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# THE AUSLANDER–REITEN QUIVER FOR POSETS OF FINITE GROWTH

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The Auslander–Reiten quiver for posets of finite growth is described. The full description of its structure and the main points of the proof (including explanation of the functors used) are presented.

The Auslander–Reiten quiver for one-parameter posets was described by Bünermann [B] and for determinative (tubular) posets by Ringel [Ri]. The aim of this article is to describe the Auslander–Reiten quiver for arbitrary posets of finite growth. We describe the regular components completely, and the nonregular ones up to a finite number of points and arrows.

Our main tool is differentiation with respect to a pair of points [Z1], for which we suggest here a new name: stratification. The stratification functor (together with a simpler functor, replenishment) permits us to carry out induction on the dimension in the proofs, reducing a given problem to that for the poset (1, 1, 1, 1) or even for trivial posets.

The full exposition of the stratification technique for posets of finite growth (with all combinatorial aspects) is presented in the preprint [Z2]. The present paper gives an account of certain parts of [Z2] devoted to a description of the Auslander–Reiten quiver.

In Section 1 we recall the basic definitions and facts which are necessary for a better understanding of the material, in Section 2 we describe without proof the combinatorial structure of the Auslander–Reiten quiver, and in Section 3 we present the main ideas of the proof including the explanation of the functors used. The main theorem is formulated at the end of Section 2.

Any nontrivial fact or statement given below without proof and reference means automatically that its proof is contained in [Z2].

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The detailed proofs of the results of this paper will appear elsewhere.

## 1. Preliminaries

Let $\mathbf{N} = \{1, 2, 3, \ldots\}$, $\mathbf{N}^- = \{-1, -2, -3, \ldots\}$, $\mathbf{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$.

Functions will usually be written to the right of variables (except the dimension function, vector functions and quadratic forms).

For a given poset $\mathfrak{M}$ denote by $\max \mathfrak{M}$ ($\min \mathfrak{M}$) the set of all maximal (minimal) points of $\mathfrak{M}$. For $a \in \mathfrak{M}$ set $a^\Delta = \{x \in \mathfrak{M} | x \geqslant a\}$, $a^\nabla = \{x \in \mathfrak{M} | x \leqslant a\}$. If $A \subset \mathfrak{M}$, then $A^\Delta = \bigcup_{a \in A} a^\Delta$, $A^\nabla = \bigcup_{a \in A} a^\nabla$.

We write $\mathfrak{M} = A_1 + \ldots + A_n$ if $A_1 \cup \ldots \cup A_n = \mathfrak{M}$ and $A_i \cap A_j = \varnothing$ for $i \neq j$ (note that the points from different $A_i$ can be comparable).

Set $\tilde{\mathfrak{M}} = \mathfrak{M} \cup \{0\}$, where $0 \notin \mathfrak{M}$ is a formal symbol. Any function $\alpha$: $\tilde{\mathfrak{M}} \to \mathbf{Z}$ will be called a *vector* (indexed by $\tilde{\mathfrak{M}}$). We write $\alpha \leqslant \beta$ if $\alpha(x) \leqslant \beta(x)$ for any $x \in \tilde{\mathfrak{M}}$. A vector $\alpha$ is said to be *nonnegative* (*positive*) if $\alpha \geqslant 0$ ($\alpha > 0$, i.e. $\alpha \geqslant 0$ and $\alpha \neq 0$). Let $\operatorname{Supp} \alpha = \{x \in \mathfrak{M} | \alpha(x) > 0\}$ for $\alpha \geqslant 0$. A positive vector $\alpha$ is called *sincere* if $\alpha(x) > 0$ for any $x \in \tilde{\mathfrak{M}}$, and *trivial* if $|\alpha(x)| \leqslant 1$ for any $x \in \tilde{\mathfrak{M}}$.

Let $(\ ,\ )$, $\langle\ ,\ \rangle$: $\mathbf{Q}^{|\tilde{\mathfrak{M}}|} \times \mathbf{Q}^{|\tilde{\mathfrak{M}}|} \to \mathbf{Q}$ denote the usual nonsymmetric and symmetric bilinear form respectively and let $\chi(x) = \langle x, x \rangle$ denote the Tits quadratic form (corresponding to $\mathfrak{M}$).

A vector $\alpha$ is said to be a *root* (an *imaginary root*) of $\mathfrak{M}$ if $\chi(\alpha) = 1$ ($\chi(\alpha) = 0$). Let $e_x$ be the *trivial root* with $e_x(x) = 1$ and $e_x(y) = 0$ for $y \neq x$.

As usual the posets $K_1 = (1, 1, 1, 1)$, $K_2 = (2, 2, 2)$, $K_3 = (1, 3, 3)$, $K_4 = (N, 4)$ and $K_5 = (1, 2, 5)$ are called *critical*. For a critical subset $K \subset \mathfrak{M}$ denote by $\mu_K$ the only simplified (i.e. with coordinates having no common divisor $\neq 1$) positive imaginary root of $\mathfrak{M}$ with support $K$.

The posets $\Psi_1, \ldots, \Psi_{20}$ listed in [Z2] (and the antiisomorphic ones) are called *determinative*.

Assume $\mathfrak{M}$ is a poset of *finite growth* [ZN], i.e. it does not contain the subsets $(1, 1, 1, 1, 1)$, $(1, 1, 1, 2)$, $(2, 2, 3)$, $(1, 3, 4)$, $(N, 5)$, $(1, 2, 6)$ and the poset of Fig. 1. A root $r > 0$ of $\mathfrak{M}$ is called *regular* if there exists an

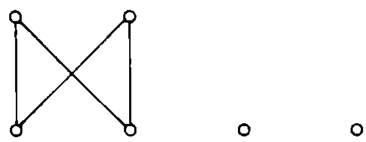


Fig. 1

imaginary root $\eta > 0$ (which is unique up to a multiplicative factor) such that $(r, \eta) = (\eta, r) = 0$; otherwise $r$ is called *nonregular*. A regular root $r$ is said to be *subordinate* to the above-mentioned $\eta$.

Let $N = \{a_1, \ldots, a_n\}$ be a subset of $\mathfrak{M}$ and suppose $a_i < a_j$ implies $i < j$. Then the *Coxeter transformations* $\Phi_N = \Phi_N^+ = \sigma_0 \sigma_{a_1} \ldots \sigma_{a_n}$ and $\Phi_N^- = \sigma_{a_n} \ldots \ldots \sigma_{a_1} \sigma_0$ are defined, where $\sigma_x$ is the usual reflection: $\alpha \sigma_x = \alpha - 2\langle e_x, \alpha \rangle e_x$.

Let $\eta > 0$ be an imaginary root, $N = \operatorname{Supp} \eta$ and let $W$ be the set of all regular roots $r$ subordinate to $\eta$ with $\operatorname{Supp} r \subset N$. A root $r \in W$ is said to be a *gene* of $\eta$ if $r \neq \varrho + \varrho\Phi_N + \ldots + \varrho\Phi_N^m$ for any $m \geqslant 1$ and any $\varrho \in W$. The set of all genes of $\eta$ is finite and splits up into three $\Phi_N$-orbits of well-known orders (see [Ri], [Z2] or Table 1 below).

If $r$ is a gene of $\eta$ denote by $\check{r}$ ($\hat{r}$) the set of all regular roots which are subordinate to $\eta$, have the form $e_x + e_0$ ($e_x$) and satisfy the condition $(r, e_x + e_0) = 1$ ($(e_x, r) = 1$). Note that $\check{r} = \hat{r} = \varnothing$ if $N$ is a determinative poset.

Write $\check{\check{r}} = \check{r} \cup \{r\}$, $\hat{\hat{r}} = \hat{r} \cup \{r\}$. For two genes $r, s$ we write $r \dashrightarrow s$ iff $s\Phi_N = r$. A finite sequence $\mathscr{L} = \{p, r_2, \ldots, r_{n-1}, q\}$ is called an *$\eta$-chain* if it satisfies the following three conditions:

(1) $r_1 \dashrightarrow r_2 \dashrightarrow \ldots \dashrightarrow r_n$ are genes of $\eta$;
(2) $p \in \check{\check{r}}_1$, $q \in \hat{\hat{r}}_n$ if $n > 1$, and $p = q \in \check{\check{r}}_1 \cup \hat{\hat{r}}_1$ if $n = 1$;
(3) $\mathscr{L} \neq \{e_x + e_0, e_y\}$ if $x < y$.

The sum $p + r_2 + \ldots + r_{n-1} + q$ is called the *$\mathscr{L}$-vector* (it can be either a regular root, or an imaginary root, or a specific vector defined in [Z2]). Any regular root is the $\mathscr{L}$-vector for a unique $\eta$-chain $\mathscr{L}$.

