. **Choice** *is true among trees.*

Proof. Let F be a tree such that for $x, y \in A\mathrm{MAX}_F$,

$$x \neq y \rightarrow \boldsymbol{F}_x \cap \boldsymbol{F}_y = \emptyset.$$

Then also

$$A\mathrm{MAX}_{F_x} \cap A\mathrm{MAX}_{F_y} = \emptyset.$$

Now assumption $F_x \neq \emptyset$ means $AMAX_{F_x} \neq \emptyset$. Take $h(x) =$ first element of $AMAX_{F_x}$. We form now

$$Y = (\bigcup_{x \in AMAX_F} F_{h(x)}) \cup \{\langle h(x), MAX_F \rangle : x \in AMAX_F\}.$$

Since we have chosen one element (in sense of Eps) from each F_x and glued them together, we have the selector.

METATHEOREM 2.36. *Let C mean the following scheme:*

$$(x)_y (Ez)\, \Phi(x, z) \to (Ez)\, (x)_y \Phi(x, z^{(x)}).$$

Then C is true among trees.

Proof. When we write down the translation C we get:

$$(X)\big(X \operatorname{Eps} Y \to (EZ)\Phi(X, Z)\big) \to (EZ)\,(X)\big(X \operatorname{Eps} Y \to \Phi(X, Z^{(x)})\big).$$

But by 2.30 we have our assumption in form

$$(x)_{AMAX_y} (EZ)\Phi$$

so by Axiom IX we can produce appropriate Z.

COROLLARY 2.37. *Let ZFC^- denote the ZF set theory with the scheme C* (cf. 2.36) *but without a powerset axiom. Let T denote the axiom: There is a strongly inaccessible family of sets. Then $ZFC^- + T$ is interpretable in M.*

Proof. By 2.29–2.34 and 2.36.

THEOREM 2.38 *M is interpretable in $ZFC^- + T$.*

Proof. Let V be an inaccessible family whose existence is guaranteed by T Let $\Phi(X) \leftrightarrow X \subseteq V$. Then it is clear that the fact that V is inaccessible implies the satisfaction of Axioms I-VII. ZF-comprehension implies M-comprehension (VIII) and the scheme of choice C implies scheme IX.

THEOREM A. *The theories M and $ZFC^- + T$ are equiconsistent.*

Remark 2.39. One should see that M is better interpretable in $ZFC^- + T$ then conversly, since in the first interpretation we just interprete "ϵ" and "$=$" identically while in the second interpretation not even "$=$" is interpreted as equality.

PROBLEM 2.40. *Does 2.38 holds when M is weakened to M_0 and $ZFC^- + T$ to $ZF^- + T$?*

Remark 2.41. M_0 is interpretable in $ZF^- + T$ by the same reasoning. So only the second direction interpretability is not known.

The interpretation from 2.38 and the above remark gives us an additional tool for indepence proofs for M and M_0. We shall use it later.

THEOREM 2.42. *Let φ be a sentence of the language of the set theory. Then $M + \varphi^{(z)}$ is consistent iff $ZFC^- + T + \varphi^{(s)}$ is consistent.*

Where S is a inaccessible family whose existence is guaranted by the axiom T.

3. Ordinal trees. Constructibility in M

DEFINITION 3.1. The tree X is called an *ordinal tree* iff

$$Z \mathrm{Eps} X \ \& \ T \mathrm{Eps} Z \to T \mathrm{Eps} X,$$
$$Z \mathrm{Eps} X \ \& \ T \mathrm{Eps} X \to Z \mathrm{Eps} T \vee T \mathrm{Eps} Z \vee T \mathrm{Eq} Z.$$

Convention 3.2. Small bold face Greek letters denote the ordinal trees.

Since the relation Eps is well founded and the axioms of ZFC^- are satisfied among the trees we can prove exactly as in ZF case the following

LEMMA 3.3. (a) $X \mathrm{Eps}\, \boldsymbol{\alpha} \to (E\boldsymbol{\beta})(X \mathrm{Eq}\, \boldsymbol{\beta})$.

(b) $\boldsymbol{\alpha} \mathrm{Eps}\, \boldsymbol{\beta} \vee \boldsymbol{\beta} \mathrm{Eps}\, \boldsymbol{\alpha} \vee \boldsymbol{\alpha} \mathrm{Eq}\, \boldsymbol{\beta}$.

(c) $\boldsymbol{\alpha} \mathrm{In}\, \boldsymbol{\beta} \to \boldsymbol{\alpha} \mathrm{Eps}\, \boldsymbol{\beta} \vee \boldsymbol{\alpha} \mathrm{Eq}\, \boldsymbol{\beta}$.

(d) *If X is a class such that for every $x \in \mathscr{D}X$, $X^{(x)}$ is an ordinal tree, then there is $\boldsymbol{\gamma}$ such that*

$$(x)_{\mathscr{D}x}(X^{(x)} \mathrm{Eps}\, \boldsymbol{\gamma}).$$

(e) *If X is a tree such that $T \mathrm{Eps} X \to (E\boldsymbol{\gamma})(T \mathrm{Eq}\, \boldsymbol{\gamma})$, then there is unique (up to* Eq *relation) $\boldsymbol{\alpha}$ such that:*

$$(\boldsymbol{\rho})\big(\boldsymbol{\rho} \mathrm{Eps}\, \boldsymbol{\alpha} \leftrightarrow (E\boldsymbol{\delta})(\boldsymbol{\delta} \mathrm{Eps} X \ \& \ \boldsymbol{\rho} \mathrm{Eps}\, \boldsymbol{\delta})\big).$$

LEMMA 3.4. *Let $\boldsymbol{\alpha}$ be a reduced ordinal tree. Define the relation in $A\,\mathrm{MAX}_{\boldsymbol{\alpha}}$ as follows: $x \leqslant y \leftrightarrow \boldsymbol{\alpha}_x \mathrm{In}\, \boldsymbol{\alpha}_y$. Then $\leqslant$ is a well-ordering. Moreover,* $\mathrm{Rank}(\boldsymbol{\alpha}, \leqslant)$.

In the situation of 3.4 we say that $\boldsymbol{\alpha}$ and all X such that $X \mathrm{Eq}\, \boldsymbol{\alpha}$ represent $\leqslant$.

LEMMA 3.5. (a) *For every well-ordering T there is $\boldsymbol{\alpha}$ such that $\boldsymbol{\alpha}$ represents T.*

(b) *If $\boldsymbol{\alpha}$ represents T and $\boldsymbol{\beta}$ represents S, then $\boldsymbol{\alpha} \mathrm{Eq}\, \boldsymbol{\beta} \leftrightarrow T \sim S$, $\boldsymbol{\alpha} \mathrm{Eps}\, \boldsymbol{\beta} \leftrightarrow T < S$, $\boldsymbol{\alpha} \mathrm{In}\, \boldsymbol{\beta} \leftrightarrow T \leqslant S$.*

LEMMA 3.6. *Let Φ be a formula and assume that $(\boldsymbol{\alpha})\big[(\boldsymbol{\beta})\big(\boldsymbol{\beta} \mathrm{Eps}\, \boldsymbol{\alpha} \to \Phi(\boldsymbol{\beta})\big) \to \Phi(\boldsymbol{\alpha})\big]$; then $(\boldsymbol{\alpha})\Phi(\boldsymbol{\alpha})$.*

Proof. Assume not; Then there is $\boldsymbol{\alpha}$ such that $\neg\, \Phi(\boldsymbol{\alpha})$ and by 3.4 and 3.5 (b) we can assume that all its Eps-elements have a property $\Phi(\cdot)$. Thence $\Phi(\boldsymbol{\alpha})$ contradition.

