# LINEAR SPACES OF RESULTS ON THE QUOTIENT FIELD OF OPERATORS

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In this paper we are going to consider two situations:

**I.** Let $S : \mathbf{X} \to Y$ be a surjection and $T : Y \to \mathbf{X}$ a primitive function, that is, $ST = id_Y$. We assume that there is a bijection $\varphi : \mathbf{X} \to G^1(\circ, e^1)$ where $G^1$ is an abelian group with addition $\circ$ and unit $e^1$ so that we have the following operation in $\mathbf{X}$:

$$x^1 \circ x^2 = \varphi^{-1}(\varphi(x^1) \circ \phi(x^2)).$$

Notice that $T(Y) = TS(\mathbf{X})$ can be a subgroup of $G^1$. For all $x \in \mathbf{X}$ there is a unique $c \in \mathbf{X}$ such that $x = c \circ TSx$ $(c = x \circ (TSx)^{-1})$. In the set of all constants $C$ we have the operation

$$c_1 \circ c_2 = (x_1 \circ x_2)[T(Sx_1) \circ T(Sx_2)]^{-1}.$$

**II.** Here assume that $S : X \to Y$ is a surjection and in addition let $X \subset Y$; the latter assumption is important so that we can consider iterations of $S$.

Now we are going to define the quotient field of operators.

Let $G$ be any group which does not contain divisors of zero. Let us consider any commutative semigroup $\Pi \subset EndX$ of injections of $X$ into itself (where in $X$ we have an operation defined as in I). $\Pi$ fulfils the following conditions:

$1^o$ $Id_X \in \Pi$.
$2^o$ If $f, g \in \Pi$ then $f \circ g \in \Pi$.
$3^o$ $f \circ g = g \circ f$ for $\forall f, g \in \Pi$.
$4^o$ $f(x) \equiv 0$ implies that $x = 0$.

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We are going to consider in more detail the case

$$f_n(x) = n \cdot x \quad \text{where } n \in \mathbf{Z}.$$

The *set of fractions* (*results*) is the set of pairs $(x, f)$ where $x \in X, f \in \Pi$ or in a more convenient notation $\frac{x}{f}$ with the operation

$$\frac{x}{f} \oplus \frac{y}{g} = \frac{g(x) \circ f(y)}{fg}.$$

If we take the field of scalars $\frac{f_n}{f_m}$ where $n, m \in \mathbf{Z}, m \neq 0$ then we also have the operation

$$\frac{f_n}{f_m}\left(\frac{x}{g}\right) = \frac{f_n(x)}{f_m g}$$

and the set of results is a linear space over the field of operators.

In situation (**I**) there are two possibilities:

**IA.** $T(Y)$ is not only a subset but also a subgroup of the set $X$. That means that in $T(Y)$ we have the operation

$$T(y_1) \circ T(y_2) = \varphi^{-1}(\varphi(T(y_1)) \circ \varphi(T(y_2)))$$

and so the limit condition can be defined as follows:

$$s(x) = x - T(S(x)) \quad \text{where } s : X \to Y$$

(here we do not have to assume that $S(X)$ is a subgroup of $X$).

**IIB.** More generally $T(Y)$ is not a subgroup but just a subset of $X$. Then we can choose a group $H$ (such that $H$ has the same number of elements as $T(Y)$) and a bijection $\psi : H \to T$.

Notice that in this case we can also calculate $T(y_1) \circ_\varphi T(y_2)$ but the set $T(Y)$ need not be closed under the operation $\circ_\varphi$, so $T(y_1) \circ_\varphi T(y_2)$ belongs to $X$ but not neccessarily to $T(Y)$.

Now let us denote $Orb_S(y) = \{x : S(x) = y\}$ and $Orb_s(c) = \{x : s(x) = c\}$. Notice that if $c$ and $y$ are fixed then $Orb_s(c) \cap Orb_S(y)$ has at most one element. We can write this in another form

$$(*) \qquad S(x) = y, \quad s(x) = c;$$

if the above system has a solution, then the solution is unique.

EXAMPLE 1. Let $S : \mathbb{R} \to \mathbb{R}_{\geq 0}$, $S(x) = x^2$, $s(x) = \operatorname{sgn} x$. Then $Orb_s(\pm 1) \cap Orb_S(r)$ has exactly one element for $r > 0$ but $Orb_s(\pm 1) \cap Orb_S(0)$ and $Orb_s(0) \cap Orb_S(r)$ where $r > 0$ are empty sets.

