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BOUNDS FOR CASTELNUOVO'S REGULARITY AND THE GENUS OF PROJECTIVE VARIETIES

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Using intrinsic geometrical properties of projective varieties we will improve Harris' bound for the geometric genus of varieties in $\mathbf{P}^n$. Furthermore, we will get new and sharp bounds for the genus of arithmetically Buchsbaum varieties and of varieties of codimension 2. In case of Buchsbaum varieties we also prove sharp bounds for Castelnuovo's regularity. Our approach in proving such bounds is to reduce the problem to the case of a collection of points in uniform position. This means the key idea here is the uniform position principle developed by J. Harris in case of space curves. Hence the present paper relies upon an analysis of the Hilbert function of the section of a subvariety V with a generic linear subspace of dimension $= \operatorname{codim}(V)$. Finally we improve some bounds in case of space curves.

0. Introduction

The study of possible genera of irreducible space curves in $\mathbf{P}^3$ has a fairly long history (see, e.g. [14–21]). A main problem is the following:

Given integers $d, k > 0$, we wish to find the maximum genus $g = G(d, k)$ of an irreducible nonsingular curve in $\mathbf{P}^3$ of degree d which is not contained in any surface of degree $< k$. This problem is still open. Our Theorem 5 of Section 5 yields contributions to solve this problem by applying new Castelnuovo bounds. Moreover, in this paper, we will study the analogous question for projective varieties of arbitrary dimension: what is the greatest possible geometric genus of an irreducible, nondegenerate variety of degree d in $\mathbf{P}^r$? This problem was solved in 1981 by J. Harris [16] (see our Corollary 6). Using

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intrinsic geometrical properties of projective varieties we will improve Harris' bound for the geometric genus of varieties in $\mathbf{P}^r$ (see our Theorem 1 of Section 2). Furthermore, we will get new and sharp bounds for the genus of arithmetically Buchsbaum varieties (see Theorem 3) and of varieties of codimension 2 (see Theorem 4). In case of Buchsbaum varieties we also prove sharp bounds for Castelnuovo's regularity. We therefore strengthen and extend main results of [32] and [35] (see Theorem 2).

Our approach in proving such bounds is to reduce the problem to the case of a collection of points in uniform position. This means the key idea here is the uniform position principle developed by J. Harris [15], [17] in case of space curves. Hence the present paper relies upon an analysis of the Hilbert function of the section $V \cap L$ of a subvariety V with a generic linear subspace L of dimension $= \operatorname{codim}(V)$. This makes it possible to give new bounds on the geometric genus of projective varieties V (see Section 2). We return in Section 3 and 4 to consider arithmetically Buchsbaum varieties and varieties with codimension 2. Finally in Section 5 we study space curves.

1. Notations and preliminary results

First we will recall some basic facts on finite sets of points in uniform position.

In general, the Hilbert function h_X of a subscheme X of $\mathbf{P}^r$ is defined by letting $h_X(t)$ be the rank of the tth graded piece of the homogeneous coordinate ring $S_X := K[x_0, \ldots, x_n]/I_X$ of X, where K is an algebraically closed field. We set

$$\Delta h_X(t) = h_X(t) - h_X(t-1).$$

DEFINITION ([15], [17]). A set X of d points in $\mathbf{P}^r$ is said *to be in uniform position* if for any subset S of X consisting, say, of s points ($1 \leqslant s \leqslant d$), we have

$$h_S(t) = \min\{s, h_X(t)\} \qquad \text{for all } t \geqslant 0.$$

Before stating our results on points in uniform position, we give some general observations and collect some known results that we need.

LEMMA 1 (see [17], Corollary 3.5). *Let X be a set of d points in uniform position. Then for any integers $t_1, \ldots, t_m$ one has*

$$h_X(t_1 + \ldots + t_m) \geqslant \min\{d, \sum_{i=1}^{m} h_X(t_i) - m + 1\}.$$

For a set X of d points we set

$$r(X) := \min\{t \in \mathbf{N} : h_X(t) = d\}.$$

LEMMA 2 ([15], [22], Corollary 2). *Let X be a finite set of points in uniform position in $\mathbf{P}^2$. Then we have*

$$\Delta h_X(t) = \begin{cases} t+1 & \text{for } 0 \leqslant t \leqslant a-1, \\ a & \text{for } a \leqslant t \leqslant b-1, \\ < \Delta h_X(t-1) & \text{for } b \leqslant t \leqslant r(X), \\ 0 & \text{for } \quad t > r(X), \end{cases}$$

where $a = \min\{t \in \mathbf{N} : h_X(t) < \binom{t+2}{2}\}$, *and*

$$b = \begin{cases} a & \text{if } h_X(a) \leqslant \binom{a+2}{2} - 2, \\ \min\{t > a : \Delta h_X(t) < a\} & \text{otherwise.} \end{cases}$$

Supplement. Moreover, it follows from [22], Proposition 1, that

$$\text{degree}(X) \leqslant ab.$$

LEMMA 3. *Let* $X \subset \mathbf{P}^r$ $(r \geqslant 2)$ *be a set of* d *points in uniform position, spanning* $\mathbf{P}^r$. *We set* $m := \min\{t \in \mathbf{N} : h_X(t) \geqslant d-r+1\}$. *Then the ideal* $I(X)$ *of* X *is generated by forms of degree* $\leqslant m+1$.

Proof. We will apply Theorem 1.2 of [24]. We note that the points are also in general position. Let $U \subseteq X$ be a subset of $d-r+1$ points. Since X is in uniform position we have

$$h_U(m) = \min\{d-r+1, h_X(m)\} = d-r+1.$$

Let $\mathscr{I}_U$ denote the ideal sheaf of U in $\mathbf{P}^r$. It follows from [30] that $(d-r+1) - h_U(t) = \dim_K H^1(\mathbf{P}^r, \mathscr{I}_U(t))$, that is, $H^1(\mathbf{P}^r, \mathscr{I}_U(m)) = 0$. Since the condition (i) of Theorem 1.2 of [24] can be dropped in the statement (this is a simple consequence of [23], Lemma 2.1) we obtain our Lemma 3 from this Theorem 1.2. ∎

LEMMA 4. *Let* $X \subset \mathbf{P}^r$, $r \geqslant 2$, *be a set of* d *points in uniform position, spanning* $\mathbf{P}^r$. *We assume that* X *is not lying on a hypersurface of degree* $< k$. *Moreover, we set*

$$p := \min\left\{ t \in \mathbf{N} : \binom{r+t}{t} > d-1-\left(\binom{r+k-1}{r}-1\right)\left[\frac{d-1}{\binom{r+k-1}{r}-1}\right]\right\},$$

$$q := \min\left\{ t \in \mathbf{N} : \binom{r+t}{t} > d-r-\left(\binom{r+k-1}{r}-1\right)\left[\frac{d-1}{\binom{r+k-r}{r}-1}\right]\right\}.$$

Then we get $0 \leqslant p \leqslant k-1$ *and* $0 \leqslant q \leqslant k-1$, *and*

(i) $r(X) \leqslant (k-1)\left[\dfrac{d-1}{\binom{r+k-1}{r}-1}\right]+p,$

(ii) *The ideal* $I(X)$ *is generated by forms of degree*

$$\leqslant (k-1)\left[\frac{d-r}{\binom{r+k-1}{r}-1}\right]+q+1.$$

Proof. Since X is not lying on a hypersurface of degree $< k$ we have $h_X(t) = \binom{r+t}{t}$ for all $t = 0, \ldots, k-1$.

