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Propositional extensions of $L_{\omega_1\omega}$

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The past two decades have seen an enormous development in the theory of logics admitting expressions of infinite length. To-date the most model-theoretically successful part of this development has been the study of $L_{\omega_1\omega}$ and its countable admissible fragments. These fragments, however, as well as $L_{\omega_1\omega}$ itself, have the following rather curious feature, which has been objected to by Kreisel, in [21].(1) Namely, from among all the $2^{2^{\aleph_0}}$ possible infinitary propositional connectives for $L_{\omega_1\omega}$, only negation ($\sim$) and countable conjunction ($\bigwedge$) are used.

The historical reason for this was undoubtedly pragmatic: negation and conjunction work well for $L_{\omega\omega}$ so why not use their infinitary versions in defining $L_{\omega_1\omega}$—especially since other connectives do not immediately suggest themselves? A problem with this approach to infinitary logic is that, while all finitary connectives can be defined in terms of conjunction and negation, the corresponding statement for infinitary connectives is far from true. Thus one can ask: are there any philosophical or mathematical reasons for the particular choice of connectives for $L_{\omega_1\omega}$?

Our aim in this paper is to examine this question from a model-theoretic point of view. In so doing we hope to be able to argue that $\sim$ and $\bigwedge$ were the correct connectives to consider after all.

Unfortunately one difficulty with our program is that, because propositional connectives are essentially sets of reals, many of our results depend on the underlying set-theoretic assumptions we use. For example the Axiom of Determinacy and the Axiom of Constructibility yield two sets of somewhat opposite results. Nevertheless, as we shall see, there is much one can say with a great deal of certainty.

The specific organization of the paper is as follows. After reviewing preliminaries in Section 0 we describe precisely in the next two sections the general framework in which we work. Here for example we introduce the propositional extension $L_{\omega_1\omega}(S)$, of $L_{\omega_1\omega}$, and its propositional part $P_{\omega_1}(S)$. Section 3 develops a few techniques with a descriptive set-theoretic flavor and then relates them to $L_{\omega_1\omega}(S)$ and $P_{\omega_1}(S)$. Section 4 describes

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(1) The authors would like to thank Professor Kreisel for suggesting to them some of the questions investigated here as well as for some discussions concerning this work.

some positive results for $L_{\omega_1\omega}(S)$—results obtained by extending Keisler's development in [19] for $L_{\omega_1\omega}$.

The main technical contributions come in the following three sections. Section 5 extends to $L_{\omega_1\omega}(S)$, the Barwise–Kunen method (see [4]) for computing Hanf numbers, with interesting byproducts obtained along the way. Section 6 shows that various strong model-theoretic properties fail for most of the logics of the form $L_{\omega_1\omega}(S)$. A few "pathological" connectives are then constructed in Section 7, assuming the Axiom of Constructibility. Some partial answers to a question of Barwise in [3] are given at the end of Section 6 using results from both Sections 6 and 7.

A very natural propositional connective, having as its motivation the classical operation $\mathscr{A}$ from descriptive set theory, is examined in Section 8. The idea of considering this particular special case also occured to several other people, whose work is cited in the appropriate places.

Some general observations based on our results then conclude the paper in Section 9. It is in this section that we try to argue that $\sim$ and $\wedge$ are indeed privileged among propositional operations in languages with expressions of countable length.

As a further motivation for the results we develop here, a few comments on the relationship between the present work and the work of Friedman [13] might be useful.

Friedman's concern in [13], which is to-date the only other published study in this general area, is largely the same as ours except that Friedman works with *countable admissible fragments of* $L_{\omega_1\omega}$, instead of $L_{\omega_1\omega}$ itself, as we do here. He begins with the following two technical results of Barwise [2]:

(i) if a is countable and recursively inaccessible, then the set of valid sentences of $L_{\omega_1\omega} \cap L_a$ is a-recursive, and

(ii) if a is countable and recursively accessible, then the set of valid sentences of $L_{\omega_1\omega} \cap L_a$ is a complete a-r.e. set,

and the following question of Kreisel:

can one add new propositional connectives to $L_{\omega_1\omega} \cap L_a$ and still have an a-r.e. validity predicate.

Friedman very successfully showed that the answer to Kreisel's questions for all countable admissibles is *yes*. His method, however, is to construct (by forcing arguments) some highly artificial connectives which, firstly, depend on the admissible ordinal a one starts with, and which, secondly, are all definable from $\sim$ and $\wedge$ in $L_{\omega_1\omega}$ (although not in $L_{\omega_1\omega} \cap L_a$). Contrasted with this approach is the present development, which is concerned with essentially new connectives (i.e. connectives not $L_{\omega_1\omega}$-definable) which, when added to $L_{\omega_1\omega}$, could then be used to uniformly extend an uncountable tower of admissible fragments.

0. Preliminaries

Except when otherwise stated, all our results are proved in Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC). We will use standard set-theoretic notations. In particular, the *cardinality* of a set A will be denoted by $\overline{\overline{A}}$. The sequence of *beth numbers* is defined inductively: $\beth_0 = \aleph_0$, $\beth_{\alpha+1} = 2^{\beth_\alpha}$ and $\beth_\lambda = \bigcup_{\alpha<\lambda} \beth_\alpha$, for limit ordinals λ. The class of all sets will be denoted by V, and the class of all ordinals will be denoted by On. HC will denote the set of all hereditarily countable sets, i.e. $X \in \mathrm{HC}$ iff the transitive closure of X is countable.

We shall employ some of the notations and ideas from the abstract treatment of logics introduced by Lindstrom in [23] and [24]. Here we shall find it convenient to use some of the terminology of Barwise [3].

By a *language* L, we understand a countable set of finitary relation, function and constant symbols. An *L-structure* $\mathfrak{A}$ is a set $|\mathfrak{A}|$ together with an assignment of a suitable relation, function or constant to each symbol of L. We often say that an L-structure is *of type* L. The notions of a *reduct* of a structure to a sublanguage and an *expansion* of a structure to a super-language are standard.

By a *logic*, $\langle L^*, \models^* \rangle$, we mean what Barwise [3] calls a system of logics. An *elementary class of* $\langle L^*, \models^* \rangle$ (EC_{L^*}) is a class K of structures of some type L_1, such that for some sentence σ of L_1^* we have $\mathfrak{A} \in K$ iff $\mathfrak{A} \models^* \sigma$. A class K of L_1-structures is a *pseudoelementary class of* $\langle L^*, \models^* \rangle$ (PC_{L^*}) if there is a language $L_2 \supseteq L_1$ and a sentence $\sigma \in L_2^*$ such that $\mathfrak{A} \in K$ iff $\mathfrak{A}$ is a reduct to L_1 of some $\mathfrak{B} \models^* \sigma$. We shall often denote a logic $\langle L^*, \models^* \rangle$ simply by L^*.

A logic L^* is *L-reducible* to a logic L^{**} (in symbols, $L^* \subseteq_L L^{**}$) if each elementary class of L^* is also an elementary class of L^{**}. In this case, we shall also say that L^* is a *sublogic* of L^{**}. We write $L^* \subsetneq_L L^{**}$ to mean that $L^* \subseteq_L L^{**}$ but L^{**} is not L-reducible to L^*

Extensive use will be made of various simple facts from descriptive set theory. Here we shall assume that the reader is acquainted with the elementary properties of both the lightfaced and the boldfaced, Δ_n^1, Σ_n^1, and Π_n^1 sets of reals. Most of the pertinent facts can be found in Shoenfield [30].

Two specific facts which will be used several times are

THEOREM 0.1. *A subset S of $P(\omega)$ is Σ_2^1 iff it is Σ_1 definable over* HC.

THEOREM 0.2. *If S is a set of well-orderings of ω containing well-orderings of arbitrarily large countable order type, then S is not Σ_1^1.*

Theorem 0.1, which is a part of the folk literature, can be proved using facts about the well-known coding of HC sets of reals. Further information can be found in Section 7. Theorem 0.2, which is essentially due to Kleene, is a version of the familiar boundedness theorems from recursion theory.

A brief mention of recursion in type 2 objects is made in Section 3. Further information can be found in Busch [6].

We shall also make some use of elementary facts from admissibility theory. These can all be ascertained from Keisler [19], Barwise [2] or Gostanian [15]. In particular, the reader should be familiar with the axioms of the so called Kripke–Platek set theory (KP) and its extension by the Σ_1-separation axiom. In each of the above-cited sources one can also find a description of the Lévy hierarchy of formulas of set theory.

Section 7 will require some deeper facts about admissibility as well as an understanding of some of the properties of Gödel's constructible hierarchy. All relevant references can be found there.

As for model-theoretic preliminaries, we shall assume that the reader is familiar with the first fifteen chapters of Keisler's exposition, [19], of $L_{\omega_1\omega}$. Extensive use will be made of much of what Keisler does.

To understand Section 5, a knowledge (and ready availability) of Barwise–Kunen [4] will be essential.

Finally, we shall use the notations Craig (L^*, L^{**}), Beth (L^*, L^{**}) and Weak Beth (L^*, L^{**}) to denote, respectively, the Craig interpolation, Beth definability and weak Beth definability properties for pairs of logics L^* and L^{**}, when L^* is a sublogic of L^{**}. These are defined as follows:

Craig (L^*, L^{**}) means that for each pair of sentences φ, ψ of L^* such that $\models^* \varphi \to \psi$, there is a sentence θ of L^{**} such that $\models^{**} \varphi \to \theta$ and $\models^{**} \theta \to \psi$ and the language of θ is contained in the intersection of the language of φ and the language of ψ;

Beth (L^*, L^{**}) means that for every sentence φ of L^*, of type $L_1 \cup \{\bar{R}\}$, such that each structure $\mathfrak{A}$ of type L_1 has at most one expansion $\langle \mathfrak{A}, R\rangle$ which is a model of φ, there is a formula ψ of L^{**} of type L_1 such that

$$\models^{**} \varphi \to (\forall v_1) \dots (\forall v_n)[\bar{R}(v_1, \dots, v_n) \leftrightarrow \psi(v_1, \dots, v_n)];$$

Weak Beth (L^*, L^{**}) is defined exactly as Beth (L^*, L^{**}) except that the condition "at most one expansion" is changed to "exactly one expansion". (For the definitions of Beth (L^*, L^{**}) and Weak Beth (L^*, L^{**}) to make sense we are of course assuming that L^{**} is a logic built up from atomic formulas and variables in the usual way.)

We shall write Craig (L^*) to mean Craig (L^*, L^*). Similarly for Beth (L^*) and Weak Beth (L^*).

1. Adding propositional connectives to $L_{\omega_1\omega}$

The early writers on infinitary logic generally viewed propositional connectives, such as infinitary conjunction and disjunction ($\bigwedge$ and $\bigvee$), as functions which map *indexed sets* of formulas to formulas (see Karp [18]).

It later became customary to regard $\bigvee$ and $\bigwedge$ as functions which map *sets* of formulas to formulas (see Barwise [2]). This change in point of view, which made coding formulas somewhat easier, worked very well for conjunctions and disjunctions. However, since many connectives are actually dependent on the order of the formulas to which they are applied (as for example implication, $\rightarrow$), the newer approach is less suitable for a general study of connectives.

Our point of view will be to combine both approaches by regarding conjunction and disjunction as functions on sets of formulas, but regarding all other connectives as functions on indexed sets of formulas.

Since we shall restrict ourselves, in this paper, to languages with expressions of countable length which are extensions of $L_{\omega_1\omega}$, we can assume from the outset that *all propositional connectives will operate on sets of formulas indexed by natural numbers*. This leads naturally to the idea of identifying propositional connectives with subsets of $P(\omega)$, i.e. if $S \subseteq P(\omega)$ and $\langle\varphi_i \mid i \in \omega\rangle$ is a sequence of formulas, then the propositional connective associated with S, applied to $\langle\varphi_i \mid i \in \omega\rangle$, results in a formula which is true iff $\{i \mid \varphi_i \text{ is true}\} \in S$.

We formalize these ideas by introducing, for each language L, and each $S \subseteq P(\omega)$ the logic $L_{\omega_1\omega}(S)$. This is defined to be just like $L_{\omega_1\omega}$ except that there is an additional primitive symbol $\mathscr{S}$, and an additional formation rule for defining the set of formulas: If $\varphi_0, \varphi_1, \ldots$ is a sequence of formulas of $L_{\omega_1\omega}(S)$, then $\mathscr{S}\varphi_i$ is a formula of $L_{\omega_1\omega}(S)$.

The usual syntactic notions such as free variables, bound variables, sentences, etc. are defined for $L_{\omega_1\omega}(S)$ analogously to the way they are defined for $L_{\omega_1\omega}$. By using some standard coding of formulas by sets, we can then show that all syntactic notions for $L_{\omega_1\omega}(S)$ are Δ_1 definable over HC. The proof is a straightforward extension of the proof of Lemma 2.4 in Barwise [2].

The notion of satisfaction for $L_{\omega_1\omega}(S)$ is defined in exactly the same way as for $L_{\omega_1\omega}$ except for the additional clause

$$\mathfrak{A} \models_x^S \mathscr{S}\varphi_i \quad \text{iff} \quad \{i \mid \mathfrak{A} \models_x^S \varphi_i\} \in S.$$

Here of course $\mathfrak{A}$ is a structure and x is an assignment of elements of $|\mathfrak{A}|$ to the free variables of $\mathscr{S}\varphi_i$.