So in general

$$\mathbf{X} = \bigcup_{c \in s(\mathbf{X}), y \in S(\mathbf{X})} Orb_s(c) \cap Orb_S(y).$$

In the regular case we have not only

$$\forall x \in X \quad x = s(x) \circ TS(x)$$

but also

$$\forall c \in s(\mathbf{X}) \ \ \forall y \in Y \ \exists! x \in \mathbf{X} \text{ such that } x = c \circ Ty.$$

Now let us look more closely at the set $s(\mathbf{X})$. $s(\mathbf{X})$ is a subgroup of $\mathbf{X}$ (with the operation $\circ$). In case $s(\mathbf{X})$ is not closed under the operation $\circ$ we can introduce a group operation in $s(\mathbf{X})$ using a bijection $\omega : s(\mathbf{X}) \to K$ where $K$ is any group with the same number of elements as $s(\mathbf{X})$.

Example 2. Let $S : C^1(a,b) \to C^0(a,b)$ where $C^1(a,b)$ is an abelian group, $(a,b)$ a given open interval, $s(C^1(a,b)) \approx \mathbb{R}$ and $T(C^0) \subset C^1$ where $T(f(t)) = \int_a^t f(\tau)\,d\tau$. Here

$$Tf \circ_\psi Tg = \int_a^t f(\tau)\,d\tau \oplus_\psi \int_a^t g(\tau)\,d\tau = \int_a^t (f(\tau) + g(\tau))\,d\tau.$$

Since $Orb_s(c) \approx C^0(a,b)$ we have $Orb_s(c) \lhd C^1(a,b) \lhd C^0$. Here

$$(x^1 - T(y^1)) + (x^2 - T(y^2)) = (x^1 \oplus x^2) - (T(y^1) \oplus T(y^2)).$$

Now we are going to consider the space of fractions $\frac{y}{u}$ where $y \in Y$ and $u : Y \to Y$ is a homomorphism and injection. Now each $x \in X$ can be written as $x = \omega_0 \oplus \omega_1 \oplus \ldots \oplus \omega_{n-1} \oplus r_n$ where $\omega_1(x) = T^i S^i x \ominus T^{i+1} S^{i+1} x \in Y$ are polynomials for $i = 0, 1, \ldots, n-1$ and $r_n(x) = T^n S^n x \in Y$ is the $n$th rest in the Taylor formula (notice that we assume situation **II**, that is, $S$ can be iterated).

Now if $Y$ is a metric group and $r_n$ tends to 0 as $n \to \infty$ then we can write that $x = \oplus_{i=0}^{\infty} \omega_i$ (note $\oplus_{i=0}^{\infty} \omega_i \leftrightarrow \{\omega_i\}_{i=1}^{\infty} \in Y^\infty$).

Now we can define functions:

$$\mathbf{S} : Y^{\mathbb{N}} \to Y^{\mathbb{N}} \quad \text{by} \quad \mathbf{S}(\{\omega_i\}_0^\infty) = \{\omega_{i+1}\}_0^\infty,$$
$$\mathbf{S} : Y^{\mathbb{N}} \to Y^{\mathbb{N}} \quad \text{by} \quad \Pi((\omega_0, \omega_1, \omega_2, \ldots)) = (0, \omega_0, \omega_1, \omega_2, \ldots) \text{ and}$$
$$\mathbf{s} : Y^{\mathbb{N}} \to Y^{\mathbb{N}} \quad \text{by} \quad \mathbf{s}(\{\omega_i\}) = \{\omega_i \cdot \delta_0^n\}.$$

$\mathbf{S}, \Pi, \mathbf{s}$ are homomorphic, $\Pi$ is an injection ($\mathbf{S}$ and $\mathbf{s}$ are not). $Y^{\mathbb{N}}$ is a linear space over the field of operators.