Recall some definitions concerning representations. Let $k$ be an arbitrary field. The *category of representations* $\mathscr{R}(\mathfrak{M})$ of a finite poset $\mathfrak{M}$ over $k$ (in the sense of Roiter [NR, Ro]) is defined as follows. Its objects are collections $U = (U_0 \xleftarrow{\delta_U} U_{\mathfrak{M}}; U_x)$ where $U_0, U_{\mathfrak{M}}, U_x$ ($x \in \mathfrak{M}$) are finite-dimensional $k$-spaces, $\delta_U$ is a $k$-linear transformation and $U_{\mathfrak{M}} = \bigoplus_{x \in \mathfrak{M}} U_x$ is a graded space. Set $U_A = \bigoplus_{x \in A} U_x$ for $A \subset \mathfrak{M}$.

A morphism $\varphi\colon U \to V$ is a pair $\varphi = (\varphi_0, \varphi_{\mathfrak{M}})$, where $\varphi_0\colon U_0 \to V_0$ and $\varphi_{\mathfrak{M}}\colon U_{\mathfrak{M}} \to V_{\mathfrak{M}}$ are $k$-linear transformations, such that: 1) $U_x \varphi_{\mathfrak{M}} \subset V_{x^\vee}$ for any $x \in \mathfrak{M}$; 2) $\delta_U \varphi_0 = \varphi_{\mathfrak{M}} \delta_V$. The composition of morphisms is defined naturally: $\varphi\psi = (\varphi_0\psi_0, \varphi_{\mathfrak{M}}\psi_{\mathfrak{M}})$.

The *dimension* of $U$ is a vector $\alpha = \underline{\dim}\, U$ such that $\alpha(x) = \dim_k U_x$. Denote by $[\alpha]$ the indecomposable representation of dimension $\alpha$ (in case it exists and is unique up to isomorphism).

We use the natural constructions of extensions. Suppose $A$, $B$ are objects of the category $\mathscr{R}(\mathfrak{M})$. A representation $U$ is called an *$(A, B)$-representation* if $U \simeq V$ where $V_0 = A_0 \oplus B_0$, $V_{\mathfrak{M}} = A_{\mathfrak{M}} \oplus B_{\mathfrak{M}}$, $V_x = A_x \oplus B_x$ and $\delta_V = \begin{bmatrix} \delta_A & 0 \\ \varepsilon & \delta_B \end{bmatrix}$ with an arbitrary $\varepsilon\colon B_{\mathfrak{M}} \to A_0$. By induction $U$ is an *$(A_1, \ldots, A_n)$-representation* if $U$ is a $(U', A_n)$-representation, where $U'$ is an $(A_1, \ldots, A_{n-1})$-representation.

For any indecomposable representation $U$ of a poset of finite growth one and only one of the following three conditions holds [Z2]:

(1) $U$ is a nonregular representation (i.e. $\underline{\dim}\, U$ is a nonregular root) in general position;

(2) $U$ is a regular nonhomogeneous representation, i.e. it is a $([p], [r_2], \ldots, [r_{n-1}], [q])$-representation for a unique $\eta$-chain $\mathscr{L} = (p, r_2, \ldots$

$\ldots, r_{n-1}, q)$; in this case we call it the *$\mathcal{L}$-representation* and write $U = [\mathcal{L}] = [p, r_2, \ldots, r_{n-1}, q]$;

(3) $U$ is a regular homogeneous representation which can be reduced to a representation of the poset $K_1$ and has a system of invariants $(\eta, \lambda)$, where $\eta = \underline{\dim}\, U$ is an imaginary root and $\lambda$ is an irreducible polynomial over $k$ (with leading coefficient 1) different from $X$ and $X-1$, and having degree dividing all coordinates of $\eta$.

The notions of the Auslander–Reiten quiver, Auslander–Reiten sequence, irreducible morphism, mesh, translation are supposed to be well known (see [AR, Ri]).

By $C(U, V)$ we denote the set of all morphisms from $U$ to $V$ in a category $C$.

## 2. The structure of the Auslander–Reiten quiver

Let $\mathfrak{M}$ be a poset of finite growth. The set of all nonregular roots of $\mathfrak{M}$ is naturally divided into several parts (corresponding to the components of the Auslander–Reiten quiver) as follows.

Let $\mathcal{K}(\mathfrak{M})$ be the collection of all critical subsets of $\mathfrak{M}$. Define an order $\preccurlyeq$ on $\mathcal{K}(\mathfrak{M})$ by

$$K \preccurlyeq N \Leftrightarrow K \subset N^{\nabla} \quad (\text{or, equivalently, } N \subset K^{\Delta})$$

for $K, N \in \mathcal{K}(\mathfrak{M})$. It is not difficult to show that $\mathcal{K}(\mathfrak{M})$ is a chain with respect to this order. Denote by $K_{\min}$ ($K_{\max}$) the minimal (maximal) element of this chain. Its neighbouring elements $K$, $N$ will be called *strongly coupled* if $K \cup N$ is a determinative poset, and *weakly coupled* otherwise.

Let $R$ be the set of all positive roots of $\mathfrak{M}$. Set

$$[-, K_{\max}] = \{r \in R \mid (r, \mu_{K_{\max}}) > 0\}, \qquad [K_{\min}, -] = \{r \in R \mid (\mu_{K_{\min}}, r) > 0\}.$$

By definition, $[-, K_{\max}] = [K_{\min}, -] = R$ if $\mathcal{K}(\mathfrak{M}) = \varnothing$. Set also

$$[N, K] = \{r \in R \mid (\mu_N, r) > 0 \text{ and } (r, \mu_K) > 0\},$$

where $N \succ K$ are neighbouring weakly coupled critical sets (the set $[N, K]$ is not defined if $N \succ K$ are strongly coupled).

PROPOSITION 1. *The set of all nonregular roots of $\mathfrak{M}$ is the union $[-, K_{\max}] \cup [N, K] \cup \ldots \cup [N', K'] \cup [K_{\min}, -]$ taken over all pairs $N \succ K$ of neighbouring weakly coupled critical sets. Different sets from the union do not intersect (except when $\mathcal{K}(\mathfrak{M}) = \varnothing$). If* $\operatorname{Supp} r = E$ *then $E \subset N^{\nabla} \cap K^{\Delta}$ if $r \in [N, K]$, $E \subset K^{\Delta}_{\max}$ if $r \in [-, K_{\max}]$ and $E \subset K^{\nabla}_{\min}$ if $r \in [K_{\min}, -]$.*

As any nonregular indecomposable representation $U$ is in general position, we shall identify $U$ with the dimension vector $\underline{\dim}\, U$.

It turns out that each of the sets

$$[-, K_{\max}], \quad [N, K], \quad \ldots, \quad [N', K'], \quad [K_{\min}, -]$$

coincides with the set of all vertices of some component of the Auslander–Reiten quiver $\Gamma(\mathfrak{M})$ of the poset $\mathfrak{M}$. We call these components *nonregular* and denote them by

$$P(\mathfrak{M}) = P[-, K_{\max}], \quad QP[N, K], \quad \ldots, \quad QP[N', K'], \quad Q(\mathfrak{M}) = Q[K_{\min}, -].$$

The components $P(\mathfrak{M})$ and $Q(\mathfrak{M})$ are nothing else but the preprojective and preinjective components which were described for any tame poset by Bünermann [B]. Their shape is well known. For any $\mathfrak{M}$, the component $P(\mathfrak{M})$ coincides up to a finite number of points and arrows with one of the standard translation quivers of type $\mathbb{N}\tilde{D}_l$ ($l \geq 4$), $\mathbb{N}\tilde{E}_6$, $\mathbb{N}\tilde{E}_7$ or $\mathbb{N}\tilde{E}_8$ defined e.g. in [Ri].

In fact, any component $QP[N, K]$ can be obtained by "glueing" the components $Q(\check{\check{N}})$ and $P(\hat{\hat{K}})$ for some subsets $\check{\check{N}}, \hat{\hat{K}} \subset N^{\nabla} \cap K^{\Delta}$. These subsets are defined for any critical set $K \subset \mathfrak{M}$ by $\hat{\hat{K}} = \{x \in K^{\Delta} \mid \langle \mu_K, e_x \rangle = 0\}$, $\check{\check{K}} = \{x \in K^{\nabla} \mid \langle \mu_K, e_x \rangle = 0\}$. "Glueing" means that up to a finite number of points and arrows $QP[N, K]$ coincides with the formal union $Q(\check{\check{N}}) \cup P(\hat{\hat{K}})$.

Among sincere posets of finite growth (listed in [Z2]) only the two-parameter posets $D_1, \ldots, D_{10}$ (and their duals) have a component of type $QP[N, K]$ in $\Gamma(\mathfrak{M})$. Let $\mathfrak{M}$ be any of these posets, let $r > 0$ be its unique sincere root and $p = e_a + e_0$, $q = e_b$, where $a \in K$ ($b \in N$) is the only point for which $\langle \mu_N, e_a \rangle \neq 0$ ($\langle \mu_K, e_b \rangle \neq 0$). Then $QP[N, K]$ has one of the forms in Figs. 2–5.

