

PÓLSKA AKADEMIA NAUK, INSTYTUT MATEMATYCZNY

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KOMITET REDAKCYJNY

KAROL BORSUK *redaktor*

ANDRZEJ BIAŁYNICKI-BIRULA, BOGDAN BOJARSKI,
ZBIGNIEW CIESIELSKI, JERZY ŁOŚ, ANDRZEJ MOSTOWSKI,
ZBIGNIEW SEMADENI, MARCELI STARK, WANDA SZMIELEW

CVII

CAT-HO NGUYEN

**Generalized Post algebras and their application
to some infinitary many-valued logics**

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INTRODUCTION

Generalized Post algebras were first formulated and examined by Traczyk [30] and later by Dwinger [5]. Both authors have given in turn the notions of Post ideals and Post homomorphisms (see [33] and [6]). However, the problem of m -representation of the generalized Post algebras has not been examined so far.

One of the main aims of this paper is an application of generalized Post algebras to some many-valued logics. In [11], Karp shows that the completeness of some formal systems of infinitary propositional logic is exactly connected with the m -representability of corresponding Boolean algebra of equivalence classes of formulas. This fact suggests a discussion of the m -representability of generalized Post algebras.

In part I we shall extend T. Traczyk's and Ph. Dwinger's definitions of generalized Post algebras and introduce the notions of Post D -filters and of Post homomorphisms in our generalized Post algebras. Under these notions we can generalize almost all the results which have been proved in the case of Post algebras of order n (see [4], [17], [29] and [30]). In particular, we will prove a theorem on necessary and sufficient conditions for the m -representability of generalized Post algebras, and a generalization of the Rasiowa and Sikorski Lemma (see [19]) for the case of generalized Post algebras.

In order to obtain these results we shall use R. Sikorski's method of examination of the m -representability of Boolean algebras (see [27]). For this purpose we will extend the notion of D -filters introduced in paper [17] and construct the Stone space of a generalized Post algebra from a set of D -filters. The proof of the theorem on m -representability is analogous to that for the case of Boolean algebras.

Note that our notion of D -filters is not entirely dual to that of Post ideals of T. Traczyk and Ph. Dwinger and that it has properties necessary to develop the theory of generalized Post algebras.

In part II we shall apply the theory of generalized Post algebras to certain infinitary many-valued propositional calculi. In [12], [13], [17], [23] and [29], V. Kirin, G. Rousseau and H. Rasiowa show that the role of Post algebras of order n for the n -valued propositional and predicate calculi is similar to that of Boolean algebras in the case of propositional

calculi and predicate calculi of classical logic. The following question arises: is it possible to construct certain many-valued propositional calculi in which the generalized Post algebras play a part analogous to that of Post algebras of order n for n -valued propositional calculi? More generally, we will discuss this problem for many-valued propositional logics based on infinitary propositional languages \mathcal{L}_m^ν corresponding to infinitary propositional languages \mathcal{L}_a as introduced by Karp [11]. The cardinal m is used here in the same sense as a by Karp, and ν is the ordinal of the well-ordered set of propositional constants in \mathcal{L}_m^ν which correspond to logical values adopted in the logic under consideration.

The answer is positive in the following cases:

1° Finitary denumerably many-valued propositional logics.

2° Infinitary many-valued propositional logics in which the cardinal number of the set of logical values is less than the cardinal m .

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PART I

A GENERALIZATION OF POST ALGEBRAS

Let A be a partially ordered set. We say that the set A is connected if any elements α, β of A have either an upper bound or a lower bound, and A is said to be a *semilattice* if for any two elements of A there exists a *greatest lower bound* (g.l.b.).

We denote by V the greatest element of A , by o the least elements of A and by 1 with indices the minimal elements of the set $A^- = A - \{o\}$, whenever these elements exist in A .

If A has the zero-element, then a minimal element of A^- is said to be an *atom* of A .

A semilattice A is called *complete* if every subset $A_0 = \{a_t: t \in T\}$ of A has a g.l.b. in A . The g.l.b. of A_0 is denoted by $\bigcap_{t \in T} a_t$ or $\bigcap A_0$ or $\bigcap \{a_t: t \in T\}$. If A is a subset of a partially ordered set \bar{A} , the g.l.b. of a subset A_0 in A is denoted by $\bigcap_{t \in T}^A a_t$ or $\bigcap^A A_0$.

A semilattice A with the zero-element is called *connected* if the set $A^- = A - \{o\}$ is a connected partially ordered set.

1. Definition and characterization of generalized Post algebras

1. Definition and properties of generalized Post algebras.

DEFINITION 1.1. Let A be a connected semilattice with the zero-element o and let P be a distributive lattice with the zero-element 0 and the unit-element 1 . B denotes the set of all complemented elements in P .

If P satisfies the following conditions:

(P 1) There exists an indexed subset $E = \{e_\alpha: \alpha \in A\} \cup \{e_V\}$ of P such that for any α, β of A , $\alpha \leq \beta$ implies $e_\alpha \leq e_\beta$, and $e_o = 0$, $e_V = 1$ (Irrespectively whether there is a greatest element V of A or not, we denote the unit element of P by e_V).

(P 2) For every element p in P , there exists a decreasing sequence

$\langle p_\alpha: \alpha \in A^- \rangle$ of type A^- of elements of B (i.e. satisfying the condition: for α, β in A , if $\alpha \leq \beta$, then $p_\alpha \geq p_\beta$) such that

$$(1) \quad p = \bigcup_{\alpha \in A^-} p_\alpha \cap e_\alpha$$

(a monotonic representation of the element p).

(P 3)⁽¹⁾ For every subset A_0 of A , the join $\bigcup_{\alpha \in A_0} e_\alpha$ exists in P .

We denote the join $\bigcup_{\beta < \alpha} e_\beta$ by e_α .

(P 4)⁽²⁾ If p, q are any elements in P and p satisfies (1), then

$$q \cap p = \bigcup_{\alpha \in A^-} q \cap (p_\alpha \cap e_\alpha).$$

This condition is called the *generalized distributivity law*.

(P 5) If $b \in B$, then the condition $b \cap e_\beta \leq e_\alpha$, where $\beta \not\leq \alpha$, or $b \cap e_\beta \leq e_\beta$ implies $b = 0$,

then P is called a *generalized Post algebra of type A* , or for brevity, a *Post algebra of type A* , with the distinguished elements in E , and it is denoted by $\langle P; \{e_\alpha: \alpha \in A\}, B \rangle$.

For example we consider the set $A_{\bar{B}}$ of all decreasing functions f from A^- into a Boolean algebra B . The order in $A_{\bar{B}}$ is the usual order of functions, i.e. if $f, g \in A_{\bar{B}}$, then $f \leq g$ iff for every $\alpha \in A^-$, $f(\alpha) \leq g(\alpha)$. We know that the set $A_{\bar{B}}$ with this order is a lattice. We denote by e_0 and e_1 the constant functions, which are respectively equal to 0 and 1, and, for every $\alpha \in A^-$, we denote by e_α the function

$$e_\alpha(\beta) = \begin{cases} 1 & \text{if } \beta \leq \alpha, \\ 0 & \text{if } \beta \not\leq \alpha. \end{cases}$$

Let E be the set $\{e_\alpha: \alpha \in A^-\} \cup \{e_0, e_1\}$. It is easy to see (or in virtue of Theorem 1.6 in the sequel) that every sublattice P of $A_{\bar{B}}$ containing all constant functions and all functions of $A_{\bar{B}}$ whose values are 0 or 1 only is a Post algebra of type A . Hence we have many examples of Post algebras of type A . Thus we see that condition (P 4) is a natural generalization of the finite distributivity law for lattices.

COROLLARY 1.2. *If, for every $t \in T$, $\alpha_t \in A$ and the meet $\alpha_0 = \bigcap_{t \in T} \alpha_t$ exists in A , then $e_{\alpha_0} = \bigcap_{t \in T} e_{\alpha_t}$. In particular, $e_\alpha \cap e_\beta = e_{\alpha \cap \beta}$, whenever α, β in A .*

Proof. From $\alpha_0 \leq \alpha_t$ we have $e_{\alpha_0} \leq e_{\alpha_t}$ for every $t \in T$. Let $p = \bigcup_{\alpha \in A^-} p_\alpha \cap e_\alpha$

⁽¹⁾ This condition is equivalent to the following one: for every decreasing sequence $\langle p_\alpha: \alpha \in A^- \rangle$ of type A^- of zero or unit elements, the join $\bigcup_{\alpha \in A^-} p_\alpha \cap e_\alpha$ exists in P .

⁽²⁾ This condition may be considered as a generalization of the finite distributivity law for lattices.

be any element of P such that for every $t \in T$, $p \leq e_{a_t}$. Since $\alpha_0 = \bigcap_{t \in T} \alpha_t$, for every β in A^- , if $\beta \not\leq \alpha_0$, then there exists an element $t_0 \in T$ such that $\beta \not\leq \alpha_{t_0}$. Thus by the inequality $p \leq e_{\alpha_{t_0}}$, we have $p_\beta \cap e_\beta \leq e_{\alpha_{t_0}}$. Hence by (P 5), $p_\beta = 0$. Consequently, $p = \bigcup_{0 < \alpha \leq \alpha_0} p_\alpha \cap e_\alpha \leq \bigcup_{\alpha \leq \alpha_0} e_\alpha$. Therefore we have proved that $e_{\alpha_0} = \bigcap_{t \in T} e_{\alpha_t}$.

THEOREM 1.3. *Let p and q be any elements of P and let $p = \bigcup_{\alpha \in A^-} p_\alpha \cap e_\alpha$, $q = \bigcup_{\alpha \in A^-} q_\alpha \cap e_\alpha$. Then $p \leq q$ iff for every $\alpha \in A^-$, $p_\alpha \leq q_\alpha$. Therefore every element of P has a unique monotonic representation; in particular,*

$$e_\alpha = \bigcup_{\beta \leq \alpha} e_\beta \quad \text{and} \quad e_V = 1 = \bigcup_{\alpha \in A^-} e_\alpha.$$

Proof. The sufficiency is obvious by equality (1) in (P 2). Now suppose that $p \leq q$. It implies that for every β in A^-

$$(2) \quad q \cap -q_\beta \cap e_\beta \geq p \cap -q_\beta \cap e_\beta \geq (-q_\beta \cap p_\beta) \cap e_\beta.$$

It follows from condition (P 4) in 1.1, Corollary 1.2 and also from the equation

$$\bigcup_{\alpha \in A^-} (-q_\beta \cap q_\alpha) \cap e_{\alpha \cap \beta} = \bigcup_{\alpha \leq \beta} (-q_\beta \cap q_\alpha) \cap e_\alpha$$

that

$$\begin{aligned} q \cap -q_\beta \cap e_\beta &= \bigcup_{\alpha \in A^-} (-q_\beta \cap e_\beta) \cap (q_\alpha \cap e_\alpha) \\ &= \bigcup_{\alpha \in A^-} (-q_\beta \cap q_\alpha) \cap e_{\alpha \cap \beta} = \bigcup_{\alpha \leq \beta} (-q_\beta \cap q_\alpha) \cap e_\alpha. \end{aligned}$$

For $\alpha = \beta$ we have $-q_\beta \cap q_\alpha = 0$. Hence

$$q \cap -q_\beta \cap e_\beta = \bigcup_{\alpha < \beta} (-q_\beta \cap q_\alpha) \cap e_\alpha \leq e_{\hat{\beta}}.$$

This together with (2) implies that $-q_\beta \cap p_\beta \cap e_\beta \leq e_{\hat{\beta}}$. Since $-q_\beta \cap p_\beta \in B$, by (P 5) in (1.1), we obtain $-q_\beta \cap p_\beta = 0$, which is equivalent to $p_\beta \leq q_\beta$. This completes the proof of the necessity. The second part of the theorem follows immediately from the first.

On account of this theorem, we can introduce for every $\alpha \in A^-$ a D_α -operation defined by the equality $D_\alpha p = p_\alpha$ for every p in P , where p_α is the coefficient of the element e_α in the monotonic representation (1) of p .

For convenience we will denote, in the sequel, the set $A \cup \{V\}$ by A^+ . Of course, if A has the unit element V , then $A^+ = A$.

The fundamental properties of Post algebras are formulated in the following theorem:

THEOREM 1.4. *In every Post algebra $P = \langle P; \{e_a: a \in A\}; B \rangle$ of type A*

- (i) *For $\alpha, \beta \in A^-$, if $\alpha \leq \beta$, then $D_\alpha p \geq D_\beta p$,*
- (ii) *$p \leq q$ iff for every $\alpha \in A^-$, $D_\alpha p \leq D_\alpha q$,*
- (iii) *$D_\alpha(p \cup q) = D_\alpha p \cup D_\alpha q$; $D_\alpha(p \cap q) = D_\alpha p \cap D_\alpha q$,*
- (iv) $D_\alpha e_\beta = \begin{cases} 1 & \text{if } \alpha \leq \beta, \\ 0 & \text{if } \alpha \not\leq \beta, \end{cases} \quad D_\alpha e_\beta = \begin{cases} 1 & \text{if } \alpha < \beta, \\ 0 & \text{if } \alpha \not< \beta, \end{cases}$

where $\alpha, \beta \in A^-$ and, as we already know,

$$e_\beta = \bigcup_{\alpha < \beta} e_\alpha,$$

- (v) *$p \in B$ iff $D_\alpha p = p$, for all α in A^- ,*
- (vi) *$D_\alpha D_\beta p = D_\beta p$,*
- (vii) *if for some $p, q \in P$ the pseudo-complement of p relative to q , denoted by $p \Rightarrow q$, exists in P , then*

$$(3) \quad D_\alpha(p \Rightarrow q) = \bigcap_{0 < \beta \leq \alpha}^B (D_\beta p \Rightarrow D_\beta q).$$

In particular, if the pseudo-complement $-p = p \Rightarrow 0$ of p exists, then

$$(4) \quad D_\alpha(-p) = - \bigcup_{0 < \beta \leq \alpha}^B D_\beta p.$$

Proof. Applying Definition 1.1, Corollary 1.2 and Theorem 1.3 it is easy to show that (i), (ii) and (iv)-(vi) are true. It remains to prove (iii) and (vii).

In order to prove (iii), let $p = \bigcup_{\alpha \in A^-} D_\alpha p \cap e_\alpha$ and $q = \bigcup_{\alpha \in A^-} D_\alpha q \cap e_\alpha$. Obviously, $p \cup q = \bigcup_{\alpha \in A^-} (D_\alpha p \cup D_\alpha q) \cap e_\alpha$, and by the uniqueness of the monotonic representation of the element $p \cup q$, we get $D_\alpha(p \cup q) = D_\alpha p \cup D_\alpha q$, i.e. the first equality of (iii).

By applying the generalized distributivity law twice, we obtain

$$(5) \quad p \cap q = \bigcup_{\alpha \in A^-} q \cap D_\alpha p \cap e_\alpha = \bigcup_{\alpha \in A^-} \bigcup_{\beta \in A^-} (D_\alpha p \cap D_\beta q) \cap e_{\alpha \cap \beta}.$$

This implies that $(D_\alpha p \cap D_\alpha q) \cap e_\alpha \leq p \cap q$ for all $\alpha \in A^-$. We now assume that

$$(6) \quad p' \geq (D_\alpha p \cap D_\alpha q) \cap e_\alpha \quad \text{for every } \alpha \in A^-, \text{ where } p' \in P.$$

Since for arbitrary elements $\alpha, \beta \in A^-$, $D_{\alpha \cap \beta} p \geq D_\alpha p$ and $D_{\alpha \cap \beta} q \geq D_\beta q$,

$$(D_\alpha p \cap D_\beta q) \cap e_{\alpha \cap \beta} \leq (D_{\alpha \cap \beta} p \cap D_{\alpha \cap \beta} q) \cap e_{\alpha \cap \beta}.$$

Thus it follows from (6) that $(D_\alpha p \cap D_\beta q) \cap e_{\alpha \cap \beta} \leq p'$. Hence by (5), $p \cap q \leq p'$. So we have proved that $p \cap q = \bigcup_{\alpha \in A^-} (D_\alpha p \cap D_\alpha q) \cap e_\alpha$. By the uniqueness

of the monotonic representation of $p \cap q$, we obtain the second equality of (iii).

Now we shall prove equality (3). Suppose that the pseudo-complement of p relative to q , $p \Rightarrow q$, exists in P .

We recall that $x \leq p \Rightarrow q$ iff $x \cap p \leq q$. Then $(p \Rightarrow q) \cap p \leq q$, which implies by (ii) and (iii) that for every $\beta \in A^-$, $D_\beta(p \Rightarrow q) \cap D_\beta p \leq D_\beta q$ and consequently

$$(7) \quad D_\beta(p \Rightarrow q) \leq D_\beta p \Rightarrow D_\beta q.$$

As $D_\alpha(p \Rightarrow q) \leq D_\beta(p \Rightarrow q)$, so in view of (7) we have $D_\alpha(p \Rightarrow q) \leq D_\beta p \Rightarrow D_\beta q$ for all $0 < \beta \leq \alpha$.

Consider an element $b \in B$ such that

$$(8) \quad b \leq D_\beta p \Rightarrow D_\beta q \quad \text{for every } 0 < \beta \leq \alpha.$$

Put

$$(9) \quad p' = (p \Rightarrow q) \cup (b \cap e_\alpha).$$

We shall prove that

$$D_\gamma(p' \cap p) = D_\gamma p' \cap D_\gamma p \leq D_\gamma q \quad \text{for any } \gamma \in A^-.$$

Indeed, if $\gamma \leq \alpha$, then from (iii), (iv) and (9) it follows that $D_\gamma p' = D_\gamma(p \Rightarrow q) \cup b$. On the other hand, since, by (7) and (8), $D_\gamma(p \Rightarrow q) \leq D_\gamma p \Rightarrow D_\gamma q$ and $b \leq D_\gamma p \Rightarrow D_\gamma q$, we have $D_\gamma p' \leq D_\gamma p \Rightarrow D_\gamma q$. Consequently $D_\gamma(p' \cap p) \leq D_\gamma q$.

If $\gamma \not\leq \alpha$, then $D_\gamma e_\alpha = 0$ and so $D_\gamma p' = D_\gamma(p \Rightarrow q)$; consequently the required inequality $D_\gamma(p' \cap p) \leq D_\gamma q$ results directly by (7).

By this inequality, we conclude in virtue of (ii) in Theorem 1.4 that $p' \cap p \leq q$ and hence that $p' \leq p \Rightarrow q$. On the other hand, it is evident from the definition of p' that $p' \geq p \Rightarrow q$. Thus $p' = p \Rightarrow q$, and so $D_\alpha(p \Rightarrow q) = D_\alpha p' = D_\alpha(p \Rightarrow q) \cup b$. Hence $b \leq D_\alpha(p \Rightarrow q)$.

In such a way we have proved that equality (3) holds.

The validity of equality (4) follows easily from (3). This completes the proof of Theorem 1.4.

COROLLARY 1.5. *If the set A has a unique atom 1, then any Post algebra P of type A is a pseudo-complemented lattice, i.e. for every element $p \in P$, there exists a pseudo-complement $-p$ of p .*

Moreover, $-p = -D_1 p$ and consequently

$$(10) \quad D_\alpha(-p) = -D_1 p \quad \text{for every } \alpha \in A^-.$$

The element e_1 is the least dense element in P .

Proof. Since for every $\alpha \in A^-$, $\alpha \geq 1$, it follows from Theorem 1.4 (i) that, for every $p \in P$, $D_\alpha p \leq D_1 p$ and consequently $D_\alpha p \cap -D_1 p = 0$. Applying the generalized distributivity law (P 4), we obtain $-D_1 p \cap p = 0$.

Now let us take an element q of P such that $q \cap p = 0$. This implies by (iii), (iv) in Theorem 1.4 that $D_1 p \cap D_1 q = 0$. Hence $-D_1 p \geq D_1 q \geq D_\alpha q$ for every $\alpha \in A^-$ and so, by Theorem 1.4 (ii), (vi), $D_\alpha(-D_1 p) \geq D_\alpha D_\alpha q = D_\alpha q$ for every $\alpha \in A^-$. Applying Theorem 1.4 (ii), we get $-D_1 p \geq q$. Thus we have proved that the pseudo-complement of p exists and $-p = -D_1 p$. Consequently equation (10) holds.

For the proof of the remaining part of the corollary, we recall that the element p is dense iff $-p = 0$. According to (10) $D_\alpha(-e_1) = -D_1 e_1 = 0$ for every $\alpha \in A^-$. Hence $-e_1 = 0$ and so e_1 is a dense element. Suppose that p is any dense element, i.e. $-p = 0$. Again by (10), $-D_1 p = 0$, which implies that $D_1 p = 1$. On account of Theorem 1.4 (iv) we have $D_\alpha p \geq D_\alpha e_1$ for every $\alpha \in A^-$. Hence $p \geq e_1$, i.e. e_1 is the least dense element in P . This completes the proof.

2. A characterization of Post algebras of type A . In the first section we have seen that in any Post algebra of type A for every $\alpha \in A^-$ there is determined a unary operation D_α . This suggests the consideration of the distributive lattices P containing the zero element 0 and the unit element 1, with the set of unary operations $\{D_\alpha: \alpha \in A^-\}$. Let $E = \{e_\alpha: \alpha \in A\} \cup \{e_V\}$, where $e_V = 1$, be fixed indexed set of some elements of P . We will denote such lattices by $\langle P; E; \{D_\alpha; \alpha \in A^-\} \rangle$.

Under the above hypotheses and notations, we have the following theorem, which characterizes the Post algebras of type A by means of D_α -operations, $\alpha \in A^-$.

THEOREM 1.6. $P = \langle P; E; \{D_\alpha: \alpha \in A^-\} \rangle$ is a Post algebra of type A iff the following conditions are satisfied:

- (i) $D_\alpha(p \cup q) = D_\alpha p \cup D_\alpha q$; $D_\alpha(p \cap q) = D_\alpha p \cap D_\alpha q$,
- (ii) $D_\beta D_\alpha p = D_\alpha p$,
- (iii) $D_\alpha p \cup -D_\alpha p = 1$,
- (iv) for any subset A_0 of A , there exists a l.u.b. $\bigcup_{\alpha \in A_0} e_\alpha$ in P ,
- (v) $D_\alpha e_\beta = \begin{cases} 1 & \text{if } \alpha \leq \beta, \\ 0 & \text{if } \alpha \not\leq \beta, \end{cases} \quad D_\alpha e_{\hat{\beta}} = \begin{cases} 1 & \text{if } \alpha < \beta, \\ 0 & \text{if } \alpha \not< \beta, \end{cases}$
where $\alpha \in A^-$ and $\beta \in A^+ = A \cup \{V\}$, $e_{\hat{\beta}} = \bigcup_{\alpha < \beta} e_\alpha$,
- (vi) for every element $p \in P$

$$p = \bigcup_{\alpha \in A^-} D_\alpha p \cap e_\alpha.$$

Proof. We have proved (see Theorem 1.4) that if P is a Post algebra of type A , then $P = \langle P; E; \{D_\alpha: \alpha \in A^-\} \rangle$ satisfies conditions (i)-(vi).

Conversely, suppose that P satisfies conditions (i)-(vi). We shall show that all conditions (P 1)-(P 5) in Definition 1.1 hold in P .

First of all, we can easily show that

$$(11_{1,2}) \quad p \leq q \text{ iff } D_a p \leq D_a q \text{ for every } a \in A^-, \text{ and if } a \leq \beta, \text{ then } D_a p \geq D_\beta p.$$

Indeed, if $D_a p \leq D_a q$ for all $a \in A^-$, then, on account of (vi), $p \leq q$. Assuming that $p \leq q$, we get $p \cup q = q$. So by (i), $D_a p \cup D_a q = D_a q$. That is $D_a p \leq D_a q$, for every $a \in A^-$. To show the validity of the second assertion of (11), let $a \leq \beta$. From (vi), $p \geq D_\beta p \cap e_\beta$. By virtue of (11₁), (ii) and (v), it follows that $D_a p \geq D_\beta p$.

Let B be the set of all complemented elements in P . Conditions (P 1), (P 2) and (P 3) result immediately by (11) and (iii)-(vi).

In order to verify condition (P 4), consider the meet $p \cap q$. By (vi), $q \cap p \geq q \cap D_a p \cap e_a$ for every $a \in A^-$. Take any element p' of P satisfying the condition $p' \geq q \cap D_a p \cap e_a$ for every $a \in A^-$. From (11) and conditions (i), (ii) and (v), it follows that

$$D_a p' \geq D_a q \cap D_a p = D_a(p \cap q).$$

Thus again, by (11₁), $p \geq p \cap q$. Consequently $p \cap q = \bigcup_{a \in A^-} q \cap D_a p \cap e_a$, i.e. condition (P 4) is satisfied.

Finally, we verify (P 5). By (i) and (v), it will be sufficient to show that if b is an arbitrary element of B , then $D_a b = b$ for every $a \in A$. Indeed, since $b \cap -b = 0$ and $b \cup -b = 1$, by (i) and (v), we obtain $D_a b \cap D_a -b = 0$ and $D_a b \cup D_a -b = 1$. Hence $-D_a b = D_a -b$ for every $a \in A^-$. In virtue of (11₂), if $a \leq \beta$, then $D_a b \geq D_\beta b$ and $D_a -b \geq D_\beta -b$. Since $-D_a b = D_a -b$ and $-D_\beta b = D_\beta -b$, from the last inequality it follows that $D_a b \leq D_\beta b$. Therefore if $a \leq \beta$, then $D_a b = D_\beta b$. But A^- is a connected set, and thus for arbitrary elements a, β of A^- either there is an element $\gamma \in A^-$ such that $\gamma \leq \beta$ and $\gamma \leq a$ or there exists an element $\gamma' \in A^-$ such that $\gamma' \geq \beta$ and $\gamma' \geq a$. In the first case $D_a b = D_\gamma b = D_\beta b$, in the second case $D_a b = D_\gamma b = D_\beta b$. Consequently, by (vi), we conclude that

$$b = \bigcup_{\beta \in A^-} D_\beta b \cap e_\beta = D_a b \cap \bigcup_{\beta \in A^-} e_\beta = D_a b,$$

which completes the proof of the theorem.

The following corollary easily results from 1.6 (ii), 1.6 (iii) and from (11₁).

COROLLARY 1.7. *If $D_a p \geq p$ ($D_a p \leq p$) for every $a \in A^-$, then $p \in B$.*

THEOREM 1.8. (i) *Let $\{b_t: t \in T\}$ be an indexed subset of B . The l.u.b. $\bigcup_{t \in T}^P b_t$ (the g.l.b. $\bigcap_{t \in T}^P b_t$) exists in P iff the l.u.b. $\bigcup_{t \in T}^B b_t$ (the g.l.b. $\bigcap_{t \in T}^B b_t$) exists in B . Moreover, $\bigcup_{t \in T}^P b_t = \bigcup_{t \in T}^B b_t$ ($\bigcap_{t \in T}^P b_t = \bigcap_{t \in T}^B b_t$).*

(ii) *Let $\{p_t: t \in T\}$ be an indexed set of elements of P . The existence of the l.u.b. $\bigcup_{t \in T} p_t$ (the g.l.b. $\bigcap_{t \in T} p_t$) implies the existence of the l.u.b. $\bigcup_{t \in T} D_a p_t$*

(the g.l.b. $\bigcap_{t \in T} D_a p_t$) for every $a \in A^-$ and

$$D_a \bigcup_{t \in T} p_t = \bigcup_{t \in T} D_a p_t \quad (D_a \bigcap_{t \in T} p_t = \bigcap_{t \in T} D_a p_t).$$

(This is a generalization of Epstein's Lemma).

Conversely, if for every $a \in A^-$ the l.u.b.'s (the g.l.b.'s) concerned exist in P and $\bigcup_{t \in T} D_a p_t = b_a$ ($\bigcap_{t \in T} D_a p_t = b_a$) and $\bigcup_{a \in A^-} b_a \cap e_a = p$, then l.u.b. $\bigcup_{t \in T} p_t$ (the g.l.b. $\bigcap_{t \in T} p_t$) also exists, and $p = \bigcup_{t \in T} p_t$ ($p = \bigcap_{t \in T} p_t$).

Proof of (i). Assume that the l.u.b. $\bigcup_{t \in T}^P b_t$ exists and equals p . Thus $p \geq b_t$ for every $t \in T$, and this implies that $D_a p \geq b_t$, $t \in T$. Accordingly $D_a p \geq p$, for every $a \in A^-$. Then, by Theorem 1.7, $p \in B$. Consequently, the l.u.b. $\bigcup_{t \in T}^B b_t$ exists in B and $\bigcup_{t \in T}^P b_t = \bigcup_{t \in T}^B b_t$.

Conversely, suppose that $b = \bigcup_{t \in T}^B b_t$. We shall show that $b = \bigcup_{t \in T}^P b_t$. For this purpose let us assume that $p \geq b_t$, $t \in T$. Hence $D_a p \geq D_a b_t = b_t$, $t \in T$. This implies that $D_a p \geq b = D_a b$, for every $a \in A^-$. Therefore we get $p \geq b$, which completes the proof.

The proof for g.l.b. is dual.

Proof of (ii). If $p = \bigcup_{t \in T} p_t$, then $p \geq p_t$, and so $D_a p \geq D_a p_t$, for every $t \in T$. Suppose that $b \geq D_a p_t$, $t \in T$, where $b \in B$, and consider the element $q = b \cup \bigcup_{\substack{\gamma \neq a \\ \gamma \neq a}} e_\gamma$. By Theorem 1.6 (v), it follows easily that for $\gamma \neq a$, $D_\gamma q = 1 \geq D_\gamma p_t$, and for $\gamma \geq a$, $D_\gamma q = b \geq D_\gamma p_t$. Hence by Theorem 1.4 (ii), $q \geq p_t$, for every $t \in T$. Consequently $q \geq p$ and so $D_a q = b \geq D_a p$. This shows that the l.u.b. $\bigcup_{t \in T} D_a p_t$ exists in B (and hence in P as well) and $\bigcup_{t \in T} D_a p_t = D_a p$.

The proof for g.l.b.'s is similar. We set $q = b \cap e_a$ instead of $q = b \cup \bigcup_{\substack{\gamma \neq a \\ \gamma \neq a}} e_\gamma$.

The proof of the second part of (ii) is similar to that in the case of the Post algebras of order n .

DEFINITION 1.9 (i). A Post algebra $P = \langle P; \{e_a: a \in A\}; B \rangle$ of type A is called *full* if for every decreasing sequence $\langle b_a: a \in A^- \rangle$ of elements of B , the l.u.b. $\bigcup_{a \in A^-} b_a \cap e_a$ exists in P ; and

(ii) P is said to be *m-complete*, where m is an infinite cardinal number, if for each set P_0 of elements of P , such that $\text{card } P_0 < m$, there exist $\bigcup P_0$ and $\bigcap P_0$.

The following corollary is an immediate consequence of Theorem 1.8:

COROLLARY 1.10 (i). *If the Post algebra P of type A is m-complete, then its underlying Boolean algebra B is m-complete.*

(ii) If, in addition, the Post algebra P is full, then the l.u.b. $\bigcup_i p_i$ exists in P iff for every $a \in A^-$, the l.u.b. $\bigcup_i D_a p_i$ exists in B ;

(iii) A full Post algebra P of type A is m -complete iff its underlying Boolean algebra B is m -complete.