When no confusion can arise we shall usually supress the superscript S in the above notations.

The set of *valid sentences* of $L_{\omega_1\omega}(S)$, denoted by $\mathrm{Val}\big(L_{\omega_1\omega}(S)\big)$ is, as usual, the set of all sentences of $L_{\omega_1\omega}(S)$ which are satisfied in all structures by all assignments.

The model-theoretic notions of $L_{\omega_1\omega}$, such as elementary equivalence, elementary substructure, etc., all have obvious analogous definitions for $L_{\omega_1\omega}(S)$.

We shall use $\bigvee$, $\rightarrow$ and $\forall$ as abbreviations for disjunction, implication and universal quantification, in the usual fashion. In addition we shall introduce a new symbol, $\not{\mathscr{S}}$, *dual* to $\mathscr{S}$, such that $\not{\mathscr{S}}\varphi_i$ will abbreviate $\sim\mathscr{S}\sim\varphi_i$.

This completes the basic syntactical and semantic description of $L_{\omega_1\omega}(S)$. On a few occasions it will be convenient to consider propositional extension of $L_{\omega_1\omega}$ with more than one new connective. The relevant definitions for these cases will not be given since they should be quite obvious. For example if S and T are subsets of $P(\omega)$, then the logic $L_{\omega_1\omega}(S, T)$ will have two new connective symbols $\mathscr{S}$ and $\mathscr{T}$, with $\mathscr{S}$ interpreted according to S and $\mathscr{T}$ interpreted according to T.

Equally obvious is the extension of $L_{\omega_1\omega}$ containing all possible propositional connectives. This logic will be denoted by $L_{\omega_1\omega}(\mathbf{Full})$.

Finally we note that, just as in the $L_{\omega_1\omega}$ case, it is quite reasonable to consider *fragments* of $L_{\omega_1\omega}(S)$. Indeed this is exactly what is done by Friedman in [13]. Since this paper will not be concerned with results about fragments—fragments enter only as a technical tool for other results—we shall not give the relevant definitions. These can be readily formulated by referring to pages 17 and 18 of Keisler [19].

2. The propositional part of $L_{\omega_1\omega}(S)$

Although our main concern is with propositional extensions of $L_{\omega_1\omega}$, it turns out that the purely propositional parts of such extensions are themselves not without interest. This is so because, on the one hand, these propositional logics provide a convenient vehicle for applying methods of descriptive set theory to the study of $L_{\omega_1\omega}(S)$, and on the other hand, because some results about $L_{\omega_1\omega}(S)$ can be strengthened to results about its propositional part.

For these reasons we introduce for each $S \subseteq P(\omega)$, the propositional logic $P_{\omega_1}(S)$. Its symbols consist of a countably infinite sequence, $p_0, p_1, \ldots$ of propositional variables, the usual connective symbols $\sim$ and $\bigwedge$, and a new connective symbol, $\mathscr{S}$.

The *set of formulas* of $P_{\omega_1}(S)$ is defined to be the smallest set F containing all the propositional variables and such that

(i) if $\varphi \in F$, then $\sim\varphi \in F$,

(ii) if Φ is a countable nonempty subset of F, then $\bigwedge\Phi \in F$, and

(iii) if for each $i \in \omega$, $\varphi_i \in F$, then $\mathscr{S}\varphi_i \in F$.

A *realization* of $P_{\omega_1}(S)$ is a map from the set of propositional variables into $\{0, 1\}$. Since the propositional variables are indexed by natural numbers, realizations of $P_{\omega_1}(S)$ are nothing more than reals.

Given a real X and a formula φ of $P_{\omega_1}(S)$, we say that X *S-satisfies* φ (in symbols $\models^S \varphi[X]$) if

(i) φ is p_n and $n \in X$, or

(ii) φ is $\sim\psi$ and it is not the case that $\vDash^{\mathbf{S}} \psi[X]$, or

(iii) φ is $\bigwedge\Psi$ and $\vDash^{\mathbf{S}} \psi[X]$ for all $\psi \in \Psi$, or

(iv) φ is $\mathscr{S}\varphi_i$ and $\{i \mid \vDash^{\mathbf{S}} \varphi_i[X]\} \in \mathbf{S}$.

The set of all reals which $\mathbf{S}$-satisfy a given $P_{\omega_1}(\mathbf{S})$ formula is called the ***S*-*truth set*** of φ and is denoted by $T^{\mathbf{S}}(\varphi)$.

Those formulas of $P_{\omega_1}(\mathbf{S})$ which are $\mathbf{S}$-satisfied by all reals are called ***S*-*valid***. The set of all such formulas is denoted by $\mathrm{Val}(P_{\omega_1}(\mathbf{S}))$.

Consistent with our convention in Section 1, we shall usually suppress all references to $\mathbf{S}$ in the above notations when no confusion can arise.

On some occasions we shall need to substitute $L_{\omega_1\omega}(\mathbf{S})$ formulas into propositional formulas. For this we introduce the following notation.

If $\gamma(p_1, p_2, \ldots)$ is a formula of $P_{\omega_1}(\mathbf{S})$ and $\varphi_1, \varphi_2, \ldots$ are formulas of $L_{\omega_1\omega}(\mathbf{S})$, then $\gamma[\varphi_n]$ will denote the result of replacing p_n by φ_n for each $n \in \omega$, everywhere in γ.

We close this section with the following two substitution results which will be used several times in the sequel. Each is easily proven by induction on the complexity of γ.

LEMMA 2.1. *Let γ be a formula of $P_{\omega_1}(\mathbf{S})$ and for each $n \in \omega$, let φ_n be a formula of $L_{\omega_1\omega}(\mathbf{S})$. If $\mathfrak{A}$ is an L-structure, then*

$$\mathfrak{A} \vDash^{\mathbf{S}}_{x} \gamma[\varphi_n] \quad \textit{iff} \quad \{n \in \omega \mid \mathfrak{A} \vDash^{\mathbf{S}}_{x} \varphi_n\} \in T^{\mathbf{S}}(\gamma).$$

LEMMA 2.2. *Let γ be a $P_{\omega_1}(\mathbf{S})$ formula, and for each $n \in \omega$ let φ_n be a $P_{\omega_1}(S)$ formula. Then for any real X*

$$\vDash^{S} (\gamma[\varphi_n])[X] \quad \textit{iff} \quad \vDash^{S} \gamma\big[\{n \in \omega \mid \vDash^{S} \varphi_n[X]\}\big].$$

3. The operation S and the boolean algebra $\mathbf{B}_S$

A large portion of classical descriptive set theory can be viewed as the study of certain propositional connectives by purely set-theoretic means. For example the set-theoretic operation of union can be associated with infinitary disjunction via $\bigcup_{n\in\omega} A_n = \big\{X \mid \{n \in \omega \mid X \in A_n\} \in \mathsf{V}\big\}$ where $\mathsf{V} = \{X \subseteq \omega \mid X \neq \emptyset\}$.

This idea can be generalized to any connective $\mathbf{S}$. The resulting operation can then be used to define a boolean algebra $\mathbf{B}_S$, which is intimately related to both $P_{\omega_1}(\mathbf{S})$ and $L_{\omega_1\omega}(\mathbf{S})$.

Accordingly let $\mathbf{S}$ be any set of reals and for each $n \in \omega$, let $A_n \subseteq P(\omega)$. The *operation* $\mathbf{S}$ is defined as follows: $\mathbf{S}$ applied to $\langle A_n \mid n \in \omega\rangle$ (in symbols $\mathbf{S}A_n$) is $\big\{X \mid \{n \in \omega \mid X \in A_n\} \in \mathbf{S}\big\}$.

The ***boolean algebra* $\mathbf{B}_{\mathbf{S}}$** is then defined to be the smallest σ-algebra containing all the open sets (in the Baire topology on $P(\omega)$) and closed under the operation S. A convenient characterization of $\boldsymbol{B}_S$ is given by the following lemma which can be easily proven by induction.

LEMMA 3.1. *$\boldsymbol{B}_S$ is precisely the set of all truth sets of formulas of $P_{\omega_1}(S)$.*

A few important properties of $\boldsymbol{B}_S$ are given in the following

LEMMA 3.2. (i) *$S \in \boldsymbol{B}_S$;*

(ii) *If $\boldsymbol{T} \in \boldsymbol{B}_S$, then $\boldsymbol{B}_T \subseteq \boldsymbol{B}_S$;*

(iii) *If $\boldsymbol{T} \in \boldsymbol{B}_S$, then $f^{-1}[\boldsymbol{T}] \in \boldsymbol{B}_S$ for every total continuous functions $f\colon P(\omega)\to P(\omega)$.*

Proof. (i) Let $A_n = \{X \subseteq \omega \mid n \in X\}$. A_n is then open and $S = SA_n$.

(ii) It is enough to show that $\boldsymbol{B}_S$ is closed under the operation $\boldsymbol{T}$. Hence suppose that for each $n \in \omega$, $A_n \in \boldsymbol{B}_S$. By Lemma 3.1 there is a $P_{\omega_1}(S)$ formula, φ_n, for each $n \in \omega$, such that $A_n = T^S(\varphi_n)$. Similarly there is a $P_{\omega_1}(S)$ formula γ such that $\boldsymbol{T} = T^S(\gamma)$. Hence

$$\begin{aligned} X \in \boldsymbol{T}A_n \quad &\text{iff} \quad \{n \mid X \in A_n\} \in \boldsymbol{T},\\ &\text{iff} \quad \{n \mid X \in T^S(\varphi_n)\} \in T^S(\gamma),\\ &\text{iff} \quad \models^S \gamma\big[\{n \mid \models^S \varphi_n[X]\}\big],\\ &\text{iff} \quad \models^S \gamma[\varphi_n][X], \end{aligned}$$

where the last equivalence follows from Lemma 2.2. Thus, $\boldsymbol{T}A_n = T^S(\gamma[\varphi_n]) \in \boldsymbol{B}_S$.

(iii) By induction on the number of applications of complementation, union or the operation S.

Clearly, if $\boldsymbol{T} \in \boldsymbol{B}_S$ and $\boldsymbol{T}$ is open, then $f^{-1}[\boldsymbol{T}]$ is in $\boldsymbol{B}_S$. Equally clearly, if $\boldsymbol{T} = P(\omega) - \boldsymbol{R}$ for some $\boldsymbol{R} \in \boldsymbol{B}_S$ which is such that $f^{-1}[\boldsymbol{R}] \in \boldsymbol{B}_S$, or if $\boldsymbol{T} = \bigcup A_n$ for $A_n \in \boldsymbol{B}_S$ such that $f^{-1}[A_n] \in \boldsymbol{B}_S$ for each $n \in \omega$, then $f^{-1}[\boldsymbol{T}] \in \boldsymbol{B}_S$.

Finally, if $\boldsymbol{T} = SA_n$ for $A_n \in \boldsymbol{B}_S$ such that $f^{-1}[A_n] \in \boldsymbol{B}_S$ for each $n \in \omega$, then $f^{-1}[SA_n]$ is also in $\boldsymbol{B}_S$, since

$$\begin{aligned} X \in f^{-1}[SA_n] \quad &\text{iff} \quad f[X] \in SA_n\\ &\text{iff} \quad \{n \mid f[X] \in A_n\} \in S\\ &\text{iff} \quad \{n \mid X \in f^{-1}[A_n]\} \in S\\ &\text{iff} \quad X \in Sf^{-1}[A_n]. \ \blacksquare \end{aligned}$$

We will often identify natural numbers with pairs of natural numbers, reals with subsets of ω^2, reals with pairs of reals, etc., using the familiar pairing functions of Shoenfield [29]. Hence $n = \big((n)_0, (n)_1\big)$, $X = \big\{\big((n)_0, (n)_1\big) \mid n \in X\big\}$, $X = \big((X)_0, (X)_1\big)$ etc. In this fashion elements of $\boldsymbol{B}_S$ can be considered as subsets of $P(\omega)$, sets of subsets of ω^2 as sets of pairs of reals, etc.

Using these conventions, we shall prove a simple lemma, which will provide an important link between $P_{\omega_1}(S)$ and $L_{\omega_1\omega}(S)$. Although the lemma is true for all countable languages, we shall state it, for reasons of notational economy, only for the case of languages consisting of a single binary relation symbol $\bar{Q}$ and a single unary relation symbol $\bar{P}$.

LEMMA 3.3. *Let* $L = \{\bar{P}, \bar{Q}\}$ *as described above and let* $L^+ = L \cup \cup \{\bar{0}, \bar{1}, \bar{2}, \ldots\}$.

(a) *For each sentence* φ *of* $L^+_{\omega_1\omega}(S)$ *there is a formula* $\check{\varphi}$ *of* $P_{\omega_1}(S)$ *such that*

$$\langle \omega, X, Y, 0, 1, 2, \ldots \rangle \models \varphi \quad \textit{iff} \quad \models \check{\varphi}[(X, Y)].$$

(b) *For each formula* φ *of* $P_{\omega_1}(S)$ *there is a quantifier-free sentence* $\hat{\varphi}$ *of* $L^+_{\omega_1\omega}(S)$ *such that*

$$\models \varphi[(X, Y)] \quad \textit{iff} \quad \langle \omega, X, Y, 0, 1, 2, \ldots \rangle \models \hat{\varphi}.$$

Proof. (a) Inductively replace each occurrence of an existential subformula, $(\exists v)\psi(v)$, in φ, by an infinitary disjunction, $\bigvee_{n \in \omega} \psi(\bar{n})$. The resulting quantifier-free sentence, φ_1, clearly holds in $\langle \omega, X, Y, 0, 1, 2, \ldots \rangle$ iff φ does.

