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SEMI-INFINITE PROGRAMMING PROBLEMS

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This paper, a summary of a lecture given at the Banach Center, presents some aspects of numerical treatment of convex semi-infinite optimization problems. In the first part of this paper the theoretical foundation of numerical treatment is studied: continuity properties and the convergence of sequences of approximations of semi-infinite problems. The second part deals with the application of the method of feasible directions to semi-infinite optimization. Sufficient conditions for the convergence of this method in the case of nondifferentiable problems are stated and adaptive discretization strategies for the semi-infinite direction search problems are discussed. All results are given without proofs, which may be found in the original papers referred to.

Introduction

Semi-infinite programming deals with the problem

$$\min\{f(x):\ g(x,t)\leqslant 0,\ t\in M,\ x\in X\}.$$

In the case of a finite set M, this is nothing but an ordinary optimization problem with a finite number of inequalities. But there are many problems where the set M is not finite, for example problems with constraints depending on time or space coordinates (i.e., general approximation problems, variational inequalities, optimal control problems) (cf. [8], [13]).

While the theory of optimality conditions for semi-infinite programming has been developed to a high level (cf. [4], [8], [10]), only a few papers are concerned with the numerical treatment of such problems (cf. [6], [13]).

Since semi-infinite programming problems cannot be solved directly, discretization methods must be used. In this way, only approximate solutions are obtained. For this reason, continuity properties of such problems with respect to perturbations of the data must be studied.

The analysis of the continuity and stability of the set of optimal solutions and the optimal value of the objective function is based on continuity and stability properties of the restriction domain. If the behaviour of the restriction domain under small perturbations of the input data is bad, one cannot expect the set of optimal solutions and the optimal value of the objective function to have a good behaviour. The connections between the stability of the restriction domain and the stability of the set of optimal solutions are considered for several classes of finite and infinite-dimensional programming problems (cf. [3], [5], [7], [9], [14]).

Statements on the continuity and stability of semi-infinite programming problems can be used to ensure the convergence of an appropriate sequence of approximations of semi-infinite programming problems. These problems will be considered in the first part of this paper.

Computational methods for solving semi-infinite programming problems can be divided into two groups: methods of a priori discretization and methods of adaptive discretization. In methods of a priori discretization a grid $M' \subset M$ is chosen a priori and then the arising finite-dimensional problem is solved by usual computational algorithms for programming problems. This concept has two striking disadvantages: first, the optimal solutions of the approximate problem are not feasible for the semi-infinite problem; second, in order to obtain a sufficiently exact approximation of an optimal solution of the original semi-infinite problem, a grid M' of high density must be used. These facts cause some numerical difficulties. The idea of adaptive discretization is to improve the discretization used successively depending on the information available at each iteration step. By these methods semi-infinite programming problems can be handled in an effective manner. Methods of feasible directions belong to the class of methods with adaptive discretization. In the second part of this paper we consider methods of the type described in [15]. The discretization of the semi-infinite direction search programs depends on the precision of the feasible approximate direction. Under natural assumptions on the smoothness of the functions involved, a sequence of feasible solutions converges to an optimal solution of the semi-infinite original problem.

1. Continuity properties and convergence of sequences of approximations of semi-infinite programming problems

We consider the following problem:

$$\min\{f(x):\ g(x,t) \leqslant 0,\ t \in M,\ x \in X\} \tag{1.1}$$

under the

ASSUMPTION A 1.1.

(i) $X \subset R^n$, $\operatorname{int} X \neq \emptyset$, X is closed and convex,

(ii) M is a compact subset of a metric space Y,

(iii) for an arbitrary fixed $t \in M$ the function $g(x, t)$ is convex on a neighbourhood of X,

(iv) $g(x, t)$ is continuous on $X \times M$, f is continuous and convex on X,

(v) problem (1.1) is solvable, i.e., $X_{\mathrm{opt}} \neq \emptyset$.

ASSUMPTION A 1.2. Problem (1.1) satisfies a Slater condition, i.e., there is an $\tilde{x} \in \operatorname{int} X$ such that $g(\tilde{x}, t) < 0$ $\forall t \in M$.