Since the axioms of ZFC^- hold among the trees we are able to prove that there is a unique (up to Eq-relation) Z such that $(T)(T \mathrm{Eps} Z \leftrightarrow T \mathrm{Eq} X \vee T \mathrm{Eq} Y)$ this tree is called a *tree-pair of X and Y*. In the same way we introduce tree-ordered pairs, tree-functions and a tree-transfinite sequence.

Seq(X) means X is a tree-transfinite sequence, **Dom**(X) is unique (up to Eq-relation) Y which is the domain of the tree-function X. If X

is a tree-transfinite sequence with the domain α and $\beta \mathrm{Eps}\, \alpha$, then $X \upharpoonright \beta$ is a restriction of X to β. $X(\alpha)$ is a value of X in α. One should note that we are not able to prove that a power tree for a given tree exists. But anyhow with the help of pairing we are able to prove the existence of the substitute of a cardinal product of X and Y which is (in ZF-case) usually proved with the help of the axiom of the power set.

LEMMA 3.7 (On inductive definitions). *Let* Φ *be a formula of the language of set theory such that* $(X)(EY)(Z)\big(\Phi(X,Z) \leftrightarrow Z \mathrm{Eq}\, Y\big)$; *then* $(A)\,(\alpha)\,(EY)\,\big[\mathbf{Seq}(Y)\ \&\ \mathbf{Dom}(Y)\mathrm{Eq}\,\alpha\ \&\ Y(0)\mathrm{Eq}\,A\ \&\ (\gamma)\,(\gamma \mathrm{Eps}\,\alpha) \to \Phi\big(Y \upharpoonright \gamma, Y(\gamma)\big)\big]$, *$Y$ is unique up to* Eq *relation.*

Proof. The proof is almost identical as in the ZF case; only in the limit α in view of the fact that Eq is not really an equality relation, we have to use the scheme IX. (Note that Φ was introduced in 2.29 and 2.30.) With the help of 3.7 we introduce the notion of Rank of the tree (analogously as it was done in Section 2) and we have the predicate $\mathbf{Rank}(X, \alpha)$ which describes the situation that α is the rank of X. We list here some of the properties of $\mathbf{Rank}(\cdot, \cdot)$.

PROPOSITION 3.8. (a) $\mathbf{Rank}(X, \alpha)\ \&\ \mathbf{Rank}(Y, \beta)\ \&\ X \mathrm{Eps}\, Y \to \alpha \mathrm{Eps}\, \beta$.

(b) $\mathbf{Rank}(X, \alpha)\ \&\ \mathbf{Rank}(Y, \beta)\ \&\ X \mathrm{In}\, Y \to \alpha \mathrm{In}\, \beta$.

(c) $\mathbf{Rank}(X, \alpha)\ \&\ \mathbf{Rank}(Y, \beta)\ \&\ X \mathrm{Eq}\, Y \to \alpha \mathrm{Eq}\, \beta$.

(d) *If* α *represents* T, *then* $\mathbf{Rank}(X, \alpha) \leftrightarrow \mathbf{Rank}(X, T)$.

(e) *If* X *is a tree and* Z *is a tree formed in the following way*: $\alpha \mathrm{Eps} Z \leftrightarrow (EY)\big(Y \mathrm{Eps} X\ \&\ \mathbf{Rank}(Y, \alpha)\big)$ *and* β *is an ordinal tree corresponding to* Z *by* 3.3 (e), *then* $\mathbf{Rank}(X, \beta)$.

Since we have an almost complete machinery of the set theory we are able to interpret formulas as certain trees. Then we introduce a satisfaction relation $\models$ inductively such a way that it satisfies the following condition:

$$X \models v_i \in v_j [Y] \leftrightarrow Y(i) \mathrm{Eps} X\ \&\ Y(j) \mathrm{Eps} X\ \&\ Y(i) \mathrm{Eps}\, Y(j),$$

$$X \models v_i = v_j [Y] \leftrightarrow Y(i) \mathrm{Eps} X\ \&\ Y(j) \mathrm{Eps} X\ \&\ Y(i) \mathrm{Eq}\, Y(j),$$

$$X \models \neg\Phi[Y] \leftrightarrow \neg X \models \Phi[Y],$$

$$X \models \Phi\ \&\ \Psi[Y] \leftrightarrow X \models \Phi[Y \upharpoonright \mathrm{Fr}\,\Phi]\ \&\ X \models \Psi[Y \upharpoonright \mathrm{Fr}\,\Psi],$$

$$X \models (Ev_i)\Phi[Y] \leftrightarrow \big[\neg(i \mathrm{Eps} \mathrm{Fr}\,\Phi)\ \&\ X \models \Phi[Y]\big] \vee \big[i \mathrm{Eps} \mathrm{Fr}\,\Phi\ \&\ (EZ)\big(Z \mathrm{Eps} X\ \&\ X \models \Phi[a(Y, Z, i)]\big)\big],$$

where $a(Y, Z, i)$ is a sequence formed from Y by adding the tree-ordered pair $\langle i, Z\rangle$.

With the help of 3.7 we form as in ZF case the formula $\mathbf{Stsf}(\cdot, \cdot, \cdot)$ such that $\mathbf{Stsf}(X, n, Y) \leftrightarrow X \models \Phi[Y]$ (n is the Gödel number of Φ).

DEFINITION 3.9. Let X be a w.f.t., Φ a formula of set theory and the tree-sequence of Eps-elements of X indexed by $\mathrm{Fr}\Phi - \{0\}$, then the tree T is said to have a *property* $W(X, \Phi, Y)$ iff

$$(Z)\left[(Z \mathrm{Eps} T) \leftrightarrow \left(Z \mathrm{Eps} X \ \& \ X \models \Phi[a(Y, Z, 0)]\right)\right].$$

LEMMA 3.10. (i) *For a given tree X the formula Φ and the sequence Y there is a tree T with the property $W(X, \Phi, Y)$.*

(ii) *A tree like this is unique up to* Eq-*relation.*

Proof. (i) is proved by cutting out of X appropriate tree.

(ii) follows from the fact that the axiom of extensionality is true among the trees.

DEFINITION 3.11. Let X be a w.f.t. The tree Y is said to *have the property* $\mathbf{Def} X$ iff $(T)\left(T \mathrm{Eps} Y \leftrightarrow \text{There are } \Phi \text{ and } Z \text{ such that } T \text{ has a property } W(X, \Phi, Z)\right)$.

LEMMA 3.12. (i) *For a given tree X there is a tree Y such that Y has the property* $\mathrm{Def} X$.

(ii) *A tree like that is unique up to* Eq *relation.*

Proof. (i) First we remind that by 2.30 if $X \models \Phi[Y]$ and $Y \mathrm{Eq} Z$, then $X \models \Phi[Z]$. Now we are able to represent every finite sequence of trees, Eps elements of X by a sequence of elements of $A\mathbf{MAX}_x$. Now we use the Hilbert Hotel method and the argument of 0.29.

(ii) as usual by Extensionality.

DEFINITION 3.13. (a) A union tree of a tree X (which exists by an axiom of union and is unique up to Eq) is denoted as $\bigcup X$.

(b) Let α be an ordinal tree. The unique up to Eq relation ordinal tree β such that $(\gamma)\,(\gamma \mathrm{Eps} \beta \leftrightarrow \gamma \mathrm{Eps}\, \alpha \vee \gamma \mathrm{Eq}\, \alpha)$ is denoted $\alpha + 1$.

(c) The ordinal tree α is called a *limit* if there is no β such that $\alpha \mathrm{Eq}\, \beta + 1$. In this case we write $\mathbf{Lim}(\alpha)$.

In view of Lemma 3.7 there is a formula $L(\cdot, \cdot)$ such that $L(X, \emptyset) \leftrightarrow \emptyset \mathrm{Eq} X$, $L(X, \alpha+1) \leftrightarrow (EY)\left(L(Y, \alpha) \ \& \ X \mathrm{Eq} \mathrm{Def}\, Y\right)$ and for α limit $\alpha(X, \alpha) \leftrightarrow X \mathrm{Eq} \bigcup Y$, where Y is a tree diagram of function α. The unique (up to Eq relation) tree X such that $L(X, \alpha)$ we denote L_α.