2. Post subalgebras and Post homomorphisms

The notions of Post subalgebras and of Post homomorphisms as defined by Dwinger in [5] does not seem quite natural.

In this section Post subalgebras, Post homomorphisms and some of their essential properties will also be discussed. Since in the definition of Post algebras of type A the hypotheses concerning set A are as weak as possible, our definitions of Post subalgebras and of Post homomorphisms are a little complicated. However, we will see later that by the additional assumption that the semilattice A is complete (as in the case of Ph. Dwinger), or is a partially well-ordered set, these definitions become simple and natural. Moreover, we will also see that they are more general than those due to Ph. Dwinger.

The following remarks will be useful in defining these notions.

Let $P = \langle P; \{e_a: a \in A\}; B \rangle$ be a given Post algebra of type A . We recall that E denotes the set $\{e_\nu\} \cup \{e_a: a \in A\}$. Let us set $\bar{E} = E \cup \{e_{\hat{a}}: a \in A\}$. It will be convenient to establish that $\hat{a} = \hat{\beta}$ if $e_{\hat{a}} = e_{\hat{\beta}}$ for a certain e_β in E and if elements $e_{\hat{a}} \notin E$ and $e_\beta \notin E$ are distinct, then the indices a and $\hat{\beta}$ are regarded as distinct elements which are not in $A^+ = A \cup \{V\}$. Now let us denote by \bar{A} (the extension of A) the set $A^+ \cup \{\hat{a}: a \in A\}$. In view of the above observation, we have a one-to-one correspondence between \bar{E} and \bar{A} . Hence, by this correspondence, the ordering in \bar{E} induces the ordering in \bar{A} . Then \bar{A} and \bar{E} are isomorphic partially ordered sets, and A is obviously a substructure (in the category of order) of \bar{A} .

Therefore, the elements of \bar{A} will be denoted, except for the special cases, also by the letters α, β, \dots . Thus the monotonic representations of elements of \bar{E} and the two expressions of condition (v) in Theorem 1.6 have the following common forms:

$$(1_1) \quad e_\alpha = \bigcup_{\substack{\beta \leq \alpha \\ \beta \in A}} e_\beta \quad \text{for every } \alpha \in \bar{A}$$

and

$$(1_2) \quad D_\beta e_\alpha = \begin{cases} 0 & \text{if } \beta \not\leq \alpha, \beta \in A^-, \\ 1 & \text{if } \beta \leq \alpha, \beta \in A^-, \end{cases} \quad \alpha \in \bar{A},$$

respectively.

Since \bar{A} and \bar{E} are isomorphic partially ordered sets, by (1₁) the following equation is true:

$$a = \bigcup^{\bar{A}} \{\beta \in A : \beta \leq a\} \quad \text{for every } a \in \bar{A}.$$

Moreover, observe that, for every $e_a \in \bar{E}$, e_a is obviously the predecessor of e_a in \bar{E} . Conversely, if $e_a \in \bar{E} - E$, then e_a is the successor of e_a in \bar{E} . In fact, let $e_\beta \in \bar{E}$ and $e_\beta > e_a$. We need to show that $e_\beta \geq e_a$. There are two cases only: either $e_\beta \in E$, or $e_\beta \notin E$. It suffices to consider the case of $e_\beta \notin E$, since the other case is similar and easier. Because of $e_\beta \notin E$, e_β has the form $e_\beta = e_\delta$ for a certain $\delta \in A^-$. Since $e_\delta > e_a$ and $e_\delta = e_\beta > e_a$, we have

$$e_\delta > e_a = \bigcup_{\substack{\gamma < a \\ \gamma \in A}} e_\gamma.$$

Thus for every $\gamma \in A$ and $\gamma < a$, $e_\gamma < e_\delta$ or equivalently $\gamma < \delta$. Hence we obtain $\gamma \leq a \cap \delta$ for every $\gamma \in A$ and $\gamma < a$. Assume that $a \not\leq \delta$. Then $a \cap \delta < a$, and since $a \cap \delta \in A$, it follows that $a \cap \delta$ is the greatest element in A which is smaller than a . Therefore

$$e_a = \bigcup_{\substack{\gamma < a \\ \gamma \in A}} e_\gamma = e_{a \cap \delta} \in E,$$

which contradicts the assumption that $e_a \notin E$. Hence $a \leq \delta$. Furthermore, since $e_a \neq e_\delta$, $a \neq \delta$ and so $a < \delta$. Consequently it follows from (1₁) and from $a \in A$ that $e_\beta = e_\delta \geq e_a$; this completes the proof.

On account of the similarity of \bar{E} and \bar{A} , we obtain the same properties for \bar{A} , namely:

Every element a of A has a predecessor a^- in \bar{A} , and every element $a \in \bar{A} - A^+$ has a successor a^+ in \bar{A} .

For the sake of convenience we assume in the sequel that, for each $a \in A^+$, $a^+ = a$. Hence we notice that, for every $a \in \bar{A}$, a^+ always belongs to A^+ .

As can easily be seen, the following properties hold in \bar{A} :

$$(2) \quad (a^+)^- \leq a \text{ for any } a \in \bar{A}, \text{ and } (a^+)^- = a \text{ iff } a \in \bar{A} - A^+;$$

$$\text{if } a \leq \beta, \text{ then } a^+ \leq \beta^+; \text{ if } \beta \in \bar{A} - A^+, \text{ then } a < \beta \text{ iff } a^+ < \beta^+.$$

LEMMA 2.1. *The partially ordered set \bar{A} is a connected semilattice and A is a subsemilattice of \bar{A} . Furthermore, if the elements α, β of \bar{A} are incomparable, then $\alpha \cap \beta = \alpha^+ \cap \beta^+$ and belongs to A . Consequently, for any elements α, β of \bar{A} , $e_\alpha \cap e_\beta = e_{\alpha \cap \beta}$.*

Proof. First of all we will show that \bar{A} is a semilattice. Obviously it is sufficient to prove that arbitrary incomparable elements α, β of \bar{A} have a g.l.b. From the incomparability of elements α, β it follows that

α^+ and β^+ are also incomparable. Indeed, suppose the contrary, i.e. either $\alpha^+ \leq \beta^+$, or $\beta^+ \leq \alpha^+$, say $\alpha^+ \leq \beta^+$. So $\alpha \leq \beta^+$.

The following two cases lead us to a contradiction:

If $\alpha = \beta^+$, then $\alpha \geq \beta$.

If $\alpha < \beta^+$, then by (2) $\alpha \leq (\beta^+)^- \leq \beta$.

Set $\alpha_0 = \alpha^+ \cap^A \beta^+$. From the incomparability of α^+ and β^+ it follows that $\alpha_0 < \alpha^+$ and $\alpha_0 < \beta^+$. Hence again by (2) $\alpha_0 \leq (\beta^+)^- \leq \beta$ and $\alpha_0 \leq (\alpha^+)^- \leq \alpha$. Now suppose that $\gamma \in \bar{A}$, $\gamma \leq \alpha$ and $\gamma \leq \beta$. In view of (2), $\gamma^+ \leq \alpha^+$ and $\gamma^+ \leq \beta^+$. Since $\gamma^+ \in A$ and $\alpha_0 = \alpha^+ \cap^A \beta^+$, we have $\alpha_0 \geq \gamma^+ \geq \gamma$. This shows that $\alpha_0 = \alpha \cap \beta$ in \bar{A} .

From this argument we can see that if $\alpha, \beta \in A$, then the g.l.b. $\alpha \cap \bar{A} \beta$ is always in A . Therefore A is a subsemilattice of \bar{A} . Evidently \bar{A} is connected, for \bar{A} contains the unit V . Thus the proof is completed.

1. Post subalgebras. Let $P = \langle P; \{e_a: a \in A\}; B \rangle$ be a given Post algebra of type A and let P_0 be a subset of the set P . Let E_0 be a subset of the set $\{e_a \in \bar{E}: e_a \in P_0\}$. In connection with E_0 , we denote by A_0 the set $\{a \in \bar{A}: e_a \in E_0\}$. Put $B_0 = B \cap P_0$.

DEFINITION 2.2. The structure $P_0 = \langle P_0; \{e_a: a \in A_0\}; B_0 \rangle$ is called a *Post subalgebra of type A_0* of the Post algebra P provided

(p₁) P_0 is a sublattice of the lattice P and contains the zero and the unit elements,

(p₂) $e_0 \in E_0$ and for every $e_a \in E_0$, $e_a > \bigcup_{a > \beta \in A_0}^P e_\beta$,

(p₃) for every subset $\{e_{a_t}: t \in T\}$ of E_0 , the l.u.b. $\bigcup_{t \in T}^P e_{a_t}$ belongs to P_0 ,

(p₄) if $p \in P_0$, then there exists a decreasing sequence of elements of $B_0 \langle b_a: a \in A_0^- \rangle$ of type A_0^- such that

$$(3) \quad p = \bigcup_{a \in A_0^-}^P b_a \cap e_a,$$

(p₅) for each $\alpha_0 \in A$, the set $\{a \in A_0: a \geq \alpha_0\}$ has a least element, which is denoted by $r(\alpha_0)$.

Note that the fact $\{a \in A_0: a \geq \alpha_0\} \neq \emptyset$ results from (p₄).

Remark 1. If A is a partially well-ordered set, then obviously condition (p₅) is satisfied and hence, in this case, it can be omitted.

Remark 2. If A is a complete semilattice, then by adding the following condition

(p'₃) for any subset $\{e_{a_t}: t \in T\}$ of E_0 , $\bigcap_{t \in T}^P e_{a_t} \in P_0$, we find that the last condition (p₅) is a consequence of conditions (p₁)-(p₄).

Proof. First of all we shall prove the following lemma

LEMMA 2.3 (i) *If A is a complete semilattice, then so is \bar{A} .*

(ii) If $a = \bigcap_{t \in T}^A a_t$, where $a_t \in \bar{A}$, then $e_a = \bigcap_{t \in T}^P e_{a_t}$.

In fact, let A be complete, and let $\{a_t: t \in T\}$ be a subset of \bar{A} . We know that $\{a_t^+: t \in T\} \subset A^+$, and so $\bigcap_{t \in T}^{A^+} a_t^+$ exists in A^+ . Set $a_0 = \bigcap_{t \in T}^{A^+} a_t^+$.

There are only two cases.

Firstly, for every $t \in T$, $a_0 \leq a_t$. We easily see that in this case $a_0 = \bigcap_{t \in T}^{\bar{A}} a_t$.

Secondly, there is at least one element $t_0 \in T$ such that $a_0 \not\leq a_{t_0}$. Since $a_0 \leq a_{t_0}^+$, we get $a_0^- < a_{t_0}^+$ and hence $a_0^- \leq (a_{t_0}^+)^- \leq a_{t_0}$. If $a_0 < a_{t_0}^+$, then $a_0 \leq (a_{t_0}^+)^- \leq a_{t_0}$ and if $a_{t_0} \in A^+$, then $a_0 \leq a_{t_0}$, since $a_0 \leq a_{t_0}^+$ and $a_{t_0}^+ = a_{t_0}$. In both these cases we obtain a contradiction of the assumption that $a_0 \not\leq a_{t_0}$. Hence $a_0 = a_{t_0}^+$ and $a_{t_0} \notin A^+$. The last implies $a_{t_0} < a_{t_0}^+$. Consequently, $a_{t_0} \leq a_0^-$ and since $a_0^- \leq a_{t_0}$, we get $a_0^- = a_{t_0}$, which implies that $a_0^- = \bigcap_{t \in T}^{\bar{A}} a_t$. Thus we have shown that \bar{A} is complete.

Proof of (ii). Suppose that $a = \bigcap_{t \in T}^{\bar{A}} a_t$. Thus $e_a \leq e_{a_t}$ for each $t \in T$. Now take an arbitrary element p of P such that $p \leq e_{a_t}$ for every $t \in T$, and consider any element β of A^- .

If $\beta \leq a$, then $D_\beta e_a = 1$. If $\beta \not\leq a$, then from $a = \bigcap_{t \in T}^A a_t$ it follows that there exists an element t_0 of T such that $\beta \not\leq a_{t_0}$. This leads to $D_\beta e_a t_0 = 0$ and so $D_\beta p = 0$, since $p \leq e_{a_{t_0}}$. Accordingly, for every $\beta \in A^-$, $D_\beta p \leq D_\beta e_a$, which implies $p \leq e_a$. Consequently $e_a = \bigcap_{t \in T}^P e_{a_t}$. Thus the lemma is proved.

We now return to the proof of Remark 2. Put $r(a_0) = \bigcap^{\bar{A}} \{a \in A_0: a \geq a_0\}$. The existence of this g.l.b. follows from the completeness of \bar{A} . Then by the above lemma $e_{r(a_0)} = \bigcap^P \{e_a \in E_0: e_a \geq e_{a_0}\}$. In virtue of (p₃) we have $e_{r(a_0)} \in P_0$. Evidently $D_{a_0} e_{r(a_0)} = 1$, since $a_0 \leq r(a_0)$. On the other hand, if $\beta \in A_0$, $\beta < r(a_0)$, then, by the definition of $r(a_0)$, $a_0 \not\leq \beta$, and hence $D_{a_0} e_\beta = 0$. From this it easily follows that

$$(r) \quad e_{r(a_0)} > \bigcup_{A_0 \ni \beta < r(a_0)} e_\beta.$$

It remains to show that $r(a_0) \in A_0$. Suppose the contrary, $r(a_0) \notin A_0$. According to (p₄), $e_{r(a_0)} = \bigcup_{\beta \in A_0^-}^P b_\beta \cap e_\beta$. Thus $e_{r(a_0)} \geq b_\beta \cap e_\beta$. For $\beta \not\leq r(a_0)$ we

easily see that $b_\beta = 0$. Hence $e_{r(a_0)} = \bigcup_{A_0 \ni \beta < r(a_0)}^P b_\beta \cap e_\beta \leq \bigcup_{A_0 \ni \beta < r(a_0)} e_\beta$.

This contradicts inequality (r). Accordingly, $r(a_0) \in A_0$ and hence $r(a_0)$ is the least element of the set $\{a \in A_0: a \geq a_0\}$, i.e. (p₅) is satisfied.

The next lemma is deduced from Definition 2.2.

LEMMA 2.4. *The set A_0 is a connected subsemilattice of \bar{A} .*

Proof. Take any two elements α, β of A_0 . Obviously it suffices to consider the case where α and β are incomparable. Then, on account of

Lemma 2.1, $\alpha \cap \beta = \alpha^+ \cap \beta^+ = \alpha_0 \in A$. We wish to show that $\alpha_0 \in A_0$. On the contrary, suppose that $\alpha_0 \in A_0$.

By (p₄) we have $e_{\alpha_0} = \bigcup_{\gamma \in A_0^-} b_\gamma \cap e_\gamma$, which implies $e_{\alpha_0} \geq b_\gamma \cap e_\gamma$. For $\gamma \neq \alpha_0$ it is easy to check by the last inequality that $b_\gamma = 0$. Hence $e_{\alpha_0} = \bigcup_{A_0 \ni \gamma \leq \alpha_0} b_\gamma \cap e_\gamma$ and so $D_{\alpha_0} e_{\alpha_0} = \bigcup_{A_0 \ni \gamma \leq \alpha_0} b_\gamma \cap D_{\alpha_0} e_\gamma$. We already know that

$D_{\alpha_0} e_{\alpha_0} = 1$ and $D_{\alpha_0} e_\gamma = 0$ for all $\gamma \leq \alpha_0$ and $\gamma \in A_0$ (since $\alpha_0 \notin A_0$). Consequently $1 = 0$, which is impossible. By this contradiction we conclude that $\alpha_0 \in A_0$. Accordingly A_0 is a subsemilattice of \bar{A} .

To prove that A_0 is connected, consider again any two incomparable elements α, β of A_0 . If $\alpha \cap \beta = 0$ and hence $\alpha^+ \cap \beta^+ = 0$, then, since A is connected, there is an element γ of A such that $\alpha^+ \leq \gamma$ and $\beta^+ \leq \gamma$. Thus $\alpha^+ \leq r(\gamma)$, $\beta^+ \leq r(\gamma)$ and since $r(\gamma) \in A_0$, it follows that A_0 is connected. The lemma is proved.

THEOREM 2.5 (i) *Every Post subalgebra $P_0 = \langle P_0; \{e_\alpha; \alpha \in A_0\}; B_0 \rangle$ is a Post algebra of type A_0 . Furthermore, for every element $p \in P_0$,*

$$(4) \quad p = \bigcup_{\alpha \in A_0^-} b_{r(\alpha)} \cap e_\alpha.$$

Consequently, if $D_{r(\alpha)}^0$ is a D-operation associated with $r(\alpha) \in A_0$, then $D_\alpha p = D_{r(\alpha)}^0 p$ for each $\alpha \in A^-$.

(ii) *P_0 is an m -regular Post subalgebra of P iff B_0 is an m -regular Boolean subalgebra of B , where m is an infinite cardinal.*

Proof. We shall show that the subalgebra P_0 satisfies all conditions (P₁)-(P₆) of Definition 1.1. It is obvious that (P₁)-(P₃) are immediate consequences of (p₁), (p₃) and (p₄). Consider now (P₄). To begin with we prove the validity of (4). Since $\alpha \leq r(\alpha) \in A_0^-$ we get from (p₄) $b_{r(\alpha)} \cap e_\alpha \leq b_{r(\alpha)} \cap e_{r(\alpha)} \leq p$ for every $\alpha \in A^-$.

Conversely, let us assume that $q \in P$ and $q \geq b_{r(\alpha)} \cap e_\alpha$ for every $\alpha \in A^-$. According to the definition of $r(\alpha)$, if $\alpha_0 \in A_0^-$ and $\alpha \leq \alpha_0$ ($\alpha \in A^-$), then $r(\alpha) \leq \alpha_0$, which implies $b_{r(\alpha)} \geq b_{\alpha_0}$. Hence $b_{\alpha_0} \cap e_\alpha \leq b_{r(\alpha)} \cap e_\alpha \leq q$, for each $\alpha \in A^-$ and $\alpha \leq \alpha_0$. It follows that $q \geq \bigcup_{A^- \ni \alpha \leq \alpha_0} b_{\alpha_0} \cap e_\alpha = b_{\alpha_0} \cap \bigcup_{\substack{\alpha \leq \alpha_0 \\ \alpha \in A^-}} e_\alpha = b_{\alpha_0} \cap e_{\alpha_0}$.

Since α_0 is an arbitrary element in A_0^- , and by (p₄) we obtain $q \geq p$. Thus equality in (4) has been proved.

To verify (P₄), we take any elements p, q of P_0 and consider the meet $q \cap p$. In view of the generalized distributivity law for algebra P , we infer from (4) that

$$q \cap p = \bigcup_{\alpha \in A^-} q \cap b_{r(\alpha)} \cap e_\alpha.$$

It is evident by (p₄) that $q \cap p \geq q \cap b_{\alpha_0} \cap e_{\alpha_0}$ for every $\alpha_0 \in A_0^-$. Now let p' be an arbitrary element of P such that $p' \geq q \cap b_{\alpha_0} \cap e_{\alpha_0}$ for every $\alpha_0 \in A_0^-$.

Since $r(\alpha) \in A_0^-$ and $\alpha \leq r(\alpha)$, it follows that $p' \geq q \cap b_{r(\alpha)} \cap e_{r(\alpha)} \geq q \cap b_{r(\alpha)} \cap e_\alpha$ for every $\alpha \in A^-$. Consequently, the last inequality gives us $p' \geq q \cap p$, which implies that

$$q \cap p = \bigcup_{\alpha_0 \in A_0^-}^P q \cap b_{\alpha_0} \cap e_{\alpha_0}.$$

Thus (P_4) is satisfied.

Finally, it remains to verify condition (P_5) . First let us suppose that $b \in B_0$ and $b \cap e_{\alpha_0} \leq e_{\beta_0}$, where $\alpha_0, \beta_0 \in A_0$ and $\alpha_0 \not\leq \beta_0$. Since, as we know, $\alpha_0 = \bigcup^A \{\alpha \in A : \alpha \leq \alpha_0\}$, there is an element α of A^- such that $\alpha \leq \alpha_0$, but $\alpha \not\leq \beta_0$. So $D_\alpha e_{\alpha_0} = 1$, $D_\alpha e_{\beta_0} = 0$ and consequently $b = D_\alpha b = 0$, since $D_\alpha b \cap D_\alpha e_{\alpha_0} \leq D_\alpha e_{\beta_0}$.

Secondly, assume that $b \in B_0$, $\alpha_0 \in A_0^-$ and $b \cap e_{\alpha_0} \leq \bigcup_{A_0^+ \beta < \alpha_0}^{P_0} e_\beta$. As a consequence of (p_3) , we have $\bigcup_{A_0^+ \beta < \alpha_0}^{P_0} e_\beta = \bigcup_{A_0^+ \beta < \alpha_0}^P e_\beta$. Recall that $e_{\alpha_0} = \bigcup_{A^- \alpha \leq \alpha_0}^P e_\alpha$ and by (p_2) , $e_{\alpha_0} > \bigcup_{A_0^+ \beta < \alpha_0}^P e_\beta$. Hence there exists an element α of A^- such that $\alpha \leq \alpha_0$ but, for each $\beta \in A_0^-$, if $\beta < \alpha_0$, then $\alpha \not\leq \beta$. For this $\alpha \in A^-$, $D_\alpha e_{\alpha_0} = 1$, $D_\alpha \left(\bigcup_{A_0^+ \beta < \alpha_0}^P e_\beta \right) = \bigcup_{A_0^+ \beta < \alpha_0}^P D_\alpha e_\beta = 0$. Therefore, similarly as above, we obtain again $b = D_\alpha b = 0$. Thus (P_5) is satisfied and consequently P_0 is a Post algebra of type A_0 .

(ii) The necessity is evident. Conversely, assume that B_0 is an m -regular Boolean subalgebra of B . Let $p, p_t \in P_0$; $t \in T$ and

$$(5) \quad p = \bigcup_{t \in T}^{P_0} p_t, \quad \text{where } \bar{T} < m.$$

We need to show that $p = \bigcup_{t \in T}^P p_t$. For this purpose take any element $q \in P$ such that $q \geq p_t$ for each $t \in T$. Thus $D_\alpha q \geq D_\alpha p_t = D_{r(\alpha)}^0 p_t$. Since B_0 is m -regular, by applying Epstein's Lemma (1.8) to the l.u.b. (5) we get

$$D_{r(\alpha)}^0 p = \bigcup_{t \in T}^{B_0} D_{r(\alpha)}^0 p_t = \bigcup_{t \in T}^B D_{r(\alpha)}^0 p_t.$$

Hence it follows that $D_\alpha q \geq D_{r(\alpha)}^0 p = D_\alpha p$ for each $\alpha \in A^-$, which implies $q \geq p$. Thus $p = \bigcup_{t \in T}^P p_t$ is proved and hence P_0 is an m -regular Post subalgebra of P .

2. Post homomorphism. Let $P = \langle P; \{e_\alpha : \alpha \in A\}; B \rangle$ and $P' = \langle P'; \{e'_\alpha : \alpha' \in A'\}; B' \rangle$ be given Post algebras of type A and A' , respectively. Let h be a mapping from P into P' .

DEFINITION 2.6. We call the mapping h a *Post homomorphism* of P into P' provided h satisfies the following conditions:

- (h₁) h is a lattice homomorphism,
- (h₂) $h(E) \subset E'$, where, as we know, $E = \{e_\alpha : \alpha \in A^+\}$ and $E' = \{e'_\alpha : \alpha' \in A'^+\}$.

Further,

$$h(0) = 0 = e'_0 \quad \text{and} \quad h(1) = 1 = e'_{v'}.$$

- (h₃) $h\left(\bigcup_{\substack{A^- \\ \alpha < \beta}}^P e_\alpha\right) = \bigcup_{\beta > \alpha \in A^-}^{P'} h(e_\alpha)$ for any $\beta \in A^-$.
- (h₄) If A'_0 denotes the set $\{\alpha' \in A' : e'_{\alpha'} = h(e_\alpha) \text{ for some } \alpha \in A\}$, then for each $\alpha' \in A'$, the set $\{\alpha'_0 \in A'_0 : \alpha'_0 \geq \alpha'\}$ has a least element denoted by $r(\alpha')$.
- (h₅) For each $\alpha'_0 \in A'_0$, the set $\{\alpha \in A^+ : h(e_\alpha) = e'_{\alpha'_0}\}$ has a least element denoted by $s\alpha'_0$. Of course,
- (6) $h(e_{s\alpha'_0}) = e'_{\alpha'_0}$ for each $\alpha'_0 \in A'_0$.

Note that from (h₂) and (h₃) we infer that the set $\{\alpha'_0 \in A'_0 : \alpha'_0 \geq \alpha'\}$ is not empty for any $\alpha' \in A'$. Indeed, if for a certain $\alpha' \in A'$ this set were empty, then for each $\alpha'_0 \in A'_0$, we would have $\alpha'_0 \not\geq \alpha'$, and so $D_{\alpha'} e'_{\alpha'_0} = 0$. By virtue of (h₂) and (h₃) we would have

$$1 = h(1) = h\left(\bigcup_{\alpha \in A^-}^P e_\alpha\right) = \bigcup_{\alpha \in A^-}^{P'} h(e_\alpha) = \bigcup_{\alpha'_0 \in A'_0}^{P'} e'_{\alpha'_0}.$$

Consequently, for such α'

$$1 = D_{\alpha'} 1 = \bigcup_{\alpha'_0 \in A'_0}^{P'} D_{\alpha'} e'_{\alpha'_0} = 0,$$

which is impossible.

Remark 3. If the sets A and A' are partially well-ordered, then conditions (h₄) and (h₁) are obviously satisfied.

Remark 4. If the semilattices A and A' are complete, then conditions (h₄) and (h₅) can be deduced from (h₁)-(h₃) and from the following condition (h'₃):

(h'₃) for every subset $\{e_{\alpha_t} : t \in T\}$ of the set E ,

$$h\left(\bigcap_{t \in T}^P e_{\alpha_t}\right) = \bigcap_{t \in T}^{P'} h(e_{\alpha_t}).$$

It follows from Remarks 3 and 4 that Definition 2.6 can be simplified by adding some hypotheses concerning the set A . Now we shall prove Remark 4. Since \bar{A}' is complete, we can put

$$r(\alpha') = \bigcap^{A'} \{\alpha'_0 \in A'_0 : \alpha'_0 \geq \alpha'\}.$$

It suffices to show that $r(\alpha') \in A'_0$. For brevity, denote by $A'_0(\alpha')$ the set $\{\alpha'_0 \in A'_0 : \alpha'_0 \geq \alpha'\}$ and (in accordance with $A'_0(\alpha')$) by $A_0(\alpha')$ the set $\{\alpha \in A : \text{there exists an } \alpha' \in A'_0 \text{ such that } h(e_\alpha) = e'_{\alpha'}\}$. By the completeness of A , we have $\bigcap^{A'} A_0(\alpha') = \alpha \in A$. Thus, as we know, $\bigcap_{\beta \in A_0(\alpha')}^P e_\beta$. On account of (h'₃) and later by Theorem 2.3 (ii), we see that

$$h(e_\alpha) = \bigcap_{\beta \in A_0(\alpha')}^{P'} h(e_\beta) = \bigcap_{\beta' \in A'_0(\alpha')}^{P'} e'_{\beta'} = e'_{r(\alpha')}.$$

This implies that $r(\alpha') \in A'_0$.

Analogously condition (h_5) easily results from the completeness of A and (h'_3) .

Put:

$$A_0 = \{a \in A : a = s\alpha' \text{ for a certain } \alpha' \in A'_0\}.$$

THEOREM 2.7. *If h is a (Post) homomorphism of P into P' , then $h(P)$ is a Post subalgebra of type A'_0 , and $h(B) = B' \cap h(P)$. Moreover, if $D_{\alpha'_0}^0$ ($\alpha'_0 \in A'_0$) is a D -operation in $h(P)$, then*

$$(7) \quad D_{\alpha'_0}^0 h(p) = h(D_{s\alpha'_0} p),$$

and also

$$(8) \quad h(p) = \bigcup_{\alpha'_0 \in A'^{-}} D_{\alpha'_0}^0 h(p) \cap e_{\alpha'_0} = \bigcup_{\alpha \in A} D_{\alpha} h(p) \cap h(e_{\alpha}).$$

Proof. First of all, we shall show that A'_0 and A_0 are similar partially ordered sets, or more precisely, that for any $\alpha'_0, \beta'_0 \in A'_0$,

$$(9) \quad \alpha'_0 < \beta'_0 \quad \text{iff} \quad s\alpha'_0 < s\beta'_0.$$

In fact, assume that $s\alpha'_0 < s\beta'_0$. Thus $e_{s\alpha'_0} < e_{s\beta'_0}$, which implies by (h_1) that $h(e_{s\alpha'_0}) \leq h(e_{s\beta'_0})$. Hence, by (6), $e'_{\alpha'_0} \leq e'_{\beta'_0}$. In view of the definition of the mapping s , the equality $e_{\alpha'_0} = e_{\beta'_0}$ cannot hold. Hence $e'_{\alpha'_0} < e'_{\beta'_0}$ and consequently $\alpha'_0 < \beta'_0$. Conversely, suppose that $\alpha'_0 < \beta'_0$ and $s\alpha'_0 \not\leq s\beta'_0$. The equality $s\alpha'_0 = s\beta'_0$ is impossible, since from $\alpha'_0 < \beta'_0$ we have $e_{\alpha'_0} < e_{\beta'_0}$ and hence by (6), $h(e_{s\alpha'_0}) \neq h(e_{s\beta'_0})$. So $s\alpha'_0 \not\leq s\beta'_0$, which implies $s\alpha'_0 \cap s\beta'_0 < s\alpha'_0$. Simultaneously, we see that

$$\begin{aligned} h(e_{s\alpha'_0 \cap s\beta'_0}) &= h(e_{s\alpha'_0} \cap e_{s\beta'_0}) = h(e_{s\alpha'_0}) \cap h(e_{s\beta'_0}) \\ &= e'_{\alpha'_0} \cap e'_{\beta'_0} = e'_{\alpha'_0} = h(e_{s\alpha'_0}). \end{aligned}$$

This contradicts the definition of $s\alpha'_0$. Therefore $s\alpha'_0 < s\beta'_0$ has been deduced from $\alpha'_0 < \beta'_0$.

It is obvious that if $b \in B$, then

$$h(b) \cap h(-b) = 0 \quad \text{and} \quad h(b) \cup h(-b) = 1.$$

Hence we can conclude that

$$(10) \quad h(b) \in B' \quad \text{and} \quad h(-b) = -h(b).$$

Now consider an arbitrary element $p' \in h(P)$, i.e.

$$p' = \bigcup_{\alpha' \in A'^{-}} D_{\alpha'} p' \cap e_{\alpha'} = h(p) \quad \text{for some } p \in P.$$

We wish to show that for each $\alpha' \in A'^{-}$

$$(11) \quad D_{\alpha'} p' = D_{\alpha'} h(p) = h(D_{s\alpha'} p),$$

where $s\alpha' = s(r(\alpha'))$.