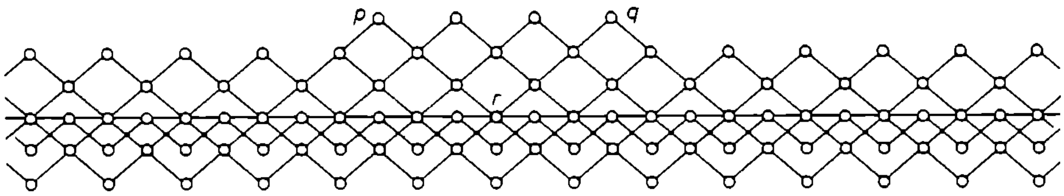



Fig. 2. The case $\mathfrak{M} = D_1$

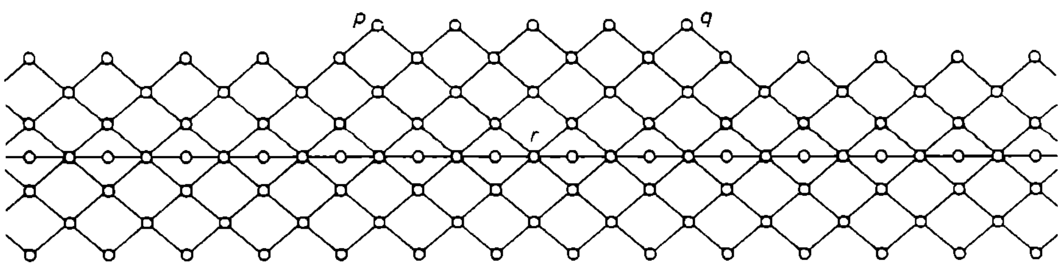



Fig. 3. The cases $\mathfrak{M} = D_2, D_3$

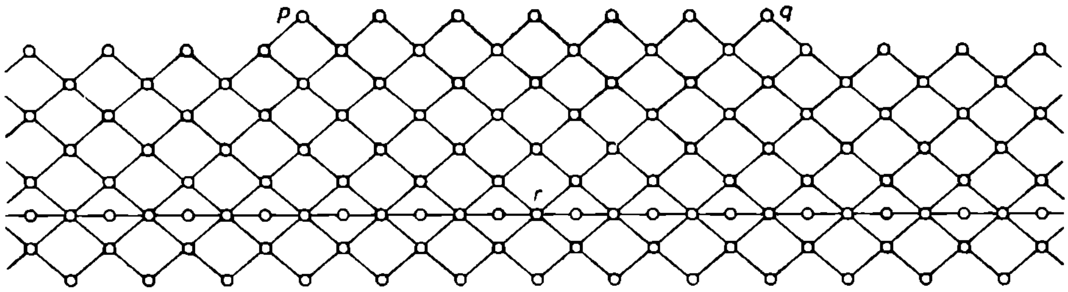



Fig. 4. The cases $\mathfrak{M} = D_4, \ldots, D_7$

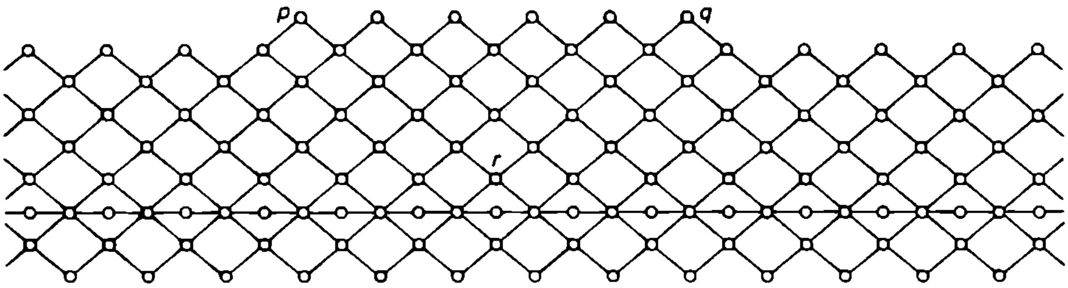



Fig. 5. The cases $\mathfrak{M} = D_8, D_9, D_{10}$

The type of the components $Q(\check{N})$, $P(\hat{N})$ in each case can be determined from Table 1 below (the type of $P(\hat{N})$ is $\mathbf{N}\tilde{X}$ where $X$ denotes the Dynkin diagram indicated in the table).

Now we turn to the description of the regular components.

The nonhomogeneous indecomposables split into components $U_i(\eta)$, where $i = 1, 2, 3$ and $\eta > 0$ is a simplified imaginary root. Each $U_i(\eta)$ bears on a certain orbit $O_i$ of genes of $\eta$. More precisely, the set of vertices of $U_i(\eta)$ is the set of all nonhomogeneous indecomposables $[p, r_2, \ldots, r_{n-1}, q]$ with $r_1, \ldots, r_n \in O_i$. We describe $U_i(\eta)$ in terms of $\eta$-chains $\{p, r_2, \ldots, r_{n-1}, q\}$.

Let $r$ be a gene of $\eta$. The set $\hat{r}$ will be considered as a poset which is a chain with minimal element $r$ where $r < e_x$ for any $e_x \in \hat{r}$ and $e_x < e_y$ whenever $x < y$. Dually, $\check{r}$ will be a chain (where $e_x + e_0 < r$ for any $e_x + e_0 \in \check{r}$ and $e_x + e_0 < e_y + e_0$ whenever $x < y$).

If $r \dashrightarrow s$ then we also compare the elements of $\check{r}$ and $\hat{s}$ as follows: the gene $r$ is incomparable with points of $\hat{s}$, the gene $s$ is incomparable with points of $\check{r}$, and for $e_x + e_0 \in \check{r}$ and $e_y \in \hat{s}$ we have $e_x + e_0 < e_y$ if $x < y$. So, the poset $\check{r} \cup \hat{s}$ is as shown in Fig. 6.

If $\alpha, \beta \in \check{r} \cup \hat{s}$ then $\beta$ is said to *cover* $\alpha$ if $\alpha < \beta$ and there is no $\gamma \in \check{r} \cup \hat{s}$ such that $\alpha < \gamma < \beta$. The following statement completely describes the components $U_i(\eta)$.

PROPOSITION 2. *The arrows of the nonhomogeneous components of the quiver $\Gamma(\mathfrak{M})$ are exhausted by the arrows of the following types (below $n$ denotes the length of an $\eta$-chain; $r \dashrightarrow s$ are arbitrary genes; $p, p' \in \check{r}$; $q, q' \in \hat{s}$):*

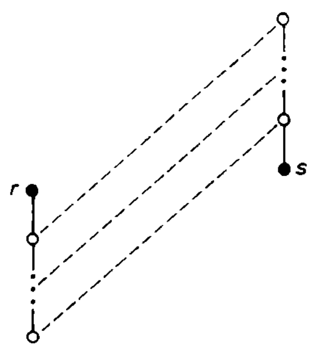



Fig. 6

1) $[\ldots, p] \to [\ldots, p, q]$, *where* $n \geqslant 1$ *and* $q$ *is the greatest element of the chain* $\hat{s}$ *incomparable with* $p$;

1*) $[p, q, \ldots] \to [q, \ldots]$, *where* $n \geqslant 2$ *and* $p$ *is the least element of the chain* $\check{r}$ *incomparable with* $q$;

2) $[\ldots, p, q'] \to [\ldots, p, q]$, *where* $n \geqslant 2$ *and* $q'$ *covers* $q$;

2*) $[p, q, \ldots] \to [p', q, \ldots]$, *where* $n \geqslant 2$ *and* $p$ *covers* $p'$;

3) $[p, q] \to [p']$, *where* $p, q$ *cover* $p'$;

3*) $[q'] \to [p, q]$, *where* $q'$ *covers* $p, q$;

4) $[p] \to [p']$, *where* $p$ *and only* $p$ *covers* $p'$;

4*) $[q'] \to [q]$, *where* $q'$ *covers* $q$ *and only* $q$.

For a given orbit $O_i$ of genes of the imaginary root $\eta$ consider the posets

$$\check{O}_i = \bigcup_{r \in O_i} \check{r}, \qquad \hat{O}_i = \bigcup_{r \in O_i} \hat{r}, \qquad O_i^{\#} = \check{O}_i \cup \hat{O}_i,$$

with the order induced by the orders on all subsets $\check{r} \cup \hat{s}$ where $r \dashrightarrow s$ are genes from $O_i$. Denote by $U_i(\eta) = U(O_i^{\#})$ the nonhomogeneous component consisting (according to Proposition 2) exactly of the vertices $[p, r_2, \ldots, r_{n-1}, q]$ with $p, r_2, \ldots, r_{n-1}, q \in O_i^{\#}$. One can easily construct the component $U(O_i^{\#})$ directly from the poset $O_i^{\#}$. It is a tube in the sense of [Ri] and consists of the meshes listed in Fig. 7 (numbers denote the types of arrows according to Proposition 2). Here $\alpha, \alpha'$ are arrows of types 1) or 2) and $\beta, \beta'$ – of types 1*) or 2*) (automatically $L$, $M$ are chains of length $\geqslant 2$).