Hence we get from Lemma 1.

$$h_X(i(k-1)+j) \geqslant \min\left\{d,\ i\left[\binom{r+k-1}{r}-1\right]+\binom{j+r}{r}\right\} \quad \text{for } j = 0, \ldots, k-1.$$

Then a trivial verification shows (i). The assertion (ii) then results immediately from Lemma 3. ∎

Remark. Lemma 4(ii) improves Corollary 1.2 of [35]. Lemma 4(i) also follows from Lemma 2.3 of [35].

Finally, we introduce some notations. Let $S := K[x_0, \ldots, x_r]$ be a polynomial ring. Let $\mathfrak{a}$ be a homogeneous ideal of S. We set $A := S/\mathfrak{a}$, that is, A is a graded K-algebra. Let $M = \bigoplus_{n \in \mathbf{Z}} M_n$ be a graded A-module. The ith local cohomology module of M with support in the irrelevant ideal $\mathfrak{m} = \bigoplus_{n>0} A_n$, denoted by $H^i_{\mathfrak{m}}(M)$, is also a graded A-module. For $i \in \mathbf{Z}$ let $[M]_i$ denote the ith graded part of M, i.e., $[M]_i = M_i$. Let p be an integer then let $M(p)$ denote the graded A-module whose underlying module is the same as that of M and whose grading is given by $[M(p)]_i = [M]_{p+i}$ for all $i \in \mathbf{Z}$. For an arbitrary graded A-module M we set

$$e(M) := \sup\{t \in \mathbf{Z} : [M]_t \neq 0\},$$
$$a(M) := \inf\{t \in \mathbf{Z} : [M]_t \neq 0\},$$
$$[h^i_{\mathfrak{m}}(M)]_t := \operatorname{rank}_k [H^i_{\mathfrak{m}}(M)]_t.$$

We set for integers $a, b \geqslant 0$

$$\left\{\frac{a}{b}\right\} := \min\{t \in \mathbf{Z} : a \leqslant tb\},$$

$$\left[\frac{a}{b}\right] := \max\{t \in \mathbf{Z} : a \geqslant tb\},$$

$$\binom{a}{b} = \frac{a(a-1)\ldots(a-b+1)}{1\ 2 \ldots b}, \quad \text{where } \binom{a}{b} := 0 \text{ for } a < b.$$

2. Hilbert functions of finite sets of points and bounds for the geometric genus of projective varieties

Before proving our key lemma we will give the following definition.

DEFINITION. Let X be a subscheme of $\mathbf{P}_K^n$ of dimension zero and degree d. A numerical function $h'_X: \mathbf{Z} \to \mathbf{Z}$ is said to be a *lower estimation* of h_X if

(i) $0 \leqslant h'_X(t) \leqslant h_X(t)$ for all $t \in \mathbf{Z}$

(ii) $h'_X(t) = d$ for all $t \gg 0$.

Moreover we set $M+1 := \min\{t \in \mathbf{N}: h'_X(t) = d\}$.

LEMMA 5. *Let $V \subset \mathbf{P}_K^r$ be a scheme of dimension $n \geqslant 1$ and degree d. Let $X := V \cap L_{r-n}$ be the points of the intersection of V with a sufficiently general linear space L_{r-n} of dimension $r-n$ where we set $L_{r-n} = H_1 \cap \ldots \cap H_n$. Let h'_X be a lower estimation of h_X. Then we have*

$$\dim_K H^n(V, \mathcal{O}_V(t)) \leqslant \begin{cases} 0 & \textit{for } t > M-n, \\ \binom{M-t}{n} d - \sum_{j=t+n}^{M} \binom{j-t-1}{n-1} h'_X(j) & \textit{for } t \leqslant M-n. \end{cases}$$

Proof. We induct on n. Let $n = 1$. Let A be the homogeneous coordinate ring of V. Since $\operatorname{depth}(A) > 0$ we have the following exact sequence:

$$0 \to [A]_t \to H^0(V, \mathcal{O}_V(t)) \to [H^1_{\mathfrak{m}}(A)]_t \to 0.$$

Hence we get

$$h^0(V, \mathcal{O}_V(t)) - h^0(V, \mathcal{O}_V(t-1)) = h_V(t) - h_V(t-1) + [h^1_{\mathfrak{m}}(A)]_t - [h^1_{\mathfrak{m}}(A)]_{t-1}$$

where we put $h^i(V, \mathcal{O}_V(t)) = \dim_K H^i(V, \mathcal{O}_V(t))$.

Moreover, the exact sequence

$$0 \to A(-1) \xrightarrow{l} A \to A/lA \to 0$$

provides

$$h^0(V, \mathcal{O}_V(t)) - h^0(V, \mathcal{O}_V(t-1))$$
$$= h_X(t) + [h^0_{\mathfrak{m}}(A/lA)]_t + [h^1_{\mathfrak{m}}(A)]_t - [h^1_{\mathfrak{m}}(A)]_{t-1} \geqslant h_X(t).$$

Riemann's half of the Riemann–Roch theorem asserts simply (see, e.g., [25], § 11):

$$h^1(V, \mathcal{O}_V(t)) = -dt + g - 1 + h^0(V, \mathcal{O}_V(t)).$$

Hence we get $h^1(V, \mathcal{O}_V(t-1)) - h^1(V, \mathcal{O}_V(t)) \leqslant d - h_X(t)$. Since $h^1(V, \mathcal{O}_V(t)) = 0$ for all $t \gg 0$ we obtain our lemma in case $n = 1$. Let $n > 1$. We set $W := V \cap H_1$. Consider the exact sequence

$$0 \to \mathcal{O}_V(t-1) \to \mathcal{O}_V(t) \to \mathcal{O}_W(t) \to 0.$$

Hence we have $h^n(V, \mathcal{O}_V(t-1)) \leqslant h^{n-1}(W, \mathcal{O}_W(t)) + h^n(V, \mathcal{O}_V(t))$. Since $h^n(V, \mathcal{O}_V(t)) = 0$ for all $t \gg 0$, by induction we obtain

$$h^n(V, \mathcal{O}_V(t)) \leqslant \begin{cases} 0 & \text{for } t \geqslant M-n \\ \sum_{i=t+1}^{M-n+1} h^{n-1}(W, \mathcal{O}_W(i)) & \text{for } t \leqslant M-n. \end{cases}$$

Using induction again we get:

$$\begin{aligned}\sum_{i=t+1}^{M-n+1} h^{n-1}(W, \mathcal{O}_W(i)) &\leqslant \sum_{i=t+1}^{M-n+1} \binom{M-i}{n-1} d - \sum_{i=t+1}^{M-n+1} \sum_{j=i+n-1}^{M} \binom{j-i-1}{n-2} h'_X(j) \\ &= \sum_{i=t+1}^{M-n+1} \binom{M-i}{n-1} d - \sum_{j=t+n}^{M} \sum_{i=t+1}^{j-n+1} \binom{j-i-1}{n-2} h'_X(j) \\ &= \binom{M-t}{n} d - \sum_{j=t+n}^{M} \binom{j-t-1}{n-1} h'_X(j).\end{aligned}$$

since $\sum_{i=s}^{b-c} \binom{b-i}{c} = \binom{b-s+1}{c+1}$ for $b, c, s \in \mathbf{N}$. ■