Now replace all atomic subformulas of φ_1 by propositional formulas as follows:

replace $\bar{P}(\bar{m})$ by p_{2m},
replace $\bar{Q}(\bar{m}, \bar{n})$ by $p_{2(m,n)+1}$,
replace $\bar{m} = \bar{n}$ by $p_0 \vee \sim p_0$ when $m = n$, and
replace $\bar{m} = \bar{n}$ by $p_0 \wedge \sim p_0$ when $m \neq n$.

Using Lemma 2.1, it can easily be shown that the resulting propositional formula satisfies the requirements on $\check{\varphi}$.

(b) To obtain $\hat{\varphi}$ replace each occurrence of p_{2m} in φ by $\bar{P}(\bar{m})$ and each occurrence of p_{2m+1} by $Q\big(\overline{(m)_1}, \overline{(m)_2}\big)$. The rest is again obvious from Lemma 2.1. ∎

The correspondence between $L_{\omega_1\omega}(S)$ and $P_{\omega_1}(S)$ described in Lemma 3.3 also extends to pseudo-elementary classes. By analogy with the definition of analytic sets as projections of Borel sets, we first define $\Sigma(\boldsymbol{B}_S)$ as

$$\{\mathrm{dom}(\boldsymbol{T}) \mid \boldsymbol{T} \in \boldsymbol{B}_S\},$$

where, of course, $\mathrm{dom}(\boldsymbol{T}) = \{X \subseteq \omega \mid (X, Y) \in \boldsymbol{T}, \text{ for some } Y \subseteq \omega\}$. Note that if $\boldsymbol{T} \in \Sigma(\boldsymbol{B}_S)$, then $\mathrm{dom}(\boldsymbol{T})$ is also in $\Sigma(\boldsymbol{B}_S)$.

Lemma 3.3 immediately yields the following

COROLLARY 3.3. $\boldsymbol{T} \in \Sigma(\boldsymbol{B}_S)$ *iff there is a sentence* φ *of* $L^+_{\omega_1\omega}(S)$ *such that*

$$\boldsymbol{T} = \{X \subseteq \omega \mid \langle \omega, X, Y, 0, 1, 2, \ldots \rangle \models \varphi, \textit{ for some } Y \subseteq \omega\}.$$

We shall next show how the size of the algebra $\boldsymbol{B}_S$ reflects a great deal of the expressive power of both $P_{\omega_1}(S)$ and $L_{\omega_1\omega}(S)$. For this we must first define a certain reducibility relation between sets of reals.

DEFINITION. If $\boldsymbol{S}$ and $\boldsymbol{T}$ are sets of reals, then $\boldsymbol{S}$ is *L-reducible* to $\boldsymbol{T}$ (in symbols $\boldsymbol{S} \leqslant_L \boldsymbol{T}$) if $\boldsymbol{B_S} \subseteq \boldsymbol{B_T}$. We write $\boldsymbol{S} \lneqq_L \boldsymbol{T}$ if $\boldsymbol{B_S}$ is a proper subset of $\boldsymbol{B_T}$. The L-degree of $\boldsymbol{S}$ is $\{\boldsymbol{T} \mid \boldsymbol{S} \leqslant_L \boldsymbol{T}$ and $\boldsymbol{T} \leqslant_L \boldsymbol{S}\}$, i.e. $\{\boldsymbol{T} \mid \boldsymbol{B_S} = \boldsymbol{B_T}\}$.

THEOREM 3.1. *If $\boldsymbol{S}$ and $\boldsymbol{T}$ are sets of reals, then $\boldsymbol{S} \leqslant_L \boldsymbol{T}$ iff $L_{\omega_1\omega}(\boldsymbol{S}) \subseteq_L L_{\omega_1\omega}(\boldsymbol{T})$. Moreover $\boldsymbol{S} \lneqq_L \boldsymbol{T}$ iff $L_{\omega_1\omega}(\boldsymbol{S}) \subsetneq_L L_{\omega_1\omega}(\boldsymbol{T})$.*

Proof. First suppose that $\boldsymbol{S} \leqslant_L \boldsymbol{T}$. We shall show, by induction on the complexity of formulas, that for each formula φ of $L_{\omega_1\omega}(\boldsymbol{S})$ there is a formula φ' of $L_{\omega_1\omega}(\boldsymbol{T})$ such that

$$\mathfrak{A} \models_x^{\boldsymbol{S}} \varphi \quad \text{iff} \quad \mathfrak{A} \models_x^{\boldsymbol{T}} \varphi'$$

for all $\mathfrak{A}$ and x.

To do so let ψ be some $P_{\omega_1}(\boldsymbol{T})$ formula such that $\boldsymbol{S} = T^{\boldsymbol{T}}(\psi)$.

The only nontrivial case arrises when φ is of the form $\mathscr{S}\varphi_n$, where we make the inductive assumption that for each $n \in \omega$ there is a formula φ_n' of $L_{\omega_1\omega}(\boldsymbol{T})$ such that $\mathfrak{A} \models_x^{\boldsymbol{S}} \varphi_n$ iff $\mathfrak{A} \models_x^{\boldsymbol{T}} \varphi_n'$. In this case we let φ' be $\psi[\varphi_n']$. Then clearly φ' is an $L_{\omega_1\omega}(\boldsymbol{T})$ formula and

$$\begin{aligned} \mathfrak{A} \models_x^{\boldsymbol{S}} \mathscr{S}\varphi_n \quad & \text{iff} \quad \{n \mid \mathfrak{A} \models_x^{\boldsymbol{S}} \varphi_n\} \in \boldsymbol{S}, \\ & \text{iff} \quad \{n \mid \mathfrak{A} \models_x^{\boldsymbol{T}} \varphi_n'\} \in T^{\boldsymbol{T}}(\psi), \\ & \text{iff} \quad \mathfrak{A} \models_x^{\boldsymbol{T}} \psi[\varphi_n'], \end{aligned}$$

where the last equivalence follows from Lemma 2.1.

Conversely suppose that $L_{\omega_1\omega}(\boldsymbol{S}) \subseteq_L L_{\omega_1\omega}(\boldsymbol{T})$. To show that $\boldsymbol{S} \leqslant_L \boldsymbol{T}$, it suffices to find a formula φ of $P_{\omega_1}(\boldsymbol{T})$ such that $T^{\boldsymbol{T}}(\varphi) = \boldsymbol{S}$.

To do so let ψ' be the conjunction of

$$\bigwedge_{m,n \in \omega} \overline{m} \neq \overline{n}, \qquad (\forall v) \bigvee_{n \in \omega} v = \overline{n},$$

and

$$\mathscr{S}\overline{X}(\overline{n}).$$

ψ' is then a $L^+_{\omega_1\omega}(\boldsymbol{S})$ sentence such that $\langle \omega, X, 0, 1, 2, \ldots \rangle \models \psi'$ iff $X \in \boldsymbol{S}$. If ψ is a $L^+_{\omega_1\omega}(\boldsymbol{T})$ sentence with the same models, it follows that $\check{\psi}$ is a $P_{\omega_1}(\boldsymbol{T})$ formula such that $T^{\boldsymbol{T}}(\check{\psi}) = \boldsymbol{S}$.

To prove the additional assertion first note that if $\boldsymbol{S} \lneqq_L \boldsymbol{T}$, then $\boldsymbol{B_T}$ contains some $P_{\omega_1}(\boldsymbol{T})$ truth set which is not in $\boldsymbol{B_S}$ so that there is $P_{\omega_1}(\boldsymbol{T})$ formula ψ, which is not equivalent to any formula of $P_{\omega_1}(\boldsymbol{S})$. $\hat{\psi}$ is then an $L_{\omega_1\omega}(\boldsymbol{T})$ sentence which, by Lemma 3.3, is not equivalent to any sentence of $L_{\omega_1\omega}(\boldsymbol{S})$. Hence $L_{\omega_1\omega}(\boldsymbol{S}) \subsetneq_L L_{\omega_1\omega}(\boldsymbol{T})$.

Conversely suppose that $L_{\omega_1\omega}(\boldsymbol{S}) \subsetneq_L L_{\omega_1\omega}(\boldsymbol{T})$. Then from the first part of the proof it follows that $\boldsymbol{B_S} \neq \boldsymbol{B_T}$, so that $\boldsymbol{S} \lneqq_L \boldsymbol{T}$. ■

Part of the significance of Theorem 3.1 is that it shows that results about the structure of the L-degrees translate immediately into results about the structure of the set of propositional extensions of $L_{1\omega\omega}$ with respect to the partial ordering $\subseteq_L$. Hence it will be important to gain information about the structure of the L-degrees. To do so we shall not only work directly with the definition of $\leqslant_L$, but also with some results from the literature concerning orderings related to $\leqslant_L$.

Two such orderings are given in the following

Definition (Wadge [34]). Let S and T be sets of reals. We say that S is *many-one reducible* to T (in symbols $S \leqslant_m T$) if $S = f^{-1}[T]$ for some total continuous function f. S is *Wadge reducible* to T (in symbols $S \leqslant_w T$) if $S \leqslant_m T$ or $S \leqslant_m P(\omega) - T$. The *Wadge degree* of S is $\{T \mid S \leqslant_w T$ and $T \leqslant_w S\}$. S is *Wadge equivalent* to T (in symbols $S \equiv_w T$) if S is in the Wadge degree if T. Similarily, the *m-degree* of S is $\{T \mid S \leqslant_m T$ and $T \leqslant_m S\}$. Finally S is *m-equivalent* to T (in symbols $S \equiv_m T$) if S is in the m-degree of T.

A third ordering is given in the following

Definition (Solovay [31]). Let S and T be sets of reals. S is called *Solovay reducible* to T (in symbols $S \leqslant_s T$) if there is a real X such that S is (Kleene) recursive in T, ${}^2E$ and X. The *Solovay degree* of S is $\{T \mid S \leqslant_s T$ and $T \leqslant_s S\}$.

These orderings are related to $\leqslant_L$ according to the following

Lemma 3.4. (a) *If $S \leqslant_w T$, then $S \leqslant_L T$.*

(b) *If $S \leqslant_L T$, then $S \leqslant_s T$.*

Proof. (a) If $S \leqslant_w T$, then either $S = f^{-1}[T]$ or $S = f^{-1}[P(\omega) - T]$ for some total continuous f. However, $T \in B_T$ (by Lemma 3.2(i)) so that $P(\omega) - T \in B_T$. Hence $S \in B_T$ (by Lemma 3.2 (ii)).

(b) It suffices here to show that each element of B_T is Solovay reducible to T. This can be done quite easily by induction on the number of applications of complementation, union or the operation T. ∎

We shall have occasion to use the following results from the literature concerning $\leqslant_w$ and $\leqslant_s$.

I (Wadge [34]). The Axiom of Determinacy (AD) implies that for all $S, T \subseteq P(\omega)$ either $S \leqslant_m T$ or $T \leqslant_m P(\omega) - S$. This of course implies that $\leqslant_w$ is a linear ordering—something which is not true about $\leqslant_m$ since a set of reals need not be many-one reducible to its complement.

II (Martin [26]). (i) AD+DC (the Axiom of Dependent Choice) implies that the Wadge degrees are well-ordered by $\leqslant_w$.

(ii) Δ_n^1 – Det (all Δ_n^1 games are determined) + DC + the Continuum Hypothesis (CH) imply that the Wadge degrees of the Δ_n^1 sets of reals are well-ordered by $\leqslant_w$.

III (Busch [6]). The Axiom of Choice (AC) implies that there is a set, of cardinality $(2^{\aleph_0})^+$, of pairwise incomparable Solovay degrees. (A weaker hypothesis is sufficient.)

The notions and results discussed so far will provide some of the important technical tools for our later work. At this point however we shall pause briefly to discuss a few simple applications concerning the structure of the entire set of propositional extensions of $L_{\omega_1\omega}$ with respect to the partial ordering $\subseteq_L$. In so doing we shall get a first glimpse of the dependence of many of the results in this area on the set-theoretic assumptions one uses.

First we note the rather remarkable organizational unity which results from assuming AD.

THEOREM 3.2 (AD+DC). *The set of propositional extensions of* $L_{\omega_1\omega}$ *forms an increasing chain well-ordered by* $\subseteq_L$.

Proof. Immediate from II and Lemma 3.4(a) and Theorem 3.1. ∎

Assuming CH however, the following contrasting result is obtained.

THEOREM 3.3 (AC+CH) *There are* $(2^{\aleph_0})^+$ *mutually* $\subseteq_L$*-incomparable propositional extensions of* $L_{\omega_1\omega}$.

Proof. Immediate from III, Lemma 3.4(b) and Theorem 3.1. ∎

The first element in the ordering of Theorem 3.2 is of course $L_{\omega_1\omega}$ itself. (One should note in this connection, that $L_{\omega_1\omega}(S)$ is always L-equivalent to $L_{\omega_1\omega}$ whenever S is Borel.) Since the question of characterizing the second element of this ordering has important technical, as well as aesthetic, consequences, we shall devote the rest of this section to a discussion of this question.

For this purpose we let $W = \{X \subseteq \omega \mid X \text{ is a well-ordering}\}$.

THEOREM 3.4 (AD). $L_{\omega_1\omega}(W)$ *is the smallest proper propositional extension of* $L_{\omega_1\omega}$.