For each fixed $t_0 \in M$ the map $g_{t_0}(x)$: $R^n \to R$ is convex and defined by $x \to g(x, t_0)$. Therefore, the set of feasible points of the finite convex programming problem

$$\min\{f(x):\ g_t(x) \leqslant 0,\ x \in X\}, \qquad t \in M \text{ fixed} \tag{1.2}$$

is convex and compact.

Now, introduce the following maps:

$$\psi:\ t \to \psi(t) = \{x \in X:\ g_t(x) \leqslant 0\},$$

the point-to-set map of feasible solutions to the parameter t;

$$\varphi:\ t \to \varphi(t) = \min\{f(x):\ x \in \psi(t)\},$$

the optimal value function and

$$\Phi:\ t \to \Phi(t) = \{x \in \psi(t):\ f(x) = \varphi(t)\},$$

which assigns the set of optimal solutions to t.

By using the definition of continuity for point-to-set maps according to [1], the following statements can be proved (cf. [14]).

THEOREM 1.3. *Let $t' \in M$ be fixed and* A 1.1 *be satisfied. Then the point-to-set map ψ is upper-semi-continuous* (u.s.c.) *at the point t'. If in addition* A 1.2 *holds, then ψ is also lower-semi-continuous* (l.s.c.).

Remark 1.4. If the Slater condition is not satisfied, the map ψ need not be l.s.c. at t' as examples in [13] show.

THEOREM 1.5. *Let $t' \in M$ be fixed and let* A 1.1 *and* A 1.2 *be satisfied. Then the point-to-set map Φ is* u.s.c. *at t'.*

THEOREM 1.6. *Let $t' \in M$ be fixed and let* A 1.1 *hold. Then the functional φ is* u.s.c. *at t'. If, together with* A 1.1, A 1.2 *is satisfied, then φ is* l.s.c. *at t'.*

Remark 1.7. In the theorems above the continuity of $g(x, t)$ on $X \times M$ is required. Since M is a compact set, the map $[x, t] \in X \times M \to g(x, t)$ is continuous on the product space if and only if

(a) the family of functionals g_t is uniformly continuous at all $x \in X$ and

(b) the map $t \in M \to g_t(x)$ is continuous $\forall x \in X$.

These statements on the continuity and stability of problem (1.1) can be used to ensure the convergence of an appropriate sequence of approximations of problem (1.1).

As such an approximation the following problem is considered:

$$\min\{f(x):\ g(x, t) \leqslant 0,\ t \in M_k,\ x \in X\}, \tag{1.3}$$

where $M_k \subset M$ is a finite set for all k and M_k converges to M in the sense of the metric space Y ($M_k \xrightarrow{Y} M$). Hence, the approximate problems (1.3) are finite-dimensional convex problems and computable ones.

As an immediate consequence of the continuity statements described in the theorems above, the convergence of problems (1.3) to the semi-infinite problem (1.1) can be proved in the following sense (cf. [14]):

THEOREM 1.8. *Let* A 1.1 *be satisfied and* $M_k \xrightarrow{Y} M$. *Then*

(i) *an optimal solution* x^k *of problem* (1.3) *exists for* k *sufficiently large, i.e.,*

$$\exists k' \forall k \geqslant k':\ f(x^k) = \min\{f(x):\ g(x, t) \leqslant 0,\ t \in M_k,\ x \in X\};$$

(ii) *the sequence* $\{x^k\}$ *of optimal solutions of* (1.3) *contains at least one convergent subsequence and all cluster points of* $\{x^k\}$ *belong to the set of optimal solutions of* (1.1).

In a computer only finitely many digits with limited precision can be stored, and therefore it is very helpful to know the error bounds for the optimal solutions of the approximate problems (1.3) in connection with the semi-infinite problem (1.1). In the following such error bounds are described in relation to the density of the set M_k and the smoothness of the functional $g(x, t)$.

We consider the following error bounds:

$$\delta(x) = \max\{g(x, t):\ t \in M\},$$

which is a measure of feasibility of any point $x \in X$ and

$$\delta^* = |f(x^*) - f(x^k)|,$$

which is the defect between the optimal values of problems (1.1) and (1.3).