Lemma 5 has interesting consequences. For instance, using this result we will describe another proof of Harris' bound on the geometric genus of projective varieties (see [16], p. 44). Following J. Harris we first recall the definition of the geometric genus. Let V be a reduced, irreducible and nondegenerate subscheme of $\mathbf{P}^r_K$. K is a field of characteristic zero. Then we can find a resolution of V, that is, a smooth abstract variety $\tilde{V}$ mapping holomorphically and birationally to V. We take the geometric genus $p_g(V)$ to be the number $h^0(\tilde{V}, \Omega^n_{\tilde{V}}) = h^n(\tilde{V}, \mathcal{O}_{\tilde{V}})$ of holomorphic forms of top degree on $\tilde{V}$ where $n = \dim V = \dim \tilde{V}$. This is a birational invariant, that is, it does not depend on the choice of a resolution. In what follows, we will maintain the terminology of projective geometry: a "hyperplane section of $\tilde{V}$" will be the pullback of a hyperplane in $\mathbf{P}^r$ via the map $\pi\colon \tilde{V} \to V$, and $\mathcal{O}_{\tilde{V}}(1) := \pi^* \mathcal{O}_{\mathbf{P}^r}(1)$ the corresponding sheaf. Likewise, an m-plane section "$\mathbf{P}^m \cap \tilde{V}$" of $\tilde{V}$ will be the intersection on $\tilde{V}$ of $r-m$ elements of the linear system $|\mathcal{O}_{\tilde{V}}(1)|$; note that since $|\mathcal{O}_{\tilde{V}}(1)|$ has no base points, by Bertini the generic m-plane section of $\tilde{V}$ will be again smooth. Therefore our Lemma 5 yields bounds for $h^n(\tilde{V}, \mathcal{O}_{\tilde{V}}(t)) = p_g(V)$. Using Lemma 5 we will apply the uniform position lemma of Harris [15]. Hence we need again that the characteristic of the basic field K is zero. We note that this uniform position lemma is not true in general if $\operatorname{char}(K) \neq 0$ (see [29]).

COROLLARY 6 (see [16], p. 44). *Let $V \subset \mathbf{P}^r$ be an irreducible, reduced and nondegenerate subvariety of dimension n and degree d. Then we have the* *following bound on the geometric genus of V:*

$$p_g(V) \leqslant \binom{M}{n+1}(r-n)+\binom{M}{n}\varepsilon$$

where M is defined by $d-1 =: M(r-n)+\varepsilon$ with $0<\varepsilon \leqslant r-n$.

Proof. We consider first a generic section $X = V \cap L_{r-n}$ such that X is a set of d points in uniform position, spanning $\mathbf{P}^{r-n}$. This is possible by Bertini's theorem and the uniform position lemma for a general hyperplane section of a curve (see [15]). Hence $h_X(1)=r-n+1$ and by Lemma 1 we have

$$h_X(t) \geqslant \min\{d, t(r-n)+1\}.$$

(This is also true, for points in general position, see, e.g., [12].) Choosing $h'_X(t) = \min\{d, t(r-n)+1\}$ we obtain our corollary from Lemma 5. Note that our M coincides with the integer M of the definition of Section 2. ∎

We may apply the same techniques to bound the genus $p_g(V)$ of V under the hypothesis that $X = V \cap L_{r-n}$ does not lie on a hypersurface of a certain degree. Knowing such property we will improve the bound of Corollary 6.

THEOREM 1. *Let $V \subset \mathbf{P}^r$ be an irreducible, reduced and nondegenerate subvariety of dimension n and degree d. Let L_{r-n} be a sufficiently general linear space such that $X := V \cap L_{r-n}$ is a set of d points in uniform position not lying on a hypersurface of degree $< k'$. Then we have*

$$p_g(V) \leqslant G(d, k', n) := \binom{M}{n} - \sum_{j=n}^{M} \binom{j-1}{n-1} h'_X(j)$$

where

$$h'_X(i(k'-1)+l) = \min\left\{d, i\left[\binom{r-n+k'-1}{r-n}-1\right]+\binom{l+r-n}{l}\right\}$$

for all $0 \leqslant l \leqslant k'-1$ and $i \in \mathbf{N}$ and M is defined by h'_X.

Proof. It follows from the proof of Lemma 4 that h'_X is a lower estimation of h_X. Hence Lemma 5 provides Theorem 1. ∎

Also, Theorem 1 has some interesting consequences for special subvarieties. First we recall that a subvariety V of degree $d \geqslant n(r-n)+2$ which achieves our bound on $p_g(V)$ of Corollary 6 is said to be a Castelnuovo variety (see [16]).

COROLLARY 7. *Let $V \subset \mathbf{P}^r$ be a Castelnuovo variety of degree $d \geqslant \binom{r-n+2}{2}$ of codimension $(r-n) \geqslant 2$. Then the generic section $V \cap L_{r-n} = X$ is lying on a quadric.*

Proof. Theorem 1 and by our proof of Corollary 6 since $\binom{r-n+2}{2} > 2(r-n)+1$. ■

COROLLARY 8. *Let $V \subset \mathbf{P}^r$ be an arithmetically Cohen–Macaulay subvariety not lying on a hypersurface of degree $< k$. Then*

$$p_g(V) \leqslant G(d, k, n).$$

Proof. Theorem 1 since in the Cohen–Macaulay case we may choose $k' = k$ (see, e.g., Lemma 5.1 of [2]). ■

EXAMPLE 1. Consider the irreducible, reduced and nondegenerate curve $C \subset \mathbf{P}^3$ of degree $d = 32$, genus $g = 109$ given in [9], Proposition 3.1. Then C is arithmetically Cohen–Macaulay and not lying on a hypersurface of degree $< k = 7$. For instance, this curve is a positive example for Hartshorne's conjecture 2.5 of [19] since Corollary 3.8 of [20] provides for the genus $g \leqslant 111$. Also, Corollary 8 gives $g \leqslant G(32, 7, 1) = 111$.

Finally, we will state a new bound for the first cohomology of a normal surface. By combining our approach (Lemma 5 and Corollary 6) with Brodmann's approach [4] (Proposition A and Lemma 3) we get the following corollary.