It is not difficult to verify that

$$p \cap e_{s\hat{r}a'} = \bigcup_{\substack{a < sra' \\ a \in \mathcal{A}^-}}^P p_a \cap e_a$$

and

$$\begin{aligned} p \cap e_{sra'} &= \bigcup_{\substack{a \leq sra' \\ a \in \mathcal{A}^-}}^P p_a \cap e_a \\ &= \left(\bigcup_{\substack{a < sra' \\ a \in \mathcal{A}^-}}^P p_a \cap e_a \right) \cup (p_{sra'} \cap e_{sra'}). \end{aligned}$$

Thus

$$(12) \quad p \cap e_{sra'} = (p \cap e_{s\hat{r}a'}) \cup (p_{sra'} \cap e_{sra'}).$$

According to (h₅), if $a < sra'$, then $h(e_a) = ea'_0 < h(e_{sra'}) = e'_{ra'}$, or equivalently $a'_0 < ra'$. Hence by (h₃) and (9), $h(e_{s\hat{r}a'}) = \bigcup_{sra' > a \in \mathcal{A}^-}^{P'} h(e_a) =$

$\bigcup_{\substack{A'_0 \triangleright a'_0 \\ a'_0 < ra'}}^{P'} e'_{a'_0}$. Accordingly, it follows from (12) and (h₁) that

$$p' \cap e'_{ra'} = h(p) \cap h(e_{sra'}) = h(p \cap e_{sra'}) = (h(p) \cap h(e_{s\hat{r}a'})) \cup (h(p_{sra'}) \cap h(e_{sra'})),$$

and so

$$(13) \quad p' \cap e'_{ra'} = (h(p) \cap \bigcup_{\substack{A'_0 \triangleright a'_0 < ra'}}^{P'} e'_{a'_0}) \cup (h(p_{sra'}) \cap e'_{ra'}).$$

Since $a' \leq ra'$, it follows from the definition of ra' that if $a'_0 < ra'$, then $a' \not\leq a'_0$. Thus we have

$$(14) \quad D'_a e'_{ra'} = 1 \quad \text{and} \quad D'_a \left(\bigcup_{\substack{a'_0 < ra' \\ a'_0 \in A'_0}}^{P'} e'_{a'_0} \right) = \bigcup_{A'_0 \triangleright a'_0 < ra'}^{P'} D'_a e'_{a'_0} = 0.$$

Consequently, by applying D'_a to both sides of equation (13), we obtain

$$D'_a p' = D'_a h(p) = h(p_{sra'}) = h(D_{sra'} p),$$

i.e. formula (11) is valid.

Notice that if $a'_0 \in A'_0$, $a'_0 \geq a'$, then $ra' \leq a'_0$, and so by (9) $sra' \leq sa'_0$. Thus in view of (11) it follows that

$$D'_a p' = h(D_{sra'} p) \geq h(D_{sa'_0} p) \quad \text{for } a' \leq a'_0.$$

This yields

$$\begin{aligned} p' &\geq \bigcup_{\substack{a' \leq a'_0 \\ a' \in \mathcal{A}'^-}}^{P'} D'_a p' \cap e'_{a'} \geq \bigcup_{\substack{a' \leq a'_0 \\ a' \in \mathcal{A}'^-}}^{P'} h(D_{sa'_0} p) \cap e'_{a'} \\ &= h(D_{sa'_0} p) \cap \bigcup_{A' \triangleright a \leq a'_0}^{P'} e'_{a'} = h(D_{sa'_0} p) \cap e'_{a'_0}. \end{aligned}$$

i.e. for every $a'_0 \in A'_0$

$$(15) \quad p' \geq h(D_{sa'_0} p) \cap e'_{a'_0}.$$

Now let q' be any element of P' such that $q' \geq h(D_{sa'_0}p) \cap e'_{a'_0}$ for every $a'_0 \in A'_0$. We see that $ra' \in A'_0$ for every $a' \in A'$. Hence by the assumption on q' together with (11) and the inequality $a' \leq ra'$, we get

$$q' \geq h(D_{sra'}p) \cap e'_{ra'} \geq D'_{a'}p' \cap e'_{a'}$$

for any $a' \in A^-$. This result shows that $q' \geq p'$. Therefore we have proved the following formula

$$(16) \quad p' = \bigcup_{a'_0 \in A'_0} h(D_{sa'_0}p) \cap e'_{a'_0}.$$

Consider an arbitrary element a of A^- , and let $h(e_a) = e'_{a'_0}$. Of course $a'_0 \in A'_0$. On account of the definition of sa'_0 , we have $a \geq sa'_0$ and so $D_a p \leq D_{sa'_0}p$, which implies that $h(D_a p) \leq h(D_{sa'_0}p)$. Thus (15) leads to

$$p' \geq h(D_{sa'_0}p) \cap e'_{a'_0} \geq h(D_a p) \cap h(e_a)$$

for each $a \in A^-$. On the other hand, suppose that $q' \in P'$ and $q' \geq h(D_a p) \cap h(e_a)$ for each $a \in A^-$. Then since $sa'_0 \in A^-$ for each $a'_0 \in A'_0$, we obtain $q' \geq h(D_{sa'_0}p) \cap h(e_{sa'_0}) = h(D_{sa'_0}p) \cap e'_{a'_0}$. From this and (16), $q' \geq p'$ follows. Accordingly

$$(17) \quad p' = h(p) = \bigcup_{a \in A} h(D_a p) \cap h(e_a).$$

Now we come to prove the main assertion of the theorem, namely that $h(P) = \langle h(P); \{e'_{a'}: a' \in A'_0\}; h(B) \rangle$ is a Post subalgebra of type A'_0 of P' . It is easily observed that (p₁) follows from (h₁), (p₂) — from the definition of A'_0 , (p₄) — from (9), (10) and (16), and (p₆) follows from (h₄) and the fact that $ra' \in A'_0$ for every $a' \in A'$.

It still remains to check condition (p₃). For this purpose consider an arbitrary subset $\{e'_{a'_i}: t \in T\}$ of the set E'_0 . Certainly, $\{e_{sa'_i}: t \in T\} \subset E$. Hence from the definition of a Post algebra, the l.u.b. $\bigcup_{t \in T} e_{sa'_i} = p$ exists in P . Let $p = \bigcup_{a \in A^-} D_a p \cap e_a$, and let us denote by M the set $\{a \in A: a \leq sa^t$ for some $t \in T\}$. It is evident that

$$D_a p = D_a \left(\bigcup_{t \in T} e_{sa'_i} \right) = \bigcup_{t \in T} D e_{sa'_i} = \begin{cases} 1 & \text{if } a \in M, \\ 0 & \text{if } a \notin M. \end{cases}$$

So

$$h(D_a p) = \begin{cases} 1 & \text{if } a \in M, \\ 0 & \text{if } a \notin M. \end{cases}$$

Accordingly, from (17) we have

$$h(p) = \bigcup_{a \in M} h(e_a).$$

Since obviously $sa'_i \in M$, it is easy to show from (6) and the definition of M that

$$\bigcup_{a \in M}^{P'} h(e_a) = \bigcup_{t \in T}^{P'} h(e_{sa'_i}) = \bigcup_{t \in T}^{P'} e'_{a'_i}.$$

Consequently $\bigcup_{t \in T}^{P'} e'_{a'_i} = h(p) \in h(P)$, i.e. (p_8) is satisfied in $h(P)$.

By virtue of Theorem 2.5 (i), $h(P)$ is a Post algebra of type A'_0 , and so if $D_{a'_0}^0$ is a D -operation associated with $a'_0 \in A'_0$, then by (16) $D_{a'_0}^0 p' = h(D_{sa'_0} p)$, which is essentially formula (7).

Finally, to complete the proof of the theorem, we show that $h(B) = B' \cap h(P)$. Indeed, (10) leads to $h(B) \subset B' \cap h(P)$. Conversely, let us take any element $b' = h(p) \in B' \cap h(P)$. Because of (11), we have $b' = D_{a'} h(p) = h(D_{sa'} p) \in h(B)$. This means that $h(B) \supset h(P) \cap B'$.

Let m be an arbitrary infinite cardinal, and let P, P' be, as before, the Post algebras of type A and A' , respectively. We call a Post homomorphism of P into P' the m -homomorphism if h preserves all the existing l.u.b.'s and g.l.b.'s of subsets of P of power less than m . Moreover, if h is a mapping from P into P' and P_0 is a subset of P , then $h_{|P_0}$ denotes the mapping h restricted to P_0 .

THEOREM 2.8. *Let h be a homomorphism from P into P'*

(i) *h is an m -homomorphism iff $h_{|B}$ is an m -homomorphism from B into B' .*

(ii) *h is an isomorphism from P into P' iff $h_{|B \cup \bar{B}}$ is a one-to-one mapping.*

Proof (i). Obviously it is enough to prove the sufficiency. Let h be an m -homomorphism from B into B' . Let $p = \bigcup_{t \in T}^P p_t$ and let the power of T be smaller than m . According to the above theorem, $h(P) = \langle h(P); e'_{a'_0}: a'_0 \in A'_0; h(B) \rangle$ is a Post algebra and $D_{a'_0}^0 h(p) = h(D_{sa'_0} p)$. Then since by Epstein's Lemma $D_{sa'_0} p = \bigcup_{t \in T}^B D_{sa'_0} p_t$ and since $h_{|B}$ is an m -homomorphism, we obtain

$$D_{a'_0}^0 h(p) = h(D_{sa'_0} p) = \bigcup_{t \in T}^{B'} h(D_{sa'_0} p_t) = \bigcup_{t \in T}^{B'} D_{a'_0}^0 h(p_t)$$

for every $a'_0 \in A'_0$.

Since $h(D_{sa'_0} p) \in h(B)$, obviously $D_{a'_0}^0 h(p) = \bigcup_{t \in T} h(B) D_{a'_0}^0 h(p_t)$ for each $a'_0 \in A'_0$. Applying the sufficiency of Theorem 1.8 (ii) to the set $\{h(p_t): t \in T\}$ and $h(p)$ in the algebra $h(p)$, we have

$$h(p) = \bigcup_{t \in T}^{h(P)} h(p_t).$$

By virtue of Theorem 2.5 (ii), the m -regularity of $h(P)$ can be deduced from the m' -regularity of $h(B)$. Hence $h(p) = \bigcup_{t \in T}^{P'} h(p_t)$. Analogously we

can show that if $p = \bigcup_{t \in T} p_t$, then $h(p) = \bigcup_{t \in T} h(p_t)$. Consequently, h is an m -homomorphism.

Proof of (ii). Since the necessity of (ii) is clear, we show only the sufficiency. In view of the assumption it is easy to check that $A_0 = sA'_0 = A$, where sA'_0 denotes the image of the set A'_0 under the mapping s . It suffices to show that h is a one-to-one mapping, i.e. if $h(p) = h(q)$, then $p = q$ for any two elements $p, q \in P$. Indeed, $h(p) = h(q)$ yields $D_{\alpha'_0}^0 h(p) = D_{\alpha'_0}^0 h(q)$, or by (7), $h(D_{s\alpha'_0} p) = h(D_{s\alpha'_0} q)$ for each $\alpha'_0 \in A'_0$. Because of $sA'_0 = A$, $h(D_\alpha p) = h(D_\alpha q)$ for each $\alpha \in A^-$. Since h is a one-to-one mapping over B , we have $D_\alpha p = D_\alpha q$, $\alpha \in A^-$. This implies $p = q$.

The theorem has been entirely proved.

To conclude this section, we shall prove a remark which will be useful later on.

Remark 5. The set A_0 defined above by Theorem 2.7 has the following property: for every element $a \in A$, the set $\{\alpha_0 \in A_0: \alpha_0 \leq a\}$ has a greatest element.

In fact, we will show that sa' , where $a' \in A'_0$ is such that $h(e_a) = e'_a$ is the greatest element of the set $I = \{\alpha_0 \in A_0: \alpha_0 \leq a\}$. From (h₅) we evidently have $sa' \leq a$, i.e. $sa' \in I$. Consider an arbitrary element $\alpha_0 \in A_0$, i.e. $\alpha_0 = s\beta'$ for some $\beta' \in A'_0$, and $\alpha_0 = s\beta' \leq a$. Then obviously by (6) we have

$$h(e_{s\beta'}) = e'_{\beta'} \leq h(e_a) = e'_a.$$

So $\beta' \leq a'$, which implies by (9) that $\alpha_0 = s\beta' \leq sa'$. This is the conclusion required. Thus the proof is completed.

3. Filters in the Post and semi-Post algebras. Quotient algebras

1. D -filters. The notions of an ideal in a generalized Post algebra as introduced by Traczyk [33] and by Dwinger [6] are not sufficient in order to develop the theory of the generalized Post algebras.

In this section we shall generalize the notion of D -filter, which has been defined by H. Rasiowa in [17] in the case of Post algebras of finite order. The notion of D -filter (of order α) will be defined in such a way that the following requirements will be fulfilled:

1° Every D -filter is the shell of a certain Post homomorphism and vice-versa.

2° Every D -filter \mathcal{F} of order α ($\alpha \in A^-$) is a prime lattice filter iff the set $\mathcal{F}_0 = \mathcal{F} \cap B$ is a prime (lattice) filter in the underlying Boolean algebra B .

3° If h is a Post homomorphism of P into P' , then there is a D -filter ∇ in P such that the algebra $h(P)$ is isomorphic to the quotient algebra P/∇ .

We already know that these three conditions are easily satisfied in Post algebras of finite order.

Now let $P = \langle P; \{e_\alpha: \alpha \in A\}; B \rangle$ be a Post algebra of type A , and let \bar{A} be the extension of the semilattice A defined as in the previous section.

DEFINITION 3.1 (i) Let $a \neq 0$ be an arbitrary element of \bar{A} . A lattice filter $\nabla \subset P$ is said to be a D_a -filter if there is a subset A_0 of A such that

(d₁) A_0 is a connected semilattice containing zero (but not necessarily a subsemilattice of A) for which

$$(1) \quad \bigcup^{\bar{A}} \{\alpha_0 \in A_0\} = a$$

and for every $\beta \in A$ the set $\{\alpha_0 \in A_0: \alpha_0 \leq \beta\}$ has a greatest element denoted by $S_0\beta$

(d₂) Element $p \in \nabla$ iff for all $\alpha_0 \in A_0^-$, $D_{\alpha_0}p \in \nabla$.

D -filter is a common name of all D_a -filters.

(ii) By a prime D -filter we understand a D_a -filter ∇ such that $\nabla_0 = \nabla \cap B$ is a prime (lattice) filter in B .

A D_a -filter ∇ with the set A_0 , satisfying conditions (d₁) and (d₂) is said to be of type A_0 and is denoted by $\nabla(A_0)$ or by $\langle \nabla_0, A_0 \rangle$, where $\nabla_0 = \nabla \cap B$.

Remark 1. It will easily be seen that if $\alpha \in A^-$, then equation (1) implies that $\alpha \in A_0$. Consequently condition (d₂) is equivalent to the following one:

$$d'_2) \quad p \in \nabla \quad \text{iff} \quad D_a p \in \nabla.$$

(COROLLARY 3.2. If $\nabla = \langle \nabla_0, A_0 \rangle$ is a D_a -filter ($a \in \bar{A}$), then $e_\beta \in \nabla$ iff $\beta \geq a$.

Indeed, if $\beta \geq a$, then (1) gives $\beta \geq \alpha_0$, which implies that $D_{\alpha_0}e_\beta = 1$, for $\alpha_0 \in A_0^-$. Hence $e_\beta \in \nabla$ follows from (d₂). Conversely, if $e_\beta \in \nabla$, then by (d₂), $D_{\alpha_0}e_\beta = 1$ or equivalently $\alpha_0 \leq \beta$, for all $\alpha_0 \in A_0^-$. Consequently, (1) yields $\beta \geq a$.

THEOREM 3.3. (i) Let h be a (Post) homomorphism of $P = \langle P; \{e_\alpha: \alpha \in A\}; B \rangle$ into $P' = \langle P'; \{e'_\alpha: \alpha' \in A'\}; B' \rangle$ and let A_0 be the image of the set A'_0 under the mapping s (see Definition 2.7). Then the shell $\nabla = \{p \in P: h(p) = 1\}$ of the homomorphism h is a D -filter. Further, ∇ is of type A_0 , i.e. A_0 satisfies (d₁) and (d₂).

(ii) For each $a \in A^-$, the D_a -filter ∇ is a prime (lattice) filter iff the filter $\nabla_0 = \nabla \cap B$ is prime in B .

(iii) If ∇ is a prime (lattice) filter in P , the set $\{e_\alpha \in \bar{E}: e_\alpha \in \nabla\}$ has a least element e_{α_0} and furthermore $\alpha_0 \in A^-$, then ∇ is a D_{α_0} -filter.

Proof. First we shall show that the l.u.b. $\bigcup^{\bar{A}} A_0$ exists, where $A_0 = \{s\beta'_0: \beta'_0 \in A'_0\}$. From (h₅) in Definition 2.6, it is known that $\alpha_0 = sV'$ (V' is the unit element of A') is a least element in A^+ such that $h(e_{sV'}) = 1 = e'_{V'}$. By formula (9) in section 2, we have $s\beta'_0 \leq sV' = \alpha_0$ for every $s\beta'_0 \in A_0$.

Hence it is evident that if $\alpha_0 \in A_0$, then $\bigcup^{\bar{A}} A_0 = \alpha_0$. Now assume that $\alpha_0 \notin A_0$. Thus $\alpha_0 > s\beta'_0$ and this obviously implies that $\alpha_0^- \geq s\beta'_0$ for every $s\beta'_0 \in A_0$. Let us consider an arbitrary element α of \bar{A} such that $\alpha \geq s\beta'_0$ for every $s\beta'_0 \in A_0$. So we see that $e_\alpha \geq e_{s\beta'_0}$ and hence, because of formula (6) in Definition 2.6, $h(e_\alpha) \geq h(e_{s\beta'_0}) = e'_{\beta'_0}$ for every $\beta'_0 \in A'_0$. From this and by (8) in Theorem 2.7 it follows that $h(e_\alpha) \geq \bigcup^{\beta'_0 \in A'_0} e'_{\beta'_0} = 1$. Then,

since $e_{\alpha^+} \geq e_\alpha$, we get $h(e_{\alpha^+}) = 1$, which implies that $\alpha^+ \geq sV' = \alpha_0$. Consequently $\alpha \geq (\alpha^+)^- \geq \alpha_0^-$. Thus we have proved that $\alpha_0^- = \bigcup^{\bar{A}} A_0$ and, accordingly, the l.u.b. $\bigcup^{\bar{A}} A_0$ always exists.

Put $\alpha = \bigcup^{\bar{A}'} A_0$.

This together with Remark 5, Section 2, shows that A_0 satisfies (d₁) in Definition 3.1. Further, it is easy to see that

$$p \in \mathcal{V} \text{ iff } h(p) = 1 \text{ iff for every } \beta'_0 \in A'_0, D_{\beta'_0}^0 h(p) = 1$$

iff for every $s\beta'_0 \in A_0$, $h(D_{s\beta'_0} p) = 1$ (because of (7), Section 2), iff for every $s\beta'_0 \in A_0$, $D_{s\beta'_0} p \in \mathcal{V}$.

Hence \mathcal{V} with the set A_0 satisfies (d₂) in 3.1. Therefore \mathcal{V} is a D_{α_0} -filter of type A_0 in P .

(ii) The necessity is evident. To prove the sufficiency, let us assume that \mathcal{V}_0 is a prime filter in B and $p \cup q \in \mathcal{V}$. By virtue of (d'₂) we have $D_\alpha(p \cup q) = D_\alpha p \cup D_\alpha q \in \mathcal{V}_0$. Thus either $D_\alpha p \in \mathcal{V}_0$ or $D_\alpha q \in \mathcal{V}_0$, say $D_\alpha p \in \mathcal{V}_0$. Again by (d'₂) we have $p \in \mathcal{V}$. Consequently, \mathcal{V} is a prime (lattice) filter.

(iii) In order to prove that \mathcal{V} is a D_{α_0} -filter, we will show that \mathcal{V} satisfies condition (d'₂). If $D_{\alpha_0} p \in \mathcal{V}$, then obviously $p \in \mathcal{V}$, since $e_{\alpha_0} \in \mathcal{V}$ and $p \geq D_{\alpha_0} p \cap e_{\alpha_0}$. Conversely, suppose that $p \in \mathcal{V}$. Since \mathcal{V} is prime and $D_{\alpha_0} p \cup -D_{\alpha_0} p = 1 \in \mathcal{V}$, then at least one of the elements $D_{\alpha_0} p$ and $-D_{\alpha_0} p$ belongs to \mathcal{V} . If $-D_{\alpha_0} p \in \mathcal{V}$, then $-D_{\alpha_0} p \cap p \cap e_{\alpha_0} \in \mathcal{V}$. By using twice the generalized distributivity law we obtain

$$-D_{\alpha_0} p \cap p \cap e_{\alpha_0} = \bigcup_{\substack{\alpha \in A^- \\ \alpha < \alpha_0}} -D_{\alpha_0} p \cap D_\alpha p \cap e_\alpha \leq \bigcup_{\substack{\alpha \in A^- \\ \alpha < \alpha_0}} e_\alpha = e_{\alpha_0}.$$

Consequently $e_{\alpha_0} \in \mathcal{V}$, while $e_{\alpha_0} < e_{\alpha_0}$.

This contradicts the assumption on e_{α_0} adopted in the theorem. The contradiction shows that $-D_{\alpha_0} p \notin \mathcal{V}$ and hence $D_{\alpha_0} p \in \mathcal{V}$. Therefore \mathcal{V} satisfies (d'₂). This completes the proof of the theorem.

2. Semi-Post algebras- α -filters.

DEFINITION 3.4 (i) By a semi-Post algebra of type A we mean a lattice $P = \langle P; \{e_a: a \in A\}; B \rangle$ satisfying conditions (P₁)-(P₄) in Definition 1.1.

(ii) Let a_0 and \mathcal{V}_0 be an arbitrary element of A^- and a lattice filter in B , respectively. The set $\{p \in P: \text{there exists a monotonic representation } p = \bigcup_{a \in A^-} p_a \cap e_a \text{ of the element } p \text{ such that } p_{a_0} \in \mathcal{V}_0\}$ is called an a_0 -filter in P determined by the filter \mathcal{V}_0 .

LEMMA 3.5. A set \mathcal{V} is an α -filter in P determined by a filter \mathcal{V}_0 in B iff it is a lattice filter generated by \mathcal{V}_0 and the element e_α .

Proof. We denote the lattice filter in P generated by the filter \mathcal{V}_0 in B and the element e_α by $[\mathcal{V}_0, e_\alpha]$. To prove the lemma, it is enough to show that $\mathcal{V} = [\mathcal{V}_0, e_\alpha]$. In fact, if $p \in \mathcal{V}$, then from the definition of \mathcal{V} there is a representation of the element $p = \bigcup_{a \in A^-} p_a \cap e_a$ such that $p_a \in \mathcal{V}_0$.

Obviously $p \geq p_a \cap e_a$. Hence $p \in [\mathcal{V}_0, e_\alpha]$. Conversely, suppose that $p \in [\mathcal{V}_0, e_\alpha]$, i.e. there exists an element $b \in \mathcal{V}_0$ such that $p \geq b \cap e_\alpha$. Let $p = \bigcup_{a \in A^-} p_a \cap e_a$ be an arbitrary representation of the element p . Thus

$$p = p \cup (b \cap e_\alpha) = \left(\bigcup_{\beta \in A^-} p_\beta \cap e_\beta \right) \cup \left(\bigcup_{\beta \in A^-} b \cap e_\beta \right) = \bigcup_{\substack{\beta \in A^- \\ \beta \leq \alpha}} c_\beta \cap e_\beta,$$

where

$$c_\beta = \begin{cases} p_\beta \cup b & \text{if } \beta \leq \alpha, \\ p_\beta & \text{if } \beta \not\leq \alpha. \end{cases}$$

Of course, $c_\alpha = p_\alpha \cup b \in \mathcal{V}_0$. Thus $p \in \mathcal{V}$. Accordingly we have proved that $\mathcal{V} = [\mathcal{V}_0, e_\alpha]$.

Henceforth, by \mathcal{X} we will always denote the set of all α -filters ($\alpha \in A^-$) in P which are defined by prime lattice filters in B .

Before stating the next theorem we introduce some notations. By a strictly decreasing sequence of type A_s of α -filters in P , where A_s is a partially ordered set, we shall understand the indexed subset $\mathfrak{G} = \{\mathcal{V}_\alpha: \alpha \in A_s\}$ of \mathcal{X} satisfying the following condition:

For every $\alpha, \beta \in A_s$, $\mathcal{V}_\alpha \cap B = \mathcal{V}_\beta \cap B = \mathcal{V}_0$ and $\alpha < \beta$ iff $\mathcal{V}_\alpha \supsetneq \mathcal{V}_\beta$.

Let us denote by \mathfrak{F} the set of all such strictly decreasing sequences \mathfrak{G} . We introduce in \mathfrak{F} a partially ordering relation, as follows:

For any two elements $\mathfrak{G}_1, \mathfrak{G}_2$ in \mathfrak{F} , \mathfrak{G}_1 of type A_{s_1} is regarded as smaller than \mathfrak{G}_2 of type A_{s_2} , and symbolically we write $\mathfrak{G}_1 \rightarrow \mathfrak{G}_2$, if there exists an order isomorphism f of A_{s_1} into A_{s_2} such that

$$\mathcal{V}_\alpha = \mathcal{V}_{f(\alpha)} \quad \text{for every } \alpha \in A_{s_1}.$$

Obviously, if $\mathfrak{G}_1 \rightarrow \mathfrak{G}_2$, then $\mathfrak{G}_1 \subset \mathfrak{G}_2$. Further, as will easily be seen, the set \mathfrak{F} satisfies the hypothesis of the Kuratowski-Zorn Lemma, i.e. each chain of elements in \mathfrak{F} has an upper bound in \mathfrak{F} .

THEOREM 3.6. *Let $P = \langle P: \{e_\alpha: \alpha \in A\}; B \rangle$ be a semi-Post algebra of type A . The following conditions are equivalent:*

- (S1) P is a Post algebra of type A .
- (S2) If $b \in B$, each of the two conditions: $b \cap e_\alpha \leq \bigcup_{\beta < \alpha} e_\beta$, $\alpha \in \bar{A}$ and $b \cap e_\alpha \leq e_\beta$, where $\alpha, \beta \in A$ and $\alpha \not\leq \beta$, implies $b = 0$.
- (S3) Each element $p \in P$ has a unique monotonic representation.
- (S4) For every α -filter $\mathcal{V}(\alpha \in A^-)$ in P , $\bigcup \{e_{\alpha_t}: \alpha_t \in A, t \in T\} \in \mathcal{V}$ iff there exists an element $t_0 \in T$ such that $\alpha_{t_0} \geq \alpha$.
- (S5) For every filter \mathcal{V}_0 in B and every element e_α , $\alpha \in A^-$, the lattice filter $\mathcal{V} = [\mathcal{V}_0, e_\alpha]$ generated by \mathcal{V}_0 and e_α contains the l.u.b. $\bigcup_{t \in T} e_{\alpha_t}$ iff there exists an element $t_0 \in T$ such that $\alpha_{t_0} \geq \alpha$.
- (S6) For every prime filter \mathcal{V}_0 in B , there exists at least one element $\mathcal{V} \in \mathcal{X}$ containing \mathcal{V}_0 . Moreover, every maximal sequence \mathcal{G} in \mathfrak{F} is of type A^- , and there is an enumeration $\{\mathcal{V}_\alpha: \alpha \in A^-\}$ of \mathcal{G} such that
- (2) $\bigcup_{t \in T} e_{\alpha_t} \in \mathcal{V}_\alpha$ iff there exists an element $t_0 \in T$ such that $\alpha_{t_0} \geq \alpha$.

Proof. (S1) is evidently equivalent to (S2). (S1) implies (S3), by 1.3, Section 1.

(S3) \Rightarrow (S4). For every $p \in P$ we denote by $D_\alpha p$ the coefficient of the element e_α in the unique representation of the element p . Now assume that $p = \bigcup_{t \in T} e_{\alpha_t} \in \mathcal{V}$, and \mathcal{V}_0 is the filter in B , which determines the α -filter \mathcal{V} .

Let us denote by M the set $\{\beta \in A^-: \beta \leq \alpha_t \text{ for some } t \in T\}$. Then it is easily checked that $p = \bigcup_{t \in T} e_{\alpha_t} = \bigcup_{\beta \in M} e_\beta$. This is the monotonic representation of p , and hence either $D_\beta p = 0$, or $D_\beta p = 1$. On account of the definition of α -filter, $p \in \mathcal{V}$ implies $D_\alpha p \in \mathcal{V}$. From this we infer that $D_\alpha p = 1$. Thus $\alpha \in M$, since $D_\beta p = 1$ iff $\beta \in M$. Consequently, there exists a $t_0 \in T$ such that $\alpha_{t_0} \geq \alpha$. Evidently, if there is an $\alpha_{t_0} \geq \alpha$, then $p = \bigcup_{t \in T} e_{\alpha_t} \in \mathcal{V}$.

Implication (S4) \Rightarrow (S5) is clearly an immediate consequence of Lemma 3.5.

(S5) \Rightarrow (S6). Let us denote the set $\{\mathcal{V}_\alpha = [\mathcal{V}_0, e_\alpha]: \alpha \in A^-\}$ by $\mathcal{X}(\mathcal{V}_0)$. If $\alpha < \beta$, then by virtue of (S5), $e_\alpha = \bigcup_{\gamma < \alpha} e_\gamma$ does not belong to \mathcal{V}_β . Consequently, $e_\alpha < e_\beta$, and $\mathcal{V}_\alpha \not\supseteq \mathcal{V}_\beta$. Hence $\mathcal{X}(\mathcal{V}_0)$ is a strictly decreasing sequence of type A^- satisfying condition (2). We note that

$$(2') \quad \alpha < \beta \text{ iff } e_\alpha < e_\beta \text{ iff } \mathcal{V}_\alpha \not\supseteq \mathcal{V}_\beta.$$

Let \mathcal{G} be an arbitrary element maximal in \mathfrak{F} and let $\mathcal{V}_0 = \mathcal{V} \cap B$ for some $\mathcal{V} \in \mathcal{G}$. From the definition of a sequence in \mathfrak{F} we see that \mathcal{V}_0 is uniquely defined by \mathcal{G} . Hence it follows from 3.5 that for every $\mathcal{V} \in \mathcal{G}$, $\mathcal{V} = [\mathcal{V}_0, e_\alpha]$ for a certain $\alpha \in A^-$. Thus we conclude that $\mathcal{G} \subset \mathcal{X}(\mathcal{V}_0)$.

In view of (2') it is easy to show that $\mathfrak{G} \rightarrow \mathfrak{X}(\mathcal{V}_0)$. Since \mathfrak{G} is a maximal element, we obtain $\mathfrak{G} = \mathfrak{X}(\mathcal{V}_0)$. Consequently \mathfrak{G} is a sequence of type A^- and satisfies condition (2).