Proof. We first show that for any $S \subseteq P(\omega)$, either $W \leqslant_w S$ or S is Borel. For suppose that $W \nleqslant_w S$, then neither $W \leqslant_m S$ nor $P(\omega)-W \leqslant_m S$, and hence (by I) both $S \leqslant_m P(\omega)-W$ and $P(\omega)-S \leqslant_m P(\omega)-W$. Moreover since W is Π_1^1, $P(\omega)-W$ is Σ_1^1 and since the inverse image of a Σ_1^1 set under a total continuous function is Σ_1^1, it follows that both S and $P(\omega)-S$ are Σ_1^1 and hence Borel. The theorem now follows immediately by Lemma 3.4 and Theorem 3.1. ∎

In the absence of AD we can still get the following result.

THEOREM 3.5. *If* $(\forall X)(X^{\#}$ *exists*), *then* $L_{\omega_1\omega}(W)$ *is a minimal propositional extension of* $L_{\omega_1\omega}$.

Proof. Modify the proof of Theorem 3.3 using the following observations.

(1) If $S \leqslant_w W$, then either S or $P(\omega)-S$ is Π_1^1.

(2) Wadge's proof of the crucial fact (I) for sets which are either Σ_1^1 or Π_1^1 requires only the assumption of determinacy for sets which are unions of a Π_1^1 and a Σ_1^1 set. (Such determinacy will be labelled $\Pi_1^1 \vee \Sigma_1^1$-Det.)

(3) Friedman [12] has shown that $\Pi_1^1 \vee \Sigma_1^1$-Det follows from $(\forall X)(X^{\#}$ exists). ■

Under the assumption of $V = L$ (Gödel's Axiom of Constructibility), however, W is not the smallest, or even minimal, propositional extension of $L_{\omega_1\omega}$. Indeed, the L-degrees of Π_1^1 sets strictly between the L-degree of W and the L-degree of the Borel sets are densely ordered by $\leqslant_L$. These results will be discussed further in Section 7.

4. General model-theoretic properties of $L_{\omega_1\omega}(S)$

Although $L_{\omega_1\omega}(S)$ is a *proper extension of* $L_{\omega_1\omega}$ when S is non-Borel, there is a number of properties of $L_{\omega_1\omega}$ which are shared by $L_{\omega_1\omega}(S)$, for all S. These include a completeness and Löwenheim–Skolem property as well as a weak interpolation and Beth definability property.

Since the proofs of these properties involve little more than generalizing, in a quite straightforward fashion, the corresponding proofs for the $L_{\omega_1\omega}$ case, we do not regard these results as particularly deep. We have, however, obtained a few sharper results about $L_{\omega_1\omega}(S)$, concerning bounds on the cardinality of the set of premises needed for the various deduction rules, which seem to require some tricks from descriptive set theory.

In this section we shall work out all of this systematically. As the development will more or less parallel the development for $L_{\omega_1\omega}$, we shall discuss in detail only those parts which differ from the $L_{\omega_1\omega}$ case. In so doing we shall rely heavily on Keisler's exposition in [19].

We begin with a model existence theorem, from which many of the results will follow. For this purpose, it turns out to be convenient to regard $\bigvee$, $\forall$ and $\mathscr{B}$ as well as $\bigwedge$, $\exists$ and $\mathscr{S}$ as primitive symbols of $L_{\omega_1\omega}(S)$. Hence for the remainder of this section (and Section 5) we shall do precisely this.

We first define the set, sub(φ), of subformulas of φ, for any formula φ of $L_{\omega_1\omega}(S)$. This is done by using Keisler's definition for the analogous notion in $L_{\omega_1\omega}$ and adding the following inductive clauses:

$$\mathrm{sub}(\mathscr{S}\varphi_i) = \Big(\bigcup_{i\in\omega} \mathrm{sub}(\varphi_i)\Big) \cup \{\mathscr{S}\varphi_i\},$$

and

$$\mathrm{sub}(\mathscr{B}\varphi_i) = \Big(\bigcup_{i\in\omega} \mathrm{sub}(\varphi_i)\Big) \cup \{\mathscr{B}\varphi_i\}.$$

We also extend Keisler's notation $(\varphi)\urcorner$, to $L_{\omega_1\omega}(S)$ by adding to his inductive conditions

$$(\mathscr{S}\varphi_i)\urcorner \quad \text{is} \quad \mathscr{B}\sim\varphi_i,$$

and

$$(\not{\mathscr{S}}\varphi_i)\urcorner \quad \text{is} \quad \mathscr{S} \sim \varphi_i.$$

Now let L be any countable language and let C be a countably infinite set of new constant symbols. We let $M_C(L)$ (or simply M) be the result of adding each $\bar{c} \in C$ to L.

A consistency property for $L_{\omega_1\omega}(\mathbf{S})$ will be a certain set of countable sets of sentences of $M_{\omega_1\omega}(\mathbf{S})$. The notion could be defined by blindly generalizing the definition in [19] for the $L_{\omega_1\omega}$ case. However, rather than do this we shall first introduce a new technical concept, which will allow us to obtain more refined results than would otherwise be possible.

Definition. Let $\mathbf{S} \subseteq P(\omega)$ and let $\{\gamma_a \mid a \in A\}$ be any set of formulas of P_{ω_1}. If $\mathbf{S} = \bigcup_{a\in A} T(\gamma_a)$, then $\{\gamma_a \mid a \in A\}$ is called a *cover* of $\mathbf{S}$. Moreover, if $\{\delta_b \mid b \in B\}$ is a cover of $\mathbf{S}$, where $\mathbf{S} = \{X \mid P(\omega) - X \notin \mathbf{S}\}$, then $(\{\gamma_a \mid a \in A\}, \{\delta_b \mid b \in B\})$ is called a *covering pair* for $\mathbf{S}$, and $\max\{\overline{\overline{A}}, \overline{\overline{B}}\}$ is called the *cardinality of the pair.*

It should be noted that every $\mathbf{S}$ has a trivial cover of cardinality $\overline{\overline{\mathbf{S}}}$, namely,

$$\{\bigwedge_{n\in X} p_n \wedge \bigwedge_{n\notin X} \sim p_n \mid X \in \mathbf{S}\}.$$

Further examples of covers will be given later.

We next define the notion of a consistency property with respect to a particular cover.

Definition. Let $\mathbf{S} \subseteq P(\omega)$ and let $(\{\gamma_a \mid a \in A\}, \{\delta_b \mid b \in B\})$ be a covering pair for $\mathbf{S}$. A set S of countable sets of sentences of $M_{\omega_1\omega}(\mathbf{S})$ is called a *consistency property for $L_{\omega_1\omega}(\mathbf{S})$ with respect to the given covering pair* if for each $s \in S$, all of the following hold.

(C 1)–(C 7) on pages 11 and 12 of Keisler [19], as well as,

(C 8) if $(\mathscr{S}\varphi_n) \in s$, then $s \cup \{\gamma_a[\varphi_n]\} \in S$ for some $a \in A$, and

(C 9) if $(\not{\mathscr{S}}\varphi_n) \in s$, then $s \cup \{\delta_b[\varphi_n]\} \in S$ for some $b \in B$.

In the following we shall often speak of consistency properties for $L_{\omega_1\omega}(\mathbf{S})$. When we do it should always be understood that we have some fixed covering pair for $\mathbf{S}$ in mind.

Theorem 4.1 (Model Existence Theorem). *If S is a consistency property for $L_{\omega_1\omega}(\mathbf{S})$ and $s_0 \in S$, then s_0 has a model.*

Proof. This will be similar to the proof of Theorem 2 of Keisler [19] except that some modifications will be necessary since it is not possible to begin with a countable set of formulas of $M_{\omega_1\omega}(\mathbf{S})$ containing all of the formulas which will eventually be choosen by (C 8) and (C 9). Hence we proceed as follows.

For each set s of formulas of $M_{\omega_1\omega}(\mathbf{S})$ let $Y(s)$ be the smallest set

Y of $M_{\omega_1\omega}(S)$ formulas such that

$s \subseteq Y$,

Y is closed under subformulas,

if τ is a term, $\bar{c} \in C$ and $\varphi(\tau) \in Y$, then $\varphi(\bar{c}) \in Y$,

if $(\sim\varphi) \in Y$, then $(\varphi\urcorner) \in Y$, and if $\bar{c}, \bar{d} \in C$, then $(\bar{c} = \bar{d}) \in Y$.

Let $\{t_0, t_1, \ldots\}$ enumerate all the basic terms of $M_{\omega_1\omega}(S)$.

As in [19] we shall construct a sequence $s_0, s_1, s_2, \ldots$ of elements of S, by induction, as follows.

Let s_0 be as given and suppose that s_n has been defined. Let $\{\psi_{n,0}\ \psi_{n,1}, \psi_{n,2}, \ldots\}$ enumerate all sentences in $Y(s_n)$. Let $s'_n = s_n \cup \{\bar{c} = t_n\}$ for some $\bar{c} \in C$ such that $s'_n \in S$. If $s'_n \cup \{\psi_{(n)_0,(n)_1}\} \notin S$, then $s_{n+1} = s'_n$. Otherwise

$$s_{n+1} = \begin{cases} s'_n \cup \{\psi_{(n)_0,(n)_1}\} & \text{if } \psi_{(n)_0,(n)_1} \text{ is atomic, negated atomic, conjunctive, or universally quantified;} \\ s'_n \cup \{\psi_{(n)_0,(n)_1}, \theta\} & \text{if } \psi_{(n)_0,(n)_1} \text{ is of the form } \bigvee \Phi \text{ and } \theta \in \Phi \text{ is such that } s'_n \cup \{\psi_{(n)_0,(n)_1}, \theta\} \in S; \\ s'_n \cup \{\psi_{(n)_0,(n)_1}, \varphi(\bar{c})\} & \text{if } \psi_{(n)_0,(n)_1} \text{ is of the form } (\exists x)\varphi \text{ and } \bar{c} \in C \text{ is such that } s'_n \cup \{\psi_{(n)_0,(n)_1}, \varphi(\bar{c})\} \in S; \\ s'_n \cup \{\psi_{(n)_0,(n)_1}, \gamma_a[\varphi_n]\} & \text{if } \psi_{(n)_0,(n)_1} \text{ is of the form } \mathscr{S}\varphi_n \text{ and } a \in A \text{ is such that } s'_n \cup \{\psi_{(n)_0,(n)_1}, \gamma_a[\varphi_n]\} \in S; \\ s'_n \cup \{\psi_{(n)_0,(n)_1}, \delta_b[\varphi_n]\} & \text{if } \psi_{(n)_0,(n)_1} \text{ is of the form } \mathscr{B}\varphi_n \text{ and } b \in B \text{ is such that } s'_n \cup \{\psi_{(n)_0,(n)_1}, \delta_b[\varphi_n]\} \in S. \end{cases}$$

Note that (C 5), (C 6), (C 7), (C 8) and (C 9) insure that s_{n+1} can always be chosen to be in S. Also note that the pairing functions of Shoenfield [30] have the property that $(n)_0 \leqslant n$ so that $\psi_{(n)_0,(n)_1}$ is defined at stage $n+1$.

Now let $s_\omega = \bigcup_{n \in \omega} s_n$ and let $\mathfrak{A}$ be as on the bottom of page 13 of Keisler [19]. To complete the proof it suffices to show that

(1) $$\mathfrak{A} \models \varphi \quad \text{if} \quad \varphi \in s_\omega.$$

This is done by induction on $\varrho(\varphi)$ where

$$\varrho(\varphi) = \begin{cases} \sup\{\varrho(\psi) \mid \psi \in \mathrm{sub}(\varphi), \psi \neq \varphi\} + 1, & \text{if } \varphi \text{ is of the form } \mathscr{S}\varphi_n \text{ or } \mathscr{B}\varphi_n; \\ \sup\{\varrho(\psi) \mid \psi \in \mathrm{sub}(\varphi), \psi \neq \varphi\}, & \text{otherwise.} \end{cases}$$

Hence assume that (1) holds for all ψ with $\varrho(\psi) < a$ and suppose that φ is such that $\varrho(\varphi) = a$. If φ is of the form $\mathscr{S}\varphi_n$ and $\varphi \in s_\omega$, then clearly $\gamma_a[\varphi_n]$, for some $a \in A$, must belong to s_ω. Since $\varrho(\gamma_a[\varphi_n]) = \sup\{\varrho(\varphi_n)\} < \varrho(\varphi)$, it follows that $\mathfrak{A} \models \gamma_a[\varphi_n]$, and hence $\mathfrak{A} \models \mathscr{S}\varphi_n$.

If φ is of the form $\mathscr{B}\varphi_n$, the proof is similar. The remaining cases for φ, with $\varrho(\varphi) = a$, are handled by induction in the usual manner using (C 1)-(C 7). ■

Our first use of the Model Existence Theorem will be to show that $L_{\omega_1\omega}(S)$ always possesses a (not totally trivial) complete axiomatization. The axiomatization will be the one given on page 15 of [19] augmented by rules for $\mathscr{S}$ and $\mathcal{S}$.

As first approximation we might wish to consider the following rules:

(I) From $(\bigwedge_{n\in X}\varphi_n\wedge\bigwedge_{n\notin X}\sim\varphi_n)\to\psi$, for all $X\in S$, deduce $\mathscr{S}\varphi_n\to\psi$.