COROLLARY 9. *Let $F \subset \mathbf{P}^r$ be an irreducible, reduced, nondegenerate and normal surface of degree d. Let $X := F \cap L_{r-2}$ be the points of the intersection of F with a sufficiently general linear space L_{r-2} of codimension 2. Let h'_X be a lower estimation of the Hilbert function h_X. We recall that $M+1 := \min\{t \in \mathbf{N} : h'_X(t) = d\}$. If $M > 0$, then we have*

$$h^1(F, \mathcal{O}_F(t)) \leqslant \begin{cases} 0 & \text{for } t < 0 \\ d\left[\binom{M+1}{2} - \binom{M-t}{2}\right] - \sum_{j=1}^{t+1} h'_X(j)j - (t+1)\left(\sum_{j=t+2}^{M} h'_X(j)\right) - \delta \cdot r, & \text{for } 0 \leqslant t < M, \\ d\binom{M+1}{2} - \sum_{j=1}^{M} h'_X(j)j - \delta \cdot r =: \gamma & \text{for } M \leqslant t < p, \\ \max\{0, \gamma - r(t-p+1)\} & \text{for } t \geqslant p. \end{cases}$$

If $M = 0$, then we get for all integers t: $h^1(F, \mathcal{O}_F(t)) = 0$, where

$$p = Md - \sum_{j=1}^{M} h'_X(j) \quad \text{and} \quad \delta = \begin{cases} 0, & \text{if } h^2(F, \mathcal{O}_F) = 0, \\ 1, & \text{otherwise}. \end{cases}$$

3. Bounds for Castelnuovo's regularity and the geometric genus of arithmetically Buchsbaum varieties

We recall that a subvariety $V \subset \mathbf{P}_K^r$ is said to be arithmetically Buchsbaum if the graded K-algebra $K[x_0, \ldots, x_r]/I(V)$ is a Buchsbaum algebra where $I(V)$ is the defining ideal of V in the polynomial ring $S := K[x_0, \ldots, x_r]$. The readers may consult the book [31] for a comprehensive introduction to the subject on Buchsbaum rings and modules and also for the recent development of the theory. Hence let us note only the next criterion which we need in the sequel:

LEMMA 10. *Let $\mathfrak{a}$ be a homogeneous ideal of $S := K[x_0, \ldots, x_r]$. Assume that $A := S/\mathfrak{a}$ is a Buchsbaum K-algebra of Krull-dimension $n+1 \geqslant 2$ and $\mathfrak{a}$ does not contain a form of degree $< k$ where without loss of generality k is an integer $\geqslant 2$. Let $l_1, \ldots, l_e$ be a part of a system of parameters of A of degree 1 with $0 \leqslant e \leqslant n$. Then we have*

$$[H_{\mathfrak{m}}^0(A/(l_1, \ldots, l_e)A)]_i = 0 \quad \textit{for all } i < k-1.$$

Proof. After a suitable change of the homogeneous coordinates we may assume that $l_1 = x_0, \ldots, l_e = x_{e-1}$. Assume that there is a form F of degree $i < k-1$ such that

$$\bar{F} \in [H_{\mathfrak{m}}^0(A/(x_0, \ldots, x_{e-1})A)]_i \subseteq [K[\bar{x}_e, \ldots, \bar{x}_r]]_i.$$

Since A is Buchsbaum we also have that $A/(x_0, \ldots, x_{e-1})A$ is Buchsbaum. Therefore we get $x_e F \in \mathfrak{a} + (x_0, \ldots, x_{e-1})S$. Since $x_e F$ is a form of degree $i+1 < k$ and $x_e F \in K[x_e, \ldots, x_r]$ we obtain the contradiction $x_e F \in \mathfrak{a}$. ■

Remark. Lemma 10 does not remain true if A is only locally Cohen–Macaulay but not Buchsbaum. For this we will consider the curve C of $\mathbf{P}_K^3$ given parametrically by $\{s^{11}, s^8 t^3, st^{10}, t^{11}\}$.

Then the defining prime ideal $I(C)$ of C in $S := K[x_0, \ldots, x_3]$ is generated by the following forms of degree $\geqslant 4$: $I(C) = (x_1^7 - x_0^5 x_2 x_3,\ x_2^8 - x_1 x_3^7,\ x_0^2 x_2^2 - x_1^3 x_3,\ x_0 x_2^5 - x_1^2 x_3^4,\ x_1^4 x_2 - x_0^3 x_3^2,\ x_1 x_2^3 - x_0 x_3^3)$. Since the primary decomposition of $(I(C) + x_3) = (x_1^7, x_2^2, x_3, x_1^4 x_2) \cap (x_0^2, x_1^7, x_2^8, x_3, x_0 x_2^5, x_1^4 x_2, x_1 x_2^3)$ we see that $[H_{\mathfrak{m}}^0(S/I(C) + x_3 S)]_2 \neq 0$. Consider the ideal $I(C) + x_1$ then $[H_{\mathfrak{m}}^0(S/I(C) + x_1 S)]_2 = 0$, that is, the integer $a(H_{\mathfrak{m}}^0(A/lA))$ depends on the parameter l.

If A is Buchsbaum with $\operatorname{depth}(A) > 0$ then $a(H_{\mathfrak{m}}^0(A/lA)) = 1 + a(H_{\mathfrak{m}}^1(A))$ for any parameter l of degree 1. This follows from the exact sequence

$$0 \to H_{\mathfrak{m}}^0(A/lA) \to H_{\mathfrak{m}}^1(A)(-1) \xrightarrow{l} H_{\mathfrak{m}}^1(A).$$

Moreover, we claim that there are examples for our Lemma 10 with $[H_{\mathfrak{m}}^0(A/(l_1, \ldots, l_e)A)]_{k-1} \neq 0$. For this we will study the following example.

EXAMPLE 11. Consider the Hilbert scheme H_8^5 of curves of degree 8 and genus 5. Then a general curve of H_8^5 has the following free resolution (see [14]):

$$0 \to \mathcal{O}_{\mathbf{P}^3}^2(-6) \to \mathcal{O}_{\mathbf{P}^3}^8(-5) \to \mathcal{O}_{\mathbf{P}^3}^7(-4) \to \mathcal{O}_{\mathbf{P}^3} \to \mathcal{O}_C \to 0.$$

Hence $k = 4$ and the Castelnuovo regularity of C, denoted by $\operatorname{reg}(C) = 4$. Let A be the K-algebra $K[x_0, \ldots, x_3]/I(C)$. Then $e(H_{\mathfrak{m}}^1(A)) \leqslant 2$. Moreover, C is arithmetically Buchsbaum with invariant $i(C) = 2$ since the Hartshorne–Rao module of C is a vector space of dimension 2 (see [28]). Lemma 10 and the above remark show that $3 = k-1 \leqslant a(H_{\mathfrak{m}}^0(A/lA)) = a(H_{\mathfrak{m}}^1(A)) + 1$ for any parameter l of degree 1. Hence $\dim_K [H_{\mathfrak{m}}^0(A/lA)]_3 = \dim_K [H_{\mathfrak{m}}^1(A)]_2 = 2$.

Also, Lemma 10 has some interesting consequence even for the degree of subvarieties.