Example. If $\mathfrak{M}$ is the poset of Fig. 8 and

$$O = \{a \dashrightarrow b \dashrightarrow c \dashrightarrow d \dashrightarrow a\}$$

with $a = e_8 + e_9 + e_0$, $b = e_5 + e_7 + e_0$, $c = e_6 + e_{11} + e_0$, $d = e_5 + e_{10} + e_0$, then $O^{\#}$ is as shown in Fig. 9. Hence the component $U(O^{\#})$ is as in Fig. 10, where the vertical dotted lines have to be identified and where we write simply $i$ instead of $e_i$ or $e_i + e_0$ in chains.

The pair $\dfrac{m_i}{n_i}$, where $m_i = |\hat{O}_i|$, $n_i = |\check{O}_i|$, will be called the *type* of the

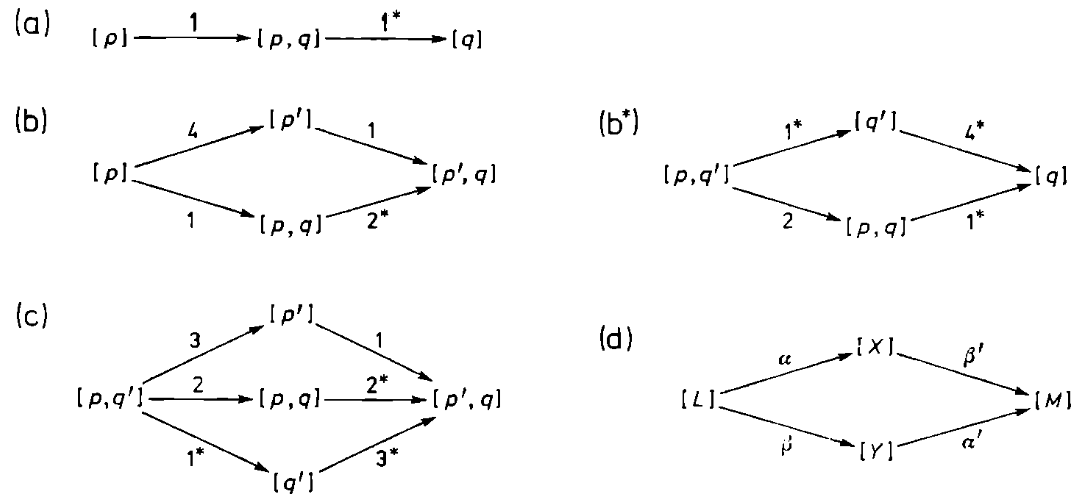



Fig. 7

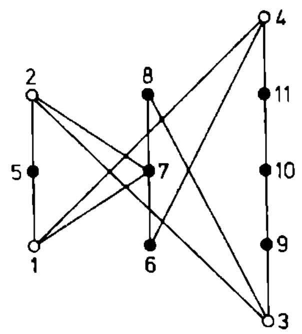



Fig. 8

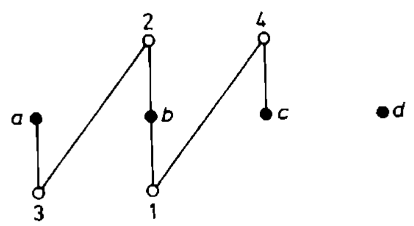



Fig. 9

component $U(O_i^{\#})$. Starting from any point and making $m_i$ steps in one direction and $n_i$ in the other, one will arrive at the same point. The tube considered above has type $\frac{6}{6}$.

As the imaginary root $\eta$ has three orbits of genes, it is natural to attach to it the triple $(m_1, m_2, m_3)$, where $m_i = |\hat{O}_i|$, and call this triple the *upper type* of $\eta$. Analogously we define the *lower type* $(n_1, n_2, n_3)$ (the orbits are considered to be fixed). Note that $\hat{O}_i = \check{O}_i = O_i$ if $N = \operatorname{Supp} \eta$ is a determinative poset, and in this case the upper (lower) type gives the usual orders of orbits of genes.

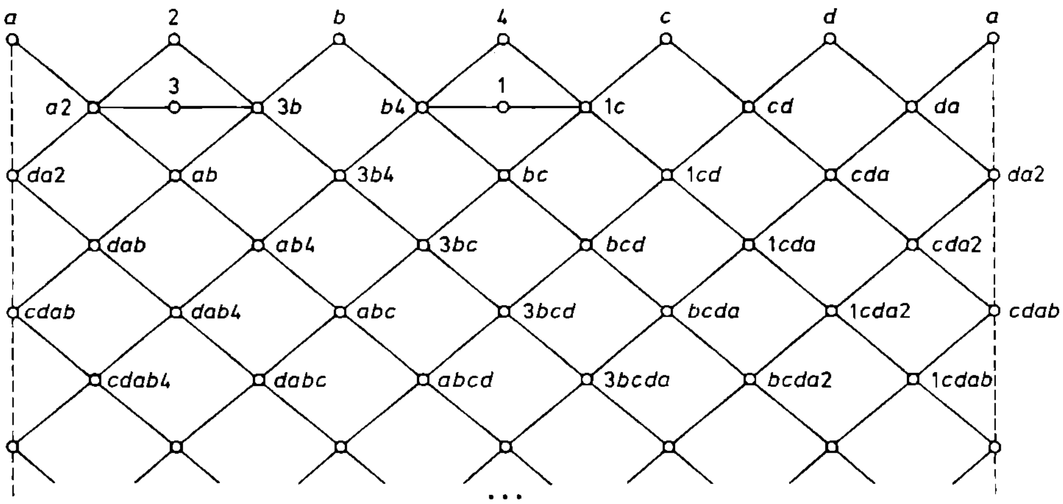



Fig. 10

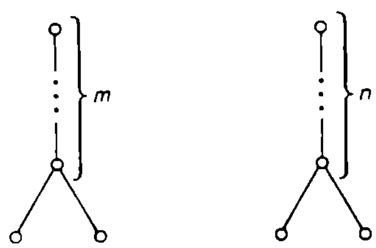



Fig. 11

Table 1 enumerates all possible upper types (see also [Ri]). Here the $A_i$, $i = 1, \ldots, 11$, are sincere one-parameter posets, and $L(m, n)$ $(m, n \geqslant 0)$ is shown in Fig. 11.

Table 1

| $\hat{N}$ | Upper type | Diagram |
|---|---|---|
| $K_1$ | (2, 2, 2) | $D_4$ |
| $L(m, n)$ | $(m+n+2, 2, 2)$ | $D_{4+m+n}$ |
| $K_2, A_2$ | (3, 3, 2) | $E_6$ |
| $K_3, A_3, A_4, A_7$ | (4, 3, 2) | $E_7$ |
| $K_4, K_5, A_5, A_6, A_8, \ldots, A_{11}$ | (5, 3, 2) | $E_8$ |
| $\psi_1, \psi_3$ | (3, 3, 3) | $\tilde{E}_6$ |
| $\psi_2, \psi_4, \psi_5, \psi_9, \psi_{11}$ | (4, 4, 2) | $\tilde{E}_7$ |
| $\psi_6, \ldots, \psi_8, \psi_{10}, \psi_{12}, \ldots, \psi_{20}$ | (6, 3, 2) | $\tilde{E}_8$ |

Combining any of these posets with the antiisomorphic one, i.e. considering different unions $\check{N} \cup \hat{N}$ (with due regard for possible permutations of orbits) one can obtain posets having components $U_i(\eta)$ of all possible types $\frac{m_i}{n_i}$.

Finally, consider homogeneous components. In contrast to the nonregular and nonhomogeneous cases, the homogeneous components are not changed by stratification and replenishment and can be reduced to some components of the poset $K_1$. But homogeneous components of this poset are well known (see [Ri]). Any of them has the form shown in Fig. 12, where the translation acts

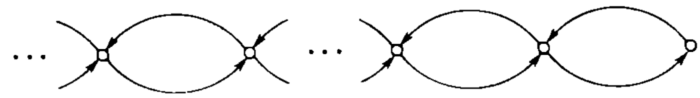



Fig. 12

identically, and their number is equal to the number of irreducible polynomials over $k$ (with leading coefficient 1) different from $X$ and $X-1$.