(S6) \Rightarrow (S2). Let b_0 be an arbitrary element of B different from zero, and let \mathcal{V}_0 be a prime filter in B containing b_0 . In view of (S6), there is an element $\mathcal{V} \in \mathfrak{X}$, which contains \mathcal{V}_0 . Since the filter \mathcal{V}_0 is maximal, it is clear that $\mathcal{V}_0 = \mathcal{V} \cap B$. According to the Kuratowski-Zorn Lemma, there is a maximal element \mathfrak{G} in \mathfrak{F} greater than \mathcal{V} . Evidently $\mathcal{V}_0 = B \cap \mathcal{V}'$ for every $\mathcal{V}' \in \mathfrak{G}$. Let the enumeration $\mathfrak{G} = \{\mathcal{V}_\alpha : \alpha \in A^-\}$ satisfy the condition mentioned in (S6). Since $b_0 \in \mathcal{V}_0$, and $e_\alpha \in \mathcal{V}_\alpha$, we infer that if $b_0 \cap e_\alpha \leq e_\beta = \bigcup_{\gamma \leq \beta} e_\gamma$, where $\alpha \not\leq \beta$, then by (2) there exists an $\gamma \leq \beta$ such that $\gamma \geq \alpha$. Consequently $\beta \geq \alpha$, which contradicts $\alpha \not\leq \beta$. Similarly, it may be shown that $b_0 \cap e_\alpha \leq \bigcup_{\beta < \alpha} e_\beta$ leads to a contradiction. Consequently each of the two conditions $b \cap e \leq e_\beta$, where $\alpha \not\leq \beta$, and $b \cap e_\alpha \leq \bigcup_{\beta < \alpha} e_\beta$ yields the conclusion $b = 0$.

Thus (S2) is deduced from (S6), which completes the proof of our theorem.

3. Quotient Post algebras. Let a Post algebra $P \langle P; e_\alpha : \alpha \in A; B \rangle$ of type A be given, and let $\mathcal{V} = \langle \mathcal{V}_0, A_0 \rangle = \mathcal{V}(A_0)$ be a D_α -filter of type A_0 in P .

We introduce a relation defined as follows:

$$p \approx_{\mathcal{V}} q \text{ iff for all } \beta \in A_0, (D_\beta p \Rightarrow D_\beta q) \cap (D_\beta q \Rightarrow D_\beta p) \in \mathcal{V}(A_0).$$

It is not difficult to show that the relation $\approx_{\mathcal{V}}$ is an equivalence. Denote by $|p|$ the equivalence class defined by an element p , and by $P/\mathcal{V}(A_0)$ or, for brevity, by P/\mathcal{V} the set of all equivalence classes in P , i.e.

$$P/\mathcal{V}(A_0) = P/\mathcal{V} = \{|p| : p \in P\}.$$

We now define an ordering relation \leq in $P/\mathcal{V}(A_0)$ in the following way:

$$|p| \leq |q| \text{ iff for every } \beta \in A_0^-, D_\beta p \Rightarrow D_\beta q \in \mathcal{V}.$$

Then $P/\mathcal{V}(A_0)$ becomes a partially ordered set and the equations $|p \cup q| = |p| \cup |q|$ and $|p \cap q| = |p| \cap |q|$ hold. Moreover, on account of $D_\gamma D_\beta p = D_\beta p$, the formula $p \approx_{\mathcal{V}} p'$ implies that $D_\beta p \approx_{\mathcal{V}} D_\beta p'$, for every $\beta \in A_0^-$. Thus we can define the operations D_β , $\beta \in A_0^-$ in $P/\mathcal{V}(A_0)$ by the equation

$$D_\beta |p| = |D_\beta p| \text{ for every } |p| \in P/\mathcal{V}(A_0).$$

It will be observed that if $|p| \leq |q|$, then $D_\beta |p| \leq D_\beta |q|$ for all $\beta \in A_0^-$, and

$$(3) \quad p = \bigcup_{t \in T}^P e_{a_t} \text{ implies that } |p| = \bigcup_{t \in T}^{P/\mathcal{V}} |e_{a_t}|.$$

Indeed, clearly $|p| \geq |e_{\alpha_t}|$, since $p \geq e_{\alpha_t}$. Now let $|q| \geq |e_{\alpha_t}|$ for every $t \in T$. If $\beta \in A_0^-$ and $\beta \leq \alpha_{t_0}$ for a certain $t_0 \in T$, then $D_\beta |q| = |D_\beta q| \geq |D_\beta e_{\alpha_{t_0}}| = |1| = 1$. So it follows that $D_\beta q \in \mathcal{V}(A_0)$.

If $\beta \in A_0^-$ but $\beta \not\leq \alpha_t$ for every $t \in T$, then we have $D_\beta p = \bigcup_{t \in T} D_\beta e_{\alpha_t} = 0$.

Hence in both cases the formulas $D_\beta p \Rightarrow D_\beta q \in \mathcal{V}(A_0)$, $\beta \in A_0^-$, always hold. Therefore, according to the definition of order in $P/\mathcal{V}(A_0)$, $|p| \leq |q|$, and in this way formula (3) is proved.

We adopt the notation $|e|_\beta = |e|$ for each $\beta \in A_0^+$ and $B/\mathcal{V}(A_0) = \{|b| : b \in B\}$.

THEOREM 3.7. (i) *The lattice $P/\mathcal{V} = \langle P/\mathcal{V}; \{|e|_\beta : \beta \in A_0\}; B/\mathcal{V} \rangle$, where \mathcal{V} is a D_α -filter of type A_0 , is a Post algebra of type A_0 , and for each $|p| \in P/\mathcal{V}$*

$$|p| = \bigcup_{\beta \in A_0^-} D_\beta |p| \cap |e|_\beta.$$

Moreover, the mapping $h(p) = |p|$ is a Post homomorphism from P onto P/\mathcal{V} and $\mathcal{V} = \mathcal{V}(A_0)$ is the shell of h , i.e.

$$\mathcal{V} = \{p \in P : |p| = 1\}.$$

(ii) *If \mathcal{V} is a prime D_α -filter, then $P/\mathcal{V}(A_0) = \langle P/\mathcal{V}; |e|_\beta : \beta \in A_0; B2 \rangle$, where $B2$ denotes the two element Boolean algebra, i.e. the underlying Boolean algebra of P/\mathcal{V} is $B2$.*

If, in addition, A_0 is a complete linearly ordered subset of A , then the set P/\mathcal{V} consists exactly of all elements $\{|e|_\beta : \beta \in \bar{A}_0\}$, where \bar{A}_0 is the extension of A_0 .

Proof. To prove the first assertion in (i), we will show that the lattice P/\mathcal{V} satisfies all conditions (i)-(vi) of Theorem (1.6).

In fact, it is obvious that

$$D_\beta(|p| \cup |q|) = D_\beta |p| \cup D_\beta |q|;$$

$$D_\beta(|p| \cap |q|) = D_\beta |p| \cap D_\beta |q| \quad \text{and} \quad D_\beta D_\gamma |p| = D_\gamma |p|.$$

Since $|D_\beta p| \cup |-D_\beta p| = |D_\beta p \cup -D_\beta p| = 1$ and $|D_\beta p| \cap |-D_\beta p| = 0$, it follows at once that the complement of $|D_\beta p| = |D_\beta p|$ exists and $-D_\beta |p| = |-D_\beta p|$ and simultaneously $D_\beta |p| \cup -D_\beta |p| = 1$. Thus conditions (i)-(iii) are fulfilled. Condition (iv) results directly by (3). Consider condition (v). Since

$$D_\beta e_{\beta_0} = \begin{cases} 1 & \text{if } \beta \leq \beta_0, \\ 0 & \text{if } \beta \not\leq \beta_0, \end{cases} \quad D_\beta \left(\bigcup_{\substack{\gamma \in A_0 \\ \gamma < \beta_0}} e_\gamma \right) = \begin{cases} 1 & \text{if } \beta < \beta_0, \\ 0 & \text{if } \beta \not< \beta_0, \end{cases}$$

we have

$$D_\beta |e|_{\beta_0} = \begin{cases} 1 & \text{if } \beta \leq \beta_0, \\ 0 & \text{if } \beta \not\leq \beta_0, \end{cases} \quad D_\beta |e|_{\hat{\beta}_0} = \begin{cases} 1 & \text{if } \beta < \beta_0, \\ 0 & \text{if } \beta \not< \beta_0. \end{cases}$$

Consequently, condition (v) is satisfied.

Finally we verify (vi). Take an arbitrary element $|p|$ of P/\mathcal{V} and let $p = \bigcup_{\alpha \in A^-} D_\alpha p \cap e_\alpha$. Then $p \geq D_\beta p \cap e_\beta$ and so $|p| \geq |D_\beta p| \cap |e|_\beta = D_\beta |p| \cap |e|_\beta$ for every $\beta \in A_0^-$. Now suppose that $|q| \geq D_\beta |p| \cap |e|_\beta = |D_\beta p \cap e_\beta|$ for every $\beta \in A_0^-$. It follows from the definition of order in P/\mathcal{V} that $D_\beta(D_\beta p \cap e_\beta) \Rightarrow D_\beta q = D_\beta p \Rightarrow D_\beta q \in \mathcal{V}$ for every $\beta \in A_0^-$. This implies that $|p| \leq |q|$. Therefore we have proved that $|p| = \bigcup_{\beta \in A_0^-} D_\beta |p| \cap |e|_\beta$, i.e. (vi) is satisfied.

We shall show that h is a Post homomorphism of P onto P/\mathcal{V} . Obviously, h satisfied conditions (h₁) and (h₃) in Definition 2.6 ((h₃) follows directly from (3)).

For any $\alpha \in A$ by virtue of the definition of $s_0 \alpha$ in Definition 3.1 (d₁) the equation $D_\beta |e_\alpha| = D_\beta |e_{s_0 \alpha}|$ holds for all $\beta \in A_0^-$. Hence $|e_\alpha| = |e_{s_0 \alpha}| = |e|_{s_0 \alpha}$, i.e. the image $h(E)$ is contained in $\{|e|_\beta : \beta \in A_0\}$ and so (h₂) is satisfied.

Since for every $\beta \in A_0$, $|e|_\beta$ is the image of a certain element of E , then obviously (h₄) is satisfied and $r(\beta) = \beta$ for every $\beta \in A_0$.

To verify (h₅), we consider for every $\beta \in A_0$ the set $I = \{\alpha \in A : h(e_\alpha) = |e|_\beta\}$. Of course $\beta \in I$. Moreover, $h(e_\alpha) = |e|_\beta$ implies that $D_\beta |e_\alpha| = |D_\beta e_\alpha| = D_\beta |e|_\beta = 1$. So $D_\beta e_\alpha = 1$ and consequently $\beta \leq \alpha$. This shows that the set I has a least element β , and consequently (h₅) holds.

It is easy to show that the last assertion in (i) is also true.

(ii) If \mathcal{V} is a prime D_α -filter, i.e. $B \cap \mathcal{V} = \mathcal{V}_0$ is a prime lattice filter in B , then clearly the set B/\mathcal{V} consists of two elements 0 and 1 only. That is $B/\mathcal{V} = B2$.

From this we note that for each $|p| \in P/\mathcal{V}$ either $D_\beta |p| = 1$ or $D_\beta |p| = 0$. Let A_0 be a complete linearly ordered set. Put $\beta_0 = \bigcap^{A_0} \{\beta \in A_0^- : D_\beta |p| = 0\}$. There are only two cases:

Case I: $D_{\beta_0} |p| = 1$. Since A_0 is a linearly ordered set and $\langle D_\beta |p| : \beta \in A_0^- \rangle$ is a decreasing sequence, we see that $D_\beta |p| = D_\beta |e|_{\beta_0}$ for every $\beta \in A_0^-$. Consequently $|p| = |e|_{\beta_0}$ (because P/\mathcal{V} is a Post algebra of type A_0).

Case II: $D_{\beta_0} |p| = 0$. In the same way we have $D_\beta |p| = D_\beta |e|_{\beta_0}$ for all $\beta \in A_0^-$ and hence $|p| = |e|_{\beta_0}$. From this argument we conclude that the set P/\mathcal{V} consists only of elements $|e|_\beta$, where $\beta \in \bar{A}_0$. The proof of the theorem is thus completed.

COROLLARY 3.8. *If h is a Post homomorphism of P into P' and $\mathcal{V}(A_0)$ (see Theorem 3.3 (i)) is the shell of h , then $h(P)$ is isomorphic with the quotient algebra $P/\mathcal{V}(A_0)$.*

Proof. It is enough to show that $h(p) = h(q)$ iff $p \approx_{\mathcal{V}(A_0)} q$.

Since h/B is a homomorphism of B into B' , $h(b) = h(b')$ for $b, b' \in B$ implies that $(b \Rightarrow b') \cap (b' \Rightarrow b) \in \mathcal{V}(A_0)$. By virtue of this remark we have $h(p) = h(q)$ iff, for every $\alpha'_0 \in A'_0$, $D_{\alpha'_0} h(p) = D_{\alpha'_0} h(q)$ (see Theorem 2.7, formula (7)) the last equation holds iff, for every $\alpha'_0 \in A'_0$, $h(D_{\alpha'_0} p)$

$= h(D_{sa'_0}q)$, which on account of $A_0 = SA'_0$ is equivalent to the condition that, for every $\alpha_0 = sa'_0 \in A_0^-$, $(D_{\alpha_0}p \Rightarrow D_{\alpha_0}q) \cap (D_{\alpha_0}q \Rightarrow D_{\alpha_0}p) \in \mathcal{V}(A_0)$. Consequently, by the definition of the relation $\approx_{\mathcal{V}(A_0)}$ the required conclusion follows.

DEFINITION 3.9. We say that a D -filter $\mathcal{V} = \mathcal{V}(A_0)$ preserves the l.u.b. (g.l.b.)

$$(4) \quad p = \bigcup_{t \in T} p_t \quad (p = \bigcap_{t \in T} p_t),$$

if

$$|p| = \bigcup_{t \in T}^{P/\mathcal{V}} |p_t| \quad (|p| = \bigcap_{t \in T}^{P/\mathcal{V}} |p_t|),$$

and that it preserves the pseudo-complement of p relative to q , $p \Rightarrow q$ (whenever it exists in P) if $|p \Rightarrow q| = |p| \Rightarrow |q|$.

THEOREM 3.10 (i) A prime D -filter $\mathcal{V} = \mathcal{V}(A_0)$ preserves the l.u.b. (g.l.b.) (4) if the following condition is satisfied:

$$(5) \quad \text{for all } \alpha_0 \in A_0^-, D_{\alpha_0}p \in \mathcal{V} \text{ iff there is a } t_0 \in T \text{ such that } D_{\alpha_0}p_{t_0} \in \mathcal{V} \text{ (} D_{\alpha_0}p \in \mathcal{V} \text{ iff for all } t \in T, D_{\alpha_0}p_t \in \mathcal{V}\text{)}.$$

(ii) Let $\mathcal{V}(A_0)$ be a D -filter in P and let A_0 be a subset of A such that if A_0 contains an element α_0 , then A_0 contains all elements of A that precede α_0 . Then the existence of $p \Rightarrow q$ in P implies the existence of $|p| \Rightarrow |q|$ in $P/\mathcal{V}(A_0)$ and the D -filter $\mathcal{V}(A_0)$ preserves $p \Rightarrow q$ iff it preserves g.l.b.'s.

$$(6) \quad D_{\alpha_0}(p \Rightarrow q) = \bigcap_{\beta \leq \alpha_0} D_{\beta}p \Rightarrow D_{\beta}q; \quad \alpha_0 \in A_0^-.$$

Proof. (i) To prove the necessity, we assume that $|p| = \bigcup_{t \in T}^{P/\mathcal{V}} |p_t|$. Applying Theorem 1.8 to this l.u.b. in P/\mathcal{V} , we obtain $|D_{\alpha_0}p| = \bigcup_{t \in T}^{P/\mathcal{V}} |D_{\alpha_0}p_t|$, for all $\alpha_0 \in A_0^-$. Since $\mathcal{V}(A_0)$ is a prime D -filter, the values of the elements $|D_{\alpha_0}p|$ and $|D_{\alpha_0}p_t|$ are either 0 or 1. Accordingly the last equation yields conclusion (5). Conversely, suppose that (5) holds for all $\alpha_0 \in A_0^-$. Consequently, it will easily be observed that, for every $\alpha_0 \in A_0^-$, $|D_{\alpha_0}p| = \bigcup_{t \in T}^{P/\mathcal{V}} |D_{\alpha_0}p_t|$, or equivalently, $D_{\alpha_0}|p| = \bigcup_{t \in T}^{P/\mathcal{V}} D_{\alpha_0}|p_t|$. This shows, by the second assertion of (ii) in Theorem 1.8 applied to $|p|$ and $|p_t|$, $t \in T$ in P/\mathcal{V} , that $|p| = \bigcup_{t \in T}^{P/\mathcal{V}} |p_t|$. Therefore the D -filter $\mathcal{V}(A_0)$ preserves the l.u.b. (4).

Analogously we can prove this assertion for the g.l.b. (4).

Now consider (ii). Let us assume that the D -filter $\mathcal{V}(A_0)$ preserves $p \Rightarrow q$, i.e. $|p \Rightarrow q| = |p| \Rightarrow |q|$. According to (vii) (1.4), for every $\alpha_0 \in A_0^-$ we have

$$|D_{\alpha_0}(p \Rightarrow q)| = D_{\alpha_0}(|p| \Rightarrow |q|) = \bigcap_{\beta \leq \alpha_0}^{P/\mathcal{V}} D_{\beta}|p| \Rightarrow D_{\beta}|q|.$$

These equations lead to

$$(7) \quad D_{\alpha_0} |p \Rightarrow q| = |D_{\alpha_0}(p \Rightarrow q)| = \bigcap_{\beta \leq \alpha_0}^{P/\mathcal{V}} |D_\beta p \Rightarrow D_\beta q|,$$

because

$$\begin{aligned} D_\beta |p| \Rightarrow D_\beta |q| &= -D_\beta |p| \cup D_\beta |q| = |-D_\beta p \cup D_\beta q| \\ &= |D_\beta p \Rightarrow D_\beta q|. \end{aligned}$$

Formula (7) means that the D -filter $\mathcal{V}(A_0)$ preserves the g.l.b.'s (6).

Now assume that the D -filter $\mathcal{V}(A_0)$ preserves all g.l.b.'s (6), i.e. that (7) holds for all $\alpha_0 \in A_0^-$. We need to show that $|p| \Rightarrow |q| = |p \Rightarrow q|$. Since $p \Rightarrow q$ exists in P , we have $(p \Rightarrow q) \cap p \leq q$, which implies that $|p \Rightarrow q| \cap |p| \leq |q|$.

Let $|p'| \in P/\mathcal{V}$ and $|p'| \cap |p| \leq |q|$. Thus $D_\beta |p'| \cap D_\beta |p| \leq D_\beta |q|$, for all $\beta \in A_0^-$. Hence for every $\beta \leq \alpha_0$, $D_{\alpha_0} |p'| \leq D_\beta |p'| \leq D_\beta |p| \Rightarrow D_\beta |q|$, and consequently $D_{\alpha_0} |p'| \leq |D_\beta p \Rightarrow D_\beta q|$, since $D_\beta |p| \Rightarrow D_\beta |q| = |D_\beta p \Rightarrow D_\beta q|$. Therefore from (7) it follows that $D_{\alpha_0} |p'| \leq D_{\alpha_0} |p \Rightarrow q|$ for every $\alpha_0 \in A_0^-$. This implies that $|p'| \leq |p \Rightarrow q|$. Accordingly, $|p| \Rightarrow |q|$ exists in P/\mathcal{V} and $|p| \Rightarrow |q| = |p \Rightarrow q|$.

This completes the proof.

4. Post algebras of type ν

In this section we shall discuss a special case of Post algebras.

Let ν be an arbitrary ordinal number and let $A_\nu = \{0\} \cup \{\alpha \leq \nu : \alpha \notin \text{Lim}\}$, where $\alpha \notin \text{Lim}$ means that α is the successor of a certain ordinal numbers, i.e. $\alpha = \beta + 1$.

Then the Post algebra P of type A_ν is called the *Post algebra of type ν* .

It is easy to see that the extension \bar{A}_ν (see Section 2) coincides with the set of all ordinal numbers smaller than $\nu + 1$, i.e. $\bar{A}_\nu = \{\alpha \leq \nu\}$. Consequently, we will denote any Post algebra P of type ν by $\langle P; \{e_\alpha : \alpha \leq \nu\}; B \rangle$ instead of $\langle P; \{e_\alpha : \alpha \in A_\nu\}, B \rangle$.

Notice that the ordinal number $\alpha \leq \nu$ has the corresponding operation D_α iff $\alpha \notin \text{Lim}$.

1. Uniqueness of type and of elements e_α of the Post algebras of type ν .

This problem was discussed by Dwinger [4] in the case of the Post algebras of order n .

THEOREM 4.1. *Let ν and μ be arbitrary ordinal numbers. If a Post algebra $P = \langle P; \{e_\alpha : \alpha \leq \nu\}; B \rangle = \langle P; \{e'_\alpha : \alpha \leq \mu\}; B \rangle$, then $\nu = \mu$ and $e_\alpha = e'_\alpha$ for all $\alpha \leq \nu = \mu$.*

Proof. Without loss of generality we can assume that $\nu \leq \mu$. We shall prove by transfinite induction that $e_\alpha \leq e'_\alpha$ for all $\alpha \leq \nu$. Of course,

$0 = e_0 \leq e'_0$, i.e. for $\alpha = 0$, the assertion is true. Suppose that $e_\alpha \leq e'_\alpha$ is true for all $\alpha < \alpha_0 \leq \nu$. If $\alpha_0 \in \text{Lim.}$, i.e. $\alpha_0 \in \bar{A}_\nu - A_\nu$, then we have $e_{\alpha_0} = \bigcup_{\alpha < \alpha_0} e_\alpha \leq \bigcup_{\alpha < \alpha_0} e'_\alpha = e'_{\alpha_0}$.

Now consider the case of $\alpha_0 \notin \text{Lim.}$ Since $e'_{\alpha_0} \in P$, we have the monotone representation of e'_{α_0} :

$$e'_{\alpha_0} = \bigcup_{\nu \geq \alpha \notin \text{Lim.}} D_\alpha e'_{\alpha_0} \cap e_\alpha.$$

For $\alpha \geq \alpha_0$, $\alpha \notin \text{Lim.}$, $D_\alpha e'_{\alpha_0} \leq D_{\alpha_0} e'_{\alpha_0}$, or equivalently, $D_\alpha e'_{\alpha_0} \cap -D_{\alpha_0} e'_{\alpha_0} = 0$. Hence

$$-D_{\alpha_0} e'_{\alpha_0} \cap e'_{\alpha_0} = \bigcup_{\alpha_0 > \alpha \notin \text{Lim.}} (-D_{\alpha_0} e'_{\alpha_0} \cap D_\alpha e'_{\alpha_0}) \cap e_\alpha,$$

which implies that

$$D_\alpha (-D_{\alpha_0} e'_{\alpha_0} \cap e'_{\alpha_0}) = 0 \quad \text{for } \alpha \geq \alpha_0, \alpha \notin \text{Lim.}$$

In view of the induction hypotheses, we have $e_{\alpha_0-1} \leq e'_{\alpha_0-1}$, which yields $D_\alpha e'_{\alpha_0-1} = 1$, because $D_\alpha e_{\alpha_0-1} = 1$ for all $\alpha < \alpha_0$, $\alpha \notin \text{Lim.}$ Consequently, it is easily seen that $D_\alpha (-D_{\alpha_0} e'_{\alpha_0} \cap e'_{\alpha_0}) = 0 \leq D_\alpha e'_{\alpha_0-1}$ for $\alpha \geq \alpha_0$, and $D_\alpha (-D_{\alpha_0} e'_{\alpha_0} \cap e'_{\alpha_0}) \leq D_\alpha e'_{\alpha_0-1} = 1$ for $\alpha < \alpha_0$. From this it follows that $-D_{\alpha_0} e'_{\alpha_0} \cap e'_{\alpha_0} \leq e'_{\alpha_0-1}$. Thus by (P5) of the definition of a Post algebra, we get $-D_{\alpha_0} e'_{\alpha_0} = 0$ or $D_{\alpha_0} e'_{\alpha_0} = 1$. Accordingly, $e'_{\alpha_0} \geq D_{\alpha_0} e'_{\alpha_0} \cap e_{\alpha_0} = e_{\alpha_0}$. Thus we have proved that $e_\alpha \leq e'_\alpha$ for every $\alpha \leq \nu \leq \mu$. In particular, $1 = e_\nu \leq e'_\nu$. This implies that $e'_\nu = e'_\mu = 1$ and consequently $\nu = \mu$.

We can similarly prove that $e'_\alpha \leq e_\alpha$ for every $\alpha \leq \nu = \mu$. Thus $e'_\alpha = e_\alpha$, which completes the proof.

2. Measure in a Post algebra of type ν . Let a Post algebra $P = \langle P; \{e_\alpha: \alpha \leq \nu\}; B \rangle$ of type ν be given, and let us consider an arbitrary element p of P such that the g.l.b.

$$(1) \quad \bigcap_{\beta \leq \alpha; \beta \notin \text{Lim.}} D_\beta p$$

exists in P for every $\alpha \leq \nu$.

For such an element p we introduce the following notation:

$$(2) \quad \begin{aligned} C_\alpha p &= \left(\bigcap_{\alpha \geq \beta \notin \text{Lim.}} D_\beta p \right) \cap -D_{\alpha+1} p, \quad \text{if } \alpha < \nu, \\ C_\nu p &= \bigcap_{\nu \geq \beta \notin \text{Lim.}} D_\beta p. \end{aligned}$$

Now we shall show that

$$(3) \quad D_\alpha p = \bigcup_{\beta \geq \alpha} C_\beta p \quad \text{for each } \alpha \leq \nu; \alpha \notin \text{Lim.}$$

In fact, it is obvious that $D_\alpha p \geq C_\beta p$ for all $\beta \geq \alpha$. Let b be an arbitrary

element of B satisfying condition: $b \geq C_\beta p$, for all $\beta \geq a$. We put $a = D_a p \cap \cap -b$. Thus

$$(4) \quad a \cap C_\beta p = 0 \quad \text{for all } \beta \geq a.$$

We wish to prove that $D_\gamma p \geq a$ for every $\gamma \leq \nu$ and $\gamma \notin \text{Lim}$.

Assume that $D_\gamma p \geq a$ for every $\gamma \leq \gamma_0$, $\gamma \notin \text{Lim}$., where $a \leq \gamma_0 < \nu$. Hence $\bigcap_{\gamma_0 > \gamma \notin \text{Lim.}} D_\gamma p \geq a$ and thus, by (2), $C_{\gamma_0} p \geq a \cap -D_{\gamma_0+1} p$. Moreover, (4) leads to $-C_{\gamma_0} p \geq a \geq a \cap -D_{\gamma_0+1} p$, and consequently, $a \cap -D_{\gamma_0+1} p \leq C_{\gamma_0} p \cap -C_{\gamma_0} p = 0$, which implies that $a \leq D_{\gamma_0+1} p$.

By transfinite induction we obtain $D_a p \geq a$ for all $a \leq \nu$, $a \notin \text{Lim}$. and so, by (2), $a \leq C_a p$. But formula (4) gives $a \leq -C_a p$. It follows from the last two inequalities that $a = D_a p \cap -b = 0$ and hence $b \geq D_a p$. This shows the validity of formula (3).

It can easily be observed in view of (3) that $p = \bigcup_{\nu \geq \alpha \notin \text{Lim.}} D_\alpha p \cap e_\alpha = \bigcup_{\alpha \leq \nu} C_\alpha p \cap e_\alpha$ for each element p satisfying (1). Indeed, evidently $p \geq C_\alpha p \cap e_\alpha$. On the other hand, if $q \geq C_\alpha p \cap e_\alpha$ for all $\alpha \leq \nu$, then $q \geq D_\alpha p \cap e_\alpha$ for all $\alpha \leq \nu$ and $\alpha \notin \text{Lim}$., because $D_\alpha p \cap e_\alpha = \left(\bigcup_{\beta \geq \alpha} C_\beta p \right) \cap e_\alpha = \bigcup_{\beta \geq \alpha} C_\beta p \cap e_\alpha$. Hence $q \geq p$.

Let m_0 be an m -measure defined on the underlying Boolean algebra B , see [27], and for every $b \in B$: $0 \leq m_0(b) \leq +\infty$. Let $\{a_\alpha: \alpha \leq \nu\}$ be a fixed increasing sequence of real numbers

$$0 = a_0 \leq a_1 \leq \dots \leq a_\alpha \leq \dots \leq a_\nu \leq +\infty.$$

We extend this measure m_0 to a function m defined as follows:

$$(5) \quad m(p) = \begin{cases} \sum_{\alpha \leq \nu} m_0(C_\alpha p) a_\alpha \quad (1) & \text{if } p \text{ satisfies (1),} \\ +\infty & \text{in the opposite case.} \end{cases}$$

We wish to show that the non-negative function m is an m -measure on P . For this purpose we shall first prove the next two lemmas.

LEMMA 4.2. *If $\{a_t: t \in T\}$ and $\{b_t: t \in T\}$ are any indexed sets of elements in B such that $a_t \cap a_{t'} = 0$ for $t \neq t'$, $a_t \geq b_t$ for all $t \in T$, and the joins in question exist, then $\bigcup_{t \in T} a_t = \bigcup_{t \in T} b_t$ iff $a_t = b_t$ for all $t \in T$.*

The easy proof of this lemma is omitted.

LEMMA 4.3. *Let $\{a_{t\alpha}: t \in T, \alpha \in L\}$ be an indexed subset of B satisfying the following conditions:*

(1) Where we mean that if the cardinal of the set of the non-zero terms of this sum is greater than \aleph_0 , then the sum is equal to $+\infty$.

- 1) L is a linearly ordered set and if $a \geq a'$, then $a_{t\alpha} \leq a_{t\alpha'}$ for all $t \in T$.
- 2) $t \neq t'$ implies $a_{t\alpha} \cap a_{t'\alpha} = 0$ for all $\alpha, \alpha' \in L$,
- 3) $\bigcup_{t \in T} a_{t\alpha}$ exists in B for every $\alpha \in L$.

Then $\bigcap_{\alpha \in L} \bigcup_{t \in T} a_{t\alpha}$ exists in B iff $\bigcap_{\alpha \in L} a_{t\alpha}$, $t \in T$ and $\bigcup_{t \in T} \bigcap_{\alpha \in L} a_{t\alpha}$ exist in B .

Furthermore, $\bigcap_{\alpha \in L} \bigcup_{t \in T} a_{t\alpha} = \bigcup_{t \in T} \bigcap_{\alpha \in L} a_{t\alpha}$.

Proof. First, it is easy to see from Lemma 4.2 that:

- (6) If $\bigcup_{t \in T} a_{t\alpha} \geq b$ for all $\alpha \in L$, then there exists a disjoint decomposition of the element b : $b = \bigcup_{t \in T} b_t$ (i.e. $b_t \cap b_{t'} = 0$, for $t \neq t'$) such that $b_t \leq a_{t\alpha}$ for every $t \in T$ and $\alpha \in L$.

Indeed, in view of the hypothesis of (6), we see that $b \cap \bigcup_{t \in T} a_{t\alpha} = b \cap \bigcup_{t \in T} a_{t\alpha'}$, and equivalently $\bigcup_{t \in T} b \cap a_{t\alpha} = \bigcup_{t \in T} b \cap a_{t\alpha'}$. Since from condition 1) $b \cap a_{t\alpha} \leq b \cap a_{t\alpha'}$ for $\alpha > \alpha'$, it follows immediately from Lemma 4.2 that $b \cap a_{t\alpha} = b \cap a_{t\alpha'}$ for all $t \in T$ and $\alpha, \alpha' \in L$. Consequently, we can put $b_t = b \cap a_{t\alpha}$ for all $\alpha \in L$. Obviously $b = \bigcup_{t \in T} b_t$ and this is a disjoint decomposition of b satisfying conclusion (6).