COROLLARY 12. *Let $V \subset \mathbf{P}^r$ be an irreducible, reduced and nondegenerate subvariety not lying on a hypersurface of degree $< k$. Assume that V is arithmetically Buchsbaum then we have*

$$\operatorname{degree}(V) \geqslant \binom{\operatorname{codim}(V) + k' - 1}{k' - 1} \quad \textit{where } k' = \max\{k-1, 2\}.$$

Proof. Consider a generic section $X := V \cap L_{r-n}$ where $n = \dim(V)$. It follows from Lemma 10 that X is not lying on a hypersurface of degree $< k-1$. On the other hand, X is not lying on a hyperplane by Bertini's theorem (see also [33], Lemma 3 or [13] on p. 174). Thus X is not lying on a hypersurface of degree $< k'$. Hence

$$\operatorname{degree}(V) = \deg(X) \geqslant h_X(k'-1) = \binom{r-n+k'-1}{r-n}. \quad \blacksquare$$

COROLLARY 13. *In addition to the hypothesis of Corollary* 12, *suppose that* $\operatorname{codim}(V) = 2$. *Then we have*

$$\operatorname{degree}(V) \geqslant \begin{cases} \dbinom{4\frac{r-2}{r-1} i(V)}{2} & \textit{if } r \textit{ is odd}, \\[2ex] \dbinom{4\frac{r-1}{r} i(V)}{2} & \textit{if } r \textit{ is even}, \end{cases}$$

where $i(V)$ is the Buchsbaum invariant of V.

Proof. Corollary 12 shows $\operatorname{degree}(V) \geqslant \binom{k}{2}$. Applying the following beautiful result of Mei-Chu Chang [5, 6] we get Corollary 13:

$$t \geqslant \begin{cases} \dfrac{r-2}{r-1} 4i(V) & \text{if } r \text{ is odd,} \\ \dfrac{r-1}{r} 4i(V) & \text{if } r \text{ is even,} \end{cases}$$

where t is the smallest degree of a form contained in $I(V)$. ∎

Before stating the main result of this section we will prove a lemma. We introduce first the following notation:

Let $\mathfrak{a}$ be an ideal of a ring A and let N be a submodule of an A-module M. We set $N :_M \mathfrak{a} = \{m \in M : \mathfrak{a} \cdot m \subseteq N\}$. Moreover, we recall the definition of Castelnuovo's regularity.

Definition. Let A be a graded K-algebra and let M be a finitely generated graded A-module. Let m be an integer. We say that M is *m-regular* (in the sense of Castelnuovo and Mumford [25]) if for the graded local cohomology modules $[H^i_{\mathfrak{m}}(M)]_j = 0$ for all i and j with $i+j > m$.

The Castelnuovo regularity $\operatorname{reg}(M)$ of M is defined by

$$\operatorname{reg}(M) = \inf\{m \in \mathbf{Z} \text{ such that } M \text{ is } m\text{-regular}\}.$$

It is well known that the regularity particularly provides upper bounds for the degrees of the syzygies of free resolutions of a homogeneous ideal of S (see for basic facts on Castelnuovo's regularity, for example, [10], [27]). If V is a subscheme of $\mathbf{P}^r$ we then set $\operatorname{reg}(V) := \operatorname{reg}(I(V))$ where $I(V)$ is the homogeneous ideal defining V.

Lemma 14. *Let $A := S/\mathfrak{a}$ be a Buchsbaum K-algebra of Krull-dimension $n+1 \geqslant 2$ and depth > 0. Let $\{l_1, \ldots, l_e\}$, $1 \leqslant e \leqslant n$, be a part of a system of parameters of A of forms of degree* 1. *Then we get*

$$\operatorname{reg}(S/\mathfrak{a}) = \operatorname{reg}(S/(\mathfrak{a} + (l_1, \ldots, l_e)S) :_A \mathfrak{m}).$$

Proof. Applying Lemma 2 of [32] we obtain

$$\begin{aligned} \operatorname{reg}(S/\mathfrak{a}) &= \operatorname{reg}(S/(\mathfrak{a} + l_1 S) : \mathfrak{m}) = \ldots = \operatorname{reg}\big(S/(\ldots((\mathfrak{a} + l_1 S) : \mathfrak{m}) + \ldots + l_e S) : \mathfrak{m}\big) \\ &= \operatorname{reg}(S/(\mathfrak{a} + (l_1, \ldots, l_e)S) : \mathfrak{m}) \end{aligned}$$

by using Proposition I.1.10 and Corollary I.1.11 of [31]. ∎

Theorem 2. *Let $V \subset \mathbf{P}^r$ be an irreducible, reduced and nondegenerate subvariety. Assume that V is arithmetically Buchsbaum and not lying on a hypersurface of degree $< k$. Then we have*

$$\operatorname{reg}(V) \leqslant (k'-1) \left[\frac{\operatorname{degree}(V) - 1}{\binom{\operatorname{codim}(V) + k' - 1}{k'-1} - 1} \right] + p + 1 =: H(d, k')$$

where $k' = \max\{k-1, 2\}$, *and*

$$p = \min\left\{t \in \mathbf{N} : \binom{\operatorname{codim}(V)+t}{t} > \operatorname{degree}(V) - 1 - \left(\binom{\operatorname{codim}(V)+k'-1}{k'-1} - 1\right)\left[\frac{\operatorname{degree}(V)-1}{\binom{\operatorname{codim}(V)+k'-1}{k'-1}-1}\right]\right\}.$$

Proof. Applying Bertini's theorem and the uniform position lemma of [15] we consider a generic section $X := V \cap L_{r-n}$ such that X is a finite set of points in uniform position. It follows again from Lemma 10 that X is not lying on a hypersurface of degree $< k'$. Therefore Lemma 4 and Lemma 14 yield our statement. ∎

Let us consider the special case of Cohen–Macaulay varieties. We then obtain the following

COROLLARY 15. *Let the situation be as described in Theorem 2, but we assume that V is arithmetically Cohen–Macaulay. Then we get*

(i) $\operatorname{reg}(V) \leq H(d, k)$.

(ii) *The defining ideal $I(V)$ of V is generated by forms of*

$$\operatorname{degree}(V) \leq E(d, k) := (k-1)\left[\frac{\operatorname{degree}(V)-\operatorname{codim}(V)}{\binom{\operatorname{codim}(V)+k-1}{k-1}-1}\right] + q + 1,$$

where

$$q := \min\left\{t \in \mathbf{N} : \binom{\operatorname{codim}(V)+t}{t} > \operatorname{degree}(V) - \operatorname{codim}(V) - \left(\binom{\operatorname{codim}(V)+k-1}{k-1} - 1\right)\left[\frac{\operatorname{degree}(V)-\operatorname{codim}(V)}{\binom{\operatorname{codim}(V)+k-1}{k-1}-1}\right]\right\}.$$

Proof. (i) follows from the proof of Theorem 2 by using $k = k'$.
(ii) results from our Lemma 4 and Lemma 5.1 of [2]. ∎

EXAMPLE. Let C_6^3 be an irreducible, reduced and nonsingular curve in $\mathbf{P}^3$ of degree 6 and genus 3 which is arithmetically Cohen–Macaulay. Then we have the following free resolution of C_6^3 (see, e.g., [31], p. 17):

$$0 \to S^3(-4) \to S^4(-3) \to S \to S/I(C_6^3) \to 0.$$

Hence $k = 3$ and the example shows that the bounds of Corollary 15 are sharp.

Remark. Our approaches of Lemma 4 and Corollary 16 also give us the possibility to improve and extend results of [35]. This results from the following estimations of our integers $H(d, k')$ and $E(d, k)$:

$$H(d, k') \leqslant \left\{\frac{d-1}{\operatorname{codim}(V)}\right\} - \left[\frac{(k'-1)(k'-2)}{4}\right],$$

$$H(d, k') \leqslant \left\{\frac{d-1}{\operatorname{codim}(V)}\right\} - \left[\frac{(k'-1)(k'-2)}{2}\right]$$

if $\operatorname{codim}(V) \geqslant 3$ or $d \geqslant k^2+k-3$, where $d = \operatorname{degree}(V)$.