So, we can formulate

THE MAIN THEOREM. *Let $\mathfrak{M}$ be a poset of finite growth (and of infinite type). Then its Auslander–Reiten quiver $\Gamma(\mathfrak{M})$ is a sum (i.e. disjoint union) of the following components:*

$$\Gamma(\mathfrak{M}) = \sum_{\eta}\Big(\sum_{\lambda} H(\eta, \lambda) + \sum_{i=1}^{3} U_i(\eta)\Big) + \sum_{[N,K]} QP[N, K] + P(\mathfrak{M}) + Q(\mathfrak{M}), \quad \textit{where}:$$

- *$\eta$ runs over the set of all positive simplified imaginary roots of the poset $\mathfrak{M}$;*
- *$\lambda$ independently runs over the set of all irreducible polynomials over $k$ (with leading coefficient 1) different from $X$ and $X-1$;*
- *$N \succ K$ are all possible pairs of neighbouring weakly coupled critical subsets of $\mathfrak{M}$;*
- *$H(\eta, \lambda)$ are homogeneous regular components of type* (H);
- *$U_i(\eta)$, $i = 1, 2, 3$, are nonhomogeneous regular components defined according to Proposition* 2;
- *$P(\mathfrak{M})$ ($Q(\mathfrak{M})$) is the preprojective (preinjective) nonregular component with the set of vertices $[-, K_{\max}]$ ($[K_{\min}, -]$);*
- *$QP[N, K]$ are nonregular components with the set of vertices $[N, K]$ obtained by "glueing" the preinjective component of the poset $\check{N}$ with the preprojective component of $\hat{K}$.*

For the proof, see the next section.

The general scheme of $\Gamma(\mathfrak{M})$ is shown in Fig. 13, where every pictured "tube" corresponds to some imaginary root and in fact is an infinite collection of tubes

$$T(\eta) = \sum_{\lambda} H(\eta, \lambda) + \sum_{i=1}^{3} U_i(\eta).$$

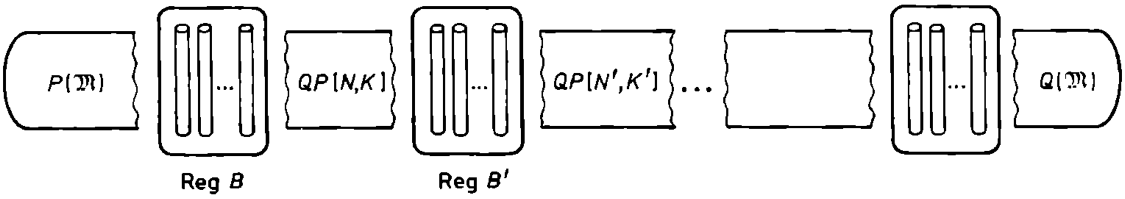



Fig. 13

The blocks of "tubes" Reg $B$, Reg $B'$, ... are determined by the "largest strongly coupled subsets" of the poset $\mathfrak{M}$, i.e. by the subsets of the type $B = \check{K} + (N^\nabla \cap K^\Delta) + \hat{N}$, where $K = X_1 \prec \ldots \prec X_n = N$ $(n \geqslant 1)$ is a chain of maximal length in which any two neighbouring critical sets are strongly coupled ($T(\eta) \in \operatorname{Reg} B$ iff $\operatorname{Supp} \eta \in N^\nabla \cap K^\Delta$).

### 3. The main points of the proof

First we give a description of the functors used: stratification (formerly called differentiation with respect to a pair of points) and replenishment (a simple operation which was formerly considered as a special case of differentiation). Although we work in the category $\mathscr{R}(\mathfrak{M})$, the description of these functors is very simple and clear when using the category $\mathfrak{M}$-sp of $\mathfrak{M}$-spaces in the sense of Gabriel [G]. That is why we prove properties of our functors in the category $\mathfrak{M}$-sp and then formulate them in the category $\mathscr{R}(\mathfrak{M})$.

Recall that an object of the category $\mathfrak{M}$-sp is a finite-dimensional $k$-space $U$ together with a collection of its subspaces $\{U_x\}_{x \in \mathfrak{M}}$ such that $U_x \subset U_y$ if $x \leqslant y$. A morphism from $U$ to $V$ is a $k$-linear map $\varphi$: $U \to V$ such that $U_x \varphi \subset V_x$ for any $x \in \mathfrak{M}$.

A pair of points $(a, b)$ is called *suitable* (for a stratification) if $\mathfrak{M} = a^\Delta + b^\nabla + \{c_1, \ldots, c_n\}$, where $c_1 < \ldots < c_n$ is a chain, $n \geqslant 1$ and the points $a$, $b$ and $c_i$ are mutually incomparable. The *stratified* (or, more exactly, *$(a, b)$-stratified*) *poset* $\mathfrak{M}'_{(a,b)}$ has a nice lattice definition:

$$\mathfrak{M}'_{(a,b)} = (\mathfrak{M} \setminus \{c_1, \ldots, c_n\}) + \{a + c_1, \ldots, a + c_n\} + \{c_1 b, \ldots, c_n b\},$$

being considered as a subposet of the free lattice generated by $\mathfrak{M}$. The order is obvious and is illustrated in Fig. 14.

Now define the *stratification functor* $'$: $\mathfrak{M}\text{-sp} \to \mathfrak{M}'_{(a,b)}\text{-sp}$ by setting $U' = U$, $U'_x = U_x$ whenever $x \in a^\Delta + b^\nabla$ and $U'_{a+c_i} = U_a + U_{c_i}$, $U'_{c_i b} = U_{c_i} \cap U_b$ for an $\mathfrak{M}$-space $U$, and $\varphi' = \varphi$ for a morphism $\varphi$: $U \to V$. Obviously, this is indeed a functor.

For convenience denote our categories briefly by $S = \mathfrak{M}\text{-sp}$ and $S' = \mathfrak{M}'_{(a,b)}\text{-sp}$. According to the definition $S(U, V) \subset S'(U', V')$.

If $a_1, \ldots, a_r$ $(r \geqslant 1)$ is a set of mutually incomparable points of some poset $\mathfrak{N}$, introduce a one-dimensional $\mathfrak{N}$-space $P(a_1, \ldots, a_r)$ by setting

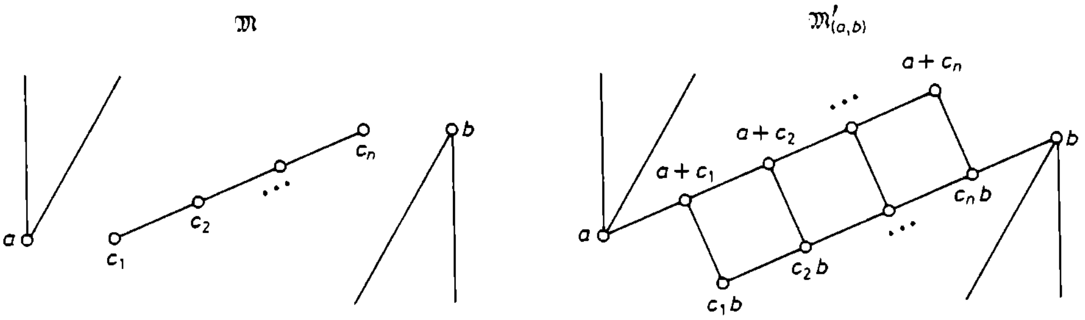



Fig. 14

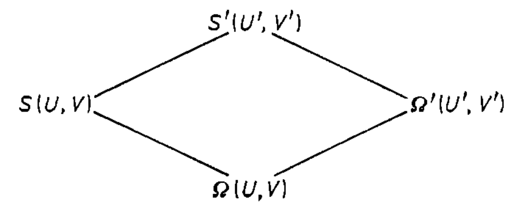
$P(a_1, \ldots, a_r) = U$, where $U = U_x = k$ if $x \in \{a_1, \ldots, a_r\}^{\Delta}$ and $U_x = 0$ otherwise (let $P(\varnothing) = k$ with $P(x)_x = 0$ for any $x \in \mathfrak{M}$). The following fact is rather trivial:

(1) A morphism $\varphi\colon U \to V$ in the category $\mathfrak{N}$-sp can be factored through a direct sum $(P(a_1, \ldots, a_r))^m$ iff $U\varphi \subset \bigcap_{i=1}^{r} V_{a_i}$ and $U_x\varphi = 0$ for $x \in \mathfrak{N} \setminus \{a_1, \ldots, a_r\}^{\Delta}$.

Now consider the following ideals of the categories $S$ and $S'$:

$$\Omega = \{1_{P(a)}, 1_{P(a,c_1)}, \ldots, 1_{P(a,c_n)}\}_S, \quad \Omega' = \{1_{P(a)}\}_{S'}.$$

Clearly, $(P(a, c_i))' \simeq P(a)$, therefore $\Omega(U, V) \subset \Omega'(U', V')$ for all $U$, $V$. So we have the following diagram in which dashes denote the usual inclusions:




LEMMA 1. $S(U, V) + \Omega'(U', V') = S'(U', V')$.