(II) From $(\bigwedge_{n\in X}\varphi_n\wedge\bigwedge_{n\notin X}\sim\varphi_n)\to\psi$, for all $X\in\mathsf{S}$, deduce $\mathcal{S}\varphi_n\to\psi$,

where $\mathsf{S}=\{X\subseteq\omega\mid P(\omega)-X\notin S\}$.

A moment's reflection on the meaning of $\mathscr{S}$ and $\mathcal{S}$, in $L_{\omega_1\omega}(S)$ shows that (I) and (II) are valid rules of inference. They however have a serious defect when compared with the $L_{\omega_1\omega}$ rules, in that, when S is non-Borel, (I) and (II) require as many premises as $\bar{\bar{S}}$ —a cardinal which is always at least as large as $\aleph_1$ (and possibly larger)—whereas the later rules require only countably many premises.

We can partially remedy this situation by exploiting the notion of a covering pair. Here the idea is to reformulate (I) and (II) so as to require only as many premises as the cardinality of some covering pair for S. In many important cases this may be less than $\bar{\bar{S}}$, as we see from the following

EXAMPLES. 1. If S is Σ^1_2, S will have a cover of cardinality $\aleph_1$. This follows from the classical result that every Σ^1_2 set is the union of $\aleph_1$ Borel sets, plus the fact that each Borel set is the truth set of some P_{ω_1} formula. It follows that Δ^1_2 connectives have covering pairs of cardinality $\aleph_1$.

2. If $(\forall X)$ ($X^{\#}$ exists), then every Σ^1_3 set of reals is the union of $\leqslant\aleph_2$ Borel sets (see Martin [25]). Hence Δ^1_3 connectives have covers of cardinality $\leqslant\aleph_2$.

3. Assuming Δ^1_2-Det, every Σ^1_4 set is a union of $\aleph_3$ Borel sets (this result is due to Moschovakis; see Martin [25]). It follows that Δ^1_4 connectives have covers of cardinality $\leqslant\aleph_3$.

The reformulated rules for $L_{\omega_1\omega}(S)$ are now as follows:

(I′) From $\gamma_a[\varphi_n]\to\psi$ for all $a\in A$ deduce $\mathscr{S}\varphi_n\to\psi$, and

(II′) From $\delta_b[\varphi_n]\to\psi$ for all $b\in B$ deduce $\mathcal{S}\varphi_n\to\psi$,

where it is understood that $(\{\gamma_a\mid a\in A\},\ \{\delta_b\mid b\in B\})$ is a covering pair for S.

These rules are then added to the $L_{\omega_1\omega}$ rules. A *theorem* of the resulting axiomatization is then defined, as usual, to be an element of the intersection of all sets containing the axioms and closed under all the rules of inference.

It might appear that different axiomatizations of $L_{\omega_1\omega}(S)$ may lead to different sets of theorems. The next result however shows that this is not the case. Indeed it shows that no matter what cover we choose, the resulting axiomatization is always sound and complete.

THEOREM 4.2 (Completeness theorem for $L_{\omega_1\omega}(\mathbf{S})$). *For any $\mathbf{S} \subseteq P(\omega)$ and any covering pair, $(\{\gamma_a \mid a \in A\},\ \{\delta_b \mid b \in B\})$, for $\mathbf{S}$*

$$\models^{\mathbf{S}} \varphi \quad \textit{iff} \quad \vdash_{L_{\omega_1\omega}(\mathbf{S})} \varphi$$

for all sentences φ of $L_{\omega_1\omega}(\mathbf{S})$.

Proof. The fact that all theorems are valid follows by an easy induction on the complexity of a proof.

For the converse we let S be the set of all finite sets of $M_{\omega_1\omega}(\mathbf{S})$ sentences such that only finitely many $\bar{c} \in C$ occur in s and $\sim \bigwedge s$ is not provable in $M_{\omega_1\omega}(\mathbf{S})$. As in Keisler [19], it is easy to show that S is a consistency property. We illustrate by showing that S satisfies (C 8).

Suppose that $\mathscr{S}\varphi_n \in s \in S$, but for all $a \in A$, $s \cup \{\gamma_a[\varphi_n]\} \notin S$. Then since each $\gamma_a[\varphi_n]$ contains only finitely many $\bar{c} \in C$, it follows that

$$\vdash_{M_{\omega_1\omega}(\mathbf{S})} \sim \bigwedge (s \cup \{\gamma_n[\varphi_n]\})$$

for all $a \in A$. But since s is finite, it follows that

$$\vdash_{M_{\omega_1\omega}(\mathbf{S})} \gamma_a[\varphi_n] \rightarrow \sim \bigwedge s$$

for all $a \in A$. (I′) then implies

$$\vdash_{M_{\omega_1\omega}(\mathbf{S})} \mathscr{S}\varphi_n \rightarrow \sim \bigwedge s.$$

However, $\mathscr{S}\varphi_n \in s$, so that

$$\vdash_{M_{\omega_1\omega}(\mathbf{S})} \bigwedge s \rightarrow \mathscr{S}\varphi_n.$$

Axiom 1 of [19] then implies that

$$\vdash_{M_{\omega_1\omega}(\mathbf{S})} \sim \bigwedge s$$

contradicting the fact that $s \in S$.

The rest of the proof follows exactly as in [19]. ■

In addition to completeness, the $L_{\omega_1\omega}(\mathbf{S})$ Model Existence Theorem can be used, in exactly the same way as in $L_{\omega_1\omega}$ to prove a downward Löwenheim–Skolem result. We merely state the theorem here since the proof is entirely routine.

THEOREM 4.3. *If Φ is a countable set of sentences of $L_{\omega_1\omega}(\mathbf{S})$ which has a model, then Φ has a countable model.*

We remark that the Skolem function considerations in Chapter 13 of [19] work equally well for $L_{\omega_1\omega}(\mathbf{S})$.

Hence the following stronger Löwenheim–Skolem theorem is also true.

THEOREM 4.4. *If φ is an $L_{\omega_1\omega}(\mathbf{S})$ sentence, $\mathfrak{A}$ is a model of φ and $X \subseteq |\mathfrak{A}|$,* *then for each cardinal* $\varkappa$, *such that* $\overline{\overline{X}} \leqslant \varkappa \leqslant \overline{\overline{\mathfrak{A}}}$, *there is a substructure* $\mathfrak{B}$ *of* $\mathfrak{A}$ *of cardinality* $\varkappa$ *such that* $X \subseteq |\mathfrak{B}|$ *and* $\mathfrak{B}$ *is a model of* φ.

We next wish to take up the question of interpolation for $L_{\omega_1\omega}(S)$. Here, as we shall see in Section 6, it is not possible to obtain the full analogue of the $L_{\omega_1\omega}$ result. However, the method used in Keisler [19], when adapted to $L_{\omega_1\omega}(S)$, does give the following weak form of interpolation.

THEOREM 4.5. *If* S *is a connective with a covering pair of cardinality* λ, *then* $\mathrm{Craig}(L_{\omega_1\omega}(S), L_{\lambda^+\omega})$.

Proof. Exactly as in [19] except that X_φ is defined to be the set of all sentences φ' of $L_{\lambda^+\omega}$ such that every relation, function or constant symbol of L which occurs in φ' also occurs in φ and only finitely many $\bar{c} \in C$ occur in φ'. X_ψ is also defined similarly. ■

Since the usual proof, in $L_{\omega\omega}$, that interpolation implies Beth definability works equally well for $L_{\omega_1\omega}(S)$, the above immediately implies

THEOREM 4.6. *If* S *is a connective with a cover of cardinality* λ, *then* $\mathrm{Beth}(L_{\omega_1\omega}(S), L_{\lambda^+\omega})$. ■

These last two results will be discussed further in Section 6 where it will be shown that any improvements are unlikely.

We close this section with the remark that it is also possible to extend the results of Chapters 6 and 7 of Keisler [19] in the style of Theorems 4.5 and 4.6. In this way, for example, we could show that, whenever S has a covering pair of cardinality λ, the sentences of $L_{\omega_1\omega}(S)$ which are preserved under homomorphisms are precisely those sentences equivalent to positive $L_{\lambda^+\omega}$ sentences. Similarly the $L_{\omega_1\omega}(S)$ sentences which are preserved under substructures are precisely those sentences which are equivalent to universal $L_{\lambda^+\omega}$ sentences.

5. Hanf number computations

The *Hanf number* of a logic L^* (in symbols $H(L^*)$) is defined to be the least cardinal λ such that if a sentence φ of L^* has a model of cardinality λ, then φ has arbitrarily large models.

$H(L_{\omega_1\omega})$ was originally shown by Morley to be $\beth_{\omega_1}$. His proof, which combined the method of indescernibles with a combinatorial result of Erdös and Rado, was later considerably extended by Barwise and Kunen (and independently Morely himself) so as to apply to much more general situations.

In this section we use the ideas of this later proof to compute the Hanf number of $L_{\omega_1\omega}(S)$. To do so, however, requires a discussion of a few notions which we have not previously used.

We first remark that the procedure, described in [19], for expanding fragments of $L_{\omega_1\omega}$ to Skolem fragments has a straightforward extension to $L_{\omega_1\omega}(\mathbf{S})$, so that the following lemma can be easily established.

LEMMA 5.1. *Each formula* φ *of* $L_{\omega_1\omega}(\mathbf{S})$ *belongs to some countable Skolem fragment,* $L_A^*(\mathbf{S})$, *of* $L_{\omega_1\omega}(\mathbf{S})$.

If $L_A(\mathbf{S})$ is a fragment of $L_{\omega_1\omega}(\mathbf{S})$ and T is a theory in $L_A(\mathbf{S})$, then T is called *complete* if for each sentence φ of $L_A(\mathbf{S})$ either $\varphi \in T$ or $\sim\varphi \in T$ (but not both). T is called *closed* if it contains all the logical axioms for $L_A(\mathbf{S})$, is closed under rules (1), (2), and (3) on page 16 of [19] and is such that

(i) if $\mathscr{S}\varphi_n \in L_A(\mathbf{S})$, then $\mathscr{S}\varphi_n \in T$ iff $\{n \in \omega \mid \varphi_n \in T\} \in \mathbf{S}$, and

(ii) if $\reflectbox{$\mathscr{S}$}\varphi_n \in L_A(\mathbf{S})$, then $\reflectbox{$\mathscr{S}$}\varphi_n \in T$ iff $\{n \in \omega \mid \varphi_n \in T\} \in \mathbf{S}$.

The notion of a complete closed theory is important for the following analogue of Lemma 1.5 from Barwise–Kunen [4]. It can be proved along the same lines as the second half of the proof of the Model Existence Theorem.

LEMMA 5.2. *If* $L_A^*(\mathbf{S})$ *is a Skolem fragment of* $L_{\omega_1\omega}(\mathbf{S})$ *and* T *is a Skolem theory in* $L_A^*(\mathbf{S})$, *then* T *has a model iff* T *is complete and closed.*

We next recall that every well-founded relation R has an *ordinal rank*, $o(R)$, defined as

$$\sup\{\varrho_R(x)+1 \mid x \in \mathrm{Fld}_R\},$$

where ϱ_R is an ordinal-valued function defined by transfinite induction on Fld_R by

$$\varrho_R(x) = \sup\{\varrho_R(y)+1 \mid R(y, x)\}.$$

Given a connective $\mathbf{S}$, we let

$$\mathrm{ord}(\boldsymbol{B}_{\mathbf{S}}) = \sup\{o(R) \mid R \in \boldsymbol{B}_{\mathbf{S}} \text{ and } R \text{ is well-founded}\},$$

and similarly,

$$\mathrm{ord}\boldsymbol{\Sigma}(\boldsymbol{B}_{\mathbf{S}}) = \sup\{o(R) \mid R \in \boldsymbol{\Sigma}(\boldsymbol{B}_{\mathbf{S}}) \text{ and } R \text{ is well-founded}\}.$$

Finally we adapt the notion of an L_A-accessible ordinal from [4]. We say that an ordinal a is *pinned down* by a sentence $\varphi(\overline{U}, \overline{<}, \dots)$ of $L_{\omega_1\omega}(\mathbf{S})$ if

(a) each model $\langle A, U, <, \dots)$ of φ is such that $\langle U, <\rangle$ is well-ordered, and

(b) there is a model $\langle B, U, <, \dots\rangle$ of φ such that $\langle U, <\rangle$ has order type at least a.

$h(\mathbf{S})$ will denote the least ordinal which cannot be pinned down by a sentence of $L_{\omega_1\omega}(\mathbf{S})$. The following theorem gives a neat characterization of $h(\mathbf{S})$. The method of proof will be crucial for the Hanf number computations.

Theorem 5.1. *For all* $S \subseteq P(\omega)$, $h(S) = \operatorname{ord}(\boldsymbol{B}_S) = \operatorname{ord}\big(\Sigma(\boldsymbol{B}_S)\big)$.

Proof. Since $\operatorname{ord}(\boldsymbol{B}_S) \leqslant \operatorname{ord}\big(\Sigma(\boldsymbol{B}_S)\big)$, it suffices to show

(a) $h(S) \leqslant \operatorname{ord}(\boldsymbol{B}_S)$, and

(b) $\operatorname{ord}\big(\Sigma(\boldsymbol{B}_S)\big) \leqslant h(S)$.