Moreover we have:

$$E(d, k) \leqslant \left\{\frac{d}{\operatorname{codim}(V)}\right\} - \left[\frac{(k-1)(k-2)}{4}\right],$$

$$E(d, k) \leqslant \left\{\frac{d}{\operatorname{codim}(V)}\right\} - \left[\frac{(k-1)(k-2)}{2}\right]$$

if $\operatorname{codim}(V) \geqslant 3$ or $d \geqslant k^2+k-3$.

Finally, we prove new bounds on the geometric genus of arithmetically Buchsbaum varieties.

THEOREM 3. *Let $V \subset \mathbf{P}^r$ be an irreducible, reduced and nondegenerate subvariety. Assume that V is arithmetically Buchsbaum and not lying on a hypersurface of degree $< k$. Then we have*

$$p_g(V) \leqslant G(d, k', n)$$

with $k' = \max\{k-1, 2\}$, $d = \operatorname{degree}(V)$ *and* $n = \dim(V)$.

Proof. Lemma 10 and Theorem 1. ∎

4. Bounds on the geometric genus of projective varieties of codimension 2

In this section we will strengthen the result of Corollary 6 in case of codimension 2. We will also consider subvarieties V which are complete intersections of two hypersurfaces of degree, say a and b. Then V is said to be a complete intersection of type (a, b).

THEOREM 4. *Let $V \subset \mathbf{P}^r$ be an irreducible, reduced and nondegenerate subvariety of codimension 2 and degree d. Assume that V is lying on an irreducible hypersurface of degree k. Then we have the following bounds on the geometric genus of V:*

(i) *If* $d > k(k-1)$, *then*

$$p_g(V) \leqslant \binom{k+c-2}{r} + \binom{k+c-\varepsilon-2}{r-1} - \binom{k-1}{r} - \binom{c-1}{r} =: F(d, k),$$

where $c = \{\frac{d}{k}\}$ *and* $\varepsilon = kc - d$.

(ii) *If* $d \leqslant k(k-1)$, *then* $p_g(V) \leqslant F(d, \{\frac{d}{k}\})$. *Moreover, if* V *is smooth and is linked to a complete intersection of type* $(1, \varepsilon)$ *by a complete intersection of type* (k, c) *then* $p_g(V) = F(d, k)$.

Proof. Consider a generic section $X = V \cap L_2$ such that X is a set of d points in uniform position. Lemma 4 shows that

$$p_g(V) \leqslant \binom{r(X)-1}{r-2} d - \sum_{j=r-2}^{r(X)-1} \binom{j-1}{r-3} h_X(j) = \binom{l}{r-2} d - \sum_{j=r-2}^{l} \binom{j-1}{r-3} h_X(j)$$

for all $l \geqslant r(X)-1$ since $h_X(l) = d$ for these l. We set $c_i := h_X(i) - h_X(i-1)$ then we get

$$(*) \qquad p_g(V) \leqslant \binom{l}{r-2} d - \sum_{j=r-2}^{l} \sum_{i=0}^{j} \binom{j-1}{r-3} c_i = \sum_{i=r-1}^{l} \binom{i-1}{r-2} c_i$$

for $l \gg 0$ by using $\sum_{j=i}^{l} \binom{j-1}{r-3} = \binom{l}{r-2} - \binom{i-1}{r-2}$. Therefore we will study $\sum_{i=r-1}^{l} \binom{i-1}{r-2} c_i$.

Case 1. $d > k(k-1)$. Since $X \subset \mathbf{P}^2$, we will apply Lemma 2. Using the notation of Lemma 2, we obtain by Bézout's theorem $a = k$ and $b \geqslant \{\frac{d}{k}\} = c$. We now claim

CLAIM. $\sum_{i=r-1}^{l} \binom{i-1}{r-2} c_i \leqslant \sum_{i=r-1}^{l} \binom{i-1}{r-2} \bar{c}_i$, *where*

$$\bar{c}_i = \begin{cases} i+1 & \text{for} \quad 0 \leqslant i < k, \\ k & \text{for} \quad k \leqslant i < c, \\ k+c-1-i & \text{for} \quad c \leqslant i \leqslant k+c-\varepsilon-2, \\ k+c-2-i & \text{for} \quad k+c-1-\varepsilon \leqslant i \leqslant k+c-2, \\ 0 & \text{for} \quad i \geqslant k+c-2. \end{cases}$$

Proof of the Claim. We first note that $\sum_{i=0}^{l} c_i = d = \sum_{i=0}^{l} \bar{c}_i$ for $l \gg 0$, and c_i and $\bar{c}_i$ satisfy the conditions stated in Lemma 2. Assume that $c_i \leqslant \bar{c}_i$ for all i; we therefore obtain $c_i = \bar{c}_i$ for all i. Let $i \geqslant c$ be the smallest integer with $c_i > \bar{c}_i$. Then there is an integer $j > i$ such that $c_j < \bar{c}_j$. We now consider two cases.

Case 1.1. $j = i+1$. We then substitute c_i by $c_i - 1$, and c_{i+1} by $c_{i+1} + 1$.

Since $c_i - 1 \geqslant \bar{c}_i > \bar{c}_{i+1} \geqslant c_{i+1} + 1$ the new c_i's satisfy also the above two conditions in the beginning of the proof of our claim. But the sum $\sum_{i=r-1}^{l} \binom{i-1}{r-2} c_i$ with the new c_i's is not smaller than the old one.

Case 1.2. $j > i+1$. Then we substitute c_i by $c_i - 1$, and c_j by $c_j + 1$. Since $c_i - 1 \geqslant \bar{c}_i > \bar{c}_{i+1} \geqslant c_{i+1}$, and $c_{j-1} \geqslant \bar{c}_{j-1} > \bar{c}_j \geqslant c_j + 1$ the new c_i's satisfy again the above two conditions. Also, the sum $\sum_{i=r-1}^{l} \binom{i-1}{r-2} c_i$ is again not smaller.

If one of the new c_i's is again $> \bar{c}_i$, say, then we repeat this consideration. It stops if $c_i = \bar{c}_i$ for all i. Hence we get our claim.

Therefore we obtain from (*)

$$p_g(V) \leqslant \sum_{i=r-1}^{k+c-2} \binom{i-1}{r-2} \bar{c}_i = F(d, k)$$

by using the following relation:

$$\sum_{i=s}^{t} (n_1 - i) \binom{i-n_2}{n_3} = (n_1 - t) \binom{t-n_2+1}{n_3+1} - (n_1 - s + 1) \binom{s-n_2}{n_3+1} + \binom{t-n_2+1}{n_3+2} - \binom{s-n_2}{n_3+2}$$

for $n_1, n_2, n_3, s, t \in \mathbf{N}$ with $t \geqslant s \geqslant n_2 + n_3$. This follows, for example, by induction on t.