DEFINITION 3.14. $L(X) \leftrightarrow (E\alpha)\,(X \mathrm{Eps} L_\alpha)$. Every X such that $L(X)$ is called *constructible.*

LEMMA 3.15. *Predicate $L(X)$ is* Eps-*transitive.*

THEOREM B. *Let* Th *be the following theory in the language of the set theory*: $ZFC^- + T + V = L$ (*a usual axiom of constructibility*). *Then* $M \vdash (\mathbf{Th})^L$, *where* $\mathbf{Th} = \{\boldsymbol{\Phi}\colon \Phi \in \mathrm{Th}\}$.

The proof needs certain amount of lemmas;

LEMMA 3.16 (Reflection principle) (Scheme). $(X)\Big\{\boldsymbol{L}(X) \to (\boldsymbol{E}a)\Big[X\,\mathrm{Eps}\,\boldsymbol{L}_a\ \&\ (Y)\Big(\mathbf{Dom}\,Y = \mathrm{Fr}\,\Phi\ \&\ \bigwedge_{i\in \mathrm{Fr}\,\Phi} Y(i)\,\mathrm{Eps}\,\boldsymbol{L}_a \to \Big(\boldsymbol{L}_a \models \Phi[Y] \leftrightarrow \Phi^{\boldsymbol{L}}\big(Y(i_0), \ldots, Y(i_n)\big)\Big)\Big)\Big]\Big\}$.

Proof. By induction on Φ by reasoning of 1.10 and 1.11. This gives the proof of the substitution scheme. $\boldsymbol{L}_{On}$ is the inaccessible family. Only $V = L$ needs some proof but this is easy by reasoning of Lemma 3.17.

LEMMA 3.17. (a) *All ordinal-trees are in* $\boldsymbol{L}$ *and they are absolute w.r.t.* $\boldsymbol{L}$.

(b) *Satisfaction predicate. Property* $W(\cdot,\cdot,\cdot)$ *and Predicate* $\mathrm{Def}(\cdot)$ *are absolute w.r.t.* $\boldsymbol{L}$.

(c) *Predicates* $\boldsymbol{L}(X, a)$ *and* $\boldsymbol{L}(X)$ *are absolute w.r.t.* $\boldsymbol{L}$.

(d) $V = L$ *holds in* $\boldsymbol{L}$.

Now with $V = L$ we are able to prove $\boldsymbol{C}$.

DEFINITION 3.18. (a) The tree Y is called *realizable* if $\mathbf{Rank}(X, a) \to a\,\mathrm{In}\,\mathbf{On}$ (where $\mathbf{On}$ is the ordinal tree representing $E \restriction On$).

(b) Let $x \in \mathscr{D}X \cup \mathscr{R}X$, X realizable; then

$$(*)\qquad \|y\|_X = \{\|x\|_X \colon X(x) = y\}, \qquad \|X\| = \|\mathrm{MAX}_x\|_x.$$

LEMMA 3.19. *Definition* 3.18 *is proper, i.e. under assumptions of* 3.18 (a) *there is exactly one class satisfying* $(*)$. *Moreover, if the notion of rank is extended as follows:* $\bar{\varrho}(X) = \varrho(X)$ *if* X *is a set and* $\bar{\varrho}(X) = On$ *if* X *is a proper class, then* $\bar{\varrho}(\|y\|_x)$ *has the property:* $\mathbf{Rank}(X_y, a) \to a$ *represents* $\bar{\varrho}(\|y\|_x)$.

Proof. Uniqueness is easily provable by induction on the rank of y in X. The second part of lemma follows because our ordinal trees behave like real ordinals.

LEMMA 3.20. (a) *If* $X \sim Y$ *and* $\|X\|$ *exists, then* $\|Y\|$ *exists and* $\|X\| = \|Y\|$.

(b) *If* $\mathrm{Red}(X, Y)$ *and* $\|X\|$ *exists, then* $\|Y\|$ *exists and* $\|X\| = \|Y\|$.

(c) *If* $X\,\mathrm{Eq}\,Y$ *and* $\|X\|$ *exists, then* $\|Y\|$ *exists and* $\|X\| = \|Y\|$.

The proof is easy; we only note that (b) and (c) are proved by simultaneous induction on the maximum of ranks of X and Y.

LEMMA 3.21. *If* X *is realisable and* $Y\,\mathrm{Eps}\,X$, *then*

(a) Y *is realisable,*

(b) $\|Y\| \in \|X\|$.

Proof. By 3.20 and definition.

LEMMA 3.22. *For every class* X *there is a tree* Y *such that* $\|Y\| = X$.

Proof. First step: we prove that every $y \in V$ has this property by induction on rank and the Hilbert Hotel method.

Second step: For proper classes we use first step and the Hilbert Hotel method.

LEMMA 3.23. *If* $X = \|Y\|$ *and* $Z = \|T\|$, *then*

(i) $X \in Z \leftrightarrow Y \operatorname{Eps} T$,

(ii) $X = Z \leftrightarrow Y \operatorname{Eq} T$.

Proof. The implications $\rightarrow$ were proved in 3.20 and 3.21, the implications $\rightarrow$ follow by induction on generalized rank.

In the sequel we shall identify ordinal trees coding ordinals smaller then or equal to *On* with the appropriate ordinals.

LEMMA 3.24. *If* α *is an ordinal, then*

(i) $\boldsymbol{L}_\alpha$ *is realizable,*

(ii) $\|\boldsymbol{L}_\alpha\| = L_\alpha$.

Proof. By induction on α. Thence $\|\boldsymbol{L}_{On}\| = L$.

LEMMA 3.25 (The Skolem–Löwenheim theorem for trees). *Let* X *be a tree,* $y \subseteq A\mathrm{MAX}_X$ *a set. Then there is a set* $z \subseteq A\mathrm{MAX}_X$ *such that* $y \subseteq z$, $\bar{\bar{z}} = \max(\bar{\bar{y}}, \aleph_0)$ *and the tree* $X(z)$ *consisting of all the elements of* X *comparable* (*in sense of* R_X) *with the elements of* z *is an elementary subtree of* X, *i.e.* $X(z) \preccurlyeq X$.

Proof. For a given tree X we define the Skolem functions and close y under them.

DEFINITION 3.26. (a) $\mathscr{L}(X) \leftrightarrow \boldsymbol{L}(X)\ \&\ X \operatorname{In} \boldsymbol{L}_{On}$.

(b) $\mathfrak{L}(X) \leftrightarrow (EY)\big(\mathscr{L}(Y)\ \&\ X = \|Y\|\big)$.

LEMMA 3.27. (a) *Predicate* $\mathscr{L}$ *is* Eps *transitive.*

(b) *Predicate* $\mathfrak{L}$ *is transitive.*

THEOREM 3.28. $V = L$ *holds in* $\mathfrak{L}$.

Proof. It is just a variant of the usual proof for the *ZF* case. All the necessary machinery was introduced in 3.23, 3.24, 3.25.

LEMMA 3.29. $M_0^{\#}$ *holds in* $\mathfrak{L}$.

Proof. Obvious.

LEMMA 3.30 (Translation Lemma). (a) *If* Φ *is formula of the set theory,* $X_1 = \|F_1\|, \ldots, X_n = \|F_n\|$, *then* $\Phi^{\mathfrak{L}}(X_1, \ldots, X_n) \leftrightarrow \boldsymbol{\Phi}^{\mathscr{L}}(F_1, \ldots, F_n)$.