*Proof.* Assume $\varphi_1 \in S'(U', V')$. Then $\varphi_1 \in S(U, V)$ iff $U_{c_i}\varphi_1 \subset V_{c_i}$ for any $i$. First consider the space $U_{c_1}$ and represent it in the form $U_{c_1} = L_1 \oplus M_1$, where $L_1 = U_{c_1} \cap (V_{c_1}\varphi_1^{-1})$ and $M_1$ is some complement. Note that $U_{c_1} \cap U_b \subset L_1$, hence $(U_b + L_1) \cap M_1 = 0$ and we have $U = (U_b + L_1) \oplus M_1 \oplus X_1$ for some $X_1$. Let $\alpha_1$ be the composition $U \to M_1 \to U$ of the natural projection and injection (as usual linear maps).

Let $V_a + V_{c_1} = A_1 \oplus V_{c_1}$ for some $A_1 \subset V_a$ and $V = A_1 \oplus V_{c_1} \oplus Y_1$ for some $Y_1$. Let $\beta_1$ be the composition $V \to A_1 \to V$ of the natural projection and injection.

Set $\omega_1 = \alpha_1\varphi_1\beta_1$. Clearly $U_b\omega_1 = 0$ and according to (1), $\omega_1 \in \Omega'(U', V')$ $(\operatorname{Im}\omega_1 \subset V_a)$.

Consider now a new morphism $\varphi_2 = \varphi_1 - \omega_1 \in S'(U', V')$ for which obviously $U_{c_1}\varphi_2 \subset V_{c_1}$. Substituting everywhere the index 2 for 1 and making the same procedure (note that $U_{c_1} \subset L_2$) we obtain a third morphism $\varphi_3 = \varphi_2 - \omega_2$ such that $U_{c_1}\varphi_3 \subset V_{c_1}$ and $U_{c_2}\varphi_3 \subset V_{c_2}$. Finally, we get $\varphi_{n+1} = \varphi_1 - (\omega_1 + \ldots + \omega_n)$, where $U_{c_i}\varphi_{n+1} \subset V_{c_i}$ for any $i$ and $\omega_1 + \ldots + \omega_n \in \Omega'(U', V')$, i.e. $\varphi_1 \in S(U, V) + \Omega'(U', V')$.

LEMMA 2. $S(U, V) \cap \Omega'(U', V') = \Omega(U, V)$.

*Proof.* Suppose $\varphi_1 \in S(U, V) \cap \Omega'(U', V')$. Set $L_1 = U_{c_1} \cap \operatorname{Ker} \varphi_1$ (clearly $U_{c_1} \cap U_b \subset L_1$) and let $U_{c_1} = L_1 \oplus M_1$ for some $M_1$ and $U = (U_b + L_1) \oplus M_1 \oplus X_1$ for some $X_1$. Let $\alpha_1$ be the composition $U \to M_1 \to U$ of the natural projection and injection (as linear maps).

Further, set $A_1 = V_a \cap V_{c_1}$ and consider the direct sum $V = A_1 \oplus Y_1$ for some complement $Y_1$. Let $\beta_1$ be the composition $V \to A_1 \to V$ of the canonical projection and injection.

Now define $\omega_1 = \alpha_1 \varphi_1 \beta_1$. Obviously $\omega_1 \in \{1_{P(a,c_1)}\}_S$ (see (1)) and $\varphi_2 = \varphi_1 - \omega_1$ satisfies $U_b + U_{c_1} \subset \operatorname{Ker} \varphi_2$. Substituting then everywhere the index 2 for 1 and repeating the process we obtain $\varphi_3 = \varphi_2 - \omega_2$, where $\omega_2 \in \{1_{P(a,c_2)}\}_S$ and $U_b + U_{c_2} \subset \operatorname{Ker} \varphi_3$. After $n$ steps we have $\varphi_{n+1} = \varphi_1 - (\omega_1 + \ldots + \omega_n)$, where $\omega_i \in \{1_{P(a,c_i)}\}_S$, $U_b + U_{c_n} \subset \operatorname{Ker} \varphi_{n+1}$ and $\operatorname{Im} \varphi_{n+1} \subset V_a$, i.e. $\varphi_{n+1} \in \{1_{P(a)}\}_S$ and $\varphi_1 \in \Omega(U, V)$.

LEMMA 3. *If $W \in \operatorname{Ob} S'$ then there exists $U \in \operatorname{Ob} S$ such that $U' \simeq W \oplus (P(a))^m$ for some $m$.*

*Proof.* Let $W_{a+c_i} \cap W_b = W_{c_i b} \oplus F_i$ for some $F_i$ and $W_{a+c_i} = W_{c_i b} \oplus F_i \oplus H_i$ for some $H_i$. Let $f_1^i, \ldots, f_{p_i}^i$ be a basis of $F_i$. Assuming that the symbols $g_j^i$ $(1 \leqslant i \leqslant n;\ 1 \leqslant j \leqslant p_i)$ form a basis of some new $k$-space $G$, define $U = W \oplus G$ and $e_j^i = f_j^i + g_j^i$. Denote by $E_i$ the $k$-space generated by $e_1^i, \ldots, e_{p_i}^i$ and define $U_x = W_x$ for $x \in b^\nabla$, $U_x = W_x \oplus G$ for $x \in a^\Delta$ and $U_{c_i} = U_{c_{i-1}} + E_i + H_i$ $(U_{c_1} = E_1 + H_1)$. It is obvious that $U' \simeq W \oplus (P(a))^m$ where $m = \dim_k G$.

From Lemmas 1–3 we immediately obtain

THEOREM 1. *The stratification $'$: $S \to S'$ induces an equivalence of the factor categories $S/\Omega \xrightarrow{\sim} S'/\Omega'$.*

COROLLARY 1. *If $\Gamma$ and $\Gamma'$ are the Auslander–Reiten quivers of the categories $S$ and $S'$, then*

$$\Gamma \backslash \{P(a), P(a, c_1), \ldots, P(a, c_n)\} \simeq \Gamma' \backslash \{P(a)\}.$$

For the replenishment, the situation is simpler. A pair of incomparable points $(a, b)$ of a poset $\mathfrak{M}$ is called *specific* if $\mathfrak{M} = a^\Delta + b^\nabla$. By definition the *replenished poset* $\bar{\mathfrak{M}}_{(a,b)}$ is obtained from $\mathfrak{M}$ by adding the only relation $a < b$. The lattice explanation can be given in one of the two ways (dual to each other), clearly illustrated in Fig. 15.

This prompts us to define the *replenishment functor* $^-$: $S \to \bar{S}$ (where

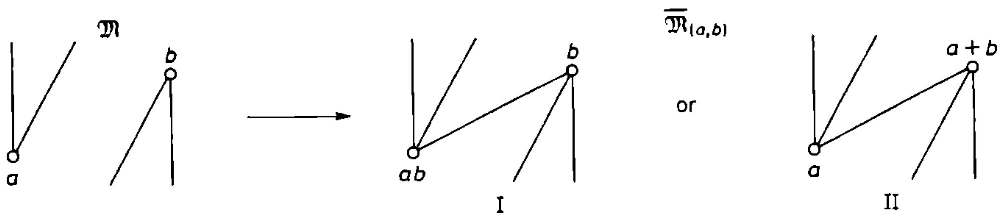



Fig. 15

$\bar{S} = \mathfrak{M}_{(a,b)}$-sp) attaching to an object $U \in \mathrm{Ob}\, S$ the object $\bar{U} \in \mathrm{Ob}\, \bar{S}$ such that $\bar{U} = U$, $\bar{U}_x = U_x$ for $x \neq a, b$ and

(I) $\bar{U}_a = U_a \cap U_b$, $\bar{U}_b = U_b$, or

(II) $\bar{U}_a = U_a$, $\bar{U}_b = U_a + U_b$.

One can easily verify the following

THEOREM 2. *The replenishment functor* $^-$: $S \to \bar{S}$ *induces an equivalence of factor categories*:

$$S/\{1_{P(a)}, 1_{P(a,b)}\} \tilde{\to} \bar{S}/\{I_{P(a)}\} \quad \textit{in case } II,$$

$$S/\{1_{P(a)}, 1_M\} \tilde{\to} \bar{S}/\{1_M\} \quad \textit{in case } I,$$

$M = P(\min(a^{\Delta} \backslash \{a\}))$ *in* $\mathfrak{M}$.

COROLLARY 2. $\Gamma \backslash \{P(a)\} \simeq \bar{\Gamma}$, *where* $\Gamma$ *and* $\bar{\Gamma}$ *are the Auslander–Reiten quivers of the categories* $S$ *and* $\bar{S}$.

Note that differentiation with respect to a maximal point $b$ in the sense of [NR] is in fact a composition of several stratifications and replenishments-I. Namely making (in any order) $(x, b)$-stratifications and $(y, b)$-replenishments-I as many times as possible, we obtain the ordinal sum of the derivative poset $\mathfrak{M}'_b$ in the sense of [NR] and of the one-point poset $\{b\}$.

Now we return to the categories $\mathscr{R} = \mathscr{R}(\mathfrak{M})$ and $\mathscr{R}' = \mathscr{R}(\mathfrak{M}'_{(a,b)})$ and explain the functors acting there.