By introducing the following notation:

$$h = \max_{t\in M} \min_{t_i\in M_k} \|t-t_i\|$$

(density of the grid M_k),

$$w_g(z) = \max\{|g(t)-g(t')|:\ \forall t, t' \in M,\ \|t-t'\| \leqslant z\}$$

(modulus of continuity), the following statements can be shown (cf. [13]):

THEOREM 1.9. *Let x^k be an optimal solution of* (1.3), *x^* an optimal solution of* (1.1), *x^0 a Slater point of* (1.1) *and* $\gamma = \max\{g(x^0, t):\ t \in M\}$.
Then

$$\delta(x^k) \leqslant w_g(h)$$

and

$$\delta^* \leqslant w_g(h)\,|\gamma^{-1} f(x^0)|$$

holds.

Remark 1.10. In the case of linear semi-infinite programming problems, i.e.,

$$g(x, t) \equiv \sum f_j(t)x_j - b(t),$$

X — polyhedral, the modulus of continuity is decomposable and therefore

$$\delta(x^k) \leqslant \sum |x_j^k|\, w_{f_j}(h) + w_b(h). \tag{1.4}$$

If X is a cube, then formula (1.4) can be used for a priori estimations.

If the modulus of continuity can be replaced by bounded partial derivatives of the function g, then the estimations can be expressed analogously in terms of the bounds of the derivatives of g.

2. Methods of feasible directions for semi-infinite programming problems

Now we consider the following problem

$$\begin{aligned} &f(x) \to \min, \\ &g(x, t) \leqslant 0 \quad \forall t \in M, \\ &x \in X = \{x:\ f_i(x) \leqslant 0,\ i = 1(1)m\}, \end{aligned} \tag{2.1}$$

under the assumptions A 1.1 and

ASSUMPTION A 2.1.

(i) $\nabla_x g(x, t)$ is continuous on $X \times M$.

(ii) $f, f_1, \ldots, f_m$ are convex and differentiable on a neighbourhood of X.

Before describing the method of feasible directions for problem (2.1), recall this method for unrestricted optimization problems of the form

$$\min\{F(x):\ x \in R^n\}. \tag{2.2}$$

Any realization of the following basic algorithm is referred to as a method of feasible directions for solving problem (2.2):

Let $x_0 \in \operatorname{dom} F$ and suppose that the points $x_1, \ldots, x_k \in \operatorname{dom} F$ have already been generated by the algorithm. Then the point x_{k+1} is determined by the following two steps:

1. Determine a direction r_k such that

$$F'(x_k, r_k) < 0$$

(direction search).

2. Determine a scalar λ_k such that

$$F(x_{k+1}) = F(x_k + \lambda_k r_k) = \min\{F(x_k + \lambda r_k):\ \lambda \geqslant 0\}$$

(step size determination by line search).

The convergence of the method of feasible directions obviously depends on the directions used, one of the difficulties being that zig-zagging must be avoided. If the direction search is made by the following program:

Determine r_k such that

$$F'(x_k, r_k) = \min\{F'(x_k, r):\ \|r\| \leqslant 1\},$$

where $F'(x, r)$ is the directional derivative in direction r at the point x, the method may not necessarily converge.

An approach to the construction of appropriate programs for the direction search using anti-zig-zagging rules will be given in the sequel. Put

$$Y = \{x:\ g(x, t) \leqslant 0 \ \ \forall t \in M\}$$

and (with $\delta(x \mid X)$ denoting the characteristic function of the set X)

$$F(x) = f(x) + \delta(x|Y) + \delta(x|X).$$

Then problem (2.1) is equivalent to problem (2.2). Next, the subdifferential of the function F must be known. It is easy to show that by A 1.1 and A 2.1 the subdifferential of F is

$$\partial F(x) = \nabla f(x) + \partial\delta(x|Y) + \partial\delta(x|X) \tag{2.3}$$

with

$$\partial\delta(x|X) = \operatorname{cone}\big\{\operatorname{conv}\{\nabla f_i(x):\ i \in I(x)\}\big\},$$

$$I(x) = \{i:\ f_i(x) = 0\}$$

and

$$\partial\delta(x|Y) = \text{cone}\{\text{cl}\,\text{conv}\{\nabla_x g(x, t):\ t \in M(x)\}\},$$
$$M(x) = \{t \in M:\ g(x, t) = 0\}.$$

The basis for the following considerations is the following general algorithm (MFD):

(MFD): Let $\|\cdot\|$ be an arbitrary norm of the Euclidean space, let $\{\varepsilon_l\}$ be any sequence of positive numbers (decreasing to 0) and let $\{h_l\}$ be a sequence of functions such that h_l: $R^n \times B \to \bar{R}$ and $h_l(x, r) \geq F'(x, r)$ $\forall (x, r) \in \text{dom}F \times B$, $B = \{r:\ \|r\| \leq 1\}$.