(b) *Let* Φ *formula of set theory,* $\boldsymbol{\Phi}$ *its translation* $\underline{\boldsymbol{\Phi}}$ *restriction of* $\boldsymbol{\Phi}$ *to predicate* $\mathscr{A}(X) \leftrightarrow X \operatorname{In} \boldsymbol{L}_{On}$; *then if* $X_1, \ldots, X_n$ *has property* $\mathscr{L}$, *then* $\underline{\boldsymbol{\Phi}}^{\boldsymbol{L}}(X_1, \ldots, X_n) \leftrightarrow \boldsymbol{\Phi}^{\mathscr{L}}(X_1, \ldots, X_n)$.

Proof. By induction.

LEMMA 3.31 (Scheme). *Each* VIII_Φ *holds in* $\mathfrak{L}$.

Proof. We use 3.30 (a), (b) and the reflection principle in order to change comprehension axioms of *M* into comprehension of *ZF* which holds in $\boldsymbol{L}$ by 3.17.

LEMMA 3.32. (Scheme) *Each* IX_Φ *holds in* $\mathfrak{L}$.

Proof. Since a definable well-ordering of all classes holds in $\mathfrak{L}$ we can choose for each x first Y such that $\Phi(x, X)$ and (by reflection principle) glue them together.

COROLLARY 3.33. *If* M *is consistent so is* $M+V=L$.

We, however, shall get a much stronger result soon.

4. Minimal model for M

This chapter will be split into two symmetric sequences of lemmatas and definitions which fit appropriately cases I and II in the proof of the theorem.

DEFINITION 4.1. (a) Let A be a class of pairs. **Trans**$(A) \leftrightarrow (X, Y)$ $(X\eta A \ \&\ Y\in X \rightarrow Y\eta A)$.

(b) Let $\mathscr{A}(\cdot)$ be one-place-predicate; then $\mathscr{A}$ is transitive if $\mathscr{A}(X)$ $\&\ Y\in X \rightarrow \mathscr{A}(Y)$.

DEFINITION 4.2. (a) Let A be a class of pairs. The well-ordering T is called the *height of* A if

(i) $(B)\,[\dot{\mathrm{B}}\mathrm{ord}(B)\ \&\ B\eta A \rightarrow (Ex)\,(B \sim T\restriction x)]$,

(ii) $(x)_{\mathscr{D}T}(EB)\,(\mathrm{Bord}(B)\ \&\ B\eta A\ \&\ T\restriction x \leqslant B)$.

(b) The same as 4.2 (a) with A changed into $\mathscr{A}$ etc.

LEMMA 4.3. (a) *For any class of pairs* A *there is* T *which is the height of* A.

(b) *If* T *is the height of* A *and* $T \sim T'$, *then* T' *is also the height of* A.

(c) *If* $A \simeq B$ *and* T *is the height of* A, *then* T *is also the height of* B.

(a′) *If* $\mathscr{A}(\cdot)$ *is a codable predicate, then* $\mathscr{A}(\cdot)$ *has the heigth.*

(b′) *If* T *is the heigth of* $\mathscr{A}(\cdot)$ *and* $T \sim T'$, *then* T' *is also the heigth of* $\mathscr{A}(\cdot)$.

(c′) *If* $(X)\,[\mathscr{A}(X) \leftrightarrow \mathscr{B}(X)]$, *then* T *is the heigth of* $\mathscr{A}(\cdot)$ *iff it is the heigth of* $\mathscr{B}(\cdot)$.

DEFINITION 4.4. Let a be an ordinal tree; then:

$$\mathfrak{L}_a(X) \leftrightarrow \mathfrak{L}(X)\ \&\ (EY)\,(\|Y\| = X\ \&\ Y\,\mathrm{Eps}\boldsymbol{L}_a).$$

I.e. $\mathfrak{L}_a$ is a restriction of $\mathfrak{L}$ to the classes constructed in at most a steps.

LEMMA 4.5. *For any* a *the predicate* $\mathfrak{L}_a(\cdot)$ *is codable.*

Proof. We prove this by induction on a. If a is smaller, than On, then $\mathfrak{L}_a \leftrightarrow X\in L_a$. For $a > On$ we use the Hilbert Hotel method and 0.29.

COROLLARY 4.6. *For each* a, $\mathfrak{L}_a(\cdot)$ *has the heigth.*

THEOREM 4.7. (a) *Let* **Trans**(A), *assume that for every axiom* Φ *of* $M_0, A \models \Phi$ *and* $On \eta A$; *then* $B \eta A \to (A \models \mathrm{Bord}(B) \leftrightarrow \mathrm{Bord}(B))$.

(b) *Let* $\mathscr{A}(\cdot)$ *be a transitive predicate such that* $\Phi^{\mathscr{A}}$ *for every* Φ *being an axiom of* M_0. *Let, moreover,* $\mathscr{A}(On)$; *then*

$$\mathscr{A}(B) \to [(\mathrm{Bord}(B))^{\mathscr{A}} \leftrightarrow \mathrm{Bord}(B)].$$

The following notions are absolute in afore named sense with respect to
(a) being an ordinal,
(b) being a set,
(c) being a function.

Proof of 4.7. We give the proof of 4.7 (a), the proof 4.7 (b) is the same. Implication from the right-hand side to the left-hand side is obvious. Assume now that $\neg \mathrm{Bord}(B)$ and $A \Vdash \mathrm{Bord}(B)$. Then there is $U \subseteq \mathrm{Dom} B \subseteq On$ such that U has no first element is the sense of B. Thence the smallest ordinal x_0 in U is not the first element in the sense of B We define now by induction x_{n+1} as the smallest ordinal in $O_{x_n}(B)$. Obviously $v = \{x_n: n \in \omega\}$ is ordered in B in type ω^*. v is a set and thence there is $\alpha \in On$ such that $v \subseteq \alpha$. Therefore v is included in a set from A. Since A is a model for M_0 therefore we have in A a function f order preserving mapping of $\mathscr{D}B \cap \alpha^2$ onto some $\beta \in On$.

But absoluteness results mentioned at the begining imply that B is not a well-ordering in A. Contradiction.

The result of 4.7 seems very important to us since it shows a very peculiar difference in the relationship between the second order arithmetics $\mathscr{A}_2$ and the first order arithmetics $\mathscr{A}_1$ on the one hand and M_0 and ZF on the other. Namely it is well known that an ω-model for $\mathscr{A}_2$ need not to be a β-model of it. Whereas we have just proved that On-model for M_0 is automatically a β-model of it. This is of course connected with the fact that in the second order arithmetics we have no ordinals and in M_0 we have them and already the first order axioms (i.e ZF axioms) allow us to perform constructions connected with well-orderings. After having proved 4.7 (a), (b) we shall prove some other absoluteness results. All of them are proved under assumption of 4.7 (a) or (b).

LEMMA 4.8. (a) *The pair is absolute.*

(b) *The ordered pair is absolute.*

(c) *The* η*-relation is absolute.*

LEMMA 4.9. (a) *Let* $X, Y \eta A$, X, Y *both well-orderings; then: If* $X \sim Y$, *then the similarity map is in* A.

(b) *Let* $\mathscr{A}(X)$, $\mathscr{A}(Y)$, X, Y *both well-orderings; then: If* $X \sim Y$, *then the similarity map has the property* $\mathscr{A}$.

Proof of (a). Assume that the similarity map is not in A Since X, Y are well-orderings they are comparable in A, and thence either Y is similar in A to the section of X or conversly. This immediately leads to contradiction.

Proof of (b). Similarly.