Case 2. $d \leqslant k(k-1)$. Then X does not lie on a hypersurface of degree $< \left\{\frac{d}{k}\right\}$, that is, $k \geqslant a \geqslant \left\{\frac{d}{k}\right\}$ and $b \geqslant \left\{\frac{d}{a}\right\}$ using the supplement of Lemma 2. Applying the methods used in case 1, we obtain

$$\sum_{i=r-1}^{l} \binom{i-1}{r-2} c_i \leqslant \sum_{i=r-1}^{l} \binom{i-1}{r-2} \bar{c}_i = F\left(d, \left\{\frac{d}{k}\right\}\right), \quad \text{where}$$

$$\bar{c}_i = \begin{cases} i+1 & \text{for} \quad 0 \leqslant i < \left\{\frac{d}{k}\right\}, \\ \left\{\frac{d}{k}\right\} & \text{for} \quad \left\{\frac{d}{k}\right\} \leqslant i < c', \\ \left\{\frac{d}{k}\right\} + c' - 1 - i & \text{for} \quad c' \leqslant i \leqslant \left\{\frac{d}{k}\right\} + c' - \varepsilon' - 2, \\ \left\{\frac{d}{k}\right\} + c' - 2 - i & \text{for} \quad \left\{\frac{d}{k}\right\} + c' - \varepsilon' - 1 \leqslant i \leqslant \left\{\frac{d}{k}\right\} + c' - 2, \\ 0 & \text{for} \quad i \geqslant \left\{\frac{d}{k}\right\} + c' - 2, \end{cases}$$

where $c' = \left\{\frac{d}{\left\{\frac{d}{k}\right\}}\right\}$ and $\varepsilon' = \left\{\frac{d}{k}\right\} c' - d$.

This shows the first part of Theorem 4.

Proof of the second part. Let U and W be the complete intersections of type (k, c) and $(1, \varepsilon)$, resp. Hence we get for the Hilbert function of U and W:

$$h_U(t) = \binom{r+t}{r} - \binom{r+t-c}{r} - \binom{r+t-k}{r} + \binom{r+t-k-c}{r},$$

that is,

$$\Delta^{r-1} h_U(t) = \binom{t+1}{1} - \binom{1+t-c}{1} - \binom{1+t-k}{1} + \binom{1+t-k-c}{1},$$

where $\Delta^m h := \Delta(\Delta^{m-1} h)$ for $m > 0$ and $\Delta^0 h = h$, and $h_W(t) = \binom{r-1+t}{r-1} - \binom{r-1+t-\varepsilon}{r-1}$, that is,

$$\Delta^{r-1} h_W(t) = \binom{t}{0} - \binom{t-\varepsilon}{0} = \begin{cases} 1 & \text{for } 0 \leqslant t \leqslant \varepsilon - 1, \\ 0 & \text{otherwise.} \end{cases}$$

It follows from liaison that V is arithmetically Cohen–Macaulay (see, e.g., [31], Theorem III.1.2). Let $l_0, \ldots, l_{r-2}$ be forms of degree 1 of the polynomial ring $S := K[x_0, \ldots, x_r]$ defining a system of parameters of $S/I(V)$ and $S/I(U)$ where $I(X)$ is the defining ideal of the subvariety X. Since for all $t \geqslant k+c-1 = \operatorname{reg}(I(U))$ we have $[S/(I(U)+(l_0, \ldots, l_{r-2})S)]_t = 0$, we obtain for all $t \geqslant k+c-1$

$$(**) \qquad \Delta^{r-1} h_V(t) = \operatorname{rank}_K [S/(I(V)+(l_0, \ldots, l_{r-2})S)]_t = 0.$$

It is easy to see that

$$\begin{aligned} h_V(t) &= \sum_{i=0}^{t} \binom{t-i+r-2}{r-2} \Delta^{r-1} h_V(i) \\ &= \sum_{i=0}^{k+c-2} \binom{t-i+r-2}{r-2} \Delta^{r-1} h_V(i) \quad \text{by } (**). \end{aligned}$$

Applying Theorem 3 of [8] we therefore obtain

$$\begin{aligned} h_V(t) &= \sum_{i=0}^{k+c-2} \binom{t-i+r-2}{r-2} [\Delta^{r-1} h_U(i) - \Delta^{r-1} h_W(k+c-2-i)] \\ &= h_U(t) - \sum_{i=k+c-1-\varepsilon}^{k+c-2} \binom{t-i+r-2}{r-2} \\ &= h_U(t) - \binom{t-k-c+\varepsilon+r}{r-1} + \binom{t-k-c+r}{r-1}. \end{aligned}$$

Considering these Hilbert functions for $t \gg 0$ we get the corresponding Hilbert polynomials:

$$p_V(t) = p_U(t) - \binom{t-k-c+\varepsilon+r}{r-1} + \binom{t-k-c+r}{r-1}.$$

Therefore we have for the arithmetical genus of V (see, e.g., [25], p. 76):

$$p_a(V) = (-1)^{r-2}(p_V(0)-1) = p_a(U) + \binom{k+c-\varepsilon-2}{r-1} - \binom{k+c-2}{r-1}$$
$$= \binom{k+c-2}{r} + \binom{k+c-\varepsilon-2}{r-1} - \binom{k-1}{r} - \binom{c-1}{r}.$$

Since V is arithmetically Cohen–Macaulay we have

$$p_V(t) = h^0(V, \mathcal{O}_V(t)) + (-1)^{r-2} h^{r-2}(V, \mathcal{O}_V(t)).$$

Since V is reduced and irreducible we get $h^0(V, \mathcal{O}_V) = 1$. Since V is smooth we finally obtain

$$p_g(V) = h^{r-2}(V, \mathcal{O}_V) = (-1)^{r-2}(p_V(0)-1) = p_a(V) = F(d, k).$$

This shows the second part of Theorem 4, q.e.d.

Remark. For curves in $\mathbf{P}^3$ our Theorem 4 yields the bounds on the genus proved by J. Harris in [15], p. 198.

5. Bounds on the genus of curves in $\mathbf{P}^r$

Let $\mathfrak{a}$ be a homogeneous ideal of the polynomial ring $S := K[x_0, \ldots, x_r]$ such that $S/\mathfrak{a}$ has Krull-dimension 2. We then write the Hilbert polynomial, say $p_\mathfrak{a}(t)$, of $\mathfrak{a}$ as follows:

$$p_\mathfrak{a}(t) = \text{degree}(\mathfrak{a})\, t - g + 1.$$

If $\mathfrak{a}$ is the defining ideal of a curve then g is said to be the genus of this curve. We set $e := e(H^2_\mathfrak{m}(S/\mathfrak{a}))$. This is a very important invariant of a curve and was known already to F. Gaeta.