Now assume that

$$(7) \quad a = \bigcap_{\alpha \in L} \bigcup_{t \in T} a_{t\alpha}$$

exists in B . Thus $\bigcup_{t \in T} a_{t\alpha} \geq a$ for all $\alpha \in L$. Applying (6) to the element a , we infer that there is a disjoint decomposition of a fulfilling (6), i.e. $a = \bigcup_{t \in T} a_t$, $a_t \cap a_{t'} = 0$ and $a_t \leq a_{t\alpha}$ for all $\alpha \in L$.

We will show that $a_{t_0} = \bigcap_{\alpha \in L} a_{t_0\alpha}$, $t_0 \in T$. In fact, suppose that $c \in B$ and $a_{t_0} \leq c \leq a_{t_0\alpha}$ for all $\alpha \in L$. Then $c \cup \bigcup_{t \in T} a_t \leq a_{t_0\alpha} \cup \bigcup_{t \in T} a_{t\alpha} = \bigcup_{t \in T} a_{t\alpha}$ for all $\alpha \in L$. Hence by (7)

$$c \cup a = c \cup \bigcup_{t \in T} a_t \leq a = \bigcup_{t \in T} a_t.$$

Since for $t \neq t_0$, $c \cap a_t \leq c \cap a_{t\alpha} \leq a_{t_0\alpha} \cap a_{t\alpha} = 0$, we have $c = c \cap a = \bigcup_{t \in T} (c \cap a_t) = c \cap a_{t_0}$, which implies $c \leq a_{t_0}$. Accordingly, we have proved that $\bigcap_{\alpha \in L} a_{t\alpha}$ exists and equals a_t . Further, $a = \bigcap_{\alpha \in L} \bigcup_{t \in T} a_{t\alpha} = \bigcup_{t \in T} a_t = \bigcup_{t \in T} \bigcap_{\alpha \in L} a_{t\alpha}$.

Conversely, suppose that for every $t \in T$, $\bigcap_{\alpha \in L} a_{t\alpha} = a_t$ exists in B and let us set $a = \bigcup_{t \in T} a_t = \bigcup_{t \in T} \bigcap_{\alpha \in L} a_{t\alpha}$.

Obviously, condition 2) gives $a_t \cap a_{t'} = 0$ for $t \neq t'$. It is well known

that $\bigcup_{t \in T} a_{t\alpha} \geq \bigcup_{t \in T} \bigcap_{\alpha \in L} a_{t\alpha} = a$ for all $\alpha \in L$. Now consider an arbitrary element b of B such that $\bigcup_{t \in T} a_{t\alpha} \geq b$ for all $\alpha \in L$. By virtue of (6) there exists a disjoint decomposition $b = \bigcup_T b_t$ of b such that $b_t \leq a_{t\alpha}$ for all $\alpha \in L$. So $b_t \leq \bigcap_{\alpha \in L} a_{t\alpha} = a_t$ and consequently $b = \bigcup_T b_t \leq \bigcup_T a_t = a$.

In this way we have shown that $\bigcap_L \bigcup_T a_{t\alpha}$ exists in B and the equation $\bigcap_L \bigcup_T a_{t\alpha} = \bigcup_T \bigcap_L a_{t\alpha}$ holds again, which completes the proof of the lemma.

THEOREM 4.4. *Let m_0 be an m -measure on a Boolean m -algebra B . The non-negative function m defined by (5) is an m -measure on P , i.e. if $p = \bigcup_{t \in T} p_t$, $p_t \cap p_{t'} = 0$ for $t \neq t'$ and the cardinal of T is less than m , then*

$$m(p) = \sum_T m(p_t).$$

Proof. Since $D_1 p_t \cap D_1 p_{t'} = 0$ for $t \neq t'$ and $D_1 p_t \geq D_\beta p_t$ for all $\beta \leq \nu$, $\beta \notin \text{Lim.}$, the set $\{D_\beta p_t: \beta \leq \nu, \beta \notin \text{Lim.}, t \in T\}$ satisfies condition (1)-(3) of Lemma 4.3. Hence, in view of Lemma 4.3, we observe that if p does not satisfy condition (1), i.e. of for a certain $\alpha \leq \nu$ the g.l.b.

$$(8) \quad \bigcap_{\alpha \geq \beta \notin \text{Lim.}} D_\beta p = \bigcap_{\substack{\beta \notin \text{Lim.} \\ \beta \leq \alpha}} \bigcup_{t \in T} D_\beta p_t$$

does not exist in B , then there is a $t_0 \in T$ such that $\bigcap_{\alpha \geq \beta \notin \text{Lim.}} D_\beta p_{t_0}$ does not exist either. So condition (1) is not satisfied for the element p_{t_0} . Consequently, by virtue of (5), $n(p_{t_0}) = +\infty = m(p) = \sum_T m(p_t)$. Now

let us assume that p satisfies condition (1), i.e. that the g.l.b. (8) exists in B for all $\alpha \leq \nu$. According to Lemma 4.3 applied to the set $\{D_\beta p_t: t \in T; \beta \leq \alpha, \beta \notin \text{Lim.}\}$ for $\alpha \leq \nu$, it follows that p_t satisfies condition (1). Hence by (5)

$$(9) \quad \begin{aligned} m(p) &= \sum_{\alpha \leq \nu} m_0(c_\alpha p) a_\alpha, \\ m(p_t) &= \sum_{\alpha \leq \nu} m_0(c_\alpha p_t) a_\alpha. \end{aligned}$$

We shall show that $C_\alpha p = \bigcup_{t \in T} C_\alpha p_t$, $\alpha \leq \nu$. Indeed, by (2) and the generalization of Epstein's Lemma, it follows that

$$\begin{aligned} C_\alpha p &= \left(\bigcap_{\alpha \geq \beta \notin \text{Lim.}} D_\beta p \right) \cap -D_{\alpha+1} p \\ &= \bigcap_{\alpha \geq \beta \notin \text{Lim.}} \bigcup_T (D_\beta p_t \cap -D_{\alpha+1} p) \\ &= \bigcap_{\alpha \geq \beta \notin \text{Lim.}} \bigcup_{t \in T} \bigcap_{t' \in T} (D_\beta p_t \cap -D_{\alpha+1} p_{t'}). \end{aligned}$$

Since, for $t \neq t'$, $D_\beta p_t \cap D_{\alpha+1} p_{t'} = 0$ and so $D_\beta p_t \cap -D_{\alpha+1} p_{t'} = D_\beta p_t$, we have $C_\alpha p = \bigcap_{\alpha \geq \beta \notin \text{Lim.}} \bigcup_{t \in T} (D_\beta p_t \cap -D_{\alpha+1} p_t)$. But it can easily be seen that the set $\{a_{t\beta} = D_\beta p_t \cap -D_{\alpha+1} p_t: \beta \leq \alpha, \beta \notin \text{Lim.}, t \in T\}$ satisfies conditions (1)-(3) of Lemma 4.2; so by this lemma we obtain

$$\begin{aligned} C_\alpha p &= \bigcup_T \bigcap_{\alpha \geq \beta \notin \text{Lim.}} (D_\beta p_t \cap -D_{\alpha+1} p_t) \\ &= \bigcup_T \left(\bigcap_{\alpha \geq \beta \notin \text{Lim.}} D_\beta p_t \right) \cap -D_{\alpha+1} p_t = \bigcup_T C_\alpha p_t. \end{aligned}$$

Thus we have reached the equality required. Since if $t \neq t'$, then $C_\alpha p_t \cap C_\alpha p_{t'} \leq D_1 p_t \cap D_1 p_{t'} = 0$, it follows from the assumption concerning m_0 that

$$m_0(C_\alpha p) = \sum_T m_0(C_\alpha p_t).$$

Hence formula (9) leads to

$$(10) \quad m(p) = \sum_{\alpha \leq \nu} \sum_T m_0(C_\alpha p_t) a_\alpha.$$

If $m(p) < +\infty$, i.e. if series (10) is convergent, then we are able to interchange the order of the sums in (10) and obtain from (9)

$$m(p) = \sum_T \sum_{\alpha \leq \nu} m_0(C_\alpha p_t) a_\alpha = \sum_T m(p_t).$$

If $m(p) = +\infty$, then also $\sum_T m(p_t) = +\infty$.

For, if $+\infty > \sum_T m(p_t) = \sum_T \sum_{\alpha \leq \nu} m_0(C_\alpha p_t) a_\alpha$, then, by interchanging the order in the last sum, we obtain

$$+\infty > \sum_{\alpha \leq \nu} \sum_T m_0(C_\alpha p_t) a_\alpha = \sum_{\alpha \leq \nu} m_0(C_\alpha p) a_\alpha = m(p),$$

contrary to $m(p) = +\infty$.

Thus the equation $m(p) = \sum_T m(p_t)$ holds again. Consequently, we always have

$$m(p) = \sum_T m(p_t).$$

Therefore m is an m -measure on P , which completes the proof.

5. m -Representability of generalized Post algebras

The main underlying idea of this section is the same as in the case of Boolean algebras, but here we will use filters instead of ideals.

1. Representation and the Stone space of a Post algebra. Consider a Post algebra $P = \langle P; \{e_\alpha: \alpha \in A\}; B \rangle$ of type A . For every $\alpha \in A^-$ we

denote the set of all prime D_α -filters in P by \mathcal{X}_α ; and the set of all prime filters in B by \mathcal{X}_0 .

Put

$$\mathcal{X} = \bigcup_{\alpha \in A^-}^0 \mathcal{X}_\alpha,$$

where \bigcup^0 denotes the set-theoretical operation of union. In general, in order to emphasize in this section that a Boolean operation 0 is a set-theoretical operation, we shall denote it, if necessary, by 0^0 . However, the set-theoretical complementation will always be denoted by \sim .

Since, in a Post algebra, the notion of a D_α -filter ($\alpha \in A^-$) coincides with that of an α -filter, every D_α -filter \mathcal{V} is, by virtue of Lemma 3.5, a lattice filter generated by $\mathcal{V}_0 = \mathcal{V} \cap B$ and by the element e_α . In symbols we shall write $\mathcal{V} = [\mathcal{V}_0, e_\alpha]$. On account of Theorem 3.6, (S5) we see that if $\alpha \neq \alpha'$; $\alpha, \alpha' \in A^-$, then $\mathcal{V} = [\mathcal{V}_0, e_\alpha] \neq \mathcal{V}' = [\mathcal{V}_0, e_{\alpha'}]$. Evidently, for every $\alpha \in A^-$, each D_α -filter \mathcal{V} is uniquely determined by the filter \mathcal{V}_0 . This enables us to introduce the following definition.

DEFINITION 5.1 (i) For every pair of elements α, β of A , we denote by $g_{\alpha\beta}$ the one-to-one mapping from \mathcal{X}_α onto \mathcal{X}_β , which associates with each D_α -filter $\mathcal{V}^\alpha \in \mathcal{X}_\alpha$ a D_β -filter $\mathcal{V}^\beta \in \mathcal{X}_\beta$ for which $\mathcal{V}^\beta \cap B = \mathcal{V}^\alpha \cap B$, i.e. $\mathcal{V}^\alpha, \mathcal{V}^\beta$ are defined by the same prime filter in B .

(ii) Let X be a subset of \mathcal{X} and let $X_\alpha = X \cap \mathcal{X}_\alpha$ for every $\alpha \in A^-$. The set X is said to be *cylindrical* if $g_{\alpha\beta}(X) = X_\beta$ for every pair $\alpha, \beta \in A^-$, or equivalently if there is a subset X_0 of \mathcal{X}_0 such that $X_\alpha = g_{0\alpha}(X_0)$ for all $\alpha \in A^-$.

It is easy to see that the class of all cylindrical subsets of \mathcal{X} constitutes a complete (Boolean) field, and that a cylindrical set X is empty iff there is at least one α in A^- such that $X_\alpha = X \cap \mathcal{X}_\alpha$ is empty.

Note that $\mathcal{X}_\alpha \cap \mathcal{X}_\beta = \emptyset$ for $\alpha \neq \beta$.

DEFINITION 5.2. A set lattice $P\mathfrak{F} = \langle P\mathfrak{F}; \{\mathfrak{E}_\alpha: \alpha \in A\}; B\mathfrak{F} \rangle$ of subsets of a set 1 is said to be a *Post field of type A* if it is a Post algebra of type A and the l.u.b. in the monotonic representation of each element X of $P\mathfrak{F}$ is a set-theoretical union i.e. $X = \bigcup_{\alpha \in A^-}^0 (D_\alpha x \cap \mathfrak{E}_\alpha)$.

We now define a mapping h from a given Post algebra P into the (Boolean) field of all subsets of the set \mathcal{X} as follows:

$$h(p) = \{\mathcal{V} \in \mathcal{X}: p \in \mathcal{V}\} \quad \text{for every } p \in P.$$

Assume the following notation:

$P\mathfrak{F}_0 = \{h(p) \subset \mathcal{X}: p \in P\}$; $\mathfrak{E}_\alpha = h(e_\alpha)$ for every $\alpha \in A^+$; $h_0(b) = \{\mathcal{V}_0 \in \mathcal{X}_0: b \in \mathcal{V}_0\}$ for every $b \in B$; and finally $h_\alpha(p) = h(p) \cap \mathcal{X}_\alpha$.

Of course

$$(1) \quad h(p) = \bigcup_{\alpha \in A^-}^0 h_\alpha(p).$$

Observe that if $p \neq q$, then there exists an $\alpha \in A^-$ such that $D_\alpha p \neq D_\alpha q$, and hence there is a D_α -filter containing exactly one of the elements $D_\alpha p$ and $D_\alpha q$, and consequently containing exactly one of the elements p and q . From this it easily follows that h is a lattice isomorphism of P into $P\mathfrak{F}_0$.

Define in $P\mathfrak{F}_0$ the operations D_α , $\alpha \in A^-$, by the following equations:

$$(2) \quad D_\alpha h(p) = h(D_\alpha p) \quad \text{for every } p \in P.$$

Under the notion and hypotheses adopted above the following representation theorem holds:

THEOREM 5.3. *The set lattice $P\mathfrak{F}_0 = \langle P\mathfrak{F}_0; \{\mathfrak{E}_\alpha: \alpha \in A\}; B\mathfrak{F}_0 \rangle$ is a Post field of type A and the mapping h is a Post isomorphism of P onto $P\mathfrak{F}_0$.*

Moreover, $B\mathfrak{F}_0 = \{h(b): b \in B\}$; $\mathfrak{E}_\alpha = \bigcup_{\beta < \alpha} \mathfrak{E}_\beta$ and $\mathfrak{E}_\alpha \cap \mathfrak{E}_\beta = \mathfrak{E}_{\alpha \cap \beta}$.

Proof. We shall show that the lattice $P\mathfrak{F}_0$ satisfies all conditions (i)-(vi) of Theorem 1.6. Since conditions (i)-(v) result directly from (vi) and formula (2), it remains to prove (vi), i.e. that the following equation holds:

$$(3) \quad h(p) = \bigcup_{\alpha \in A^-} D_\alpha h(p) \cap \mathfrak{E}_\alpha.$$

It easily follows from Corollary 3.2 that

$$(4) \quad \mathfrak{E}_\alpha = h(e_\alpha) = \bigcup_{\beta \leq \alpha} \mathfrak{X}_\beta; \quad \mathfrak{E}_0 = \emptyset \text{ and } \mathfrak{E}_1 = \mathfrak{X}.$$

Let $\mathcal{V} \in h(p)$, i.e. $p \in \mathcal{V}$. Of course, \mathcal{V} is a D_α -filter for certain $\alpha \in A^-$. Hence on account of the definition of a D_α -filter, we have $D_\alpha p \in \mathcal{V}$, and consequently $\mathcal{V} \in h(D_\alpha p) = D_\alpha h(p)$. On the other hand, by (4), $\mathcal{V} \in \mathfrak{E}_\alpha$ and hence $\mathcal{V} \in D_\alpha h(p) \cap \mathfrak{E}_\alpha$. Thus \mathcal{V} belongs to the set on the right-hand side of equality (3). Conversely, let \mathcal{V} belong to the set on the right-hand side of equality (3). Then there exists an $\alpha \in A^-$ such that $\mathcal{V} \in D_\alpha h(p) \cap \mathfrak{E}_\alpha = h(D_\alpha p) \cap \mathfrak{E}_\alpha$. It follows from this fact that $D_\alpha p \in \mathcal{V}$ and, by virtue of (4), for certain $\beta \leq \alpha$, $\mathcal{V} \in \mathfrak{X}_\beta$. From $D_\beta p \geq D_\alpha p$ we conclude that $D_\beta p \in \mathcal{V}$, which implies that $p \in \mathcal{V}$, because \mathcal{V} is D_β -filter. Accordingly, $\mathcal{V} \in h(p)$ and equation (3) holds.

It is clear that $B\mathfrak{F}_0 = \{D_\alpha h(p): p \in P, \alpha \in A^-\} = \{h(b): b \in B\}$ and that h is a Post homomorphism of P onto $P\mathfrak{F}_0$. Further, $h|_B$ is obviously a one-to-one mapping from B onto $B\mathfrak{F}_0$. Since, in virtue of (3) and (4), $\mathfrak{E}_\alpha = \bigcup_{\beta < \alpha} \mathfrak{E}_\beta = \bigcup_{\beta \in A^-} D_\beta h(e_\alpha) = h(e_\alpha)$, we see that $h|_{\bar{B}}$ is also a one-to-one mapping. Hence, according to Theorem 2.8 (ii) h is a Post isomorphism of P onto $P\mathfrak{F}_0$. This completes the proof.

Consider \mathfrak{X} as a topological space, the set $B\mathfrak{F}_0 = \{h(b); b \in B\}$ being adopted as an open basis. This topological space will be called the *Stone*

space of the Post algebra P . Every subset \mathcal{X}_α ($\alpha \in A^-$) of \mathcal{X} will be considered as a topological subspace of \mathcal{X} for which the set $\{h_\alpha(b) : b \in B\}$ is a basis. It is easily seen that every set $h(b)$ ($b \in B$) is cylindrical since $h_\alpha(b) = g_{0\alpha}(h_0(b)) = g_{\beta\alpha}(h_\beta(b))$. Hence the mapping $g_{\alpha\beta}$ is a homeomorphism of \mathcal{X}_α onto \mathcal{X}_β .

Thus each set in $B\mathfrak{F}_0$ is cylindrical and consequently, on account of the notice after Definition 5.1, every open or closed set is also cylindrical. The easy proof of the next theorem is left to the reader.

THEOREM 5.4 (i) *For every $\alpha \in A^-$, \mathcal{X}_α is the Stone space of the underlying Boolean algebra B .*

(ii) *$B\mathfrak{F}_0$ is the class of all simultaneously closed and open sets (clopen sets) in \mathcal{X} .*

(iii) *\mathcal{X} is a compact space.*

Observe that in the proofs of some theorems on homomorphism the notion of a D -filter has been used instead of the usually applied notion of a Post ideal. In connection with this remark we introduce in \mathcal{X} the dual notion of m -category.

Let m be an infinite cardinal, and let X be a subset of \mathcal{X} .

DEFINITION 5.5 (i) The set X is said to be m -open if X is a union $X = \bigcup_{t \in T} X_t$, where X_t , $t \in T$, is a clopen set in \mathcal{X} and the power of T is less than m .

(ii) The set X is said to be m -dense if X contains a certain dense m -open set.

(iii) The set X is said to be of m -cocategory if X is an intersection of n m -dense sets, where n is a cardinal number smaller than m .

By this definition, it follows that the intersection of any family \mathfrak{F} of sets of m -cocategory such that the power of \mathfrak{F} is less than m is also a set of m -cocategory.

THEOREM 5.6 (i) *If X is an m -dense set (or is a set of m -cocategory) in \mathcal{X} , then $X_\alpha = X \cap \mathcal{X}_\alpha$ is an m -dense set (is of m -cocategory) in \mathcal{X}_α for every $\alpha \in A^-$.*

(ii) *If X is a cylindrical set and, for a certain $\alpha \in A^-$, $X_\alpha = X \cap \mathcal{X}_\alpha$ is an m -dense set (or is of m -cocategory) in \mathcal{X}_α , then X is an m -dense set (or is of m -cocategory) in \mathcal{X} .*

(iii) *If the power of A is less than m , then X is a set of m -cocategory in \mathcal{X} iff for every $\alpha \in A^-$ the set $X_\alpha = X \cap \mathcal{X}_\alpha$ is of m -cocategory in \mathcal{X}_α .*

Proof. The proof of this theorem is not difficult. Therefore we shall restrict ourselves to the proof of assertion (iii) only. The necessity is obvious. Now, for every $\alpha \in A^-$, let X_α be a set of m -cocategory. By \tilde{X}_α we denote the cylindrical set generated by the set $X_\alpha \subset \mathcal{X}_\alpha$. Of course

$\tilde{X}_\alpha \cap \mathcal{X}_\alpha = X_\alpha$. According to (ii), \tilde{X}_α is a set of m -cocategory in \mathcal{X} . Since $\bar{A} < m$, the intersection $\bigcap_{\alpha \in A^-} \tilde{X}_\alpha$ is a set of m -cocategory. We will show that $X \supset \bigcup_{\alpha \in A^-} \tilde{X}_\alpha$. Indeed, let $\mathcal{V} \in \bigcap_{\alpha \in A^-} \tilde{X}_\alpha$ and let \mathcal{V} be a D_{α_0} -filter, i.e. let $\mathcal{V} \in \mathcal{X}_{\alpha_0}$, for a certain $\alpha_0 \in A^-$. Evidently \mathcal{V} belongs to \tilde{X}_{α_0} and hence $\mathcal{V} \in \mathcal{X}_{\alpha_0} \cap \mathcal{X}_{\alpha_0} = \tilde{X}_{\alpha_0} \subset X$.

2. m -representability — Canonical m -homomorphism. The notions of a Post m -algebra, of a Post m -field and of an m - D -filter are defined, in a natural way, to the case of Boolean algebras. The exact formulation of these definitions is omitted.

DEFINITION 5.7. Let m be an arbitrary infinite cardinal. A Post algebra P is said to be m -representable if there is a Post m -field \mathfrak{F} , an m - D -filter $\mathcal{V}(A_0)$ of type A_0 in \mathfrak{F} and a Post m -isomorphism of P into $\mathfrak{F}/\mathcal{V}(A_0)$.

Note that if P is a Post m -algebra, then m -representability of P is usually defined as follows:

P is m -representable if P is an image of a Post m -field \mathfrak{F} under an m -homomorphism. But this definition, by virtue of Corollary 3.8, is a particular case of Definition 5.7.

Let $B\mathfrak{F}_m$ be the least Boolean m -field of subsets of containing the field $B\mathfrak{F}_0$. Obviously every set in $B\mathfrak{F}_m$ is cylindrical.

By $P\mathfrak{F}_m$ we denote the following set

$$(5) \quad P\mathfrak{F}_m = \{X \subset \mathcal{X} : X = \bigcup_{\alpha \in A^-}^0 X_\alpha \cap \mathcal{C}_\alpha\}, \quad \text{where } \langle X_\alpha : \alpha \in A^- \rangle$$

is a decreasing sequence (with respect to the inclusion) of type A^- . It is clear that $P\mathfrak{F}_m \supset P\mathfrak{F}_0$.

THEOREM 5.8. $P\mathfrak{F}_m = \langle P\mathfrak{F}_m, \{\mathcal{C}_\alpha : \alpha \in A\}; B\mathfrak{F}_m \rangle$ is the least full Post m -field of type A of subsets of \mathcal{X} containing the Post field $P\mathfrak{F}_0$.

Proof. First we shall show that $P\mathfrak{F}_m$ is a set lattice. In fact, since

$$\begin{aligned} X \bigcup^0 Y &= \left(\bigcup_{\alpha \in A^-}^0 X_\alpha \cap \mathcal{C}_\alpha \right) \bigcup^0 \left(\bigcup_{\alpha \in A^-}^0 Y_\alpha \cap \mathcal{C}_\alpha \right) \\ &= \bigcup_{\alpha \in A^-}^0 (X_\alpha \bigcup^0 Y_\alpha) \cap \mathcal{C}_\alpha, \end{aligned}$$

it follows that if X and $Y \in P\mathfrak{F}_m$, then $X \bigcup^0 Y \in P\mathfrak{F}_m$. Now we shall show that

$$(6) \quad X \bigcap^0 Y = \bigcup_{\alpha \in A^-}^0 (X_\alpha \bigcap^0 Y_\alpha) \cap \mathcal{C}_\alpha.$$

Clearly, $X \bigcap^0 Y$ contains the set constituting the right-hand side of equation (6). For the remaining inclusion let $x \in X \bigcap^0 Y$. Then, for certain α and β in A^- , we have $x \in X_\alpha \cap \mathfrak{E}_\alpha$ and $x \in X_\beta \cap \mathfrak{E}_\beta$. Since, by Theorem 5.3, $\mathfrak{E}_\alpha \cap \mathfrak{E}_\beta = \mathfrak{E}_{\alpha \cap \beta}$ and $X_{\alpha \cap \beta} \supset X_\alpha$, $Y_{\alpha \cap \beta} \supset Y_\beta$, we obtain $x \in (X_{\alpha \cap \beta} \cap Y_{\alpha \cap \beta}) \cap \mathfrak{E}_{\alpha \cap \beta}$. Of course $\alpha \cap \beta \neq 0$, since $x \in \mathfrak{E}_{\alpha \cap \beta} \neq \emptyset$. Thus x belongs to the set on the right-hand side of (6). Consequently, (6) holds, and $X \bigcap^0 Y$ belongs to $P\mathfrak{F}_m$ provided that $X, Y \in P\mathfrak{F}_m$. Now we will verify conditions (P₁)-(P₅) of Definition 1.1. It is obvious that (P₁)-(P₄) are fulfilled in $P\mathfrak{F}_m$. Consider (P₅). Assume that $X \in B\mathfrak{F}_m$, $\alpha \not\leq \beta$ and $X \cap \mathfrak{E}_\alpha \subset \mathfrak{E}_\beta$. Clearly $X_\alpha \cap \mathfrak{E}_\beta = \emptyset$, since $\mathfrak{E}_\beta = \bigcup_{\gamma \leq \beta} \mathfrak{E}_\gamma$ and $X_\gamma \cap \mathfrak{E}_\alpha = \emptyset$ for $\alpha \neq \gamma$.

Hence $X_\alpha = X \cap \mathfrak{E}_\alpha = X \cap \mathfrak{E}_\alpha \cap \mathfrak{E}_\alpha = \emptyset$. We know that X is a cylindrical set, and so $X_\alpha = \emptyset$ yields $X = \emptyset$.

Now assume that $X \in B\mathfrak{F}_m$ and $X \cap \mathfrak{E}_\alpha \subset \bigcup_{\beta < \alpha} \mathfrak{E}_\beta$. We again observe that $X_\alpha \cap \bigcup_{\beta < \alpha} \mathfrak{E}_\beta = \emptyset$ and hence, by the same argument, we obtain $X = \emptyset$.

Therefore (P₅) is satisfied and so $P\mathfrak{F}_m$ is a Post field of type A .

In view of (5) the Post field $P\mathfrak{F}_m$ is full, and since $B\mathfrak{F}_m$ is a Boolean m-algebra, $P\mathfrak{F}_m$ is, by Theorem 1.10 (iii), a Post m-field.

Finally suppose that $\mathfrak{F} = \langle \mathfrak{F}; \{\mathfrak{E}_\alpha: \alpha \in A\}; B\mathfrak{F} \rangle$ is an arbitrary full Post m-field of type A containing $P\mathfrak{F}_0$. As is well known, the underlying Boolean algebra $B\mathfrak{F}$ is a Boolean m-field containing $B\mathfrak{F}_0$. Consequently, on account of the definition of $B\mathfrak{F}_m$ it follows that $B\mathfrak{F} \supset B\mathfrak{F}_m$. Consequently $\mathfrak{F} \supset P\mathfrak{F}_m$, since the Post field \mathfrak{F} is full. The proof of the theorem is completed.

By \mathcal{V}^m we denote the set

$$(7) \quad \mathcal{V}^m = \{X \in P\mathfrak{F}_m: X \text{ is a set of m-cocategory}\}.$$

It can easily be seen, on account of Theorem 5.5, that \mathcal{V}^m is a lattice m-filter. Moreover, if either $V \in A$ or $\bar{A} < m$, then \mathcal{V}^m is a D_V -filter of type A (i.e. $x \in \mathcal{V}^m$ iff for every $\alpha \in A^-$, $D_\alpha x \in \mathcal{V}^m$).

Indeed, let us suppose that $x \in \mathcal{V}^m$. From the definition of a set of m-cocategory it follows that $X \supset \bigcap_{t \in T} X_t$, where X_t is a dense m-open set for all $t \in T$, and $\bar{T} < m$. It is easy to see that $X_t \in B\mathfrak{F}_m$ and so $D_\alpha X_t = X_t$. Hence, by applying the generalization of Epstein's Lemma, we get

$$D_\alpha X \supset D_\alpha \bigcap_T X_t = \bigcap_T D_\alpha X_t = \bigcap_T X_t.$$

This means that $D_\alpha X \in \mathcal{V}^m$ for every $\alpha \in A^-$.

Now assume that, for every $\alpha \in A^-$, $D_\alpha X \in \mathcal{V}^m$. If $V \in A$, then from $X \supset D_V X$ it follows that $X \in \mathcal{V}^m$. If $\bar{A} < m$, then $\bigcap_{\alpha \in A^-} D_\alpha X \in \mathcal{V}^m$. Since $X \supset \bigcap_{\alpha \in A^-} D_\alpha X$, we conclude that $X \in \mathcal{V}^m$. This completes the proof.

Let $p, p_i, i \in I$, be arbitrary elements in P . We adopt for every $\alpha \in A^-$ the following notation:

$$(8) \quad \begin{aligned} S_\alpha^1 &= \sim h(D_\alpha p) \cup \bigcup_{i \in I} h(D_\alpha p_i), \\ S_\alpha^2 &= h(D_\alpha p) \cup \bigcup_{i \in I} \sim h(D_\alpha p_i). \end{aligned}$$

Since $D_\alpha p, D_\alpha p_i, i \in I$, are Boolean elements, a prime D -filter belongs to $S_\alpha^1 (S_\alpha^2)$ iff it preserves the l.u.b. $D_\alpha p = \bigcup_{i \in I} D_\alpha p_i$ (the g.l.b. $D_\alpha p = \bigcap_{i \in I} D_\alpha p_i$).

THEOREM 5.9. *Let I be a set of power less than m .*

(i) *If $p = \bigcup_{i \in I} p_i$ ($p = \bigcap_{i \in I} p_i$) in P , then for every $\alpha \in A^-$ the set S_α^1 (S_α^2) is dense m -open in \mathcal{X} .*

(ii) *Conversely, if S_α^1 (S_α^2) is a dense m -open set for every $\alpha \in A^-$ and $h(p) \supset h(p_i)$ ($h(p) \subset h(p_i)$) for every $i \in I$, then*

$$p = \bigcup_{i \in I} p_i \quad (p = \bigcap_{i \in I} p_i).$$

The proof of this theorem is analogous to that in the case of Boolean algebras (see [27]). To prove (i) we use the generalization of Epstein's Lemma, and for (ii) we use the fact that $p = \bigcup_{i \in I} p_i$ ($p = \bigcap_{i \in I} p_i$) iff for all $\alpha \in A^-$, $D_\alpha p = \bigcup_I D_\alpha p_i$ ($D_\alpha p = \bigcap_I D_\alpha p_i$).