LEMMA 4.10. (a) *Assume* $X\eta A$, $(x)_{\mathscr{D}X}(\mathrm{Bord}\, X_{(x)})$ *then*

$$T\eta A \to (A \models \mathrm{Sup}\, X = T \leftrightarrow \mathrm{Sup}\, X = T).$$

(b) *Assume* $X\eta A$ *then* X *is w.f.t. in* A *iff* X *is w.f.t.*

(a′) *Assume* $\mathscr{A}(X)$, $(x)_{\mathscr{D}X}\, \mathrm{Bord}(X^{(x)})$ *then*

$$\mathscr{A}(T) \to [(\mathrm{Sup}\, X = T)^{\mathscr{A}} \leftrightarrow \mathrm{Sup}\, X = T].$$

(b′) $\mathscr{A}(X) \to (X$ *is w.f.t.*$)^{\mathscr{A}} \leftrightarrow X$ *is w.f.t.*

Proof. (a) by 4.9. (b) by 2.19.

LEMMA 4.11. (a) *If* $X, T\eta A$, *then:*

$$A \models \mathrm{Rank}(X, x, T) \leftrightarrow \mathrm{Rank}(X, x, T).$$

(b) *If* $\mathscr{A}(X)$, $\mathscr{A}(T)$, *then*

$$\mathrm{Rank}(X, x, T)^{\mathscr{A}} \leftrightarrow \mathrm{Rank}(X, x, T).$$

LEMMA 4.12. (a) *If* X, Y *are reduced trees,* X, $Y\eta A$, X *similar to* Y, *then the similarity map is in* A.

(b) *If* X, Y *are reduced trees,* $\mathscr{A}(X)$, $\mathscr{A}(Y)$ X *similar to* Y, *then if* f *is a similarity map, then* $\mathscr{A}(f)$.

Proof by 4.10 (b) and 2.19.

LEMMA 4.13. (a) *Reduction procedure is absolute, i.e.*

$$X, Y\eta A \to A \models \mathrm{Red}(X, Y) \leftrightarrow \mathrm{Red}(X, Y),$$

(b) *if* $\mathscr{A}(X)$ *and* $\mathscr{A}(Y)$, *then*

$$\mathrm{Red}(X, Y)^{\mathscr{A}} \leftrightarrow \mathrm{Red}(X, Y).$$

Proof. We give a sketch of proof for (a). Implication from the right-hand side to the left-hand one is obvious. Assuming now that Y is not a reduced tree we only need to show that Y is not reduced a tree in A. But this is obvious by 4.12 (a).

LEMMA 4.14 (a) $X, Y\eta A \to (A \models X\,\mathrm{Eq}\,Y \leftrightarrow X\,\mathrm{Eq}\,Y)$ & $(A \models X\,\mathrm{Eps}\,Y \leftrightarrow X\,\mathrm{Eps}\,Y)$,

(b) $\mathscr{A}(X), \mathscr{A}(Y) \to \big((X\,\mathrm{Eq}\,Y)^{\mathscr{A}} \leftrightarrow X\,\mathrm{Eq}\,Y\big)$ & $\big((X\,\mathrm{Eps}\,Y)^{\mathscr{A}} \leftrightarrow X\,\mathrm{Eps}\,Y\big)$.

Proof by 4.12 and 4.13.

Now it should be noted that Eq and Eps relations are not preserved neither under η relation for transitive A nor for transitive predicates $\mathscr{A}(\cdot)$.

Indeed $X\eta A \ \& \ X \operatorname{Eq} Y \leftrightarrow Y\eta A$, $\mathscr{A}(X) \ \& \ X \operatorname{Eq} Y \leftrightarrow \mathscr{A}(Y)$.

However, it should be compared with 4.14.

LEMMA 4.15. (a) *If* X, $Y\eta A$, *then* $A \models X$ *has property* $\operatorname{Def} Y \leftrightarrow X$ *has a property* $\operatorname{Def} Y$.

(b) *Let* $X\eta A$; *then* $A \models X$ *is ordinal tree iff* X *is an ordinal tree.*

(a′) *If* $\mathscr{A}(X)$, $\mathscr{A}(Y)$, *then* (X *has a property* $\operatorname{Def} Y)^{\mathscr{A}} \leftrightarrow X$ *has a property* $\operatorname{Def} Y$.

(b′) *Let* $\mathscr{A}(X)$; *then* (X *is an ordinal tree*)$^{\mathscr{A}} \leftrightarrow X$ *is an ordinal tree.*

LEMMA 4.16. (a) *If* X, $a\eta A$, *then* $A \models \boldsymbol{L}(X, \boldsymbol{a}) \leftrightarrow \boldsymbol{L}(X, \boldsymbol{a})$.

(b) *If* $\mathscr{A}(X)$, $\mathscr{A}(\boldsymbol{a})$, *then* $\boldsymbol{L}(X, \boldsymbol{a})^{\mathscr{A}} \leftrightarrow \boldsymbol{L}(X, \boldsymbol{a})$.

Proof. By checking the inductive definition of $\boldsymbol{L}(X, \boldsymbol{a})$.

COROLLARY 4.17. (a) *Let* T *be the heigth of* A, $X\eta A$. *Let* $\boldsymbol{a}$ *represents* T

$$A \models \boldsymbol{L}(X) \leftrightarrow X \operatorname{Eps} \boldsymbol{L}_a,$$

(b) *under assumptions of* (a)

$$A \models \mathfrak{L}(X) \leftrightarrow \mathfrak{L}_a(X).$$

(a′) *If* $\mathscr{A}(\cdot)$ *has the heigth* T *and* $\boldsymbol{a}$ *represents* T, *then*

$$\mathscr{A}(X) \to \big(\boldsymbol{L}(X)\big)^{\mathscr{A}} \leftrightarrow X \operatorname{Eps} \boldsymbol{L}_a,$$

(b′) *under the same assumptions* $\big(\mathfrak{L}(X)\big)^{\mathscr{A}} \leftrightarrow \mathfrak{L}_a(X)$.

Exactly as in ZF case one should distinguish between constructible model and satisfaction of the axiom of constructibility in the model.

DEFINITION 4.18. An ordinal tree $\boldsymbol{a}$ is called *reflexive* if $\mathbf{On} \operatorname{Eps} \boldsymbol{a}$ and reflection (3.16) holds when the first quantifier is restricted to $\boldsymbol{L}_a$ and the second quantifier is restricted to ordinal trees smaller then $\boldsymbol{a}$.

LEMMA 4.19. *If* $\boldsymbol{a}$ *is reflexive ordinal tree, then* $\boldsymbol{L}_a \models ZFC^- + T + V = L$ *and the class* B *which codes* $\mathfrak{L}_a$ *is a model for* $M + V = L$. *In the model* $\mathfrak{L}_a$ *the sentence* $(X)\mathfrak{L}(X)$ *is true.*

Proof. As in the case of $\boldsymbol{L}$ and $\mathfrak{L}$. The last sentence holds by the first part of Corollary 4.17.

LEMMA 4.20. $\big((X)\mathfrak{L}(X)\big)^{\mathfrak{L}}$ *holds.*

Proof. By the second part 4.17.

$\vartheta = \boldsymbol{L}$ denotes the sentence $(X)\mathfrak{L}(X)$.

LEMMA 4.21. *If* T *is the heigth of* A *and* $\boldsymbol{a}$ *represents* T, *then* $\boldsymbol{a}$ *is reflexive.*

Unfortunately we are unable (under the assumption that M is consistent) to prove that there are A's which are transitive On-models for M and therefore we cannot prove an existence of a reflexive ordinal tree. (It is easy — by 4.19 and 4.20 — to see that it is equivalent.) So we are following the Gandy's line and we consider two following cases.

Case I. There is a reflexive ordinal-tree α. Then we take the smallest α with this property. $\mathfrak{L}_\alpha$ is the minimal transitive On-model for M.

Case II. There is no reflexive ordinal tree α. Then $\mathfrak{L}$ is the minimal transitive On-model for M. I.e. If U is a transitive predicate $U(On)$ and $M \vdash \Phi^{(u)}$ for each Φ being an axiom of M, then $M \vdash (X)\big(\mathfrak{L}(X) \rightarrow U(X)\big)$.