Step 1° Let $x_0 \in \text{dom}F$, $k := 0$, $l := 0$

2° Determine r_k such that

$$h_l(x_k, r_k) = \min\{h_l(x_k, r):\ r \in B\}$$

3° (a) If $h_l(x_k, r_k) < -\varepsilon_l$, go to 4°

(b) otherwise go to 6°

4° Determine λ_k such that

$$F(x_k + \lambda_k r_k) = \min\{F(x_k + \lambda r_k):\ \lambda \geq 0\}$$

and

$$x_{k+1} := x_k + \lambda_k r_k$$

5° $k := k+1$ go to 2°

6° $l := l+1$ go to 2°.

An analysis of this method shows that the inner cycle (steps 2°–5°) must be carried out in a finite number of steps for each index l. Otherwise an implementation of the method cannot be guaranteed. By using the notation

$$X(h_l, \varepsilon_l) = \{x:\ h_l(x, r) \geq -\varepsilon_l \,\forall r \in B\},$$
$$X(F', 0) = \{x:\ F'(x, r) \geq 0\ \forall r \in B\},$$

the convergence of the method (MFD) can be obtained under the following two conditions:

(1) For every fixed l the method (MFD) generates an element of $X(h_l, \varepsilon_l)$ in a finite number of steps,

(2) A sequence $\{h_l\}$ of functions must be chosen such that

$$\varliminf_{l\to\infty} X(h_l, \varepsilon_l) \subseteq X(F', 0).$$

The following three variants of direction search programs satisfy these two conditions and the convergence proofs are given in [15].

The first variant arises from an extension of the subdifferential (2.3) while in the other two variants the directional derivatives of the function F are used directly.

VARIANT A. In this variant the function h_l is chosen as

$$h_l(x, r) = \sup\{\langle r, y\rangle: \ y \in P_{\varepsilon_l}(x)\}, \tag{2.4}$$

where $\varepsilon_l > 0$ can be chosen at will and

$$\begin{aligned} P_{\varepsilon_l}(x) &= \nabla f(x) + \operatorname{cone}\{\operatorname{conv}\{\nabla f_i(x) + \varepsilon_l B: \ i \in I_{\varepsilon_l}(x)\}\} + \\ &\quad + \operatorname{cone}\{\operatorname{cl}\operatorname{conv}\{\nabla_x g(x, t) + \varepsilon_l B: \ t \in M_{\varepsilon_l}(x)\}\}, \\ I_{\varepsilon_l}(x) &= \{i: \ -\varepsilon_l \leqslant f_i(x) \leqslant 0\}, \\ M_{\varepsilon_l}(x) &= \{t \in M: \ -\varepsilon_l \leqslant g(x, t) \leqslant 0\}. \end{aligned} \tag{2.5}$$

THEOREM 2.2. *Let $\{\varepsilon_l\}$ be a sequence of positive numbers decreasing to* 0 *and let $\{h_l\}$ be constructed by* (2.4). *Suppose that the sequence $\{x_k\}$ is generated by* (MFD). *Then*

(i) $F(x_{k+1}) < F(x_k) \ \forall k$,

(ii) *$\{x_k\}$ is bounded and for every cluster point x_∞ of $\{x_k\}$ we have*

$$F'(x_\infty, r) \geqslant 0 \quad \forall r \in B,$$

i.e., $x_\infty \in X_{\text{opt}}$.