Let $P(a_1, \ldots, a_r) = [e_{a_1} + \ldots + e_{a_r} + e_0]$ for mutually incomparable points $a_1, \ldots, a_r$, and let $Q(b) = [e_b]$.

Let $\mathfrak{A}$ be the set of all morphisms $P(a, c_i) \to P(y)$, where $1 \leqslant i \leqslant n$ and $y \in a^{\nabla} \cap c_i^{\nabla}$; and, dually, let $\mathfrak{B}$ be the set of all morphisms $Q(z) \to P(b, c_i)$, where $1 \leqslant i \leqslant n$ and $z \in b^{\Delta} \cap c_i^{\Delta}$.

Note that $\mathfrak{A}, \mathfrak{B} \subset \mathrm{rad}^2 \mathscr{R}$, because any morphism $P(a, c_i) \to P(y)$ $(Q(z) \to P(b, c_i))$ is a composition $P(a, c_i) \to [e_a + e_b + e_{c_i} + e_0] \to P(y)$ $(Q(z) \to [e_a + e_b + e_{c_i} + 2e_0] \to P(b, c_i))$. Set

$$\Theta = \{1_{P(a)}, 1_{Q(b)}, \mathfrak{A}, \mathfrak{B}\}_{\mathscr{R}}, \qquad \Theta' = \{1_{P(a)}, 1_{Q(b)}\}_{\mathscr{R}'},$$

and define the stratification functor $'$: $\mathscr{R} \to \mathscr{R}'$ as follows. For an object

$U \in \mathrm{Ob}\,\mathscr{R}$ with $\delta_U = f$ let $B = B_U = \mathrm{Im}\, f_{b\triangledown}$. For any subspace $X \subset U_{\mathfrak{M}}$ set $\bar{X} = X \cap (Bf^{-1})$.

Let $U' \in \mathrm{Ob}\,\mathscr{R}'$ be any object having the same $U'_0 = U_0$, $U'_{\mathfrak{M}} = U_{\mathfrak{M}}$ and $\delta_{U'} = \delta_U$ but new $U'_x$ $(x \in \mathfrak{M}')$ defined (nonuniquely) by the conditions

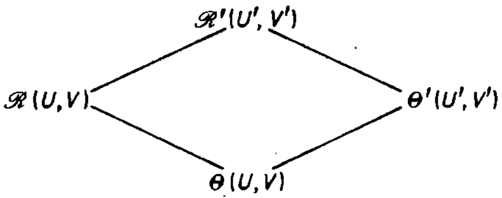
$$U'_x = U_x \quad \text{for } x \in \mathfrak{M} \backslash \{c_1, \ldots, c_n\},$$
$$\bar{U}_{\{c_1, \ldots, c_i\}} = \bar{U}_{\{c_1, \ldots, c_{i-1}\}} \oplus U'_{c_i b},$$
$$U_{\{c_1, \ldots, c_i\}} = U_{\{c_1, \ldots, c_{i-1}\}} \oplus U'_{c_i b} \oplus U'_{a+c_i}.$$

Obviously, the object $U'$ does not depend (up to isomorphism) on the choice of the complements $U'_{c_i b}$ and $U'_{a+c_i}$. We emphasize that the object $U'$ contains the trivial objects $P(a)$ and $Q(b)$ as direct summands. We define $f' = f$ for any morphism $f$.

Examples.

$$(P(x))' \simeq P(x), \quad (Q(x))' \simeq Q(x) \quad \text{if } x \in \mathfrak{M} \backslash \{c_1, \ldots, c_n\},$$
$$(P(a, c_i))' \simeq P(a) \oplus Q(a+c_i), \quad (P(b, c_i))' \simeq P(c_i b) \oplus Q(b),$$
$$[e_a + e_b + e_{c_i} + e_0]' \simeq P(a, c_i b) \oplus Q(b), \quad [e_a + e_b + e_{c_i} + 2e_0]' \simeq P(b, a+c_i) \oplus P(a),$$
$$[e_a + e_b + e_{c_1} + e_{c_2} + 2e_0]' \simeq P(a+c_1, c_2 b) \oplus P(a) \oplus Q(b).$$

Considering the diagram of inclusions




and proving statements analogous to Lemmas 1–3, one can obtain the following

Theorem 3. *The stratification functor* $'$: $\mathscr{R} \to \mathscr{R}'$ *induces an equivalence of the factor categories* $\mathscr{R}/\Theta \xrightarrow{\sim} \mathscr{R}'/\Theta'$.

Corollary 3. $\Gamma \backslash \{P(a), Q(b)\} \simeq \Gamma' \backslash \{P(a), Q(b)\}$, *where* $\Gamma$ *and* $\Gamma'$ *are the Auslander–Reiten quivers of the categories* $\mathscr{R}$ *and* $\mathscr{R}'$.

An object of the category $\mathscr{R}$ or $\mathscr{R}'$ will be called *reduced* (or $(a, b)$-*reduced*) if it does not contain direct summands $P(a)$ and $Q(b)$. For a given $U \in \mathrm{Ob}\,\mathscr{R}$ let $U^{\downarrow} \in \mathrm{Ob}\,\mathscr{R}'$ be a reduced object such that $U' \simeq U^{\downarrow} \oplus (P(a))^m \oplus (Q(b))^n$.

One can also attach (see [Z2]) to any $W \in \mathrm{Ob}\,\mathscr{R}'$ an object $W^{\uparrow} \in \mathrm{Ob}\,\mathscr{R}$ such that $W^{\uparrow\downarrow} \simeq W \oplus (P(a))^m \oplus (Q(b))^n$.

The correspondences $\downarrow$ and $\uparrow$ (nonfunctorial) are widely used in our proof since they determine an "almost" bijection between indecomposables. More exactly:

(a) $W^{\uparrow\downarrow} \simeq W$ iff $W$ is a reduced object;
(b) $U^{\downarrow\uparrow} \simeq U$ iff $U$ is a reduced object;
(c) $W$ is indecomposable iff $W^{\uparrow}$ is indecomposable;
(d) if $U$ is a reduced object, then $U$ is indecomposable iff $U^{\downarrow}$ is indecomposable (the correspondence $\downarrow$ is also called stratification).

Now let us add a few words about replenishment in the category $\mathscr{R}$. Let $\mathfrak{M}$ be a poset with a specific pair of points $(a, b)$ and $\bar{\mathscr{R}} = \mathscr{R}(\bar{\mathfrak{M}}_{(a,b)})$. Suppose $\operatorname{Ob}\mathscr{R} = \operatorname{Ob}\bar{\mathscr{R}}$ and $\operatorname{Mor}\mathscr{R} \subset \operatorname{Mor}\bar{\mathscr{R}}$ (i.e. $\mathscr{R}$ is not a full subcategory of $\bar{\mathscr{R}}$). Then the replenishment functor is nothing else but the natural inclusion functor $^-$: $\mathscr{R} \to \bar{\mathscr{R}}$. Set

$$\mathscr{I} = \{I_{P(a)}, I_{Q(b)}, I_{P(a,b)}\}_{\mathscr{R}}, \qquad \bar{\mathscr{I}} = \{I_{P(a)}, I_{Q(b)}\}_{\bar{\mathscr{R}}}.$$

THEOREM 4. *The replenishment functor* $^-$: $\mathscr{R} \to \bar{\mathscr{R}}$ *induces an equivalence between factor categories* $\mathscr{R}/\mathscr{I} \xrightarrow{\sim} \bar{\mathscr{R}}/\bar{\mathscr{I}}$.

COROLLARY 4. $\Gamma\backslash\{P(a), Q(b), P(a, b)\} \simeq \bar{\Gamma}\backslash\{P(a), Q(b)\}$, *where* $\Gamma$ *and* $\bar{\Gamma}$ *are the Auslander–Reiten quivers of the categories* $\mathscr{R}$ *and* $\bar{\mathscr{R}}$.

It follows from Corollaries 3 and 4 that the Auslander–Reiten quiver is changed insignificantly by the action of stratifications and replenishments. These changes can be easily described (see [Z2], § 5), in particular an arrow $U \to V$ survives a stratification or replenishment if $U, V \not\simeq P(a), Q(b), P(a, b)$. So, in order to describe some component one has to decrease dimensions of its indecomposables by applying the introduced functors and reducing the task to trivial objects (or to some standard poset, for example (1, 1, 1, 1)) for which the corresponding arrows are calculated trivially (or are known). At the same time one has to deal with singularities connected with the objects $P(a)$, $Q(b)$, $P(a, b)$.

The full description of nonhomogeneous components (Proposition 2) is rather bulky and includes many singularities; that is why we do not touch it here.