Recall that if either $V \in A$ or $\bar{A} < m$, then \mathcal{V}^m is a m - $D_{\mathcal{V}}$ -filter of type A . Hence, $P\mathfrak{F}_m/\mathcal{V}^m$ is, according to Theorem 3.7, a Post algebra of type A . We introduce the mapping h of P into $P\mathfrak{F}_m/\mathcal{V}^m$ defined by the equation

$$(9) \quad \tilde{h}(p) = |h(p)| \quad \text{for every } p \in P.$$

The mapping \tilde{h} is called the *canonical m -homomorphism of the Post algebra P* .

THEOREM 5.10. *\tilde{h} is an m -homomorphism of P into $P\mathfrak{F}_m/\mathcal{V}^m$. Moreover, if the Post algebra P is m -complete and $\bar{A} < m$, then \tilde{h} is also an epimorphism onto $P\mathfrak{F}_m/\mathcal{V}^m = \langle P\mathfrak{F}_m/\mathcal{V}^m; \{|\mathfrak{E}|_\alpha: \alpha \in A\}, B\mathfrak{F}_m/\mathcal{V}^m \rangle$.*

Proof. It is clear that \tilde{h} is a Post homomorphism. On account of Theorem 2.8 (ii) it remains to show that \tilde{h} is an m -homomorphism of B into $B\mathfrak{F}_m/\mathcal{V}^m$. Let $\bigcup_{i \in I} a_i = 1$, $a_i \in B$ and $\bar{I} < m$. Thus by Theorem 5.9 (i), $S^1 = \bigcup_{i \in I} h(D_\alpha a_i) = \bigcup_I h(a_i)$ is a dense m -open set, and so it belongs to \mathcal{V}^m . Since \mathcal{V}^m is an m -filter, \mathcal{V}^m preserves all m -l.u.b.'s in $B\mathfrak{F}_m$. Hence $\bigcup_I \tilde{h}(a_i) = \bigcup_I |h(a_i)| = |\bigcup_I h(a_i)| = |\mathfrak{E}_{\mathcal{V}}| = 1 = h(1)$. This shows that h preserves the l.u.b. $\bigcup_I a_i = 1$. Consequently, \tilde{h} is an m -homomorphism.

We now prove the second part of the theorem. First of all, it will be shown that \tilde{h} is an epimorphism of B onto $B\mathfrak{F}_m/\mathcal{V}^m$. For this purpose, we consider the family $\mathfrak{F} = \{X \in B\mathfrak{F}_m: \text{there exists an element } b \in B \text{ such that } (h(b) \Rightarrow X) \cap (X \Rightarrow h(b)) \in \mathcal{V}^m\}$. Obviously, $\mathfrak{F} \supset B\mathfrak{F}_0$ and $X \in \mathfrak{F}$ yields $\sim X \in \mathfrak{F}$, since $a \Rightarrow b \leq -b \Rightarrow -a$. We wish to show that \mathfrak{F} is a Boolean m -field. In fact, let $X_i \in \mathfrak{F}$, $i \in I$ and $\bar{I} < m$. Let $b_i \in B$ be such that

$$(10) \quad (h(b_i) \Rightarrow X_i) \cap (X_i \Rightarrow h(b_i)) \in \mathcal{V}^m, \quad i \in I.$$

Put $b = \bigcup_I b_i$ and $X = \bigcup_I^0 X_i$. Of course, $b \in B$ and $X \in B\mathfrak{F}_m$. Since $h(b) \supset \bigcup_I^0 h(b_i)$ and the condition $\bigcap_I (a_i \Rightarrow c_i) \leq (\bigcup_I a_i) \Rightarrow (\bigcup_I c_i)$ holds in each Boolean algebra, it follows that

$$X \Rightarrow h(b) \geq \bigcup_I^0 X_i \Rightarrow \bigcup_I^0 h(b_i) \geq \bigcap_I^0 (X_i \Rightarrow h(b_i)).$$

On account of (10) we get $X_i \Rightarrow h(b_i) \in \mathcal{V}^m$. Hence, by the inequalities proved above, we get

$$(11) \quad X \Rightarrow h(b) \in \mathcal{V}^m.$$

On the other hand, by using the inequalities $a \Rightarrow d \geq (a \Rightarrow c) \cap (c \Rightarrow d)$ and $\bigcup a_i \Rightarrow \bigcup c_i \geq \bigcap (a_i \Rightarrow c_i)$, we infer that

$$(12) \quad h(b) \Rightarrow X \geq (h(b) \Rightarrow \bigcup_I^0 h(b_i)) \cap \bigcap_I^0 (h(b_i) \Rightarrow X_i).$$

Since \tilde{h} is a m -homomorphism, we have $\tilde{h}(b) = \tilde{h}(\bigcup_I b_i) = \bigcup_I \tilde{h}(b_i)$, and consequently $|h(b)| = \bigcup_I |h(b_i)| = |\bigcup_I h(b_i)|$, because \mathcal{V}^m is an m -filter. This equation shows that $h(b) \approx_{\mathcal{V}^m} \bigcup_I h(b_i)$, which yields $D_a h(b) \Rightarrow D_a \bigcup_I h(b_i) \in \mathcal{V}^m$. Thus by the generalization of Epstein's Lemma and by the equation $D_a h(b_i) = h(D_a b_i) = h(b_i)$, we get: $h(b) \Rightarrow \bigcup_I^0 h(b_i) \in \mathcal{V}^m$. On the other hand, by virtue of (10), it is obvious that $\bigcap_I^0 (h(b_i) \Rightarrow X_i) \in \mathcal{V}^m$. Consequently, the set on the right-hand side of (12) belongs to \mathcal{V}^m , and hence $h(b) \Rightarrow X \in \mathcal{V}^m$. This formula together with (11) gives $(h(b) \Rightarrow X) \cap (X \Rightarrow h(b)) \in \mathcal{V}^m$. By the definition of \mathfrak{F} , this shows that $X \in \mathfrak{F}$.

Similarly, we can show that the m -intersection of sets in \mathfrak{F} also belongs to \mathfrak{F} . Thus we have proved that \mathfrak{F} is a Boolean m -field containing $B\mathfrak{F}_0$. This implies that $\mathfrak{F} \supset B\mathfrak{F}_m$, and consequently $\mathfrak{F} = B\mathfrak{F}_m$. Accordingly it is easy to see that

$$(13) \quad \tilde{h}(B) = \{|h(b)|: b \in B\} = \{X: X \in B\mathfrak{F}_m\} = B\mathfrak{F}_m/\mathcal{V}^m,$$

i.e. that h is an epimorphism of B onto $B\mathfrak{F}_m/\mathcal{V}^m$.

Now let p' be an arbitrary element of $P\mathfrak{F}_m/\mathcal{V}^m$. Thus p' has the following form:

$$p' = \bigcup_{\alpha \in A^-} b'_\alpha \cap |\mathfrak{E}|_\alpha, \quad \text{where } b'_\alpha \in B\mathfrak{F}_m/\mathcal{V}^m.$$

Let $b_\alpha \in B$ satisfy the condition $h(b_\alpha) = b'_\alpha$. The existence of such an element b_α in B follows from (13). Without any loss of generality we can assume that $\langle b_\alpha : \alpha \in A^- \rangle$ is a decreasing sequence of type A satisfying the condition $h(b_\alpha) = b'_\alpha$. For, instead of $\langle b_\alpha : \alpha \in A^- \rangle$, we may consider the decreasing sequence $\langle \bar{b}_\alpha : \alpha \in A^- \rangle$, where $\bar{b}_\alpha = \bigcap_{\beta < \alpha} b_\beta$ (these g.l.b.'s exists, since $\bar{A} < m$ and B is m -complete). Take the element $p = \bigcup_{\alpha \in A^-} b_\alpha \cap e_\alpha$. It is clear that $h(p) = p'$. Consequently, the mapping h is an epimorphism. This completes the proof.

3. Characterization of the m -representable Post algebras. In this section we will always assume that the semilattice A has a unit element V or that A is of power less than m , where m is an arbitrary infinite cardinal.

The following theorem is an analogue of the well-known theorem which characterizes the m -representable Boolean algebras. The proof is similar to that in the case of Boolean algebras.

THEOREM 5.11. *For every Post algebra $P = \langle P; \{e_\alpha : \alpha \in A\}; B \rangle$ of type A , the following conditions are equivalent:*

- (R₁) P is m -representable.
- (R₂) The canonical m -homomorphism is an m -isomorphism.
- (R₃) For every m -system $\langle a_{ij} : i \in I, j \in J \rangle$ (i.e. $\bar{I} < m, \bar{J} < m$) of elements of B , satisfying the conditions:

$$(14) \quad \bigcap_J a_{ij}, i \in I, \text{ and } \bigcup_I \bigcap_J a_{ij} \text{ exist in } B,$$

and

$$(15) \quad \bigcup_I \bigcap_J a_{ij} \neq 1,$$

there is a function $\varphi \in J^I$ such that

$$\bigcup_{i \in I'} a_{i\varphi(i)} \neq 1 \quad \text{for each finite subset } I' \text{ of } I.$$

- (R₄) For every m -system $\langle a_{ij} : i \in I, j \in J \rangle$ of elements of B satisfying condition (14) and the condition

$$\bigcup_I \bigcap_J a_{ij} = 0;$$

and for every element $a \in B$, $a \neq 1$, there exists a function $\varphi \in J^I$ such that

$$a \cup \bigcup_{i \in I'} a_{i\varphi(i)} \neq 1 \quad \text{for each finite subset } I' \subset I.$$

(R₅) For every m-system $\langle a_{ij} : i \in I, j \in J \rangle$ of elements of B satisfying the condition: $\bigcup_j a_{ij}, i \in I$ and $\bigcap_I \bigcup_J a_{ij}$ exist in B , and $\bigcap_I \bigcup_J a_{ij} = 1$, and for every $a \in B, a \neq 0$, there is a function $\varphi \in J^I$ such that

$$(16) \quad a \cap \bigcap_{I'} a_{i\varphi(i)} \neq 0 \text{ for every finite subset } I' \text{ of } I.$$

(R₆) For every m-system $\langle a_{ij} : i \in I, j \in J \rangle$ of elements in B such that, for each function $\varphi \in J^I$, the set $\{a_{i\varphi(i)} : i \in I\}$ contains a complementary pair of elements $b, -b$, the condition $\bigcup_J a_{ij} = 1$ for every $i \in I - \{i_0\}$ implies $\bigcup_J a_{i_0j} = 0$.

(R₇) For every m-system $\langle a_{ij} : i \in I, j \in J \rangle$ of elements in B if for each function $\varphi \in J^I$ the set $\{a_{i\varphi(i)} : i \in I\}$ contains a complementary pair of elements, then $\bigcup_I \bigcap_J a_{ij} = 1$.

(R₈) If an element $a \in B$ is such that there exists an m-system $\langle a_{ij} : i \in I, j \in J \rangle$ satisfying the following condition: for every function $\varphi \in J^I$ the set $\{a_{i\varphi(i)} : i \in I\}$ contains either the element a or else a complementary pair of elements, then $\bigcup_J a_{ij} = 1$ for all $i \in I$ implies $a = 1$.

(R₉) In the Stone space \mathcal{X} of the Post algebra P , any m-intersection of dense m-open sets is also a dense set.

(R₁₀) In the Stone space \mathcal{X} of P , every set of m-cocategory is dense.

Proof. The proofs of implications $(R_3) \Rightarrow (R_4) \Rightarrow (R_5)$ are similar to those in the case of Boolean algebras (see [27]).

$(R_5) \Rightarrow (R_6)$. Let an m-system $\langle a_{ij} : i \in I, j \in J \rangle$ satisfy the assumption required in (R_6) . If $a = \bigcup_J a_{i_0j} \neq 0$, then in view of (R_5) we should infer that for the m-system $\langle a_{ij} : j \in J \rangle, i \in I - \{i_0\}$ and $\langle -a, a_{i_0j} : j \in J \rangle$, there exists a function $\varphi \in J^{I - \{i_0\}}$ and $a_{i_0\varphi(i_0)} \in \{-a, a_{i_0j} : j \in J\}$ such that $a \cap \bigcap_{i \in I'} a_{i\varphi(i)} \neq 0$, for every finite subset I' of I . From the hypothesis in (R_6) it follows that there exists a finite subset I_0 of I for which the set $\{a_{i\varphi(i)} : i \in I_0\}$ contains either the element $-a$ or a complementary pair of elements. Hence we obtain a contradictory conclusion $a \cap \bigcap_{i \in I_0} a_{i\varphi(i)} = 0$. This shows that $a = \bigcup_J a_{i_0j} = 0$. Thus (R_6) results from (R_5) .

$(R_6) \Rightarrow (R_7)$. Let $\langle a_{ij} : i \in I, j \in J \rangle$ fulfil all assumptions in (R_7) and let $a = \bigcap_I \bigcup_J -a_{ij} = -\bigcup_I \bigcap_J a_{ij}$. We wish to show that $a = 0$. Since $\bigcup_J -a_{ij} \geq a$, we have $-a \cup \bigcup_J -a_{ij} = 1$ for all $i \in I$. Now consider the m-system $\langle -a, -a_{ij} : j \in J \rangle, i \in I$ and $\langle a_{i_0j} : j \in J \rangle$, where $i_0 \notin I$ and $a_{i_0j} = a$ for all $j \in J$. It is easily seen that this m-system satisfies the assumptions required in (R_6) . Hence according to (R_6) , $\bigcup_J a_{i_0j} = a = 0$.

$(R_7) \Rightarrow (R_8)$. Suppose that $a \in B$, and that $\langle a_{ij}: i \in I, j \in J \rangle$ is an m -system satisfying the hypothesis of (R_8) . Consider an m -system $\langle a'_{ij}: i \in I', j \in J \rangle$ with $I' = I \cup \{i_0\}$ ($i_0 \notin I$) defined as follows: $a'_{ij} = -a_{ij}$ for $i \in I$; $a'_{i_0j} = a$ for all $j \in J$. From the hypothesis in (R_8) it easily follows that this m -system satisfies the assumptions of (R_7) . Thus by (R_7) , $\bigcup_{I'} \bigcap_J a'_{ij} = a \cup \bigcup_I \bigcap_J -a_{ij} = a = 1$, since $\bigcup_J a_{ij} = 1$ for all $i \in I$. Consequently (R_8) follows from (R_7) .

$(R_8) \Rightarrow (R_9)$. Consider an m -system $\langle a_{ij}: i \in I, j \in J \rangle$ such that $\bigcap_I \bigcup_J a_{ij} = 1$. Let a be an arbitrary element of B different from 0. Suppose, contrary to the conclusion of (R_9) that for every $\varphi \in J^I$ there is a finite subset I' of I such that $a \cap \bigcap_{i \in I'} a_{i\varphi(i)} = 0$, or $-a \cup \bigcup_{i \in I'} -a_{i\varphi(i)} = 1$. We denote by $\varphi_{I'}$ the restriction of φ to the set I' . Thus $\varphi_{I'} \in J^{I'}$. It is clear that $J^{\bar{I}'} < m$ and that the power of the set of all finite subsets I' of I is less than m . Since the cardinal number m is regular, the set I^* of all ordered pairs $\langle I', \varphi_{I'} \rangle$ has a cardinality less than m . Now we define an m -system $\langle \tilde{a}_{ij}: i \in \tilde{I}, j \in J \rangle$ in such a way that

$$\begin{aligned} \tilde{a}_{ij} &= a_{ij} \quad \text{for all } i \in I, j \in J, \\ \{\tilde{a}_{i_0j}: j \in J\} &= \{-a, -a_{i\varphi_{I'}(i)}: i \in I'\} \end{aligned}$$

for each pair $i^* = \langle I', \varphi_{I'} \rangle \in I^*$,

$$\{\tilde{a}_{i_0j}: j \in J\} = \{-a, a\},$$

where $i_0 \notin I^* \cup I$ and $\tilde{I} = I^* \cup I \cup \{i_0\}$.

Obviously, the l.u.b. of the set of elements in any row in this m -system is equal to 1 and, as can easily be shown, for every function $\varphi \in J^{\tilde{I}}$ the set $\{\tilde{a}_{i\varphi(i)}: i \in \tilde{I}\}$ contains either the element $-a$, or else a complementary pair of elements. Consequently, by (R_8) , $-a = 1$, or $a = 0$, which contradicts $a \neq 0$. Thus (R_9) follows from (R_8) .

$(R_9) \Rightarrow (R_{10})$. Let $\mathfrak{G}_i, i \in I$ be a dense m -open set in \mathfrak{X} and $\bar{J} < m$. Thus (since the cardinal m is regular) we see that for every $i \in I$

$$\mathfrak{G}_i = \bigcup_{j \in J} h(a_{ij}), \quad \text{where } a_{ij} \in B, \bar{J} < m$$

and $\bigcup_J a_{ij} = 1$. So it is clear that $\bigcap_I \bigcup_J a_{ij} = 1$. Take any non-empty open set $\mathfrak{G} \in B\mathfrak{F}_0$, i.e. a set of the form $\mathfrak{G} = h(a)$, where $a \in B$ and $a \neq 0$. From (R_9) it follows that there exists a function $\varphi \in J^I$ satisfying condition (16). Accordingly, $h(a) \cap \bigcap_{I'} h(a_{i\varphi(i)}) \neq \emptyset$ for each finite subset I' of I .

Since the space \mathfrak{X} is compact and the sets $h(a), h(a_{i\varphi(i)})$ are closed, we infer that $h(a) \cap \bigcap_I h(a_{i\varphi(i)}) = \emptyset$. Consequently, $\mathfrak{G} \cap \bigcap_I \mathfrak{G}_i \neq \emptyset$, because $\mathfrak{G}_i \supset h(a_{i\varphi(i)})$. This shows that $\bigcap_I \mathfrak{G}_i$ is a dense set.

$(R_9) \Rightarrow (R_{10})$ is easily deduced from the definition of the set of m-co-category.

$(R_{10}) \Rightarrow (R_2)$: Since $\tilde{h}(e_a) = |\mathfrak{C}|_a$ for all $a \in A^+$, i.e. h/\bar{E} is a one-to-one mapping, it remains to show that, on account of Theorem 2.8 (ii) \tilde{h}/B is a one-to-one mapping of B into $B\mathfrak{F}_m/\mathcal{V}^m$. It is obvious that if $b \neq 1$, then $h(b)$ is a non-dense set. Hence $h(b) \notin \mathcal{V}^m$, which implies that $|h(b)| = \tilde{h}(b) \neq 1$. Consequently the mapping \tilde{h}/B is in fact one-to-one.

$(R_2) \Rightarrow (R_1)$ is evident.

$(R_1) \Rightarrow (R_3)$. Let P be m-isomorphically imbedded into $\mathfrak{F}/\mathcal{V}(A_0)$, where $\mathfrak{F} = \langle \mathfrak{F}; \{\mathfrak{C}_a: a \in A^+\}; B\mathfrak{F} \rangle$ is a Post m-field with a unit element \mathfrak{X} and $\mathcal{V}(A_0)$ is an m-D-filter in \mathfrak{F} . Since condition (R_3) is invariant under the m-isomorphism, it is sufficient to show that $\mathfrak{F}/\mathcal{V}(A_0)$ satisfies (R_3) . Indeed, let $\langle a_{ij}: i \in I, j \in J \rangle$ be an m-system satisfying conditions (14) and (15), where a_{ij} belongs to the underlying Boolean algebra $B(\mathfrak{F}/\mathcal{V}) = \{ |X|: x \in B\mathfrak{F} \}$, i.e. a_{ij} has the form:

$$a_{ij} = |X_{ij}|, \quad \text{with } X_{ij} \in B\mathfrak{F}.$$

By X we denote the intersection of all sets of the form $X_{i_0 j_0} \cup \dots \cup X_{i_k j_k}$ which belongs to $\mathcal{V}(A_0)$; $k < \omega$. Since $\bar{I}, \bar{J} < m$, $B\mathfrak{F}$ is m-complete and $\mathcal{V}(A_0)$ is an m-filter, we conclude that

$$X \in B\mathfrak{F} \quad \text{and} \quad X \in \mathcal{V}(A_0).$$

Put

$$(17) \quad Y_{ij} = X \Rightarrow X_{ij} = \sim X \cup X_{ij}.$$

Obviously

$$|Y|_{ij} = |X|_{ij} = a_{ij}.$$

We shall show that

$$(18) \quad \text{if } \bigcup_{l \leq k} Y_{i_l j_l} \neq \mathfrak{X}, \text{ then } \bigcup_{l \leq k} a_{i_l j_l} \neq 1.$$

In fact, suppose on the contrary that $\bigcup_{l \leq k} a_{i_l j_l} = 1$. Thus $\bigcup_{l \leq k} |Y_{i_l j_l}| = |\bigcup_{l \leq k} Y_{i_l j_l}| = 1$, so $\bigcup_{l \leq k} Y_{i_l j_l} \in \mathcal{V}(A_0)$. Since, on account of (17) $\bigcup_{l \leq k} Y_{i_l j_l} = \sim X \cup \bigcup_{l \leq k} X_{i_l j_l} = X \Rightarrow \bigcup_{l \leq k} X_{i_l j_l}$ and $X \in \mathcal{V}(A_0)$, it follows that $\bigcup_{l \leq k} X_{i_l j_l} \in \mathcal{V}(A_0)$. In view of the definition of the set X , we see that $X \subset \bigcup_{l \leq k} X_{i_l j_l}$, and so $\bigcup_{l \leq k} Y_{i_l j_l} = \sim X \cup \bigcup_{l \leq k} X_{i_l j_l} \supset \sim X \cup X = \mathfrak{X}$. This contradicts the assumption that $\bigcup_{l \leq k} Y_{i_l j_l} \neq \mathfrak{X}$. Consequently (18) holds.

By virtue of (15), $1 \neq \bigcup_I \bigcap_J a_{ij} = \bigcup_I \bigcap_J |Y_{ij}| = |\bigcup_I \bigcap_J Y_{ij}|$. Hence $\bigcup_I \bigcap_J Y_{ij} \neq \mathfrak{X}$. This yields the existence of a function $\varphi \in J^I$ such that $\bigcup_I \bigcap_J Y_{i\varphi(j)} \neq \mathfrak{X}$, which implies, on account of (18), the inclusion of (R_3) . Thus the proof of our theorem is completed.

COROLLARY 5.12. *If A has a unique atom 1, then for every Post algebra of type A each of the following conditions is equivalent to any condition in Theorem 5.11:*

(R'₃) For every m -system $\langle p_{ij}: i \in I, j \in J \rangle$ of elements in P for which
 (19) $\bigcap_J p_{ij}, i \in I$, and $\bigcup_I \bigcap_J p_{ij}$ exist in P and $\bigcup_I \bigcap_J p_{ij}$ is non-dense, there exists a function $\varphi \in J^I$ such that $\bigcup_{i \in I'} p_{i\varphi(i)}$ is non-dense for each finite subset I' of I .

(R'₄) For every m -system $\langle p_{ij}: i \in I, j \in J \rangle$ satisfying condition (19) and $\bigcup_I \bigcap_J p_{ij} = 0$, and for every non-dense element p , there exists a function $\varphi \in J^I$ such that $p \cup \bigcup_{i \in I'} p_{i\varphi(i)}$ is non-dense for each finite subset I' of I .

(R'₅) For every m -system $\langle p_{ij}: i \in I, j \in J \rangle$ satisfying the conditions:
 $\bigcup_J p_{ij}, i \in I$ and $\bigcap_I \bigcup_J p_{ij}$ exist in P , and $\bigcap_I \bigcup_J p_{ij}$ is a dense element; and for every element $p \neq 0$, there is a function $\varphi \in J^I$ such that

$$p \cap \bigcap_{i \in I'} p_{i\varphi(i)} \neq 0 \quad \text{for every finite subset } I' \text{ of } I.$$

(R'₆) For every m -system $\langle p_{ij}: i \in I, j \in J \rangle$ such that for each function $\varphi \in J^I$, the set $\{p_{i\varphi(i)}: i \in I\}$ contains a complementary pair of elements $q, -q$, the condition that $\bigcup_J p_{ij}$ is a dense element for every $i \in I - \{i_0\}$ implies $\bigcup_J p_{i_0j} = 0$.

(R'₇) For every m -system $\langle p_{ij}: i \in I, j \in J \rangle$, if for each function $\varphi \in J^I$ the set $\{p_{i\varphi(i)}: i \in I\}$ contains a complementary pair of elements, then $\bigcup_I \bigcap_J p_{ij}$ is a dense element in P .

(R'₈) If an element $p \in P$ is such that there exists an m -system $\langle p_{ij}: i \in I, j \in J \rangle$ satisfying the following condition:
 for every function $\varphi \in J^I$ the set $\{p_{i\varphi(i)}: i \in I\}$ contains either the element p , or else a complementary pair of elements, then the condition that $\bigcup_J p_{ij}$ is a dense element for every $i \in I$ implies that p is a dense element.

We can show that $(R_k) \Leftrightarrow (R'_k)$ ($k = 3, \dots, 8$). If we use the generalization of Epstein's Lemma for operation D_1 and the fact that p is a dense element iff $D_1 p = 1$, the proof is easy and hence it is omitted.

We recall that each of conditions (R_3) and (R_4) is necessary and sufficient for the Boolean algebra B to be m -representable. Therefore we obtain the following

COROLLARY 5.13 (i) *A Post algebra $P = \langle P; \{e_a: a \in A\}; B \rangle$ of type A is m -representable iff its underlying Boolean algebra B is m -representable.*

(ii) *Every Post algebra P is \aleph_1 -representable. To complete this section, we shall establish some theorems which will be useful for certain application in logic.*

The next theorem is an immediate corollary of the Rasiowa and Sikorski Lemma.

THEOREM 5.14. *Let $P = \langle P; \{e_a: a \in A\}; B \rangle$ be a Post algebra of type A . Then for every given countable set of l.u.b.'s and g.l.b.'s of elements in B*

$$(20) \quad \bigcup_{i \in I_n} a_{ni} = a_n, \quad \bigcap_{j \in J_n} b_{nj} = b_n, \quad n < \omega,$$

and for every element $a \in B$, $a \neq 1$, there exists a prime D_V -filter V of type A in P which does not contain a and which preserves all the l.u.b.'s and g.l.b.'s in (20).

The following theorem can be regarded as a generalization of the Rasiowa and Sikorski Lemma. E.g. we consider an m-family of l.u.b.'s and g.l.b.'s:

$$(21) \quad \begin{aligned} \bigcup_{j \in J_i^1} p_{ij} &= p_i, & i \in I_1; & \quad \bar{I}_1, \bar{J}_i^1 < m, \\ \bigcap_{j \in J_i^2} q_{ij} &= q_i, & i \in I_2; & \quad \bar{I}_2, \bar{J}_i^2 < m. \end{aligned}$$

THEOREM 5.15. *Let $P = \langle P; \{e_a: a \in A\}; B \rangle$ be a Post algebra of type A . Then the following condition:*

(R₁₁) *For every m-family of l.u.b.'s and g.l.b.'s (21), and for every element $p \in P$, $p \neq 1$, there exists a prime D_V -filter of type A in P which does not contain the element p , and which preserves all the l.u.b.'s and g.l.b.'s (21);*

is sufficient for the m-representability of P . Moreover, if $\bar{A} < m$, this condition is also necessary.

Proof. It follows from (R₁₁) that the underlying Boolean algebra B is m-representable (see [27]). Hence, by Theorem 5.13, P is m-representable. Conversely, suppose that P is m-representable and $\bar{A} < m$. We consider an arbitrary m-family (21) and put

$$(22) \quad \begin{aligned} S_{ai}^1 &= \sim h(D_a p_i) \cup \bigcup_{j \in J_i^1} h(D_a p_{ij}), & i \in I_1, \\ S_{ai}^2 &= h(D_a q_i) \cup \bigcup_{j \in J_i^2} \sim h(D_a q_{ij}), & i \in I_2, \end{aligned} \quad a \in A^-.$$

According to Theorem 5.9, S_{ai}^1, S_{ai}^2 are dense m-open sets. Since $\bar{I}_{1,2} < m$ and $\bar{A} < m$, the intersection X of all sets S_{ai}^k in (22) is, by virtue of (R₉), a dense set. For every element $p \neq 1$, there exists an $a_0 \in A^-$

such that $D_{\alpha_0}p \neq 1$ or, equivalently $-D_{\alpha_0}p \neq 0$. Then $h(-D_{\alpha_0}p)$ is a non-empty open set and so $X \cap h(-D_{\alpha_0}p) \neq \emptyset$.

Take an arbitrary D -filter \mathcal{V}' in this non-empty intersection. This D -filter will contain $-D_{\alpha_0}p$ and preserve all the following l.u.b.'s and g.l.b.'s:

$$D_{\alpha}p_i = \bigcup_{j \in J_i^1} D_{\alpha}p_{ij}, \quad i \in I_1,$$

$$D_{\alpha}q_i = \bigcap_{j \in J_i^2} D_{\alpha}q_{ij}, \quad i \in I_2, \alpha \in A^-.$$

Let \mathcal{V} be a $D_{\mathcal{V}}$ -filter of type A with $\mathcal{V}' \cap B = \mathcal{V} \cap B$. Obviously \mathcal{V} also contains $-D_{\alpha_0}p$ and preserves the above l.u.b.'s and g.l.b.'s. From this we conclude that the $D_{\mathcal{V}}$ -filter \mathcal{V} preserves all the l.u.b.'s and the g.l.b.'s (21) and does not contain the element p . This completes the proof.

By virtue of Theorem 3.10 (ii), the following corollary is an immediate consequence of Theorems 5.14 and 5.15.

COROLLARY 5.16. *Let us assume that either $\bar{A} < \aleph_1$ or $\bar{A} < m$, and let us consider an m -family (21) and, moreover, an m -family*

$$(23) \quad p_i \Rightarrow q_i, \quad i \in I (\bar{I} < m).$$

If the Post algebra P of type A is m -representable, then for every element $p \neq 1$ there exists a prime $D_{\mathcal{V}}$ -filter of type A in P which does not contain the element p and which preserves all the l.u.b.'s and g.l.b.'s (21) and all pseudo-complement p_i relative to q_i ($i \in I$) in (23).

PART II

INFINITARY PROPOSITIONAL ν -VALUED LANGUAGES

In this part we shall consider infinitary ν -valued propositional languages \mathcal{L}_m^ν , where m is an infinite regular cardinal number and ν is an ordinal number for which either $\nu < m$ or $\nu \notin \text{Lim}$. The alphabet of a language \mathcal{L}_m^ν is the union of the following disjoint sets: a set \mathcal{V} of propositional variables a set \mathcal{E}_ν of propositional constants, a set L of logical connectives and a set \mathcal{A} of auxiliary elements.

The elements of an alphabet will be called *signs* of this alphabet.

The elements of \mathcal{V} will be denoted by V, U, \dots (with indices if necessary). We assume that $\overline{\mathcal{V}} \geq \aleph_0$.

The elements of \mathcal{E}_ν will be denoted by $E_0, E_1, \dots, E_i, \dots, E_\nu$.