Now let $A_i : Y \to Y^{\mathbb{N}}$ be a homomorphism (it does not have to be invertible) for $i = 0, \ldots, n-1$ and let us consider the equation

$$(*) \qquad \mathbf{S}^n \oplus A_{n-1}\mathbf{S}^{n-1}x \oplus \ldots \oplus A_1\mathbf{S} + A_0 x = f, \qquad \mathbf{s}\mathbf{S}^i x = \omega_i(x),$$

where $i = 0, \ldots, n-1$. Here $\omega_0, \omega_1, \ldots, \omega_{n-1}$ are given, $x = (\omega_0, \omega_1, \ldots, \omega_{n-1}, r_n)$. Let us assume that $A_i\mathbf{S} = \mathbf{S}A_i$ for every $i$. If we differentiate the equation $(*)$ $k$ times we obtain

$$\mathbf{S}^{n+k}x \oplus A_{n-1}\mathbf{S}^{n-1+k}x \oplus \ldots \oplus A_1\mathbf{S}^{1+k}x \oplus A_0\mathbf{S}^k x = \mathbf{S}^k f$$

(here we use the fact that $\mathbf{S}$ is a homomorphism). Now if in addition $A_i\mathbf{s} = \mathbf{s}A_i$ then

$$(**) \qquad \mathbf{s}\mathbf{S}^{n+k}x \oplus A_{n-1}\mathbf{s}\mathbf{S}^{n-1+k}x \oplus \ldots \oplus A_1\mathbf{s}\mathbf{S}^{1+k}x \oplus A_0\mathbf{s}\mathbf{S}^k x = \mathbf{s}\mathbf{S}^k f.$$

Equation $(*)$ can be solved using one of the following methods:

METHOD 1.

$$x = \frac{v_0\omega_0 + v_1\omega_1 + \ldots + v_{n-1}\omega_{n-1} + vf}{\omega(\Pi)}$$

where the solution is a fraction

$$x = \oplus_{i=0}^{n-1} \frac{v_i}{\omega}\omega_i \oplus \frac{v}{\omega} f,$$

that is,

$$x = \oplus_{i=0}^{n-1} \frac{v_i}{\omega}(T^i S^i - T^{i+1}S^{i+1}x) \oplus \frac{v}{\omega} f.$$

The solution is obtained by a recursive procedure: If the polynomials $\omega_0, \ldots, \omega_{n-1}$ are given we can calculate the subsequent polynomials by differentiation of equation $(**)$ since

$$\omega_{n+k}x \oplus A_{n-1}\omega_{n+k-1}(x) \oplus \ldots \oplus A_0\omega_{k+1}(x) = \omega_k f.$$

So, succesively we find all the polynomials of $x$ (so in fact we get $x$ itself).

METHOD 2. We have

$$\Pi^n \mathbf{S}^n x \oplus \Pi^n A_{n-1}\mathbf{S}x \oplus \ldots \oplus \Pi^n A_0 x = \Pi^n f,$$
$$(x \ominus \omega_0 \ominus \Pi A_{n-1}\omega_1 \ominus \ldots \ominus \Pi^{n-1}A_1\omega_{n-1}) \oplus \ldots \oplus (\Pi^n A_0 x) = \Pi^n f,$$
$$(Id \oplus \Pi A_{n-1} \oplus \ldots \oplus \Pi^n A_0)x = 0 \Rightarrow x = 0.$$

Now let us note that from $(Id - A\Pi)x_1 = x_2$ we can get $x_1 = \frac{x_2}{Id - a\Pi}$ if the denominator is an injection. Now if $A \in EndY$ and $\Pi A = A\Pi$ then

$$(Id - A\Pi)x = 0 \Leftrightarrow x = A\Pi x \Leftrightarrow A\Pi x = A^2\Pi^2 x \Leftrightarrow x = A^n T^n x$$

for any $n$. But $\lim_{n\to\infty} A^n \Pi^n x = 0$ in the Fréchet metric. So for all $n \in \mathbb{N}$ $(Id - \Pi^n A^n)x = 0 \Rightarrow x = 0$ and so $(Id - \Pi^n A^n)$ is an injection, and

$$x = \frac{(\omega_0 \oplus \ldots \oplus \Pi^{n-1}A_1\omega_{n-1}) \oplus \ldots \oplus \Pi^{n-1}A_1\omega_{n-1} \oplus \Pi^n f}{(Id \oplus \Pi A_{n-1} \oplus \ldots \oplus \Pi^n A_0)} = \sum_{i=0}^{\infty} B_i.$$

**References**


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