Thence we proved the following

METATHEOREM C. (a) *$M \vdash \Phi^{\mathfrak{L}}$ for every axiom Φ of M,*

$$M \vdash (\mathfrak{B} = \mathfrak{L})^{\mathfrak{L}}.$$

(b) *If there is a transitive model A for M such that $On \eta A$, then there is the smallest model like this.*

(c) *If there is no transitive A such that $On \eta A$ and A is a model for M (i.e. there is no reflexive well-ordering), then for every transitive predicate U such that Φ^U for every axiom Φ of M we have $(X)\big(\mathfrak{L}(X) \rightarrow U(X)\big)$.*

LEMMA 4.22. *Every element U of the minimal model for M is definable in terms of ordinal numbers i.e. there is a formula Φ and ordinals $\alpha_1, \ldots, \alpha_n$ such that $(X)\big(\Phi(X, \alpha_1 \ldots \alpha_n) \leftrightarrow X = U\big)$.*

Proof. By the usual Montague–Vaught reasoning we get that all definable elements of our model form an elementary submodel of it. This submodel contains all ordinals (and thence L) and thence is transitive. Therefore it is a whole model.

Remark 4.23. Under very strong assumptions i.e. in the theory based on the notion of the superclass and containing appropriate impredicative axioms of the superclass existence we are able to prove the existence of a reflexive well-ordering and of the transitive codable superclass which is a model for M.

Remark 4.24. It should be noted that in the proof of 4.7 it was used only that the cofinality character of the class $\{x\colon A \models \mathrm{Ord}(x)\}$ is bigger than ω. Thence for every transitive model A for M if $\alpha = U(On \cap A)$ and cf. $\alpha > \omega$, then A is a β-model and, moreover, there is $\mathscr{F} \subseteq \mathscr{P}(L_\alpha)$ such that $L_\alpha \cup \mathscr{F} \models M + V = L$.

Among such a families there is a smallest one.

5. Forcing in M, independence results for M

We first remind

LEMMA 5.1 (Lévy–Solovay). *If $\mathfrak{M}$ is a countable standard model for ZFC, if k is inaccessible in $\mathfrak{M}$, $\langle P, \leqslant\rangle$ is a notion of forcing in $\mathfrak{M}$ with $\mathfrak{M} \models \bar{\bar{P}} < k$, then for every G which is $\langle P, \leqslant\rangle$-generic over $\mathfrak{M}$*

$$\mathfrak{M}[G] \models \ulcorner k \textit{ is inaccessible} \urcorner.$$

Checking the details of the proof one can see that if the axiom of constructibility is satisfied in $\mathfrak{M}$ and $\mathfrak{M}$ is a model for $ZFC^- + T$, then $\mathfrak{M}[G]$ also is a model for $ZFC^- + T$. (This can be derived for instance from Zarach [13].)

LEMMA 5.2. *Let $\mathfrak{M}$ be a standard model for $ZFC^- + T$. Let k be the smallest inaccessible in $\mathfrak{M}$. Then:*

$$R_k^{\mathfrak{M}} \cup \left(\mathscr{P}(R_k^{\mathfrak{M}}) \cap \mathfrak{M}\right) \models M.$$

Proof. This is just a model theoretic version of 2.38.

LEMMA 5.3. *Let $\mathfrak{M}$ be a standard model for $ZFC^- + T$, and U the predicate defined as follows:*

$$U(X) \leftrightarrow X \in R_k.$$

Then

$$\mathfrak{M} \models \Phi^u \leftrightarrow R_k^{\mathfrak{M}} \cup \left(\mathscr{P}(R_k^{\mathfrak{M}}) \cap \mathfrak{M}\right) \models \Phi^z,$$

where $Z(X) \leftrightarrow (EY)(X \in Y)$.

Combining together 5.1, 5.2 and 5.3 we get the following method of independence proofs for sentences concerning the sets in M We start from the countable standard model $\mathfrak{M}$ for M We built inside of it the model of trees. Since $\mathfrak{M}$ is standard the relation between equivalence classes of Eq-relation is well-founded. We can use then the collapsing lemma and we get a countable standard model $\mathfrak{M}_1$ for $ZFC^- + T$.

By Lemma 5.2 $R_k^{\mathfrak{M}_1} \cup \left(\mathscr{P}(R_k^{\mathfrak{M}_1}) \cap \mathfrak{M}_1\right) \models M_0$.

Now we extend $\mathfrak{M}_1$ by appropriate means (e.g. forcing) and then we obtain a model $\mathfrak{M}_1[G]$.

Now $R_k^{\mathfrak{M}_1[G]} \cup \left(\mathscr{P}(R_k^{\mathfrak{M}_1[G]}) \cap \mathfrak{M}_1[G]\right)$ is appropriate model for M. Thence by the usual elimination of the assumption of the standard model we get following theorem:

THEOREM 5.4. *If M is consistent, then the following theorie are consistent:*

(a) $M + V \neq L + GCH + A$,

(b) $M + \neg\, GCH + A$,

(c) $M_0 + \neg\, AC$.

One word should be said about the proof of the last fact. We get a symetric extension of $\mathfrak{M}_1$ and use Remark 2.41 (we remind that A is "definable approximation principle" from Section 1).

Thence we get (by 5.4 (c)) and assuming M consistent.

THEOREM 5.5. $M_0 \not\vdash \mathrm{IX}$.

Proof. IX implies $A.C.$ Use 5.4 (c).

We will get a stronger result, however, under a stronger assumption.

THEOREM 5.6. *If $ZFC+T$ is consistent, then $M_0+V+L+\neg$ IX, is consistent.*

Proof. Let $\mathfrak{M}$ be a countable standard model for $ZFC+V = L+T$, $ZFC+V = L+T$. We take its symmetric Cohen extension in which the following is done. We add a collapsing $1-1$ map f_a of $\omega_{k+a}^{\mathfrak{M}}$ onto $\omega_k^{\mathfrak{M}}$. We do this with conditions of power $< \omega_k$ In the symmetric extension $\omega_{k+1} = \omega_{k+k}^L$.

Now we take the restriction of the symmetric model $\mathfrak{M}_1$:

$$R_k^{\mathfrak{M}_1} \cup \big(\mathscr{P}(R_k^{\mathfrak{M}_1}) \cap \mathfrak{M}_1\big) \models M_0.$$

Let $\Phi(a, T)$ be a formula saying as follows: (T represents ω_{k+a}^L) & T is ordinal tree$\urcorner$.

Then clearly $(a)_k(ET)\Phi(a, T)$ but there is no "choice" class, because then $\omega_{k+k}^{\mathfrak{M}}$ would be collapsed onto $\omega_k^{\mathfrak{M}}$.

Now we give some additional facts about M.

We know that in M the strong axiom of constructibility implies condition A from Section 1.

We have the following.

THEOREM 5.7 (Zbierski). *If M is consistent, then $M+A$, $\nvdash V = L$.*

Proof. By 5.4 (a).

Under assumption of the consistency of $ZFC+T$ we get the proof of the consistency of $M+\neg A$.

THEOREM 5.8. *If $\mathfrak{M}$ is a countable standard model for $ZFC+T$, k inaccessible in $\mathfrak{M}$, then there is a Cohen-type extension $\mathfrak{M}[G]$ with the following properties:*

(i) $\mathfrak{M}[G] \models ZFC+T$,

(ii) *k is inaccessible in $\mathfrak{M}[G]$,*

(iii) $2^k > k^+$,

(iv) $R_k^{\mathfrak{M}} = R_k^{\mathfrak{M}[G]}$.

Proof. Just add the k^{++} generic subsets of k with the conditions of power less then k Since k is inaccessible thence $\mathfrak{M} \models \overline{\overline{k}} = \overline{\overline{R}}_k$ and therefore there is no new element of R_k.