LEMMA 16. *Let $\mathfrak{a} \subset K[x_0, \ldots, x_r]$ be a homogeneous ideal of degree d with Krull-dimension* 2. *Assume that* $\text{depth}(S/\mathfrak{a}) > 0$ *and $\mathfrak{a}$ does not contain a form of degree $< k$. Then we have*

$$g \leqslant dt - \binom{r+t}{t} + 1 \quad \text{for all } t \text{ with } e < t < k.$$

Proof. The Hilbert function $h_\mathfrak{a}(t) = \text{rank}_K[S/\mathfrak{a}]_t$. It follows from [30] that $h_\mathfrak{a}(t) - p_\mathfrak{a}(t) = -\text{rank}_K[H^1_\mathfrak{m}(S/\mathfrak{a})]_t + \text{rank}_K[H^2_\mathfrak{m}(S/\mathfrak{a})]_t$. Hence $\binom{r+t}{t} - dt + g - 1 \leqslant 0$ for $e < t < k$. ∎

THEOREM 5. *Let $C \subset \mathbf{P}^r_K$ be a curve, that is, a* 1*-dimensional projective scheme which is locally Cohen–Macaulay. Assume that C is not lying on a hypersurface of degree $< k$. We set $d =$* degree(C), *and $I(C) :=$ length of $\bigoplus_{s\in\mathbf{Z}} H^1(C, \mathcal{P}_C(s))$ over $K[x_0, \ldots, x_r]$ where $\mathcal{P}_C$ is the ideal sheaf of C. Suppose* *that* $d+I(C) \leqslant \binom{k+r-2}{r-1}+1$ *then we get the following bound on the genus* g *of* C:

$$g \leqslant d\alpha - \binom{\alpha+r}{r}+1,$$

where $\alpha = d-\binom{k+r-2}{r-2}+k-2+I(C)$.

Proof. We have the following bound for Castelnuovo's index of regularity:

$$\operatorname{reg}(C) \leqslant d-\binom{k+r-2}{r-1}+k+I(C).$$

The first proof of this useful result was given in [34]. (Further proofs are developed in [26], [1].) Since $e+3 \leqslant \operatorname{reg}(C)$ we get $e+1 \leqslant d-\binom{k+r-2}{r-1}+k-2+I(C) = \alpha \leqslant k-1$, by assumption. Hence $e \leqslant k-2$, and Lemma 16 gives our theorem for $t = \alpha$. ∎

COROLLARY 17. *Let* $C \subset \mathbf{P}^3$ *be an arithmetically Buchsbaum curve not lying on a hypersurface of degree* $< k$. *Assume that* $k^2 > 2(d-1)$ *then we have*:

$$g \leqslant d\beta - \binom{\beta+3}{3}+1, \quad \textit{where } \beta = d-\binom{k}{2}+\left[\frac{k}{2}\right]-2.$$

Proof. It follows from the beautiful structure Theorem of Amasaki [3] for arithmetically Buchsbaum curves in $\mathbf{P}^3$ (for other proofs see [5], [11]) that $t \geqslant 2i(C)$ where $i(C)$ is the Buchsbaum invariant of C and t is the smallest degree of a form contained in $I(C)$. Therefore we obtain $e < \beta < t$ (see the proof of Theorem 5). Lemma 16 gives again our corollary. ∎

COROLLARY 18. *Let* $C \subset \mathbf{P}^r$ *be an irreducible, reduced and nondegenerate arithmetically Buchsbaum curve not lying on a hypersurface of degree* $< k$. *Assume that degree of* $C := d < \binom{r+k'-2}{r-1}+r-1$, *where* $k' = \max\{k-1, 2\}$. *Then we have*

$$g \leqslant d(k'-1)-\binom{r+k'-1}{r}+1.$$

Proof. Theorem 2 and our assumption on d yield $e+3 \leqslant \operatorname{reg}(C) \leqslant (k'-1)+1+1$. Hence $e < k'-1 < k$. Lemma 16 gives our assertion for $t = k'-1$. ∎

Finally, we will improve Theorem 3 in case of space curves.

THEOREM 6. *Let* $C \subset \mathbf{P}^r$ *be an irreducible, reduced, nondegenerate and arithmetically Buchsbaum curve not lying on a hypersurface of degree* $< k$. *We set* $k' = \max\{k-1, 2\}$ *and define integers* m *and* ε *such that* $d-1 = m[\binom{r+k'-2}{r-1}-1]+\varepsilon$ *with* $d = \operatorname{degree}(C)$, *and* $0 \leqslant \varepsilon \leqslant \binom{r+k'-2}{r-1}-2$. *Moreover,* *we set* $p := \min\{t \in \mathbf{N} : \binom{r-1+t}{t} > \varepsilon\}$, *and* $q := \max\{t \in \mathbf{N} : \binom{r-1+t}{t} \leqslant d\}$. *Then we get*

$$qd+1-\binom{r+q}{q}-i(C) \leqslant g \leqslant G(d, k', 1)-i(C)$$

$$= \binom{m+1}{2}(k'-1)\left[\binom{r+k'-2}{k'-1}-1\right]-m\left[\binom{r+k'-1}{r}-1\right]$$

$$+(\varepsilon+1)[m(k'-1)+p-1]+1-\binom{r+p-1}{r}-i(C),$$

where $i(C)$ *is the Buchsbaum invariant of* C.

Proof. Consider a general hyperplane section $X := C \cap H$ such that X is a set of d points in uniform position. Let A be the homogeneous coordinate ring of C. Then we have

$$h_C(t)-h_C(t-1) = h_X(t)+[h^0_{\mathfrak{m}}(A/LA)]_t,$$

where L is a linear form defining H. Hence we obtain for an integer $l \gg 0$:

$$dl-g+1 = h_C(l) = \sum_{t=1}^{l} [h_C(t)-h_C(t-1)]+1$$

$$= 1+\sum_{t=1}^{l} h_X(t)+\sum_{t=1}^{l} [h^0_{\mathfrak{m}}(A/LA)]_t,$$

that is, $g = \sum_{t=1}^{l}(d-h_X(t))-i(A/LA)$, where $i(A/LA)$ is the Buchsbaum invariant of the 1-dimensional Buchsbaum K-algebra A/LA. Since C is arithmetically Buchsbaum we have $[H^1_{\mathfrak{m}}(A)]_{t-1} \cong [H^0_{\mathfrak{m}}(A/LA)]_t$. This follows from the exact sequence

$$0 \to H^0_{\mathfrak{m}}(A/LA) \to H^1_{\mathfrak{m}}(A)(-1) \xrightarrow{L} H^1_{\mathfrak{m}}(A) \to \dots$$

Hence we thus get

$$(*) \qquad g = \sum_{t=1}^{l} (d-h_X(t))-i(C).$$

We note that this relationship (*) was also obtained by Ciliberto [7] on p. 31.

Let h'_X be a lower estimation of h_X having the property of Theorem 1. Using the notation of Theorem 1 we therefore obtain

$$g \leqslant \sum_{t=1}^{l} (d-h'_X(t))-i(C) = \sum_{t=1}^{M} (d-h'_X(t))-i(C) = G(d, k', 1)-i(C).$$

This shows the upper bound.

Proving the lower bound of Theorem 6 we first note that $h_X(t) \leqslant \binom{r-1+t}{t}$ for all $t \geqslant 0$. Hence we get from (*)

$$g+i(C)=\sum_{t=1}^{l}(d-h_X(t))\geqslant\sum_{t=1}^{q}\left(d-\binom{r+t-1}{t}\right)=qd-\binom{r+q}{q}+1. \blacksquare$$

EXAMPLE. Consider again the curve C of Example 11, that is, a curve of degree 8 and genus 5 not lying on a hypersurface of degree < 4. The lower and upper bound of Theorem 6 is 5. Indeed we have $g = 5$.

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