To prove (a), let $\varphi(\overline{U}, \overline{<}, \ldots)$ be a sentence of $L_{\omega_1\omega}(S)$ such that $<^{\mathfrak{A}}$ well-orders $U^{\mathfrak{A}}$ in all models $\mathfrak{A}$ of φ. Let $L_B^*(S)$ be a *countable* Skolem fragment of $L_{\omega_1\omega}(S)$ containing φ. Let $\{\bar{c}_0, \bar{c}_1, \ldots\}$ be a countable set of new constant symbols. Denote the language $L^* \cup \{\bar{c}_0, \ldots, \bar{c}_k\}$ by L^k.

For each $k \in \omega$, let $\mathscr{T}_k$ be the set of all complete closed theories, T, in $L_B^*(S)$ such that

(i) $\varphi \in T$,

(ii) $T_{\text{Skolem}} \subseteq T$,

(iii) $\overline{U}(\bar{c}_i) \in T$ (for $i = 0, \ldots, k$), and

(iv) $(c_{i+1} < c_i) \in T$ (for $i = 0, \ldots, k-1$).

Let $\mathscr{T} = \bigcup_{k\in\omega} \mathscr{T}_k$ and for $T, T' \in \mathscr{T}$, let $T <_{\mathscr{T}} T'$ iff T properly contains T'. Then just as in Theorem 1.4 of [4] one can show that $\langle \mathscr{T}, <_{\mathscr{T}} \rangle$ is well-founded and that $<^{\mathfrak{A}}$ is bounded in order type $o(\mathscr{T})$, whenever $\mathfrak{A}$ is a model of φ.

To complete part (a), it suffices to find a well-founded relation $R \in \boldsymbol{B}_S$ which is isomorphic to $\mathscr{T}$. To do so let f be a fixed 1-1 map of B onto ω, let N be the language consisting of a single unary relation symbol $\overline{X}$ along with constant symbols for the natural numbers, and let $\psi_k(\overline{X})$ be the conjunction of the following sentences of $N_{\omega_1\omega}(S)$:

$$\bigwedge_{n\in\omega} \Big(\overline{X}(\bar{n}) \to \bigvee_{\psi\in B^k} \big(\bar{n} = f(\psi)\big)\Big), \tag{1}$$

where B^k is the set of all formulas of $L_B^k(S)$,

$$\overline{X}\big(\overline{f(\varphi)}\big), \tag{2}$$

$$\bigwedge_{\psi\in T_{\text{Skolem}}} \overline{X}\big(\overline{f(\psi)}\big), \tag{3}$$

$$\bigwedge_{i\leqslant k} \overline{X}\Big(\overline{f\big(\overline{U}(\bar{c}_i)\big)}\Big), \tag{4}$$

$$\bigwedge_{i\leqslant k-1} \overline{X}\big(\overline{f(\bar{c}_{i+1} \mathrel{\overline{<}} \bar{c}_i)}\big), \tag{5}$$

$$\bigwedge_{\psi\in \mathrm{LA}(B^k)} \overline{X}\big(\overline{f(\psi)}\big), \tag{6}$$

where $\mathbf{LA}(B^k)$ is the set of logical axioms of $L_{\omega_1\omega}^k(S)$ which are in B,

$$\bigwedge_{\psi,\chi\in B^k} \overline{X}\big(\overline{f(\psi)}\big) \wedge \overline{X}\big(\overline{f(\psi\to\chi)}\big) \to \overline{X}\big(\overline{f(\chi)}\big), \tag{7}$$

(8) $$\bigwedge \bar{X}\left(\overline{f(\psi\to\chi)}\right)\to\bar{X}\left(\overline{f(\psi\to(\forall x)\,\chi)}\right)$$

where the conjunction is taken over all ψ and χ in B^k such that x does not occur free in ψ,

(9) $$\bigwedge\Big[\bigwedge_{\chi\in\Phi}\bar{X}\left(\overline{f(\psi\to\chi)}\right)\to\bar{X}\left(\overline{f(\psi\to\bigwedge\Phi)}\right)\Big]$$

where the outer conjunction is taken over all subsets Φ of B^k such that $\bigwedge\Phi\in B^k$,

(10) $$\bigwedge_{\mathscr{S}\varphi_n\in B^k}\left(\bar{X}\left(\overline{f(\mathscr{S}\varphi_n)}\right)\to\mathscr{S}\bar{X}\left(\overline{f(\varphi_n)}\right)\right),$$

(11) $$\bigwedge_{\mathcal{S}\varphi_n\in B^k}\left(\bar{X}\left(\overline{f(\mathcal{S}\varphi_n)}\right)\to\mathcal{S}\bar{X}\left(\overline{f(\varphi_n)}\right)\right),$$

(12) $$\bigwedge_{\psi\in B^k}\bar{X}\left(\overline{f(\psi)}\right)\vee\bar{X}\left(\overline{f(\sim\psi)}\right),$$

and

(13) $$\bigwedge_{\psi\in B^k}\sim\left(\bar{X}\left(\overline{f(\psi)}\right)\wedge\bar{X}\left(\overline{f(\sim\psi)}\right)\right).$$

Now let $\chi(\bar{X})$ be $\bigvee_{k\in\omega}\psi_k(\bar{X})\wedge(\forall v_0)\bigvee_{n\in\omega}(v_0=\bar{n})$. Note that (1)–(13) insure that if $\langle\omega, X, 0, 1, 2, \ldots\rangle\models\chi(\bar{X})$, then $f^{-1}[X]\in\mathscr{T}$, i.e. (1) insures that $f^{-1}[X]$ is a theory in $L^k_B(S)$, (2) insures that $\varphi\in f^{-1}[X]$, (3) insures that $f^{-1}[X]$ is Skolemized, (4) insures that $\bar{U}(\bar{c}_i)\in f^{-1}[X]$ (for all $i\leqslant k$), (5) insures that $(\bar{c}_{i+1}\mathrel{\bar{<}}\bar{c}_i)\in f^{-1}[X]$ (for all $i\leqslant k-1$), (6)–(11) insure that $f^{-1}[X]$ is closed and (12)–(13) insure that $f^{-1}[X]$ is complete.

Hence if $\theta(\bar{X}_0, \bar{X}_1)$ is taken to be

$$\chi(\bar{X}_0)\wedge\chi(\bar{X}_1)\wedge\bigwedge_{n\in\omega}\left(\bar{X}_0(\bar{n})\to\bar{X}_1(\bar{n})\right)\wedge\bigvee_{n\in\omega}\left(\bar{X}_1(\bar{n})\wedge\sim\bar{X}_0(\bar{n})\right),$$

then $\langle\omega, X_0, X_1, 0, 1, 2, \ldots\rangle\models\theta$ iff $f^{-1}[X_0]<_{\mathscr{T}}f^{-1}[X_1]$, so that by Lemma 3.3 (a), $<_{\mathscr{T}}$ is isomorphic to the truth set of θ. Hence $o(<_{\mathscr{T}})=o\left(T^S(\theta)\right)<\operatorname{ord}(\boldsymbol{B}_S)$.

To prove part (b) we must show that for each $R\in\Sigma(\boldsymbol{B}_S)$ which is well-founded there is an $L_{\omega_1\omega}(S)$ sentence $\varphi_R(\bar{U}, \bar{<}, \ldots)$ which pins down $o(R)$.

Hence suppose that R is well-founded and is in $\Sigma(\boldsymbol{B}_S)$. Then there is a $P_{\omega_1}(S)$ formula ψ, such that $(X, Y)\in R$ iff $\models^S\psi[(X, Y, Z)]$ for some $Z\subseteq\omega$. Hence if ψ_1 is

$$\bigwedge_{\substack{m,n\in\omega\\ m\neq n}}(\bar{m}\neq\bar{n})\wedge(\forall v_0)\bigvee_{n\in\omega}(v_0=\bar{n})\wedge\hat{\psi},$$

it follows that $\mathfrak{A}\models\psi_1$ iff $\mathfrak{A}$ is isomorphic to $\langle\omega, X, Y, Z, 0, 1, 2, \ldots\rangle$ where X, Y and Z are such that $\models^S\psi[(X, Y, Z)]$.

Now add $\{\bar{Q}, \bar{U}, \bar{<}, \bar{F}, \bar{E}\}$ to the language of ψ_1—where $\bar{Q}$ and $\bar{U}$

are unary relation symbols, $\overline{<}$ is a binary relation symbol, $\bar{F}$ is a ternary relation symbol and $\bar{E}$ is a quintary relation symbol—and call the resulting language N. It is then easy to write down a sentence φ_R of $N_{\omega_1\omega}(S)$ such that whenever $\langle A, Q, U, <, F, E, q_0, q_1, \ldots\rangle$ is a model of φ_R, then

$<$ linearly orders U,
$\langle Q, q_0, q_1, \ldots\rangle$ is isomorphic to $\langle\omega, 0, 1, \ldots\rangle$ via some map $\sim$, and
F and E are such that

(i) whenever $x \neq y$, $F_t(x) \neq F_{t'}(y)$, and

(ii) whenever $t' \in A$, $x, y \in U$ and $x < y$,

then there is a $t \in A$ such that

$$\models^S \psi[(F_t(x), F_{t'}, (y), E_{x,y,t,t'})],$$

where $F_t(x) = \{\tilde{q} \in \omega \mid F(t, x, q)\}$ and $E_{x,y,t,t'} = \{\tilde{q} \in \omega \mid E(x, y, t, t', \varphi)\}$.

We complete the proof by showing that φ_R pins down $o(R)$. First we show that φ_R has a model $\mathfrak{A}$ in which $\langle U^{\mathfrak{A}}, <^{\mathfrak{A}}\rangle$ has order type $o(R)$. To see this let λ be the cardinality of R and let a be greater than all three of λ, ω and $o(R)$. $\mathfrak{A}$ can then be taken as

$$\langle a, \omega, o(R), \in \restriction o(R), F, E, 0, 1, 2, \ldots\rangle,$$

where F is such that for each $t \in a$ and $x \in o(R)$, $\{q \mid F(t, x, q)\}$ is an element of Fld_R of rank x and for each $Y \in \mathrm{Fld}_R$ of rank x there is a $t \in a$ such that $Y = \{q \mid F(t, q, x)\}$, and, finally, $\{q \mid E(x, y, t, t', q)\}$ is chosen equal to some $Z \subseteq \omega$ for which $\models^S \psi[(F_t(x), F_{t'}(y), Z)]$, if possible, and $\emptyset$ otherwise.

Finally we show that $<^{\mathfrak{A}}$ actually well-orders $U^{\mathfrak{A}}$ in any model $\mathfrak{A}$ of φ. For suppose that $<^{\mathfrak{A}}$ were not a well-ordering. Then there would be an infinite descending chain

$$<^{\mathfrak{A}} x_3 <^{\mathfrak{A}} x_2 <^{\mathfrak{A}} x_1$$

in $U^{\mathfrak{A}}$. Hence if t_1 is any element of $|\mathfrak{A}|$, there is (by (ii)) $t_2 \in |\mathfrak{A}|$ such that $(F_{t_2}(x_2), F_{t_1}(x_1)) \in R$. Similarly (by (ii) again) there is $t_3 \in |\mathfrak{A}|$ such that $(F_{t_3}(x_3), F_{t_2}(x_2)) \in R$. Continuing inductively, we then would obtain an infinite descending chain for R, contradicting the well-foundedness of R. ∎

We remark that when S is Borel, Theorem 5.1 reduces to a well-known result from descriptive set theory—namely the fact that $\aleph_1 = \mathrm{ord}(\mathrm{Borel}) = \mathrm{ord}(\Sigma_1^1)$. We do not know if the sharper form of the classical result (i.e. the fact that the countable ordinals are precisely the order types of both the Borel and Σ_1^1 *well-orderings* of $P(\omega)$) also generalizes to any $S \subseteq P(\omega)$.

Using the proof of Theorem 5.1, the Hanf number of $L_{\omega_1\omega}(S)$ can now be computed in much the same way as Barwise–Kunen [4] use the proof of their Theorem 1.4 in computing the Hanf number of L_A. Of course one must first verify the analogues, for $L_{\omega_1\omega}(S)$, of the various results one needs concerning indiscernibles. These, however, follow quite straightforwardly from the proofs given by Keisler [19] for the $L_{\omega_1\omega}$ case. Hence, once these details are carried out, we have

THEOREM 5.2. *The Hanf number of* $L_{\omega_1\omega}(S)$ *is* $\beth_{h(S)}$.

We close this section by mentioning a small improvement of Morley's two-cardinal result (cf. Theorem 3.1 in Barwise–Kunen [4]) which can be obtained from the above considerations.

THEOREM 5.3 (in the style of Morley). *Let* φ *be a sentence of* $L_{\omega_1\omega}(S)$. *Suppose that for each* $a < h(S)$ *there is* $\lambda \geqslant \omega$ *such that* φ *admits* $\big(\beth_a(\lambda), \lambda\big)$. *Then* φ *admits* $(\varkappa, \omega)$ *for all* $\varkappa \geqslant \omega$.

The proof of Theorem 5.3 follows from the proof of Theorem 5.2 in exactly the same way as the proof of Theorem 3.2 of [4] follows from the proof of Theorem 2.1 of [4].

The way in which Theorem 5.3 improves Theorem 3.2 of [4] is that it reduces the bound $h(A)$ to $h(S)$ for those sentences of $L_{(2^{\aleph_0})^+\omega}$, which happen to be equivalent to $L_{\omega_1\omega}(S)$ sentences.