VARIANT B. Put

$$h(x, r) = \min\{z: \ (r, z) \in R(x)\}, \tag{2.6}$$

where $\delta > 0$ and

$$\begin{aligned} R(x) &= \{(r, z): \ \langle \nabla f(x), r\rangle \leqslant z, f_i(x) + \langle \nabla f_i(x), r\rangle \leqslant z, \ i \in I_\delta(x), \\ &\qquad g(x, t) + \langle \nabla_x g(x, t), r\rangle \leqslant z, \ t \in M_\delta(x)\}, \\ I_\delta(x) &= \{i: \ -\delta \leqslant f_i(x) \leqslant 0\}, \\ M_\delta(x) &= \{t \in M: \ -\delta \leqslant g(x, t) \leqslant 0\}. \end{aligned} \tag{2.7}$$

THEOREM 2.3. *Let the sequence $\{x_k\}$ be generated by* (MFD) *with arbitrary fixed $\varepsilon_l = \delta > 0 \ \forall l$, (i.e., the sequence $\{\varepsilon_l\}$ is constant and does not tend to* 0). *Then*

(i) $F(x_{k+1}) < F(x_k) \ \forall k$;

(ii) *Every cluster point x_∞ of $\{x_k\}$ is an optimal solution of* (2.1).

This approach to the direction search program is equivalent to that of Oettli (cf. [2]).

VARIANT C. A further direct realization of a function $h(x, r)$ satisfying the convergence conditions is

$$\begin{aligned} h(x, r) = \max\{&f(x+r) - f(x), \max\{f_i(x+r): \ i \in I_\delta(x)\}, \\ &\max\{g(x+r, t): \ t \in M_\delta(x)\}\} \end{aligned} \tag{2.8}$$

with $I_\delta(x)$ and $M_\delta(x)$ as in (2.7) and $\delta > 0$ fixed, too.

The convergence theorem is analogous to Variant B. Variant C works without derivatives, and hence the functions involved need not be differentiable. Difficulties in this variant occur in the numerical treatment of the direction search problem, which, according to (2.8), is

$$\min\{h(x, r): r \in B\}.$$

In this case methods without derivatives must be used, i.e., methods of random search.

In all the three variants the direction search problems are semi-infinite problems, because the sets $M_\varepsilon(x)$ and $M_\delta(x)$ may be infinite sets. The purpose of the next few pages is to show a treatment of these semi-infinite direction search problems which permits us to determine a suitable and feasible direction for the (MFD). The application of an a priori discretization concept for the semi-infinite direction search programs is possible in principle. However, a discretization which is successively improved according to the information available in each iteration step seems to be more appropriate. This means discretizing the set M in such a way that at each step discretization points are used only to an extent necessary to determine a suitable and feasible direction.

In order to describe this approach, consider the following Lipschitz condition.

ASSUMPTION A 2.4. For every $x \in X$ there exists a constant $L(x)$ with

$$|g(x, t') - g(x, t)| \leqslant L(x)\,\|t' - t\|$$

and

$$\|\nabla_x g(x, t') - \nabla_x g(x, t)\| \leqslant L(x)\,\|t' - t\| \qquad \forall t', t \in M.$$

Then the following statement holds:

LEMMA 2.5. *Let r be a direction at the point $(\bar{x}, \bar{t})$ with $r \in B$ such that*

$$\langle \nabla_x g(\bar{x}, \bar{t}), r\rangle \leqslant -\gamma.$$

Then

$$\langle \nabla_x g(\bar{x}, t), r\rangle \leqslant -\gamma/2 \qquad \forall t \in \{t \in M:\ \|t - \bar{t}\| \leqslant \gamma(2L(x))^{-1}\}.$$

Let M_h be an arbitrary finite discretization of the set M and let

$$h = \max_{t \in M} \min_{t_i \in M_h} \|t - t_i\|$$

denote the density of the point set M_h. Lemma 2.5 serves to prove the important statement of the next lemma:

LEMMA 2.6. *Let*

$$h \leqslant \min\{\gamma(2L(\bar{x}))^{-1}, \varepsilon(2L(\bar{x}))^{-1}\}$$

and

$$\tilde{M}_h(\bar{x}) = \{t \in M_h:\ g(\bar{x}, t) \geqslant -\varepsilon\}.$$

If a direction r exists with $r \in B$ and

$$\langle \nabla_x g(\bar{x}, t), r \rangle \leqslant -\gamma \quad \forall t \in \tilde{M}_h(\bar{x}),$$

then

$$\langle \nabla_x g(\bar{x}, t), r \rangle \leqslant -\gamma/2 \quad \forall t \in M_{\varepsilon/2}(x) = \{t \in M:\ -\varepsilon/2 \leqslant g(\bar{x}, t) \leqslant 0\}.$$