LEMMA 5.9. *If $\mathfrak{M}[G]$ is a model from 5.8, then*

$$R_k^{\mathfrak{M}[G]} \cup \big(\mathscr{P}(R_k^{\mathfrak{M}[G]}) \cap \mathfrak{M}[G]\big) \models M+\neg A.$$

THEOREM 5.10. *If $ZFC+T$ is consistent, then $M+V = L+\neg A$ is consistent.*

Proof. We can assume that in $\mathfrak{M}$ from 5.8. $V = L$ holds therefore in $R_k^{\mathfrak{M}[G]} \cup \big(\mathscr{P}(R_k^{\mathfrak{M}[G]}) \cap \mathfrak{M}[G]\big) \models V = L$.

COROLLARY 5.11. *If $ZFC+T$ is consistent, then so is $M+V = L+$ + there is no definable well-ordering of all classes.*

6. Hierarchy of formulas in M

The particular part aims at using the method given in Lévy [5] and extends some of his theorem proved in the ZF case to M. We remind that the formula $Z(X) \leftrightarrow (EY)(X \in Y)$ means that X is a set that small Latin letters are reserved for the variables ranging over sets. Q is any theory in the language of the set theory.

DEFINITION 6.1. (a) The formula Φ is called Σ_l (Π_l) iff it is of the following form: Φ: $(EX_{i_0}) \ldots (\mathsf{XX}X_{i_{l-1}})\Phi$ (Φ: $(X_{i_0}) \ldots (\mathsf{XX}X_{i_{l-1}})\Phi$), where Φ is a predicative formula, i.e. all quantifiers in it are restricted to sets and XX is appropriate quantifier.

(b) The formula Φ is called Σ_l^Q (Π_l^Q) iff it is provably in Q equivalent to the formula from $\Sigma_l(\Pi_l)$.

DEFINITION 6.2. (a) Z is called a *pair* of X and Y — in symbols $Z = «X, Y»$ iff $\mathscr{D}Z = 2$ and $Z^{(0)} = X$ and $Z^{(1)} = Y$.

(b) Z is called a *finite sequence* of classes iff $\mathscr{D}Z \subseteq \omega$, $\overline{\overline{\mathscr{D}Z}} < \omega$. If $\mathscr{D}Z = n$, then we write $Z = «Z^{(0)}, \ldots, Z^{(n-1)}»$.

LEMMA 6.3. *With the interpretation of the ordered pair as in* 6.2 *the axiom of pairing is provable in* M_0.

We have now almost verbatim lemma from Lévy (Lemma 1 from [5]).

LEMMA 6.4. (a) $\Sigma_0 = \Pi_0$.

(b) *If* $m \in n$, *then* $\Sigma_m^Q \cup \Pi_m^Q \subseteq \Sigma_n^Q \cap \Pi_n^Q$.

(c) *If* $\Phi \in \Sigma_n^Q$, *then* $\sim\Phi \in \Pi_n^Q$. *If* $\Phi \in \Pi_n^Q$, *then* $\sim\Phi \in \Pi_n^Q$.

(d) *If* $\Phi \in \Sigma_n^Q$, *then* $(X_i)\Phi \in \Pi_{n+1}^Q$ *and if* $n > 0$, *then* $(EX_i)\Phi \in \Sigma_n^Q$ *and if* $n = 0$, *then* $(EX_i)\Phi \in \Sigma_1^Q$. *If* $\Phi \in \Pi_n^Q$, *then* $(EX_i)\Phi \in \Sigma_{n+1}^Q$ *and if* $n > 0$, *then* $(X_i)\Phi \in \Pi_n^Q$ *and if* $n = 0$, *then* $(X_i)\Phi \in \Pi_1^Q$.

(e) *If* $\Phi, \Psi \in \Sigma_n^Q$, *then* $\Phi \,\&\, \Psi$, $\Phi \vee \Psi \in \Sigma_n^Q$. *If* $\Phi, \Psi \in \Pi_n^Q$, *then* $\Phi \,\&\, \Psi$, $\Phi \vee \Psi \in \Pi_n^Q$. *If* $\Phi \in \Sigma_n^Q$, $\Psi \in \Pi_n^Q$, *then* $\Phi \,\&\, \Psi$, $\Phi \vee \Psi \in \Sigma_{n+1}^Q \cap \Pi_{n+1}^Q$.

(f) *If* $Q \supseteq M$, $\Phi \in \Sigma_n^Q$, *then* $(Ex)\Phi, (x)\Phi \in \Sigma_n^Q$. *If* $Q \supseteq M$, $\Phi \in \Pi_n^Q$, *then* $(x)\Phi, (Ex)\Phi \in \Pi_n^Q$.

The proof of the facts (a)-(e) is clear with that only that a new definition of the ordered pair should be used. The point (f) easily follows by induction, note that Axiom IX is used (instead of the axioms of the power-set, replacement and foundation as it is done in the ZF case).

LEMMA 6.5. *Every formula is in* $\bigcup_{n \in \omega} (\Sigma_n^Q \cup \Pi_n^Q)$.

DEFINITION 6.6. The rank of a formula Φ (denoted following Lévy by $\varrho(\Phi)$) is defined recursively as follows:

(a) If Φ is atomic, then $\varrho(\Phi) = 0$.

(b) $\varrho(\sim\Phi) = \varrho(\Phi)$.

(c) $\varrho(\Phi \,\&\, \Psi) = \varrho(\Phi \vee \Psi) = \max(\varrho\Phi, \varrho\Psi)$.

(d) $\varrho\big((Ex)\Phi\big) = \varrho\big((x)\Phi\big) = \varrho\Phi$.

(e) $\varrho\big((Ev_i)\Phi\big) = \varrho\Phi+1$ provided that Φ is not of the form: $v_i \in v_j \,\&\, \Phi_1$.

(f) $\wedge_r = \{\Phi\colon\ \varrho\Phi = n\}$.

(g) $\wedge_n^Q = \{\Phi\colon\ (E\Psi)_{\wedge_n}(Q \vdash \Phi \leftrightarrow \Psi)\}$.

LEMMA 6.7 (Lemma 3 in Lévy's [5]). (a) $m \in n \rightarrow \wedge_m^Q \subseteq \wedge_n^Q$.

(b) $\wedge_0 = \Sigma_0 = \Pi_0$.

(c) $\Sigma_n \cup \Pi_n \subseteq \wedge_n$.

(d) *If* $M \subseteq Q$, *then* $\wedge_n^Q \subseteq \Sigma_{n+1}^Q \cap \Pi_{n+1}^Q$.

Proof (a), (b), (c) are obvious. (d) is proved by induction.

DEFINITION 6.8. The classes with Σ_n^Q, Π_n^Q, $\wedge_n^Q$ etc. definitions are called Σ_n^Q, Π_n^Q, $\wedge_n^Q$ etc.

Thence predicative classes are called Σ_0 (or Π_0). Now we shall use the extended language; i.e. it will be a language with the additional symbol $\{:\}$ for the class term formation.

DEFINITION 6.9. Term T of the extended language is called *predicative* iff $T = \{x\colon\ \Phi(x)\}$ for some $\Phi \in \Sigma_0$.

LEMMA 6.10. *If* T *is the predicative term and* $\Phi \in \Sigma_\mathfrak{t}^M$, *then* $\Phi(T)$ *is also* Σ_0^M.

Proof. If $\Phi\colon\ v_i = v_i$, then $\Phi(T)$ is truth (say $(x)\,(x = x)$). If $v_i \neq v_i$, then $\Phi(T)$ is falsity. If $v_i \in v_j$, then $\Phi(T)\colon\ T \in v_j$, this is equivalent to $(Ex)\,(x = T \,\&\, x \in v_j)$, other cases are obvious.

LEMMA 6.11. *If* Φ *is* Σ_k^M *and* T *is predicative term, then* $\Phi(T)$ *is* Σ_k^M.