6. Negative results for $L_{\omega_1\omega}(S)$

We turn now from the general results of the last two sections to an analysis of $L_{\omega_1\omega}(S)$ based on properties of S itself. Here we shall find that most of the propositional extensions of $L_{\omega_1\omega}$ fail to have the usual nice properties which would make them desirable for model-theoretic purposes. Among these are the Beth definability and Σ_1-validity properties.

All the negative results which we obtain will generally hinge on the fact that $h(S)$ is often greater than $\aleph_1$. Exactly when this is true will be discussed later. For the moment we shall use it as working hypothesis.

First we note the following lemma, which enables us to transfer results from $P_{\omega_1}(S)$ to $L_{\omega_1\omega}(S)$ and conversely.

LEMMA 6.1. $\mathrm{Val}\big(L_{\omega_1\omega}(S)\big)$ *is* Σ_1 *over* HC *iff* $\mathrm{Val}\big(P_{\omega_1}(S)\big)$ *is* Σ_1 *over* HC.

Proof. (a) For notational simplicity assume that L consists of only a single unary relation symbol. Given a sentence φ of $L_{\omega_1\omega}(S)$, it follows from the downward Löwenheim–Skolem theorem and Lemma 3.3(a) that

$$\models^S \varphi \quad \text{iff} \quad \langle\omega, X\rangle \models^S \varphi \text{ for all } X \subseteq \omega \quad \text{iff} \quad \models^S \check{\varphi}.$$

Since the function mapping φ to $\check{\varphi}$ is Δ_1 over HC it follows that $\mathrm{Val}\big(L_{\omega_1\omega}(S)\big)$ is Σ_1 over HC if $\mathrm{Val}\big(P_{\omega_1}(S)\big)$ is.

(b) Similarly using part (b) of Lemma 3.3, it follows that $\mathrm{Val}\big(P_{\omega_1}(S)\big)$ is Σ_1 over HC if $\mathrm{Val}\big(L^+_{\omega_1\omega}(S)\big)$ is, where L^+ is the language L augmented by countably many constant symbols $\bar{0}, \bar{1}, \bar{2}, \ldots$ (We remark that it is possible to replace L^+ by a finite language $L' = L \cup \{\bar{R}\}$ where $\bar{R}$ is a binary relation symbol: namely, we can write down a simple sentence of $L'_{\omega_1\omega}$ postulating that R is an equivalence relation having exactly one equivalence class of size n for each natural number n, and then use equivalence classes of R in place of natural numbers.) ■

THEOREM 6.1. *If* $h(S) > \aleph_1$, *then* $\mathrm{Val}(P_{\omega_1}(S))$ *is not* Σ_1 *over* HC.

Proof. We first show that $\Sigma(\boldsymbol{B}_S)$ contains all $\boldsymbol{\Sigma}^1_2$ sets of reals whenever $h(S) > \aleph_1$. To see this let $\varphi(\bar{U}, \bar{<}, \ldots)$ be an $L_{\omega_1\omega}(S)$ sentence which pins down $\aleph_1$. Let $L' = L \cup \{\bar{R}, \bar{F}\}$ where $\bar{R}$ is a binary relation symbol and $\bar{F}$ is a unary function symbol. Let ψ be the conjunction of

"$\bar{R}$ is a linear ordering",

and

"$\bar{F}$ is an isomorphism of $\bar{R}$ into $\bar{<}$",

and let χ be $\varphi \wedge \psi$.

Since a downward Löwenheim–Skolem argument easily shows that φ must have arbitrarily large countable models, it follows that

$$R \in \boldsymbol{W} \quad \text{iff} \quad (\exists F)(\exists <) \models^S \check{\chi}\,[(R, <, F)]$$

so that $\boldsymbol{W} \in \Sigma(\boldsymbol{B}_S)$. Since every $\boldsymbol{\Pi}^1_1$ set is many-one reducible to $\boldsymbol{W}$, it follows that all $\boldsymbol{\Pi}^1_1$ sets are in $\boldsymbol{\Sigma}(\boldsymbol{B}_S)$. Our claim now follows from the fact that $\boldsymbol{\Sigma}^1_2$ sets are projections of $\boldsymbol{\Pi}^1_1$ sets and the fact that $\Sigma(\boldsymbol{B}_S)$ is closed under projections.

We next show that the assumption that $\mathrm{Val}(P_{\omega_1}(S))$ is Σ_1 over HC implies that all elements of $\boldsymbol{\Sigma}(\boldsymbol{B}_S)$ are $\boldsymbol{\Pi}^1_2$. This will then contradict the previous paragraph and establish the result.

Hence assume that $\boldsymbol{T} \in \Sigma(\boldsymbol{B}_S)$. $\boldsymbol{T}$ is then $\{X \subseteq \omega \mid \models^S \varphi[(X, Y)]$ for some $Y \subseteq \omega\}$, for some $P_{\omega_1}(S)$ formula φ. If φ_X is the result of replacing p_{2i} in φ by $p_0 \wedge \sim p_0$, when $i \notin X$, and by $p_0 \vee \sim p_0$, when $i \in X$, it follows that

$$X \in \boldsymbol{T} \quad \text{iff} \quad \sim\varphi_X \text{ is not valid}.$$

Since the function mapping X to $\sim\varphi_X$ is Δ_1 over HC, it follows that $\boldsymbol{T}$ would be Π_1 over HC if $\mathrm{Val}(P_{\omega_1}(S))$ were Σ_1 over HC. Theorem 0.1 would then show that $\boldsymbol{T}$ is $\boldsymbol{\Pi}^1_2$. ■

By combining Theorem 6.1 with Lemma 6.1 below we get the following result.

COROLLARY 6.1. *If* $h(S) > \aleph_1$, *then* $\mathrm{Val}(L_{\omega_1\omega}(S))$ *is not* Σ_1 *over* HC.

THEOREM 6.2. *If* $h(S) > \aleph_1$, *then* Weak $\mathrm{Beth}(L_{\omega_1\omega}(S))$ *fails.*

Proof. We use the idea of the proof of Theorem 1 in [16] leaving out many of the details.

First let $\varphi_0(\bar{U}, \bar{<}, \ldots)$ be an $L_{\omega_1\omega}(S)$ sentence which pins down $\aleph_1$. Then let φ_1 be the conjunction of the axioms of KP (formulated with a binary relation symbol $\bar{E}$), φ_0, and $(\forall v_0)(\forall v_1)[\bar{E}(v_0, v_1) \to \bar{F}(v_0) \bar{<} \bar{F}(v_1)]$ (where $\bar{F}$ is a new unary function symbol).

Now let L be the language consisting of all the symbols of φ_1 and let L' be $L \cup \{\bar{R}\}$, where $\bar{R}$ is some new binary relation symbol. By formally specifying that $\bar{R}$ is the satisfaction relation for $L_{\omega_1\omega}(S)$, it is easy to obtain a sentence, $\varphi_2(\bar{R})$, of $L'_{\omega_1\omega}(S)$, such that each L-structure, $\langle A, E, U, <, \ldots\rangle$, has a unique expansion, $\langle A, E, U, <, R, \ldots\rangle$, to a model of $\varphi(\bar{R})$, where $\varphi(\bar{R})$ is $\big(\varphi_1 \wedge \varphi_2(\bar{R})\big) \vee \big({\sim}\varphi_1 \wedge (\forall v_0)(\forall v_1)\bar{R}(x, y)\big)$. Moreover, $\varphi(\bar{R})$ is such that whenever $\langle A, \in, U, <, R, \ldots\rangle$ is a model of $\varphi_1 \wedge \varphi(\bar{R})$, then R is *the* satisfaction relation for $L_A(S)$ over $\langle A, \in, U, <, \ldots\rangle$.

The sentence $\varphi_2(\bar{R})$ is constructed as in [16] using clauses (0)–(3), restricting (4) to finite quantification, and adding

(5′) in $\langle A, E\rangle$, $x = \mathscr{S}y_n$ and $\mathscr{S}$ applied to $\big((y_n, f) \in R_A\big)$ holds.

Finally, suppose that $\bar{R}$ is explicitly definable by some formula ψ of $L_{\omega_1\omega}(S)$. Since ψ belongs to some countable admissible fragment, $L_B(S)$ of $L_{\omega_1\omega}(S)$, and since φ_0 has models $\mathfrak{A}$ in which the order type of $\langle U^{\mathfrak{A}}, <^{\mathfrak{A}}\rangle$ can be of arbitrarily large countable length, it follows that $\langle B, \in, U, <, \ldots\rangle$ can be expanded to a model of $\varphi_1 \wedge \varphi_2(\bar{R})$. Hence $\psi \in B$ would define $L_B(S)$ satisfaction over $\langle B, \in, U, <, \ldots\rangle$—something which can easily be contradicted by a Tarski style diagonalization. ∎

Theorem 6.2 can be strengthened by showing, under the same hypothesis, that a propositional analogue of the Weak Beth property also fails. To state this propositional Beth property we shall find it convenient to denote the propositional variables p_{2i} by q_i and the propositional variables p_{2i+1} by r_i. The notation $\varphi(q_i)$ will mean that only the propositional variables $q_0, q_1, \ldots$ occur in φ.

DEFINITION. $P_{\omega_1}(S)$ has the *Weak Beth property* if for every formula φ of $P_{\omega_1}(S)$ such that for each $X \subseteq \omega$, there is a unique $Y \subseteq \omega$ for which $\models^S \varphi(q_i, r_j)[(X, Y)]$, there exist $P_{\omega_1}(S)$ formulas $\psi_j(q_i)$ such that

$$\models^S \varphi \to \bigwedge_{j \in \omega} (r_j \leftrightarrow \psi_j).$$

The *Beth property* for $P_{\omega_1}(S)$ is obtained by replacing "unique" by "at most one" in the above sentence.

We got the idea of examining the Beth and Weak Beth properties in the above form from some unpublished work of Friedman.

THEOREM 6.3. *If $h(S) > \aleph_1$, then the Weak Beth property for $P_{\omega_1}(S)$ fails. In particular, there is a set $\boldsymbol{T} \notin \boldsymbol{B}_S$ such that $\boldsymbol{T}$, $P(\omega) - \boldsymbol{T} \in \boldsymbol{\Sigma}(\boldsymbol{B}_S)$* (*i.e.* *$\boldsymbol{B}_S$ fails to have the first separation property*).

Proof. The proof is an easy consequence of the proof of Theorem 6.2. We will need only some minor changes.

First we let φ_{-1} be a sentence of $L_{\omega_1\omega}$, in a language consisting only of a single binary relation symbol $\bar{H}$, such that

$$\langle A, H\rangle \models \varphi_{-1} \quad \text{iff} \quad \langle A, H\rangle \simeq \langle \omega, \dashv\rangle$$

where $\dashv$ is the usual ordering on ω. Then we let φ_1 be as in the proof of Theorem 6.2 except that we add φ_{-1} as an additional conjunct. Finally we construct φ as in Theorem 6.2 using the new φ_1.

For the sake of definiteness let us assume that the language of φ is $\{\bar{H}, \bar{E}, \bar{U}, \bar{<}, \bar{P}, \bar{E}, \bar{F}, \bar{R}\}$, where $\bar{H}$ is from φ_{-1}, $\bar{P}$ is a unary relation symbol from the sentence $\varphi_0(\bar{U}, \bar{<}, \bar{P})$ which pins down $\aleph_1$ and $\bar{F}$ is from φ_1, but viewed this time as a binary relation symbol (instead of a unary function symbol).

Using Lemma 3.3, we can find a formula $\varphi(q_i, r_j)$ of $P_{\omega_1}(S)$ such that

$$\langle \omega, \dashv, E, U, <, P, F, R\rangle \models^{S} \varphi \quad \text{iff} \quad \models^{S} \check{\varphi}\big[\big((\dashv, E, U, <, P, F), R\big)\big].$$

Hence for each $X \subseteq \omega$ there is a unique R such that $\models^{S} \check{\varphi}[(X, R)]$.

To complete the proof, let us now assume that there are formulas $\psi_j(q_i)$ of $P_{\omega_1}(S)$ for which

$$\models^{S} \check{\varphi} \to \bigwedge_{j\in\omega} \big(r_j \leftrightarrow \psi_j(q_i)\big).$$

Then using part (b) of Lemma 3.3 we have $\langle \omega, \dashv, \ldots, R, 0, 1, \ldots\rangle \models$ $\models \bar{R}(\bar{c}_k, \bar{c}_l) \leftrightarrow \hat{\psi}_{(k,l)}$ for all $k, l \in \omega$, whenever $\langle \omega, \dashv, \ldots, R\rangle \models \varphi$. (Here, fo course, $\hat{\psi}_n$ may also contain additional constants $\bar{c}_i$ which are names for elements of ω.)

Now let $\psi'(v_1, v_2)$ be

$$\bigvee_{k,l\in\omega} [v_1 = \bar{c}_k \wedge v_2 = \bar{c}_l \wedge \psi_{(k,l)}].$$

Then ψ' explicitly defines R over all models of φ, when $\bar{c}_i$ is interpreted as the ith element in the ordering $\dashv$. However, since the ith element of $\dashv$ can be defined in $L_{\omega_1\omega}$, we can replace these constants by their definitions. The resulting formula ψ of $L_{\omega_1\omega}(S)$ is in the language of φ_1 and is such that

$$\models^{S} \varphi \to (\forall v_1)(\forall v_2)[\bar{R}(v_1, v_2) \leftrightarrow \psi],$$

which contradicts the original choice of φ.