For the description of homogeneous components note that the operation $\downarrow$ can be defined for positive vectors (stratification for vectors) in a natural way (see [Z2], § 3), and the transformation rule for positive imaginary roots (by the action of $\downarrow$) coincides with that for dimensions of homogeneous indecomposables. It is not necessary to formulate this rule now; we only note that the dimension $\eta$ decreases if $a, b \in \operatorname{Supp}\eta$. And the replenishment acts on imaginary roots as the identity.

LEMMA 4. *Let* $\eta > 0$ *be an imaginary root of a poset* $\mathfrak{M}$ *of finite growth. Then with the help of a finite number of stratifications and replenishments (without restrictions to supports!) one can transform* $\eta$ *to an imaginary root* $\eta_1$ *with support* (1, 1, 1, 1).

*Proof.* Set $\operatorname{Supp}\eta = N$. Suppose $N \neq K_1$ and proceed by induction on $\omega(\eta) = \sum_{x\in\mathfrak{M}} \eta(x)$. As $N \neq K_1$ there exists a suitable pair of points $(a, b)$ of the poset $N$. Let $N = a^{\Delta}+b^{\nabla}+\{c_1, \ldots, c_n\}$. We can assume $(a, b)$ to be chosen in such a way (see the lists of critical and determinative posets) that the poset $a^{\Delta}\cap N$ $(b^{\nabla}\cap N)$ contains a chain $\{a_1 < a_2\}$ $(\{b_1 < b_2\})$ which is not comparable with the point $c_n$ $(c_1)$. Set

$$P = P(\mathfrak{M}) = \{p\in\mathfrak{M} \mid p < a,\ p \nless b\}, \qquad Q = Q(\mathfrak{M}) = \{q\in\mathfrak{M} \mid q > b,\ q \ngtr a\}.$$

Clearly $P\cap N = Q\cap N = \emptyset$. If $p\in P$, $q\in Q$ then one can verify that $(p, q)$ is a specific pair of $\mathfrak{M}$, and $\mathfrak{M}$ can be $(p, q)$-replenished. Assume all such replenishments have been done, i.e. $P = \emptyset$ or $Q = \emptyset$.

Suppose $P = Q = \emptyset$ and consider the poset $L = \mathfrak{M}\backslash(a^{\Delta}+b^{\nabla})$. If $x\in L \cap(N^{\Delta}\backslash N)$, then $x > c_n$ because otherwise $\mathfrak{M} \supset \{a_1 < a_2, c_n, x, b\} = (1, 1, 1, 2)$. Analogously, $y < c_1$ whenever $y\in L\cap(N^{\nabla}\backslash N)$. If $x, y\in L$ is a pair of incomparable points and e.g. $x, y > c_n$, then again $\mathfrak{M} \supset \{a_1 < a_2, x, y, b\} = (1, 1, 1, 2)$. Hence $L$ is a chain and $(a, b)$ is a suitable pair of points of $\mathfrak{M}$ with respect to which we can stratify, decreasing $\omega(\eta)$.

Suppose $Q \neq \emptyset$. Let $q\in\max Q$. Set $L = \mathfrak{M}\backslash(a^{\Delta}+q^{\nabla})$. It follows from an analogous reasoning that $L$ is a chain and $(a, q)$ is a suitable pair for $\mathfrak{M}$. Applying the $(a, q)$-stratification and $(a, q)$-replenishment we get a poset $\mathfrak{M}'$ and its imaginary root $\eta'$ with the same support $N$ and the same sum $\omega(\eta') = \omega(\eta)$, but with a smaller set $Q' = Q(\mathfrak{M}')$. Continuing this process we finally obtain the case $P = Q = \emptyset$ considered above. This finishes the proof.

It follows from Lemma 4 that any homogeneous component of a poset of finite growth is isomorphic to a homogeneous component of $K_1$ and therefore has the form (H).

Finally, the description of nonregular components $QP[N, K]$ is based on the following lemma, where $\Phi_l$ $(l \geqslant 0)$ is the poset of Fig. 16.

LEMMA 5. *Let $K \prec N$ be two neighbouring weakly coupled critical subsets of a poset $\mathfrak{M}$ of finite growth and suppose $N^{\nabla}\cap K^{\Delta} \neq \Phi_l$. Then there exists a finite sequence of stratifications and replenishments which transforms $\mathfrak{M}$ into an*

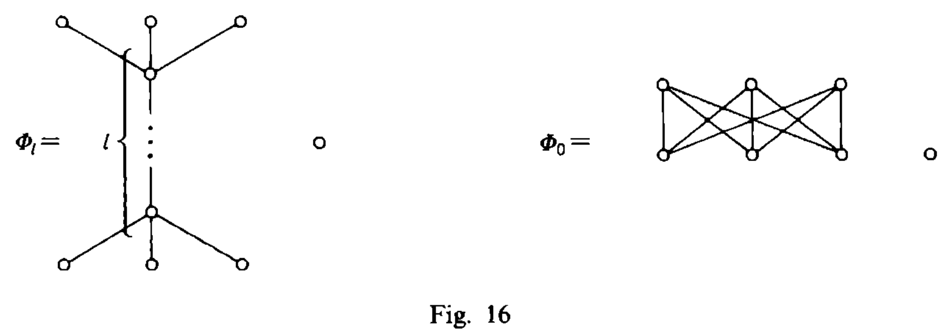



Fig. 16

*ordinal sum* $\mathfrak{M}' = A + B$, *where* $a < b$ *for any* $a \in A$, $b \in B$ *and, moreover,* $K' \subset A$, $N' \subset B$ *where* $K'$, $N'$ *are the images of the subsets* $K$, $N$ *in this sequence of operations.*

*Remark.* The image of a critical set $K$ by a stratification or replenishment is to be understood as the support of the image of the imaginary root $\mu_K$ (which is again a critical set).

*Proof.* Set $N^{\nabla} \cap K^{\Delta} = W$. According to Lemma 4 we can assume $N \simeq K \simeq K_1$ (note that $W \neq \Phi_l$ even if we have applied the operations as in Lemma 4). Making all possible replenishments we obtain $x < y$ for any $x \in K^{\nabla} \setminus K$, $y \in K^{\Delta} \setminus K$ and any $x \in N^{\nabla} \setminus N$, $y \in N^{\Delta} \setminus N$.

Let $K = \{u_1, \ldots, u_4\}$, $N = \{v_1, \ldots, v_4\}$. If $K \cap N \neq \emptyset$ and e.g. $u_1 = v_1$, then $K \cap N = \{u_1\}$, because $\mathfrak{M}$ does not contain the poset shown in Fig. 1, and from the inequality $W \neq \Phi_l$ one can conclude that there exists a suitable pair $(a, b)$ with $a, b \in \{u_2, u_3, u_4\}^{\Delta} \cap \{v_2, v_3, v_4\}^{\nabla}$ such that the $(a, b)$-stratification gives two critical subsets $\{u_1 b, u_2, u_3, u_4\}$ and $\{u_1 + a, v_2, v_3, v_4\}$.

Assume now that $K \cap N = \emptyset$. Set $A = W \setminus (K + N)$. After every operation (i.e. stratification or replenishment) we obtain new posets $K, N, W, A, \ldots$ which will be denoted by the same letters. Note that after any step $W$ does not contain critical subsets different from $K$ and $N$.

Now, do the following in consecutive order:

1) Operating with respect to pairs $(u_i, v_j)$, obtain $u_i < v_j$ for all $i, j$.

2) Operating with respect to pairs $(x, v_j)$, $x \in A$, obtain a point $a \in A$ with $N \subset a^{\Delta}$.

3) Operating with respect to pairs $(u_i, x)$, $x \in a^{\Delta}$, obtain also $K \subset a^{\nabla}$. Let $B = A \setminus (a^{\Delta} \cup a^{\nabla})$.

4) Operating with respect to pairs $(x, b)$, $x \in a^{\nabla}$, $b \in B \cap u_1^{\Delta}$, after a finite number of steps obtain the case $B \cap u_1^{\Delta} = \emptyset$ (in fact we operate with the poset $W \setminus (N \cup \{u_1\})$ of finite type decreasing the number of its positive roots).

5) As $B \cap u_1^{\Delta} = \emptyset$ and the width of $B$ is $\leqslant 2$, one of $u_2$, $u_3$, $u_4$, say $u_2$, satisfies $\min B \subset u_2^{\Delta}$. Operating now with respect to pairs $(x, b)$, $x \in a^{\nabla}$, $b \in B$, after a finite number of steps obtain the case $B = \emptyset$ (in fact we operate with the poset $W \setminus (N + \{u_2\})$ of finite type). But $W$ is ordinally decomposable if $B = \emptyset$. This finishes the proof.

Applying this lemma we can always (except in the trivial case $N^{\nabla} \cap K^{\Delta} = \Phi_l$) transform every weak coupling into an ordinal sum and "glue" the component $QP[N, K]$ from the known components $Q(\check{N})$ and $P(\hat{K})$.

*Final remark.* A detailed explanation of the functors used in the paper is contained also in the paper of D. Simson [S], where many related questions are discussed.

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