In the sequel such a discretization concept is described for Variant A. By using the results in Variant A the semi-infinite direction search problem

$$\min_{r \in B} \sup\{\langle r, y \rangle:\ y \in P_{\varepsilon_l}(x)\}$$

is obtained. By definition (2.5) of $P_{\varepsilon_l}(x)$ this problem can be expressed by

$$(2.9)\qquad \begin{aligned} &\langle r, \nabla f(x) \rangle \to \min, \\ &\langle r, \nabla f_i(x) \rangle \leqslant -\varepsilon_l \quad \forall i \in I_{\varepsilon_l}(x), \\ &\langle r, \nabla_x g(x, t) \rangle \leqslant -\varepsilon_l \quad \forall t \in M_{\varepsilon_l}(x), \\ &\|r\| \leqslant 1. \end{aligned}$$

If r_x is an optimal solution of problem (2.9) with

$$\langle r_x \nabla f(x) \rangle < -\varepsilon_l,$$

then, according to step 3° of (MFD), r_x can be used as a suitable direction for (MFD).

Now, by using Lemmas 2.5 and 2.6 it is possible to discretize the set $M_{\varepsilon_l}(x)$ in such a way that an optimal solution of the discretized problem satisfies the restrictions of (2.9). By using A 2.4, a discertization M_h with

$$h = \varepsilon_l \big(2L(x)\big)^{-1}$$

can be chosen. Then, consider the finite set

$$(2.10)\qquad \tilde{M}^h_{\varepsilon_l}(x) = \{t \in M_h:\ g(x, t) \geqslant -2\varepsilon_l\}.$$

This set can be determined without difficulty, since the set M_h has only finitely many points. Using (2.10), (2.9) yields the following finite programming problem:

$$(2.11)\qquad \begin{aligned} &\langle r, \nabla f(x) \rangle \to \min, \\ &\langle r, \nabla f_i(x) \rangle \leqslant -\varepsilon_l, \quad i \in I_{\varepsilon_l}(x), \\ &\langle r, \nabla_x g(x, t) \rangle \leqslant -2\varepsilon_l, \quad t \in \tilde{M}^h_{\varepsilon_l}(x), \\ &\|r\| \leqslant 1. \end{aligned}$$

Let r^* be an optimal solution of (2.11) with

$$\langle r^*, \nabla f(x) \rangle < -\varepsilon_l.$$

Then this solution satisfies all conditions of the (MFD). The choice of the norm in problem (2.11) determines the class of optimization problems to which this problem belongs. By choosing the L_∞-norm linear problems are obtained. This approach is equivalent to Zoutendijk's P1-Algorithm [16] for finite-dimensional problems.

The step-size problem

$$\min\{F(x+\lambda r):\ \lambda \geqslant 0\}$$

can be handled by classical one-dimensional methods for convex functions. The essential problem is the determination of feasibility for the point $x+\lambda r$ for problem (2.1). The determination of the feasible step-length can be computed by using only the finite set M_h.

Remark 2.7. 1. The fact that no nonconvex programming problem of the form

$$\max\{g(x,t):\ t \in M(x)\}$$

has to be solved (as in cutting plane methods) is an important advantage of this discretization concept.

2. The Lipschitz condition of the function g in t (with respect to its gradients) allows the determination of a bound of density for the set M_h. Without difficulty, an estimation for the dimension of the finite-dimensional direction search problem can be given according to the parameter ε_l. If a minimal precision ε_l for the method of feasible direction is given, the maximal number of restrictions for the direction search problem using formula (2.10) can be estimated. In addition, it is an advantage that at the beginning of the algorithm we use a grid of relatively low density which is subsequently enlarged only if, due to the decreased ε_l, no suitable direction can be found by the present approximation.

3. In the discretization concept described above it was assumed that the Lipschitz constant of the function g is known for fixed x. The method is also practicable if this constant is unknown. Then begin with an estimation L_0 of this constant and determine the grid constant h with it. If the resulting direction, which is determined by the given ε_l, is not feasible, or if the step-size is too small, then the chosen L_0 is too small. Therefore, L_0 must be increased in an appropriate manner.

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