Proof. By induction.

Now we are going to evaluate certain formulas and terms.

LEMMA 6.12. (a) $X^{(x)}$ *is a predicative term.*

(b) $\mathscr{D}X$ *is a predicative term.*

(c) $\mathrm{CLP}(X) \overset{\mathrm{df}}{\longleftrightarrow} \ulcorner X$ *is the class of pairs*$\urcorner$ *is* Σ_0^M.

(d) $\mathrm{PAIR}(X) \overset{\mathrm{df}}{\longleftrightarrow} \ulcorner \mathrm{CLP}(X) \,\&\, \mathscr{D}X = 2\urcorner$ *is* Σ_0^M.

(e) $\mathrm{REL}(X) \overset{\mathrm{df}}{\longleftrightarrow} \ulcorner \mathrm{CLP}(X) \,\&\, {}_{\mathscr{D}X}\mathrm{PAIR}(X^{(x)})\urcorner$ *is* Σ_0^M.

(f) $\mathrm{FUNC}(X) \overset{\mathrm{df}}{\longleftrightarrow} \ulcorner \mathrm{REL}(X) \,\&\, (x, y)_{\mathscr{D}X}(X^{(x)})^{(0)} = \big((X)^{(y)}\big)^{(0)} \rightarrow \big((X)^{(x)}\big)^{(1)} = (X^{(y)})^{(1)}\urcorner$ *is* Σ_0^M.

(g) $\mathrm{FUNC}^2(X) \overset{\mathrm{df}}{\longleftrightarrow} \ulcorner \mathrm{FUNC}(X) \,\&\, (x)_{\mathscr{D}X}\mathrm{PAIR}\big((X^{(x)})^{(0)}\big)\urcorner$ *is* Σ_0^M.

(h) $\mathrm{RG}(X) \subseteq 2 \overset{\mathrm{df}}{\longleftrightarrow} \ulcorner \mathrm{FUNC}(X) \,\&\, (x)\,\big((X^{(x)})^{(1)} \in Z\big)\urcorner$ *is* Σ_0^M.

Now using Lévy's Lemma 10 (about representability of the *PR* formulas in the set theory the only change is that in our case they are Σ_0) we are able to define by simultanous induction the formulas.

$\mathrm{Satfun}^j(\cdot)$ and $\mathrm{Stsf}^i(\cdot, \cdot)$ which are going to express correspondingly: H is a satisfaction function for the formulas of the rank $\leqslant j$, finite sequence of classes f satisfies the formula with the Gödel number n.

They are written down almost exactly as in Lévy's [5]:

$$\mathrm{Satfun}^j(H) \leftrightarrow \mathrm{FUNC}^2(H) \,\&\, \mathrm{RG}(H) \subseteq 2 \,\&\, (x)\Big(x \in \mathscr{D}(H) \to \varrho\Big(\big((H^{(x)})^{(0)}\big)^{(0)}\Big) \leqslant j\Big) \,\&$$
$$\&\, \mathscr{D}\Big(\big((H^{(x)})^{(0)}\big)^{(1)}\Big) = \mathrm{Fr}\Big(\big((H^{(x)})^{(0)}\big)^{(0)}\Big) \,\&\, [\Phi_1 \vee \ldots \vee \Phi_7],$$

where $\Phi_1, \ldots, \Phi_7$ are inductive conditions corresponding to the inductive definition of satisfaction of atomic formulas $v_i = v_j$, atomic formulas $v_i \in v_j$, negation, conjunction, alternative, predicative existential quantifier, full existential quantifier.

We shall not write them all down; indeed the alternative of the first six is described in Mostowski's [10]. Note that (unlike in the *ZF* case) they are still in Σ_0^M We shall write the 7-th formula in Φ_7 case:

$$(E_{i,k})\Big(\big((H^{(x)})^{(0)}\big)^{(0)} = 2^3 \cdot 3^i \cdot 5^k \,\&\, \Big\{\Big[(EX)\,\mathrm{Stsf}^{j-1}\Big(\big((H^{(x)})^{(0)}\big)^{(1)} \cup \text{«}i, X\text{»} \upharpoonright \mathrm{Fr}(k), k\Big)$$
$$\&\, (H^{(x)})^{(1)} = 1\Big] \vee \Big[\sim (EX)\,\mathrm{Stsf}^{j-1}\Big(\big((H^{(x)})^{(0)}\big)^{(1)} \cup \text{«}i, X\text{»}\,\mathrm{Fr}(k), k\Big) \,\&\, (H^{(x)})^{(1)} = 0\Big]\Big\}.$$

Now

$$\mathrm{Stsf}^j(F, n) \leftrightarrow (EH)\,[\mathrm{Satfun}^j(H) \,\&\, \text{««}n, F\text{», } 1\text{»}\,\eta H].$$

We are able to find the lemmas corresponding to Lévy's Lemmas 13-16. It is even easier since we work in the full system M (it should be noted that if in the Lemma 16 in Lévy, *ZF* is taken instead of S then the lemma is almost obvious).

In this way we get the following theorem which corresponds to Lévy's theorem 17.

THEOREM 6.13. *Let Φ be a formula with the Gödel number n, of the rank $\leqslant j$ and free variables $v_{i_1}, \ldots, v_{i_m}$; then*

$$M \vdash \mathrm{Stsf}^j(F, n) \leftrightarrow \Phi,$$

where F is the predicative term with the domain $\{i_1, \ldots, i_m\}$ and such that $F^{(i_m)} = v_{i_m}$.

Proof. By induction on j.

THEOREM 6.14. $\mathrm{Stsf}^j(F, n) \in \Sigma_{j+1}^M \cap \Pi_{j+1}^M$.

Proof. We prove that by induction and calculation of components. We note here that further results (like Lévy's Semantical hierarchy theorem) can be proved.

DEFINITION 6.15. (a) The class Y is called Σ_i (Π_i, Δ_i) in X iff it has a definition of the form $\{x\colon \Phi(x, X^{(a_1)}, \ldots, X^{(a_k)}, X)\}$ for some $a_1, \ldots, a_k \in \mathscr{D}X$, where Φ is Σ_i (Π_i, Δ_i) formula.

(b) The class Y is called Σ_i (Π_i, Δ_i) iff it is Σ_i (Π, Δ_i) in the class $E = \{\langle x, y\rangle\colon x \in y\}$.

LEMMA 6.16. *For each $i \in \omega$ and X the predicate Σ, $(X)\big(\Pi_i(X), \Delta_i(X)\big)$ is codable, in particular Σ_i is codable.*

Proof. We give here the proof for the case of Σ_i but this can be easily seen to be generalisable to the general case.

It is clear that Σ_i are exactly these of the form $\{x\colon \Phi(x, x_1, \ldots, x_n)\}$, where $x_1, \ldots, x_n$ are sets, Φ is a Σ_i^M formula and Φ contains no other parameters. Now by 6.13 we can form the class

$$\{\langle p, x, x_1 \ldots x_n\rangle\colon \mathrm{Stsf}^j(«x, x_1, \ldots, x_n», p)\}.$$

It is clear how to transform it into the required class.

COROLLARY 6.17. *The consistency of the theory $PZF_1^+ +$ Axiom of powerset is provable in M* (where PZF_1^+ is a theory of Moschovakis [9]).

Since we can assume that each class is Σ_i for some i (by 4.22).

Therefore we have

COROLLARY 6.18. *Let W_i be the "code" for all Σ_i classes. Then the mapping $i \to W_i$ is not definable in M.*

At the same time we get once more the following

THEOREM 6.19 (Montague). (a) *The theory M is essentially reflexive.*

(b) *M is not finitely axiomatisable,*

(c) *M is not axiomatisable by the axioms of bounded depth, i.e.*

(i) $(T)\,(T \subseteq \Sigma_i \to T$ *is not equiconsistent with* $M)$.

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