The set L consists of: unary connectives denoted by \bar{D}_α , $0 < \alpha \leq \nu$ and $\alpha \notin \text{Lim}$.; one unary connective denoted by \neg and called the *negation sign*; one binary connective denoted by \rightarrow and called the *implication sign*; two infinitary connectives denoted by \vee, \wedge and called the *disjunction sign* and the *conjunction sign*, respectively.

The set \mathcal{A} consists of two elements denoted by $(,)$ and called *parentheses*.

By an expression of a length l , where $0 < l < m$, we mean any well-ordered sequence $\langle G_0, G_1, \dots, G_i, \dots : i < l \rangle$ of elements in the alphabet of \mathcal{L}_m^ν .

The set of formulas of \mathcal{L}_m^ν is the least set \mathcal{F}_m^ν of expressions of any length $0 < l < m$ satisfying the following conditions:

1° $\mathcal{V} \cup \mathcal{E}_\nu \subset \mathcal{F}_m^\nu$.

2° If $0 < l < m$ and $F_i \in \mathcal{F}_m^\nu$ for all $i < l$, then $(\vee F_0 \dots F_i \dots)$ and $(\wedge F_0 \dots F_i \dots)$ belong to \mathcal{F}_m^ν .

3° If $F_0, F_1 \in \mathcal{F}_m^\nu$, then $(F_0 \rightarrow F_1) \in \mathcal{F}_m^\nu$.

4° If $F \in \mathcal{F}_m^\nu$, then $\neg F$ and $\bar{D}_\alpha F$ belong to \mathcal{F}_m^ν , for every $\alpha \leq \nu$, $\alpha \notin \text{Lim}$.

For convenience we shall write $(F_0 \vee F_1)$ for $(\vee F_0 F_1)$; $(F_0 \wedge F_1)$ for $(\wedge F_0 F_1)$; $(F_0 \leftrightarrow F_1)$ for $((F_0 \rightarrow F_1) \wedge (F_1 \rightarrow F_0))$ $(\bigvee_{i < l} F_i)$ for $(\vee F_0 \dots F_i \dots)$ and similarly $(\bigwedge_{i < l} F_i)$ for $(\wedge F_0 \dots F_i \dots)$.

Since m is a regular cardinal, every formula of \mathcal{L}_m^ν is evidently of a length less than m .

The formulas of \mathcal{L}_m^ν will be denoted by F, G, \dots with indices if necessary.

1. A fundamental formal system $S(\mathcal{L}_m^\nu)$

The logical axioms for this system are all formulas of the following form, where $F_0, F_1, \dots, F_i, \dots$ denote arbitrary formulas in \mathcal{L}_m^ν .

Group (A)

- (a₁) $((F_0 \rightarrow F_1) \rightarrow ((F_1 \rightarrow F_2) \rightarrow (F_0 \rightarrow F_2)))$.
- (a₂) $((F_0 \rightarrow (F_1 \rightarrow F_2)) \rightarrow ((F_0 \wedge F_1) \rightarrow F_2))$.
- (a₃) $((F_0 \wedge F_1) \rightarrow F_2) \rightarrow (F_0 \rightarrow (F_1 \rightarrow F_2))$.
- (a₄) $((\bigwedge_{i < l} (F_i \rightarrow F_l)) \rightarrow ((\bigvee_{i < l} F_i) \rightarrow F_l))$.
- (a₅) $((\bigwedge_{i < l} (F_l \rightarrow F_i)) \rightarrow (F_l \rightarrow (\bigwedge_{i < l} F_i)))$.
- (a₆) $(F_l \rightarrow (\bigvee_{i < l} F_i))$ for every $l < \nu$.
- (a₇) $((\bigwedge_{i < l} F_i) \rightarrow F_l)$ for every $l < \nu$.
- (a₈) $((F_0 \wedge \neg F_0) \rightarrow F_1)$.
- (a₉) $((F_0 \rightarrow (F_1 \wedge \neg F_1)) \rightarrow \neg F_0)$.

Group (B)

- (b₁) $\bar{D}_\alpha E_\beta$ for all $\beta \geq \alpha$, $\alpha \leq \nu$, $\alpha \notin \text{Lim}$.
- (b₂) $\neg \bar{D}_\alpha E_\beta$ for all $\beta < \alpha$, $\alpha \leq \nu$, $\alpha \notin \text{Lim}$.
- (b₃) $(\bar{D}_1 F \vee \neg \bar{D}_1 F)$.
- (b₄) $(\bar{D}_\alpha (\bigvee_{i < n} F_i) \leftrightarrow (\bigvee_{i < n} \bar{D}_\alpha F_i))$
- (b₅) $(\bar{D}_\alpha (\bigwedge_{i < n} F_i) \leftrightarrow (\bigwedge_{i < n} \bar{D}_\alpha F_i))$
- (b₆) $(\bar{D}_\alpha (F_0 \rightarrow F_1) \rightarrow (\bar{D}_\beta F_0 \rightarrow \bar{D}_\beta F_1))$ for any $\beta \leq \alpha$.
- (b₇) $(\bar{D}_\beta \bar{D}_\alpha F \leftrightarrow \bar{D}_\alpha F)$ for all $\alpha, \beta \leq \nu$ and $\alpha, \beta \notin \text{Lim}$.
- (b₈) $(\bar{D}_\alpha \neg F \leftrightarrow \neg \bar{D}_1 F)$ for all $\alpha \leq \nu$, $\alpha \notin \text{Lim}$.

In the case $\nu < m$ we have

- (b₉) $(F \leftrightarrow (\bigvee_{\alpha \leq \nu, \alpha \notin \text{Lim}} (\bar{D}_\alpha F \wedge E_\alpha)))$.

In the case $\nu \geq m$ and $\nu \notin \text{Lim}$ we have

- (b'₉) $((\bar{D}_\alpha F \wedge E_\alpha) \rightarrow F)$ for every $\alpha \leq \nu$, $\alpha \notin \text{Lim}$.

The groups (A) and (B) of axioms are simple generalizations of the axioms of the intuitionistic propositional calculi in [21]; Chapter IX,

and of the axioms of the formal system of m -valued predicate calculi in [17], respectively.

We adopt the following rules of inference:

Modus ponens:

$$\frac{F, (F \rightarrow G)}{G}$$

Conjunction:

$$\frac{F_i: i < l < m}{(\bigwedge_{i < l} F_i)}$$

In order to introduce the next rule (RD) we denote by $\bar{D}_\nu^0 F$ the formula $\bar{D}_\nu F$ provided $\nu \notin \text{Lim.}$ or else the formula $\bigwedge_{\alpha \leq \nu, \alpha \notin \text{Lim.}} \bar{D}_\alpha F$

$$(RD) \quad \frac{F}{\bar{D}_\nu^0 F}.$$

In the case $\nu \geq m$, $\nu \notin \text{Lim.}$ we have also the following rule:

$$(R_\nu) \quad \frac{((\bar{D}_\alpha F \wedge E_\alpha) \rightarrow G); \alpha \leq \nu, \alpha \notin \text{Lim.}}{(F \rightarrow G)}$$

More generally, we shall consider formal systems $S(L_m^v, \Sigma)$ the axioms of which are the formulas in group (A) and (B) and, moreover, the formulas in a set Σ .

Let Γ be a set of formulas. By a proof from Γ in $S(\mathcal{L}_m^v, \Sigma)$ we understand a well-ordered sequence $\langle F_i; i < l \rangle$ of formulas, where $l < \max(m, (\bar{\nu})^+)$, for which either F_i is an axiom or a formula in Γ , or F_i a consequence of some previous (in this sequence) formulas by one of the rules of inference.

A formula F is said to be *provable* from Γ in $S(\mathcal{L}_m^v, \Sigma)$, and we write $\vdash_{\Gamma} F$, provided F is the last formula in a certain proof from Γ .

Of course the sets Σ and Γ can be empty. If $\Sigma = \emptyset$, we have $S(\mathcal{L}_m^v, \Sigma) = S(\mathcal{L}_m^v)$, and if $\Gamma = \emptyset$, we write $\vdash F$ for $\vdash_{\Gamma} F$.

On account of the axioms in the group (A), the following statement holds:

(I.C) Let F and $G_i (i \leq n)$ be formulas in a language \mathcal{L}_m^2 such that $G_0, \dots, G_n \vdash F$ in the intuitionistic propositional calculus (see [21]) and let f be an arbitrary mapping from the set of propositional variables in \mathcal{L}_m^2 into \mathfrak{F}_m^2 . Then $\vdash S^f G_i (i < n)$ in $S(\mathcal{L}_m^v, \Sigma)$ implies that $\vdash S^f F$ in $S(\mathcal{L}_m^v, \Sigma)$, where $S^f G$ denotes the result of simultaneous replacement of all occurrences of the propositional variables v_1, \dots, v_n, \dots in G by $f(v_1), \dots, f(v_n), \dots$, respectively.

For example, the following formulas are provable in $S(\mathcal{L}_m^\nu, \Sigma)$ (see [21], p. 338).

$$(a_{10}) \quad (F \rightarrow F).$$

$$(a_{11}) \quad (F_0 \rightarrow (F_1 \rightarrow F_0)).$$

$$(a_{12}) \quad ((F_0 \rightarrow (F_1 \rightarrow F_2)) \rightarrow ((F_0 \rightarrow F_1) \rightarrow (F_0 \rightarrow F_2))).$$

THEOREM 1.1. *In any formal system $S(\mathcal{L}_m^\nu, \Sigma)$*

- (i) $\vdash (\bar{D}_\alpha F \rightarrow \bar{D}_\beta F)$ for all $\alpha \geq \beta$, $\alpha, \beta \notin \text{Lim}$.
- (ii) $\vdash (E_\alpha \rightarrow E_\beta)$ and $\vdash E_\nu$ for all $\alpha \leq \beta$.
- (iii) $\vdash (F_0 \rightarrow F_1)$ iff $\vdash (\bar{D}_\alpha F_0 \rightarrow \bar{D}_\alpha F)$ for all $\alpha \leq \nu$, $\alpha \notin \text{Lim}$.
- (iv) $\vdash (\bar{D}_\alpha (\bigvee_{i < l} F_i) \leftrightarrow (\bigvee_{i < l} \bar{D}_\alpha F_i))$ and $\vdash (\bar{D}_\alpha (\bigwedge_{i < l} F_i) \leftrightarrow (\bigwedge_{i < l} \bar{D}_\alpha F_i))$.
- (v) For every formula F , $\vdash (\bar{D}_\nu^0 F \rightarrow F)$ and $\vdash (F \rightarrow \bar{D}_1 F)$.
- (vi) $\vdash ((E_1 \rightarrow F) \leftrightarrow \neg \neg F)$ and $\vdash (\bar{D}_1 F \leftrightarrow \neg \neg F)$.
- (vii) $\vdash ((F \wedge (\bigvee_{i < l} F_i)) \leftrightarrow (\bigvee_{i < l} (F \wedge F_i)))$.
- (viii) $\vdash (\neg (\bigvee_{i < l} F_i) \leftrightarrow (\bigwedge_{i < l} \neg F_i))$.

Proof. We consider, for instance, the case $\nu \geq m$. The proofs for the case $\nu < m$ are similar and the axiom (b_0) is used instead of the rule (R_ν) and the axiom (b'_9) .

Proof of (i): $1^\circ \vdash ((\bar{D}_\alpha F \wedge E_\alpha) \rightarrow F)$ (by b'_9).

$2^\circ \vdash (\bar{D}_\beta (\bar{D}_\alpha F \wedge E_\alpha) \rightarrow \bar{D}_\beta F)$ (1° , (RD) and (b_0)).

$3^\circ \vdash (\bar{D}_\beta (\bar{D}_\alpha F \wedge E_\alpha) \leftrightarrow (\bar{D}_\beta \bar{D}_\alpha F \wedge \bar{D}_\beta E_\alpha))$ (by (b_4)).

$4^\circ \vdash ((\bar{D}_\beta \bar{D}_\alpha F \wedge \bar{D}_\beta E_\alpha) \leftrightarrow \bar{D}_\alpha F)$ for $\beta \leq \alpha$ ((b_7) and (b_1)).

$5^\circ \vdash (\bar{D}_\alpha F \rightarrow \bar{D}_\beta F)$ for $\beta \leq \alpha$ (3° , 4° , (a_1) and 2°).

Proof of (ii). $1^\circ \vdash ((\bar{D}_\alpha E_\beta \wedge E_\alpha) \rightarrow E_\beta)$ (b'_9).

$2^\circ \vdash ((\bar{D}_\alpha E_\beta \wedge E_\alpha) \leftrightarrow E_\alpha)$ (by (I.C.))

$3^\circ \vdash ((\bar{D}_\alpha E_\beta \wedge E_\alpha) \leftrightarrow E_\alpha)$ for $\alpha \leq \beta$ (2° , (b_1) , (I.C.)).

$4^\circ \vdash (E_\alpha \rightarrow E)$ for $\alpha \leq \beta$ (1° , 3° , (a_1)).

Proof of $\vdash E_\nu$: $1^\circ \vdash (E_\alpha \rightarrow E_\nu)$ (by above).

$2^\circ \vdash ((\bar{D}_\alpha \bar{D}_\beta E_\nu \wedge E_\alpha) \rightarrow E_\nu)$ (by 1° and (I.C.)) for any $\alpha \leq \nu$, $\alpha \notin \text{Lim}$.

$3^\circ \vdash (\bar{D}_\beta E_\nu \rightarrow E_\nu)$ (2° , (R_ν)).

$4^\circ \vdash E_\nu$ (3° , (b_1) , modus ponens).

(iii) The necessity is clear (by (RD) and (b_0)).

Now we prove the sufficiency:

$1^\circ \vdash (\bar{D}_\alpha F_0 \rightarrow \bar{D}_\alpha F_1)$ for all $\alpha \leq \nu$, $\alpha \notin \text{Lim}$. (by assumption).

$2^\circ \vdash ((\bar{D}_\alpha F_0 \wedge E_\alpha) \rightarrow (\bar{D}_\alpha F_1 \wedge E_\alpha))$ (by 1° , (I.C.)).

$3^\circ \vdash ((\bar{D}_\alpha F_1 \wedge E_\alpha) \rightarrow F_1)$ (by (b'_9)).

4° $\vdash ((\bar{D}_\alpha F_0 \wedge E_\alpha) \rightarrow F_1)$ for all $\alpha \leq \nu$, $\alpha \notin \text{Lim}$. (2°, 3°, (a₁)).

5° $\vdash (F_0 \rightarrow F_1)$ (4°, (R_ν)).

(iv) We prove the first part of (iv). Let us write $F = (\bigvee_{i < l} F_i)$ and $G = (\bigvee_{i < l} \bar{D}_\alpha F_i)$.

1° $\vdash (F_i \rightarrow F)$ (by (a₆)).

2° $\vdash (\bar{D}_\alpha F_i \rightarrow \bar{D}_\alpha F)$ (1° and (iii)).

3° $\vdash (\bigwedge_{i < l} (\bar{D}_\alpha F_i \rightarrow D_\alpha F))$ (2°, conjunction).

4° $\vdash ((\bigvee_{i < l} \bar{D}_\alpha F_i) \rightarrow \bar{D}_\alpha F)$ or $\vdash (G \rightarrow \bar{D}_\alpha F)$ (3°, (a₄), modus ponens).

5° $\vdash (\bar{D}_\beta F_i \rightarrow (\bar{D}_\beta E_{\alpha-1} \vee \bar{D}_\beta \bar{D}_\alpha F_i))$ for all $\beta \leq \nu$, $\beta \notin \text{Lim}$. (by (b₁) if $\beta < \alpha$ and by (b₇), (ii) if $\beta \geq \alpha$).

6° $\vdash (F_i \rightarrow (E_{\alpha-1} \vee \bar{D}_\alpha F_i))$ (5°, (b₄) and (iii)).

7° $\vdash (\bar{D}_\alpha F_i \rightarrow G)$ (a₆).

8° $\vdash (\bar{D}_\alpha F_i \rightarrow \bar{D}_\beta G)$ for all $\beta \leq \nu$, $\beta \notin \text{Lim}$. (7°, (iii), (b₇)).

9° $\vdash (F_i \rightarrow (E_{\alpha-1} \vee \bar{D}_\beta G))$ (6°, 8°) for all $i < l$.

10° $\vdash ((\bigvee_{i < l} F_i) \rightarrow (E_{\alpha-1} \vee \bar{D}_\beta G))$ or $\vdash (F \rightarrow (E_{\alpha-1} \vee \bar{D}_\beta G))$ (9°, conjunction, (a₄) and modus ponens).

11° $\vdash (\bar{D}_\alpha F \rightarrow \bar{D}_\beta G)$ (10°, (iii), (b₂), (b₄) and (b₇)).

12° $\vdash (\bar{D}_\beta \bar{D}_\alpha F \rightarrow \bar{D}_\beta G)$ for all $\beta \leq \nu$, $\beta \notin \text{Lim}$. (11°, (iii) and (b₇)).

13° $\vdash (\bar{D}_\alpha F \rightarrow G)$ (12° and (iii)).

14° $\vdash (\bar{D}_\alpha F \leftrightarrow G)$ or $\vdash (\bar{D}_\alpha (\bigvee_{i < l} F_i) \leftrightarrow (\bigvee_{i < l} \bar{D}_\alpha F_i))$ (by 13°, 4°).

(v) easily follows from (iii).

(vi) First we show that $\vdash (\neg \neg F \rightarrow \bar{D}_1 F)$.

1° $\vdash (\bar{D}_\alpha (\neg \neg F) \leftrightarrow \bar{D}_\alpha \neg \neg \bar{D}_1 F)$ for all $\alpha \leq \nu$, $\alpha \notin \text{Lim}$. (by (b₇) and (b₈)).

2° $\vdash (\neg \neg F \leftrightarrow \neg \neg \bar{D}_1 F)$ (1° and (iii)).

3° $\vdash (\neg \neg \bar{D}_1 F \leftrightarrow \bar{D}_1 F)$ ((b₃) and (I.C.)).

4° $\vdash (\neg \neg F \leftrightarrow \bar{D}_1 F)$ (2° and 3°).

Proof of $\vdash ((E_1 \rightarrow F) \leftrightarrow \neg \neg F)$.

1° $\vdash ((E_1 \rightarrow F) \rightarrow (\neg \neg E_1 \rightarrow \neg \neg F))$ (by (I.C.)).

2° $\vdash (((E_1 \rightarrow F) \rightarrow \neg \neg E_1) \rightarrow ((E_1 \rightarrow F) \rightarrow \neg \neg F))$ ((a₁₂), 1° and modus ponens).

3° $\vdash \neg \neg E_1$ (by $\vdash (\neg \neg E_1 \leftrightarrow \bar{D}_1 E_1)$), ((b₁), modus ponens).

4° $\vdash ((E_1 \rightarrow F) \rightarrow \neg \neg E_1)$ (3°, (a₁₁), modus ponens).

5° $\vdash ((E_1 \rightarrow F) \rightarrow \neg \neg F)$ (2°, 4°, modus ponens).

6° $\vdash ((\bar{D}_1 F \wedge E_1) \rightarrow F)$ (b₉).

7° $\vdash (\bar{D}_1 F \rightarrow (E_1 \rightarrow F))$ (6°, (a₃), modus ponens).

$8^\circ \vdash (\neg\neg F \rightarrow (E_1 \rightarrow F))$ (by $\vdash (\neg\neg F \leftrightarrow \bar{D}_1 F)$ and 7°).

$9^\circ \vdash (\neg\neg F \leftrightarrow (E_1 \rightarrow F))$ (5° , 8°).

The proof of (vii) is easy, and therefore omitted.

(viii) $1^\circ \vdash (F_i \rightarrow (\bigvee_{i < l} F_i))$ (a_6).

$2^\circ \vdash (\neg(\bigvee_{i < l} F_i) \rightarrow \neg F_i)$ for all $i < l$ (1° and (I.C.)).

$3^\circ \vdash (\neg(\bigvee_{i < l} F_i) \rightarrow (\bigwedge_{i < l} \neg F_i))$ (2° , conjunction, (a_5) and modus ponens).

$4^\circ \vdash ((\bigwedge_{i < l} \neg F_i) \rightarrow \neg F_i)$ (a_7).

$5^\circ \vdash ((F_i \wedge (\bigwedge_{i < l} \neg F_i)) \rightarrow (F_i \wedge \neg F_i))$ (4° , (I.C.)).

$6^\circ \vdash ((F_i \wedge (\bigwedge_{i < l} \neg F_i)) \rightarrow (E_1 \wedge \neg E_1))$ ((a_8) , (a_1), 5° and modus ponens)

for all $i < l$.

$7^\circ \vdash ((\bigvee_{i < l} (F_i \wedge (\bigwedge_{i < l} \neg F_i))) \rightarrow (E_1 \wedge \neg E_1))$ (6° conjunction, and (a_4)).

$8^\circ \vdash (((\bigvee_{i < l} F_i) \wedge (\bigwedge_{i < l} \neg F_i)) \rightarrow (E_1 \wedge \neg E_1))$ (7° , (vii)).

$9^\circ \vdash ((\bigvee_{i < l} F_i) \wedge (\bigwedge_{i < l} \neg F_i))$ (8° , (a_9), modus ponens).

$10^\circ \vdash ((\bigwedge_{i < l} \neg F_i) \rightarrow \neg(\bigvee_{i < l} F_i))$ (9° , $\vdash \neg(p \wedge q) \leftrightarrow (q \rightarrow \neg p)$ and (I.C.)).

$11^\circ \vdash (\neg(\bigvee_{i < l} F_i) \leftrightarrow (\bigwedge_{i < l} \neg F_i))$ (3° and 10°).

The following definition will be useful in the sequel: $\mathfrak{F}_{m_0}^\nu$ is the least class of formulas in \mathcal{L}_m^ν satisfying the following:

1° for every $F \in \mathfrak{F}_m^\nu$ $\neg F$ and $\bar{D}_a F \in \mathfrak{F}_{m_0}^\nu$,

2° if $F_i \in \mathfrak{F}_{m_0}^\nu$ for all $i < l < m$, then $(\bigwedge_{i < l} F_i)$ and $(\bigvee_{i < l} F_i)$ belong to $\mathfrak{F}_{m_0}^\nu$,

3 if $F_0, F_1 \in \mathfrak{F}_{m_0}^\nu$, then $(F_0 \rightarrow F_1) \in \mathfrak{F}_{m_0}^\nu$.

THEOREM 1.2. *In every formal system $\mathcal{S}(\mathcal{L}_m^\nu, \Sigma)$*

- (i) $\vdash_{\{(F_0 \vee \neg F_0)\}} ((F_0 \rightarrow F_1) \leftrightarrow (\neg F_0 \vee F_1))$.
- (ii) *If $F \in \mathfrak{F}_{m_0}^\nu$, $\vdash (F \leftrightarrow \bar{D}_a F)$ for all $a \leq \nu$, $a \notin \text{Lim}$.*
- (iii) *If $F \in \mathfrak{F}_{m_0}^\nu$, $\vdash (F \vee \neg F)$.*
- (iv) *If $F_i \in \mathfrak{F}_{m_0}^\nu$ for $i < l < m$, then*

$$\vdash (\neg(\bigwedge_{i < l} F_i) \leftrightarrow (\bigvee_{i < l} \neg F_i)).$$

Proof of (ii). By virtue of (iii) in (1.1) and of axiom (a_1), it is easy to show that if $\vdash (F \leftrightarrow \bar{D}_1 F)$, then $\vdash (F \leftrightarrow \bar{D}_a F)$. Hence it is sufficient to prove that $\vdash (F \leftrightarrow \bar{D}_1 F)$. This will be proved by induction on the length of the formulas in $\mathfrak{F}_{m_0}^\nu$.

Case 1: If $F = \bar{D}_\alpha F_0$ for a certain $\alpha \leq \nu$, then we have $\vdash (\bar{D}_1 \bar{D}_\alpha F_0 \leftrightarrow \bar{D}_\alpha F_0)$ by axiom (b₇).

Case 2: If $F = \neg F_0$, then we have $\vdash (\bar{D}_1 \neg F_0 \leftrightarrow \neg \neg \neg F_0)$ (by (vi) of (1.1)) and hence $\vdash (\bar{D}_1 \neg F_0 \leftrightarrow \neg F_0)$ (by $\vdash (\neg p \leftrightarrow \neg \neg \neg p)$).

Case 3. Assume that $F = (F_0 \rightarrow F_1)$.

1° $\vdash (F_0 \leftrightarrow \bar{D}_1 F_0)$ (by the induction assumption).

2° $\vdash (F_1 \leftrightarrow \bar{D}_1 F_1)$ (by the induction assumption).

3° $\vdash ((\bar{D}_1 F_0 \rightarrow \bar{D}_1 F_1) \rightarrow (\bar{D}_1 F_0 \rightarrow \bar{D}_1 F_1))$ ((a₁₀)).

4° $\vdash ((\bar{D}_1 F_0 \rightarrow \bar{D}_1 F_1) \rightarrow (F_0 \rightarrow F_1))$ (by 3°, 1°, 2° and (I.C.)).

5° $\vdash (\bar{D}_1 (F_0 \rightarrow F_1) \rightarrow (\bar{D}_1 F_0 \rightarrow \bar{D}_1 F_1))$ (axiom (b₆)).

6° $\vdash (\bar{D}_1 F \rightarrow F)$ (4° and 5°).

7° $\vdash (\bar{D}_1 F \leftrightarrow F)$ (6° and (v) of Theorem 1.1).

Case 4: Assume that $F = (\bigvee_{i < l} F_i)$, where $F_i \in \mathfrak{F}_{m_0}^r$ and

1° $\vdash (\bar{D}_1 F_i \leftrightarrow F_i)$ for all $i < l < m$ (by the induction assumption).

2° $\vdash ((\bigvee_{i < l} \bar{D}_1 F_i) \leftrightarrow (\bigvee_{i < l} F_i))$ (by 1°).

3° $\vdash (\bar{D}_1 F \leftrightarrow (\bigvee_{i < l} \bar{D}_1 F_i))$ (2°, (iv) of (1.1)).

4° $\vdash (\bar{D}_1 F_i \rightarrow F)$ (1°, (a₆) and (a₁)) for all $i < l$.

5° $\vdash (\bar{D}_1 F \rightarrow F)$ (4°, conjunction, (a₄), 3° and (a₁)).

6° $\vdash (\bar{D}_1 F \leftrightarrow F)$ (5°, (v) of (1.1)).

Case 5: If $F = (\bigwedge_{i < l} F_i)$, then the proof is similar to that in case 4.

The proofs of (ii), (iii) and (iv) are easy and therefore left to the reader.

THEOREM 1.3. *In every formal system $S(\mathcal{L}_m^r, \Sigma)$, if $\vdash (F \rightarrow (\bar{D}_\alpha F_0 \rightarrow \bar{D}_\alpha F_1))$ for all $\alpha \leq a_0$, $\alpha \notin \text{Lim.}$, then $\vdash (F \rightarrow \bar{D}_{a_0} (F_0 \rightarrow F_1))$.*

Moreover, if $a_0 < m$, then $\vdash (\bar{D}_{a_0} (F_0 \rightarrow F_1) \leftrightarrow (\bigwedge_{\alpha \leq a_0, \alpha \notin \text{Lim.}} (\bar{D}_\alpha F_0 \rightarrow \bar{D}_\alpha F_1)))$.

Proof. To prove this theorem, we write

$$G = ((F_0 \rightarrow F_1) \vee ((\bar{D}_1 F \vee \bar{D}_{a_0} (F_0 \rightarrow F_1)) \wedge E_{a_0})).$$

1° $\vdash (F \rightarrow (\bar{D}_\alpha F_0 \rightarrow \bar{D}_\alpha F_1))$ for all $\alpha \leq a_0$, $\alpha \notin \text{Lim.}$ (by hypothesis).

2° $\vdash (\bar{D}_1 F \rightarrow (\bar{D}_\alpha F_0 \rightarrow \bar{D}_\alpha F_1))$ (1°, (iii) of (1.1), (b₆) and (b₇)).

3° $\vdash (\bar{D}_{a_0} (F_0 \rightarrow F_1) \rightarrow \bar{D}_\alpha (F_0 \rightarrow F_1))$ for $\alpha \leq a_0$ (by (ii) of Theorem 1.1).

4° $\vdash (\bar{D}_\alpha G \leftrightarrow (\bar{D}_\alpha (F_0 \rightarrow F_1) \vee \bar{D}_1 F \vee \bar{D}_{a_0} (F_0 \rightarrow F_1)))$ for $\alpha \leq a_0$ (by (b₄), (b₁), (b₇)).

5° $\vdash (\bar{D}_\alpha G \rightarrow (\bar{D}_\alpha (F_0 \rightarrow F_1) \vee \bar{D}_1 F))$ (by 3°, 4°, (a₁)).

6° $\vdash (\bar{D}_\alpha G \rightarrow (\bar{D}_\alpha F_0 \rightarrow \bar{D}_\alpha F_1))$ for $\alpha \leq a_0$ (2°, (b₆), 5°, (a₁)).

7° $\vdash (\bar{D}_\alpha G \leftrightarrow \bar{D}_\alpha (F_0 \rightarrow F_1))$ ((b₂), (b₄), (b₆)) for $\alpha > a_0$.

- $8^\circ \vdash (\bar{D}_\alpha G \rightarrow (\bar{D}_\alpha F_0 \rightarrow \bar{D}_\alpha F_1))$ ((b_6) , (a_1) and 7°).
 $9^\circ \vdash (\bar{D}_\alpha (G \wedge F_0) \rightarrow \bar{D}_\alpha F_1)$ for all $\alpha \leq \nu$, $\alpha \notin \text{Lim}$. (6° , 8° , (a_2) , (b_5)).
 $10^\circ \vdash ((G \wedge F_0) \rightarrow F_1)$ (9° and (iii) of Theorem 1.1).
 $11^\circ \vdash (G \rightarrow (F_0 \rightarrow F_1))$ (10° , (a_3)).
 $12^\circ \vdash (\bar{D}_{\alpha_0} G \rightarrow \bar{D}_{\alpha_0} (F_0 \rightarrow F_1))$ (11° , (iii) of Theorem 1.1).
 $13^\circ \vdash (\bar{D}_1 F \rightarrow \bar{D}_{\alpha_0} G)$ (4° for $\alpha = \alpha_0$).
 $14^\circ \vdash (F \rightarrow \bar{D}_{\alpha_0} (F_0 \rightarrow F_1))$ (12° , 13° , (\forall) of Theorem 1.1, (a_1)).

The second part of the theorem is clear.

THEOREM 1.4. *In every formal system $S(\mathcal{L}_m^\nu, \Sigma)$*

- (i) $\vdash ((\bigvee F \dots F \dots) \leftrightarrow F)$ and $\vdash ((\bigwedge F \dots F \dots) \leftrightarrow F)$.
(ii) If $\alpha = \Sigma \langle \alpha_i : i < l \rangle$ and $\beta_i = \Sigma \langle \alpha_j : j < i \rangle$ for all $i < l$, then

$$\vdash \left(\left(\bigvee_{i < \alpha} F_i \right) \leftrightarrow \left(\bigvee_{i < 1} \left(\bigvee_{k < \alpha_i} F_{\beta_i + k} \right) \right) \right)$$

and

$$\vdash \left(\left(\bigwedge_{i < \alpha} F_i \right) \leftrightarrow \left(\bigwedge_{i < 1} \left(\bigwedge_{k < \alpha_i} F_{\beta_i + k} \right) \right) \right).$$

- (iii) If $\{F_i : i < l\} = \{F'_j : j < l'\}$, then

$$\vdash \left(\left(\bigvee_{i < l} F_i \right) \leftrightarrow \left(\bigvee_{j < l'} F'_j \right) \right)$$

and

$$\vdash \left(\left(\bigwedge_{i < l} F_i \right) \leftrightarrow \left(\bigwedge_{j < l'} F'_j \right) \right).$$

- (iv) Let $\alpha = \alpha_0 + \alpha_1$. Then

$$\vdash \left(\left(\bigwedge_{i < \alpha_1} F_{\alpha_0 + i} \right) \rightarrow \left(\left(\bigwedge_{i < \alpha} F_i \right) \leftrightarrow \left(\bigwedge_{i < \alpha_0} F_i \right) \right) \right)$$

and

$$\vdash \left(\neg \left(\bigvee_{i < \alpha_1} F_{\alpha_0 + i} \right) \rightarrow \left(\left(\bigvee_{i < \alpha} F_i \right) \leftrightarrow \left(\bigvee_{i < \alpha_0} F_i \right) \right) \right).$$

The proof of (i)-(iii) is not difficult.

To prove (iv) we use the following statements on intuitionistic propositional calculi (see [14]):

If $\vdash (p \leftrightarrow (p \wedge p_1))$, then $\vdash (p_1 \rightarrow (p_0 \leftrightarrow p))$, and if $\vdash (p \leftrightarrow (p_0 \vee p_1))$, then $\vdash (\neg p_1 \rightarrow (p_0 \leftrightarrow p))$.

DEFINITION 1.5. Let Γ be a set of formulas in \mathcal{L}_m^ν . Then we shall write $\Gamma \vdash F$ in $S(\mathcal{L}_m^\nu, \Sigma)$ either if $\vdash F$ in $S(\mathcal{L}_m^\nu, \Sigma)$, or if there are formulas G_0, \dots, G_l, \dots ($i < l < m$) in Γ such that $\vdash ((\bigwedge_{i < l} G_i) \rightarrow F)$ in $S(\mathcal{L}_m^\nu, \Sigma)$.

By $D\Gamma$ we shall denote the set $\Gamma \cup \{\bar{D}_i^\nu G : G \in \Gamma\}$.

THEOREM 1.6. *In every formal system $S(\mathcal{L}_m^\nu, \Sigma)$*

- (i) if $\Gamma \vdash F_0$ and $\Gamma \vdash (F_0 \rightarrow F_1)$, then $\Gamma \vdash F_1$;

- (ii) if $\vdash F$, then $D\Gamma \vdash \bar{D}_\alpha F$ for all $\alpha \leq \nu$ and $\alpha \notin \text{Lim.}$,
 (iii) if $\Gamma \vdash F_i$, $i < l < m$, then $\Gamma \vdash (\bigwedge_{i < l} F_i)$,
 (iv) in the case $\nu \geq m$, we assume in addition that $\bar{l} < m$.

Then $\Gamma \vdash ((\bar{D}_\alpha F \wedge E_\alpha) \rightarrow F')$ for all $\alpha \leq \nu$, $\alpha \notin \text{Lim.}$ implies $\Gamma \vdash (F \rightarrow F')$.

Proof. The proofs of (i)-(iii) are left to the reader. To prove (iv) assume that $\langle G_i: i < l \rangle$ is the sequence of all formulas from Γ and let $G = (\bigwedge_{i < l} G_i)$. According to Definition 1.5, we see that

- 1° $\vdash (G \rightarrow ((\bar{D}_\alpha F \wedge E_\alpha) \rightarrow F'))$;
 2° $\vdash ((\bar{D}_\alpha F \wedge E_\alpha) \rightarrow (G \rightarrow F'))$ for all $\alpha \leq \nu$, $\alpha \notin \text{Lim.}$ (by 1°, (a₂) and (a₃));
 3° $\vdash (F \rightarrow (G \rightarrow F'))$ (2° and the rule (R_v));
 4° $\vdash (G \rightarrow (F \rightarrow F'))$ (3°, (a₂), (a₃));
 5° $\Gamma \vdash (F \rightarrow F')$ (4° and Definition 1.5).

THEOREM 1.7. In any system $S(\mathcal{L}_m^\nu, \Sigma)$

- (i) $D\Gamma \vdash F$ iff $\vdash_{\Gamma} F$ (In the case $\nu \geq m$, we assume that $\bar{l} < m$).
 (ii) If $\Gamma = \langle G_i: i < l \rangle$, where $l < m$, then

$$\vdash_{\Gamma} F \text{ iff } \vdash (\bar{D}_\nu (\bigwedge_{i < l} G_i) \rightarrow F).$$

Proof. We prove only (i), since (ii) easily follows from (i). It is clear that $D\Gamma \vdash F$ implies $\vdash_{\Gamma} F$. Suppose now that $\vdash_{\Gamma} F$ and let $\langle G_i: i \leq l \rangle$ be a proof of F from Γ in $S(\mathcal{L}_m^\nu, \Sigma)$. We shall show by transfinite induction that $D\Gamma \vdash G_i$ for all $i \leq l$.

Suppose that we have proved $D\Gamma \vdash G_i$ for $i < i_0$. We have to show that $D\Gamma \vdash G_{i_0}$. Consider the following three possible cases:

- 1° If G_{i_0} either is an axiom or belongs to Γ , then evidently $D\Gamma \vdash G_{i_0}$.
 2° If G_{i_0} is deduced from the preceding formulas by the rules modus ponens, conjunction or (RD), then $D\Gamma \vdash G_{i_0}$ as a consequence of (i)-(iii) in (1.6).

3° Finally let G_{i_0} be deduced from the preceding formulas by the rule (R_v). Thus G_{i_0} has the form $G_{i_0} = (F \rightarrow F')$. In this case we had to assume that $\bar{l} < m$ and hence $\bar{\bar{l}} < m$. On account of (iv) in (1.6), it follows that

$$D\Gamma \vdash G_{i_0}.$$

Therefore we have proved that $D\Gamma \vdash F$, since $G_l = F$. This completes the proof.

The following theorem is evident.

THEOREM 1.8. In any formal system $S(\mathcal{L}_m^\nu, \Sigma)$

- (i) $\vdash \left(\left(\bigwedge_{i < l} (F_i \leftrightarrow F'_i) \right) \rightarrow \left(\left(\bigvee_{i < l} F_i \right) \leftrightarrow \left(\bigvee_{i < l} F'_i \right) \right) \right)$;

- (ii) $\vdash \left(\left(\bigwedge_{i < l} (F_i \leftrightarrow F'_i) \right) \rightarrow \left(\left(\bigwedge_{i < l} F_i \right) \leftrightarrow \left(\bigwedge_{i < l} F'_i \right) \right) \right)$;
- (iii) $\vdash \left(\left(\bigwedge_{i < 2} (F_i \leftrightarrow F'_i) \right) \rightarrow \left((F_0 \rightarrow F_1) \leftrightarrow (F'_0 \rightarrow F'_1) \right) \right)$;
- (iv) $\vdash \left((F \leftrightarrow F') \rightarrow (\neg F \leftrightarrow \neg F') \right)$;
- (v) $\vdash (F \leftrightarrow F') \text{ iff } \vdash (\bar{D}_a F \leftrightarrow \bar{D}_a F') \text{ for } a \leq \nu, a \notin \text{Lim.}$

2. Completeness of some formal systems based on languages \mathcal{L}_m^ν

1. Definition and completeness conditions. Consider a ν -valued language \mathcal{L}_m^ν . The elements of every ν -element Post algebra $P_\nu = \langle \{e_a : a \leq \nu\}; \text{B2} \rangle$, where B2 is the two-element Boolean algebra, can be considered as the truth values associated with the language \mathcal{L}_m^ν . The value $1 = e_\nu$ is called the *truth* and the value $0 = e_0$ — the *falsity*.

Let $\mathcal{E}_\nu = \{e_a : a \leq \nu\}$. Then every mapping $v \in \mathcal{E}_\nu^\nu$, where \mathcal{V} denotes the set of all propositional variables in \mathcal{L}_m^ν , is called a *valuation of the language \mathcal{L}_m^ν* . Each valuation v can be extended to a mapping v^* , which assigns to every formula in \mathcal{L}_m^ν a truth value in \mathcal{E}_ν . The mapping v^* is uniquely defined by the following recursive process:

- (V) $v^*(E_a) = e_a$ for every propositional constant E_a ,
- $v^*\left(\bigvee_{i < l} F_i\right) = \bigcup_{i < l} v^*F_i; \quad v^*\left(\bigwedge_{i < l} F_i\right) = \bigcap_{i < l} v^*F_i,$
- $v^*(F_0 \rightarrow F_1) = v^*F_0 \Rightarrow v^*F_1,$
- $v^*(\neg F) = -v^*F \quad \text{and} \quad v^*(\bar{D}_a F) = D_a(v^*F).$

A set Γ of formulas in \mathcal{L}_m^ν is said to be *satisfiable* if there is a valuation v such that $v^*F = 1$ for every formula F in Γ and then we say that v satisfies Γ .

A set Γ of formulas is said to be *semantically consistent* provided every subset of Γ having power less than m is satisfiable.

A formula F is said to be a *semantic consequence* of a set Γ of formulas, and we write $\vdash_{\Gamma} F$, iff every valuation satisfying Γ satisfies also the formula F . In the case $\Gamma = \emptyset$ such a formula F is called *valid*, i.e. $v^*F = 1$ for every valuation.

In general as in the case of the two-valued logic — we can interpret the language \mathcal{L}_m^ν in an arbitrary m -complete Post algebra P_ν of type ν which is a pseudo-Boolean algebra. Every valuation $v \in P_\nu^\nu$ can be uniquely extended to a mapping v^* satisfying conditions (V). If $v^*F = 1$ for all formulas F in Γ , then we say that v satisfies the set Γ of formulas or that Γ is satisfiable in P_ν . If $v^*F = 1$ for every $v \in P_\nu^\nu$, then F is said to be *valid* in P_ν .

A formal system $S(\mathcal{L}_m^\nu, \Sigma)$ is said to be *complete* if for every formula F , $\vdash F$ in $S(\mathcal{L}_m^\nu, \Sigma)$ iff $\models_{\Sigma} F$.

Let Γ be a set of formulas in \mathcal{L}_m^ν . Γ is said to be *formally consistent* with respect to $S(\mathcal{L}_m^\nu, \Sigma)$, provided there exists a formula F such that non $\vdash F$ in $S(\mathcal{L}_m^\nu, \Sigma)$.

Note that in an infinitary language a formally consistent set can be non-satisfiable (see [11]).

Given a formal system $S(\mathcal{L}_m^\nu, \Sigma)$ and a set of formulas Γ formally consistent with respect to $S(\mathcal{L}_m^\nu, \Sigma)$, we introduce an equivalence relation \sim_{Γ} between formulas in \mathfrak{F}_m^ν defined as follows:

$$F \sim_{\Gamma} F' \text{ iff } \vdash_{\Gamma} (F \leftrightarrow F') \text{ in } S(\mathcal{L}_m^\nu, \Sigma).$$

By $|F|$ we denote the equivalence class represented by a formula F . Let $P(S(\mathcal{L}_m^\nu, \Sigma); \Gamma)$ be the set of all equivalence classes in \mathfrak{F}_m^ν . On account of (iii) in Theorem 1.8 and of (iii) in Theorem 1.1 we can define in $P(S(\mathcal{L}_m^\nu, \Sigma); \Gamma)$ the following ordering relation and D -operations:

$$|F| \leq |F'| \text{ iff } \vdash_{\Gamma} F \rightarrow F' \text{ in } S(\mathcal{L}_m^\nu, \Sigma),$$

$$D_{\alpha}|F| = |\bar{D}_{\alpha}F|.$$

Using these definitions and the groups (A) and (B) of axioms in Section 1 we can show, on account of Theorem 1.6, Part I, that the set $P(S(\mathcal{L}_m^\nu, \Sigma); \Gamma)$ is an m -complete Post algebra of type ν with the constants $0 = |E_0| \leq \dots \leq |E_{\alpha}| \leq \dots \leq |E_{\nu}| = 1$, and that the following equations hold:

$$\bigcap_{i < l} |F_i| = |\bigwedge_{i < l} F_i|, \quad \bigcup_{i < l} |F_i| = |\bigvee_{i < l} F_i|,$$

$$|F_0| \Rightarrow |F_1| = |F_0 \rightarrow F_1|, \quad -|F| = |\neg F|,$$

where $0 < l < m$. The proofs of these statements are analogous to that in the case of the classical logic (cf. [21], Chapter VI).

We call

$$(1) \quad \langle F_{ij}; i < l, j < l \rangle, \quad 0 < l < m,$$

a doubly indexed m -system of formulas. By a choice sequence for this m -system we understand any set containing at least one formula from each row of this m -system.

We say that a doubly indexed m -system is *contradictory* if its every sequence contains at least one certain contradictory pair formulas F and $\neg F$.

A doubly indexed m -system is said to be *contradictory* for a formula F_0 if it contains the formula F_0 and every choice sequence for it contains either F_0 or else a contradictory pair of formulas F and $\neg F$.

DEFINITION. Let $P = \langle P; e_\alpha: \alpha \leq \nu; B \rangle$ be an m -complete Post algebra of type ν . We say that a set \mathfrak{G} of elements m -generates the Post algebra P if every m -complete Post subalgebra of type ν of P containing the set \mathfrak{G} and all elements e_α coincides with P . The elements in \mathfrak{G} are called m -generators.

LEMMA 2.1. Let \mathcal{L}_m^ν be a language having π propositional variables and let $S(\mathcal{L}_m^\nu, \Sigma)$ be an arbitrary formal system, where π is an infinite cardinal number and Σ is a set of valid formulas. Then every m -complete Post algebra P generated by a set \mathfrak{G} such that $\mathfrak{G} = \pi$ is isomorphic to a Post algebra $P(S(\mathcal{L}_m^\nu, \Sigma); \Gamma)$ with a suitable set Γ of formulas formally consistent with respect to $S(\mathcal{L}_m^\nu, \Sigma)$. Consequently, P , is an image of $P(S(\mathcal{L}_m^\nu, \Sigma))$ under an m -homomorphism.

The proof of this lemma is easy (cf. [11], p. 59).

THEOREM 2.2. Let \mathcal{L}_m^ν be a language having π propositional variables, where either $\nu < m$, or $m = \omega_0$ and $\nu < \omega_1$, and let Σ be a set of valid formulas in \mathcal{L}_m^ν . Then the following conditions are equivalent:

(i) The formal system $S(\mathcal{L}_m^\nu, \Sigma)$ is complete.

(ii) For every contradictory doubly indexed m -system (1) for a formula F , the condition

$$(2) \quad \vdash (E_1 \rightarrow (\bigvee_{i < l} F_{ij})) \quad \text{in } S(\mathcal{L}_m^\nu, \Sigma) \quad \text{for all } i < l$$

implies that

$$\vdash (E_1 \rightarrow F) \quad \text{in } S(\mathcal{L}_m^\nu, \Sigma).$$

(iii) For every contradictory doubly indexed m -system (1), the condition

$$(3) \quad \vdash (E_1 \rightarrow (\bigvee_{i < l} F)) \quad \text{in } S(\mathcal{L}_m^\nu, \Sigma) \quad \text{for all } 0 < i < l$$

implies that $\vdash \bigwedge_{i < l} \bigvee F_{0j}$ in $S(\mathcal{L}_m^\nu, \Sigma)$.

(iv) The Post algebra $P(S(\mathcal{L}_m^\nu, \Sigma))$ is m -representable.

(v) Every m -complete Post algebra of type ν generated by a set \mathfrak{G} of power π in which every formula from Σ is valid is m -representable.

(v) If Γ is a set of formulas formally consistent with respect to $S(\mathcal{L}_m^\nu, \Sigma)$ and the power of Γ is less than m , then Γ is satisfiable.

Proof. To prove that (i) implies (ii), we take a contradictory m -system (1) for a formula F satisfying condition (2). Then, by the rule (RD), $\vdash \bigvee_{i < l} \bar{D}_1 F_{ij}$, which implies that, for every valuation $\nu, \nu^*(\bigvee_{i < l} \bar{D}_1 F_{ij}) = \bigcup_{i < l} D_1 \nu^* F_{ij} = 1$, where i is an arbitrary ordinal number smaller than l .

Hence

$$1 = \bigcap_{i < l} \bigcup_{j < l} D_1 \nu^* F_{ij} = \bigcup_{\sigma \in l^l} \bigcap_{i < l} D_1 \nu^* F_{i\sigma(i)},$$

because the algebra $B2 = \{0, 1\}$ is infinitely distributive. From this equation it follows that there exists a function $g_0 \in \mathcal{V}^l$ such that $\bigcap_{i < l} D_1 v^* F_{i g_0(i)} = 1$, i.e. $v^* \bar{D}_1 F_{i g_0(i)} = 1$ for every $i < l$. Thus the choice sequence $\langle F_{i g_0(i)} : i < l \rangle$ does not contain any contradictory pair of formulas, and so it must contain the formula F . Hence $v^* \bar{D}_1 F = 1$ for every valuation v . Since the system $S(\mathcal{L}_m^\nu, \Sigma)$ is complete, we infer that $\vdash \bar{D}_1 F$ or equivalently by (vi) of Theorem 1.1, $\vdash (E_1 \rightarrow F)$ in $S(\mathcal{L}_m^\nu, \Sigma)$.

Proof of (ii) \Rightarrow (iii). Take into consideration a contradictory m -system (1) satisfying condition (3). We write $F = \neg \bigvee_{i < l} F_{0j}$ and replace the zero-row $\langle F_{0j} : j < l \rangle$ by the row $\langle F, F_{0j} : j < l \rangle$. Then the resulting m -system for the formula F is clearly contradictory and satisfies condition (2). On account of (ii) it follows that $\vdash (E_1 \rightarrow F)$ or $\vdash (E_1 \rightarrow \neg (\bigvee_{i < l} F_{0j}))$ in $S(\mathcal{L}_m^\nu, \Sigma)$. By (vi) of (1.1) we have $\vdash \neg \neg \neg (\bigvee_{i < l} F_{0j})$, which implies $\vdash \neg (\bigvee_{i < l} F_{0j})$ in $(S\mathcal{L}_m^\nu, \Sigma)$. Accordingly, (iii) follows from (ii).

Proof of (iii) \Rightarrow (iv). Consider the Post algebra $P(S(\mathcal{L}_m^\nu, \Sigma))$. According to (R'_6) , in Corollary 5.12 in Part I, to prove the m -representability of this Post algebra, it is sufficient to show that for every m -system $\langle |F_{ij}| : i < l, j < l \rangle$ such that every choice set $\{|F_{i g(i)}| : i < l\}$, with $g \in \mathcal{V}^l$ contains a contradictory pair of elements $|F_{i g(i)}|$ and $\neg |F_{i g(i)}|$, for some $i < l$, the condition that $\bigcup_{i < l} |F_{ij}|$ is a dense element for $0 < i < l$ implies that $\bigcup_{i \in I} |F_{0j}| = 0$.

Indeed, consider an m -system of formulas which consists of the following rows:

$$\begin{aligned} & \langle F_{ij} : j < l \rangle, \quad i < l, \\ & \langle \neg F_{ij}, \neg F_{i'j'} \rangle \quad \text{for every pair } (i', j') \text{ such that } |F_{ij}| = \neg |F_{i'j'}| \\ & \quad \text{and for every } i < l \text{ and } j < l. \end{aligned}$$

It can easily be seen that this m -system is contradictory and satisfies condition (3), since an element $|F|$ is dense in $P(S(\mathcal{L}_m^\nu, \Sigma))$ iff $\vdash (E_1 \rightarrow F)$ in $S(\mathcal{L}_m^\nu, \Sigma)$. Hence it follows from (iii) that $\vdash \neg (\bigvee_{i < l} F_{0j})$ in $S(\mathcal{L}_m^\nu, \Sigma)$ and consequently that $\bigcup_{i < l} |F_{0j}| = 0$.

Implication (iv) \Rightarrow (v) is a consequence of Lemma 2.1.

Proof of (v) \Rightarrow (vi). Let Γ be a set of formulas formally consistent with respect to $S(\mathcal{L}_m^\nu, \Sigma)$ and let $\bar{\Gamma} < m$. Since by (v), the Post algebra $P(S(\mathcal{L}_m^\nu, \Sigma); \Gamma)$ is m -representable, on account of Corollary 5.16 in Part I, there exists a maximal D_ν -filter \mathcal{V} in $P(S(\mathcal{L}_m^\nu, \Sigma); \Gamma)$ which preserves the l.u.b.'s, the g.l.b.'s and the implications corresponding to all the disjunctions, conjunctions and implications that occur in the formulas from Γ . We know that $P(S(\mathcal{L}_m^\nu, \Sigma); \Gamma)/\mathcal{V} = P_\nu$ is a ν -element Post algebra

and so the mapping ν , such that $\nu V_i = |V_i|_F|_\nu$ for every $V_i \in \mathcal{V}$, is a valuation of \mathcal{L}_m^ν . It is not difficult to show that $\nu^* F = |F|_F|_\nu = 1$ for every formula F in Γ . Hence Γ is satisfiable.

Proof of (vi) \Rightarrow (i). Let F be a formula such that $\text{non} \vdash F$ in $S(\mathcal{L}_m^\nu, \Sigma)$. Then there is an α , $\alpha \leq \nu$, $\alpha \notin \text{Lim}$, such that $\text{non} \vdash \bar{D}_\alpha F$ and hence the set $\Gamma = \{\neg \bar{D}_\alpha F\}$ is formally consistent with respect to $S(\mathcal{L}_m^\nu, \Sigma)$. Therefore, according to (vi), there is a valuation ν such that $\neg \nu^* \bar{D}_\alpha F = 1$ and hence $\nu^* F \neq 1$. Thus if F is a valid formula, then $\vdash F$ in $S(\mathcal{L}_m^\nu, \Sigma)$.

2. Complete formal systems. From the equivalence of (i) to (iv) in Theorem 2.2 and from (ii) in Corollary 5.13, Part I, it follows that

THEOREM 2.3. *The formal system $S(\mathcal{L}_{\omega_1}^\nu)$ and $S(\mathcal{L}_{\omega_0}^\nu)$, with $\nu < \omega_1$, are complete.*

Now we consider the following two sets of formulas:

$$(II_1) \quad (E_1 \rightarrow (\bigvee_{i < l} (\bigwedge_{j < l} F_{ij}))),$$

where $0 < l < m$ and $\langle F_{ij} : i < l, j < l \rangle$ is a contradictory m -system of formulas;

$$(D_1) \quad ((\bigwedge_{i < l} (\bigvee_{j < l} F_{ij})) \rightarrow (\bigvee_{k < 2^l} (\bigwedge_{i < l} F_{i\sigma_k(i)}))),$$

where $\langle \sigma_k : k < 2^l \rangle$ is an established ordering of the set l^l .

We denote by II_m the set $\{II_l : 0 < l < m\}$ and by \mathcal{D}_m the set $\{D_l : 0 < l < m\}$.

The following theorem is an immediate consequence of the equivalence of (i) and (iv) in Theorem 2.2 and of (R') in Corollary 5.12, Part I:

THEOREM 2.4. *The formal system $S(\mathcal{L}_m^\nu, II_m)$, where $\nu < m$, is complete.*

THEOREM 2.5. *If m is a strongly inaccessible cardinal and $\nu < m$, then the formal system $S(\mathcal{L}_m^\nu, \mathcal{D}_m)$ is complete.*

Proof. By virtue of (2.2), it is sufficient to show that condition (ii) holds in $S(\mathcal{L}_m^\nu, \mathcal{D}_m)$. For this purpose we consider an arbitrary contradictory m -system (1) for a formula F satisfying condition (2), i.e. $\vdash (E_1 \rightarrow (\bigvee_{j < l} F_{ij}))$ in $S(\mathcal{L}_m^\nu, \mathcal{D}_m)$, for $i < l$. This implies by the conjunction law, axiom (a₅) and modus ponens that $\vdash (E_1 \rightarrow (\bigwedge_{i < l} (\bigvee_{j < l} F_{ij})))$, which together with a suitable instance of (D₁) yields

$$(4) \quad \vdash (E_1 \rightarrow (\bigvee_{k < 2^l} (\bigwedge_{i < l} F_{i\sigma_k(i)}))).$$

If the choice sequence $\langle F_{i\sigma_k(i)} : i < l \rangle$ contains the formula F , then we have $\vdash (\bigwedge_{i < l} F_{i\sigma_k(i)} \rightarrow F)$. Suppose that the choice sequence $\langle F_{i\sigma_k(i)} : i < l \rangle$ contains a contradictory pair $F_{i_0\sigma_{k_0}(i_0)}$ and $\neg F_{i_0\sigma_{k_0}(i_0)}$. Then we again have

$$(5) \quad \vdash ((\bigwedge_{i < l} F_{i\sigma_k(i)} \rightarrow F).$$

Consequently, (5) holds for all $k < 2^l$. Hence by applying the conjunction law to (5), axiom (a₄) and modus ponens, we obtain $\vdash \left(\left(\bigvee_{k < 2^l} \left(\bigwedge_{i < l} F_{i\sigma_k(i)} \right) \right) \rightarrow F \right)$. This together with (4) yields $\vdash (E_1 \rightarrow F)$ which completes the proof of Theorem 2.5.

DEFINITION. A formal system $S(\mathcal{L}_m^v, \Sigma)$ is said to be *strongly complete* provided for every set Γ of formulas, $\vdash_{\Gamma} F$ in $S(\mathcal{L}_m^v, \Sigma)$ iff $\models_{\Gamma} F$.

In order to prove the next theorem we state the following lemmas, the proofs of which are omitted.

LEMMA 2.6. *An m-complete Post algebra of type A is isomorphic with an m-complete Post field iff for every element $p \neq 1$ there is a maximal m-complete D-filter which does not contain the element p.*

LEMMA 2.7. *Let Γ be a set of formulas in \mathcal{L}_m^v . Then the following two conditions are equivalent:*

- (i) Γ is semantically satisfiable, i.e. satisfiable in a ν -element Post algebra of type ν .
- (ii) There exists an m-complete Post algebra P_ν of type ν isomorphic with an m-complete Post field and a valuation $v \in P_\nu^v$ such that the set $\{v^*(\bar{D}_\alpha F) : F \in \Gamma, \alpha \leq \nu \text{ and } \alpha \notin \text{Lim}\}$ has a non-zero lower bound.

THEOREM 2.8. *Let \mathcal{L}_m^v be a language having n propositional variables, where n is an infinite cardinal number. Let $S(\mathcal{L}_m^v, \Sigma)$ be a formal system with a set Σ of valid formulas in \mathcal{L}_m^v . The following conditions are equivalent:*

- (i) $S(\mathcal{L}_m^v, \Sigma)$ is strongly complete;
- (ii) Every set Γ formally consistent with respect to $S(\mathcal{L}_m^v, \Sigma)$ is satisfiable;
- (iii) If Γ is formally consistent with respect to $S(\mathcal{L}_m^v, \Sigma)$, then $P(S(\mathcal{L}_m^v, \Sigma); \Gamma)$ is isomorphic with a m-complete Post field of type ν ;
- (iv) Every m-complete Post algebra of type ν generated by a set of generators of power n in which every formula in Σ is valid is isomorphic with an m-complete Post field of type ν .

Proof. (i) implies (ii) is obvious. Now we shall show that (ii) implies (iii). Let Γ be a formally consistent set of formulas. Consider the Post algebra $\mathbf{P} = P(S(\mathcal{L}_m^v, \Sigma); \Gamma)$ and an arbitrary element $|F_0| \in \mathbf{P}$ and $|F_0| \neq 1$. The last inequality implies that there exists an $\alpha \leq \nu$ such that $D_\alpha |F_0| = |\bar{D}_\alpha F_0| \neq 1$. Hence, as can easily be seen, the set $\Gamma' = \Gamma \cup \{\neg \bar{D}_\alpha F_0\}$ is formally consistent with respect to $S(\mathcal{L}_m^v, \Sigma)$. By virtue of (ii) there is a valuation v_0 satisfying the set Γ' . We set $\mathcal{V} = \{|F| \in \mathbf{P} : v_0^* F = 1\}$. It is easy to verify that the set \mathcal{V} is well defined and that it is a maximal m-complete D -filter. Moreover, \mathcal{V} does not contain the element $|F_0|$, for $v_0^*(\bar{D}_\alpha F_0) = 0$ and hence $v_0^* F_0 \neq 1$. Consequently, on account of Lemma 2.6, $P(S(\mathcal{L}_m^v, \Sigma); \Gamma)$ is isomorphic with an m-complete Post field.

(iii) implies (iv), by Lemma 2.1 (iv) implies (ii) by Lemma 2.7 applied to the Post algebra $P(S(\mathcal{L}_m^\nu, \Sigma), \Gamma)$.

Finally, we shall show that (ii) implies (i). Let Γ be a set of formulas formally consistent with respect to $S(\mathcal{L}_m^\nu, \Sigma)$. Suppose that $\Vdash_{\Gamma} F$ and, on the contrary, that non $\Vdash_{\Gamma} F$. Then there is an $\alpha_0 \leq \nu$, $\alpha_0 \notin \text{Lim.}$ such that non $\Vdash_{\Gamma} \bar{D}_{\alpha_0} F$, which implies that the set $\Gamma' = \Gamma \cup \{\neg \bar{D}_{\alpha_0} F\}$ is formally consistent. On account of (ii), Γ' is satisfiable, i.e. there is a valuation ν_0 such that $\nu_0^* G = 1$ for all $G \in \Gamma$ and $\nu_0^* \bar{D}_{\alpha_0} F = 0$. Since $\Vdash_{\Gamma} F$ we get $\Vdash_{\Gamma} \bar{D}_{\alpha_0} F$, which implies that $\nu_0^* \bar{D}_{\alpha_0} F = 1$. This contradicts $\nu_0^* \bar{D}_{\alpha_0} F = 0$ and completes the proof.

LEMMA 2.9. *Let κ be an infinite cardinal such that $2^\kappa < m$. Every m -complete m -distributive Post algebra $\mathbf{P} = \langle P; \{e_a: a \in A\}; B \rangle$ of type A , where A is a connected semilattice with the zero-element 0 and $\bar{A} \leq \kappa$, generated by κ m -generators is isomorphic with an m -complete Post Field. Moreover, it is isomorphic with a complete Post field.*

Proof. It is not difficult to show that the underlying Boolean algebra B is generated by κ m -generators b_i , $i < \kappa$. From the m -distributivity of B it follows that B is an atomic algebra and every atom has the form

$$\bigcap_{i < \kappa} \varepsilon(i) b_i, \quad \text{where } \varepsilon(i) = \pm 1.$$

Therefore the set of all atoms is of a power less than m . From this and from the m -completeness of B we conclude that B is complete. Consequently, algebra B is isomorphic with a complete Boolean field, which implies that P is isomorphic with a complete Post field.

As a consequence of 2.9 and the equivalence of (i) and (iv) in 2.8 we obtain the following

COROLLARY 2.10. *The formal system $S(\mathcal{L}_m^\nu, \mathcal{D}_\kappa)$, where $2^\kappa < m$ and $\nu \leq \kappa$, based on a language \mathcal{L}_m^ν having at most κ propositional variables is strongly complete.*

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