Since the failure of the Weak Beth property for $P_{\omega_1}(S)$ implies the failure of the interpolation property for $P_{\omega_1}(S)$, the second part of the theorem, which simply restates this failure in descriptive set theoretic form, follows immediately. ■

It would have been much more elegant had we been able to prove Theorem 6.3 directly by some descriptive set theoretic argument and then use the translation procedure of Lemma 3.3 (a) to derive Theorem 6.2. Unfortunately, however, we were not able to accomplish this. It would also be nice to know if Weak Beth fails for $P_{\omega_1}(S)$ *exactly when* Weak Beth fails for $L_{\omega_1\omega}(S)$.

For $L_{\omega_1\omega}(S)$ the Weak Beth property actually fails in a much more striking way than is indicated by Theorem 6.2. Namely, we have the following result which should be contrasted with Theorem 4.5.

THEOREM 6.4. *If* $h(S) > \aleph_1$, *then* Weak Beth$(L_{\omega_1\omega}(S), L_{\omega_1\omega}(S, T))$ *fails for all* $T \subseteq P(\omega)$.

Proof. We begin as in the proof of Theorem 6.2 except that we now require that L contain an additional new unary relation symbol $\bar{T}$, and that $\varphi_2(\bar{R})$ is now intended to formally express the satisfaction relation for $L_{\omega_1\omega}(S, T)$. This is done with the help of a unary relation symbol $\bar{T}$, by adding the following inductive clause to the ones in the proof of Theorem 6.2:

(6) in $\langle A, E, \ldots, T\rangle$, $x = \mathscr{T}y_n$ and

$$(\exists v_0 \subseteq \omega)[\bar{T}(v_0) \wedge (\forall v_1)(v_1 \in v_0 \leftrightarrow (y_{v_1}, f) \in R_A)].$$

Since (6) is expressible in $L'_{\omega_1\omega}(S)$, $\varphi_2(\bar{R})$ (and hence the corresponding $\varphi(\bar{R})$) is a sentence of $L'_{\omega_1\omega}(S)$.

Unlike in the proof of Theorem 6.2, it is not true in this case that R is the actual satisfaction relation for $L_A(S, T)$, whenever $\langle A, \in, \ldots \ldots, T, R\rangle$ is a model of $\varphi_1 \wedge \varphi_2(\bar{R})$. However, if we take the unique expansion of $\langle \mathrm{HC}, \in, \ldots, T\rangle$ which is a model of $\varphi(\bar{R})$, the R in this case will be the satisfaction relation for $L_{\omega_1\omega}(S, T)$ over $\langle \mathrm{HC}, \in, \ldots, T\rangle$. The combination of this fact with a Tarski style diagonalization then completes the proof as before. ∎

Since one can add countably many clauses of type (6) and still have the resulting $\varphi(\bar{R})$ as a sentence of $L_{\omega_1\omega}(S)$, the above proof can be extended to show that

Weak Beth $(L_{\omega_1\omega}(S), L_{\omega_1\omega}(S, T_0, T_1, \ldots))$ *fails for all* S *and* $\langle T_i \mid i \in \omega\rangle$ *such that* $h(S) > \aleph_1$.

Moreover, since any sentence of $L_{\omega_1\omega}$(**Full**) can contain only countably many different connective symbols, the above considerations imply

THEOREM 6.5. Weak Beth$(L_{\omega_1\omega}(S), L_{\omega_1\omega}$(**Full**)) *fails whenever* $h(S) > \aleph_1$.

These stronger Weak Beth property failures for $L_{\omega_1\omega}(S)$ are, however, never shared by $P_{\omega_1}(S)$ as shown by the following

THEOREM 6.6. *For every* $S \subseteq P(\omega)$ *there is a* $T \subseteq P(\omega)$ *such that any* *valid* $P_{\omega_1}(S)$ *formula of the form* $\varphi \to \psi$, *can be interpolated by a formula in* $P_{\omega_1}(T)$, *i.e.* Craig$\big(P_{\omega_1}(S),\, P_{\omega_1}(T)\big)$.

Proof. Let $\{(\varphi_X, \psi_X) \mid X \in P(\omega)\}$ be an enumeration (by reals) of all pairs of formulas of $P_{\omega_1}(S)$ such that $T^S(\varphi_X) \cap T^S(\psi_X) = \emptyset$. Let $T_X \subseteq P(\omega)$ be such that $T^S(\varphi_X) \subseteq T_X$ and $T^S(\psi_X) \subseteq P(\omega) - T_X$, and let $\boldsymbol{T} = \{(X, Y) \mid Y \in T_X\}$. It is then quite easy to see that $\boldsymbol{T}$ works. ■

Now that we have seen some of the consequences of the fact that $h(S) > \aleph_1$, we turn to the question of which connectives have this property. Our strategy here is to show that the connective $\boldsymbol{W}$, defined in Section 3, has the property and then use this fact as a tool for obtaining further information.

LEMMA 6.2. $h(\boldsymbol{W}) > \aleph_1$.

Proof. Let L be a language with unary relation symbols $\bar{U}$ and $\bar{N}$, a binary relation symbol $\bar{<}$, ternary relation symbols $\bar{R}$ and $\bar{F}$ and constant symbols $\bar{0}, \bar{1}, \bar{2}, \ldots$ Let ψ be the conjunction of the following sentences of $L_{\omega_1\omega}(\boldsymbol{W})$:

(i) $\bigwedge\limits_{\substack{m \neq n \\ m, n \in \omega}} \bar{m} \neq \bar{n}$,

(ii) $(\forall v_0)\big(\bar{N}(v_0) \leftrightarrow \bigvee\limits_{n \in \omega} v_0 = \bar{n}\big)$,

(iii) "$\bar{<}$ linearly orders $\bar{U}$",

(iv) "$(\forall x)\ [\bar{U}(x) \to \{(y, z) \mid \bar{F}(x, y, z)\}$ is an isomorphism between $\{(y, z) \mid \bar{N}(y)$ and $\bar{N}(z)$ and $\bar{R}(x, y, z)\}$ and $\{(y, z) \mid \bar{U}(x)$ and $\bar{U}(z)$ and $y \mathrel{\bar{<}} z$ and $z \mathrel{\bar{<}} x\}]$",

(v) $(\forall v_0)[\bar{U}(v_0) \to \mathscr{S}\bar{R}(v_0, \overline{(n)_0}, \overline{(n)_1})]$.

From (v) it follows that if $\langle A, U, N, <, R, F, a_0, a_1 \ldots\rangle \models^{W} \psi$, then $\{(y, z) \mid y \in N$ and $z \in N$ and $(x, y, z) \in R\}$ is well-ordered for each $x \in U$. Hence $\langle U, <\rangle$ must be well-ordered in each model of ψ.

Since it is easy to get a model of ψ in which $\langle U, <\rangle$ has order type $\aleph_1$, it follows that ψ pins down $\aleph_1$. ■

Since we know that $L_{\omega_1\omega}(\boldsymbol{W})$ is a rather weak propositional extension of $L_{\omega_1\omega}$, it now follows that most connectives $\boldsymbol{S}$ are such that $h(\boldsymbol{S}) > \aleph_1$. Moreover, by assuming various forms of determinacy we get (using Theorem 3.4) the following more precise information.

LEMMA 6.3. (a) (AD) $h(\boldsymbol{S}) > \aleph_1$ *for all* $\boldsymbol{S} \subseteq P(\omega)$, $\boldsymbol{S} \in \Delta_1^1$,

(b) (Δ_n^1–Det, *for* $n \geqslant 2$) $h(\boldsymbol{S}) > \aleph_1$ *for all* $\boldsymbol{S} \subseteq P(\omega)$, $\boldsymbol{S} \in \Delta_n^1 - \Delta_1^1$.

This combined with Theorems 6.1, 6.3, and 6.5 gives our main negative results assuming various kinds of determinacy.

THEOREM 6.8. (AD) *For every non-Borel* $\boldsymbol{S} \subseteq P(\omega)$,

(a) *neither* $\mathrm{Val}\big(P_{\omega_1}(\boldsymbol{S})\big)$ *nor* $\mathrm{Val}\big(L_{\omega_1\omega}(\boldsymbol{S})\big)$ *is* Σ_1*-definable over* HC, *and*

(b) *neither* Weak Beth $(P_{\omega_1}(S))$ *nor* Weak Beth $(L_{\omega_1\omega}(S), L_{\omega_1\omega}(\mathbf{Full}))$.

THEOREM 6.9. (Δ^1_n—Det, *for* $n \geqslant 2$) *For every* Δ^1_n $S \subseteq P(\omega)$ *which is not Borel,*

(a) *neither* $\mathrm{Val}(P_{\omega_1}(S))$ *nor* $\mathrm{Val}(L_{\omega_1\omega}(S))$ *is* Σ_1*-definable over* HC, *and*

(b) *neither* Weak Beth $(P_{\omega_1}(S))$ *nor* Weak Beth $(L_{\omega_1\omega}(S), L_{\omega_1\omega}(\mathbf{Full}))$.

We thus see that determinacy completely destroys any possibility of having propositional extensions of $L_{\omega_1\omega}$ with nice model-theoretic properties. In the next section we shall see that V = L (Gödel's Axiom of Constructibility) has a partially offsetting influence in that it at least allows for the existence of proper propositional extensions of $L_{\omega_1\omega}$ with Σ_1-definable sets of valid sentences.

Before leaving our discussion of $h(S)$, we might note the following descriptive set-theoretic characterization.

THEOREM 6.10. *If* S *is* Π^1_1 *and not Borel, then* $h(S) = \delta^1_2$ *where* δ^1_2 *is the supremum of lengths of all* Δ^1_2 *prewellorderings of reals.*

Proof. From the well-known fact that all Π^1_1 sets are Δ^1_1 in S we conclude easily that $\Sigma(\mathbf{B}_S) = \Sigma^1_2$. By our Theorem 5.1, $h(S)$ is then the supremum of ordinal ranks of all Σ^1_2 well-founded relations on reals. The latter ordinal has been identified with δ^1_2 by Moschovakis. ■

We conclude this section with a discussion of a question of Barwise.

Recall that in [3], Barwise introduces the notion of a logic *absolute with respect to a true set theory.* After proving that any logic L^* which

(a) is absolute with respect to a true set theory,

(b) is coded by HC,

(c) effectively contains $L_{\omega_1\omega}$,

(d) is effectively closed under $\sim$ and $\wedge$, and

(e) is such that either the set of valid sentences of L^* is Σ_1 over HC, or L^* has the interpolation property,

is a sublogic of $L_{\infty\omega}$, Barwise asks if there can be a logic with properties (a)–(e) which *properly* extends $L_{\omega_1\omega}$.

Although we don't know the answer to this question, our results do provide some partial solutions. Before we can give them, however, we must first have the following

LEMMA. *If* $L_{\omega_1\omega}(S)$ *is absolute with respect to some set theory, then* S *is* Δ^1_2.

Proof. Since

$$X \in S \quad \text{iff} \quad \langle \omega, X, 0, 1, \ldots, \rangle \models^S \mathscr{S}\bar{X}(\bar{n}),$$

it follows from the absoluteness of $L_{\omega_1\omega}(S)$ that S is Σ_1 over V. From Lévy's theorem (see Barwise [3], page 339) S is then Σ_1 over HC, and hence Σ^1_2, by Theorem 0.1.

Since

$$X \notin S \quad \text{iff} \quad \langle \omega, X, 0, 1, 2, \ldots \rangle \models^{S} \sim\mathscr{S}\bar{X}(\bar{n}),$$

the same ideas show that $P(\omega)-S$ is Σ^1_2. Hence S is Δ^1_2. ∎

By combining this lemma with Theorem 6.9, we thus see that if Δ^1_2-Det is true, then there is no proper propositional extension of $L_{\omega_1\omega}$ having properties (a)–(e)—thereby giving a partial negative answer to Barwise's question.

On the other hand, the usual dichotomy which results from alternately assuming AD (or some form of it) and V = L is once again evidenced *vis-à-vis* this question. For in the next section we shall show, assuming that V = L is true, that there is an $S \subseteq P(\omega)$ such that $L_{\omega_1\omega}(S)$ satisfies (a)–(d), and is such that $\mathrm{Val}(L_{\omega_1\omega}(S))$ is Σ_1 over HC—thereby providing a strong positive answer to Barwise's question.

Unfortunately, we have not been able to settle the related question of whether there is an $S \subseteq P(\omega)$ such that Craig $(L_{\omega_1\omega}(S))$ or even Weak Beth $(L_{\omega_1\omega}(S))$ holds. One might expect that V = L should imply the existence of such an S. We conjecture, however, that this is not the case. Indeed we conjecture that

$$\mathrm{Craig}(L_{\omega_1\omega}(S)) \text{ fails for all } S \subseteq P(\omega)$$

is even provable in ZF.

7. Propositional extensions of $L_{\omega_1\omega}$ in the constructible universe

We have seen that AD provides some striking answers to questions about propositional extensions of $L_{\omega_1\omega}$. The burden of this section is to show how the assumption of the Axiom of Constructibility (V = L) provides, in some cases, answers opposite to those obtained by assuming AD. Specifically we shall prove the following three results.

THEOREM 7.1. (V = L). *There is a Π^1_1 connective S_0 such that $L_{\omega_1\omega}(S_0)$ is a proper extension of $L_{\omega_1\omega}$ strictly weaker than $L_{\omega_1\omega}(W)$.*

THEOREM 7.2. ($2^{\aleph_0} = \aleph_1$). *There is a connective S_1 such that S_1 is not Borel but $h(S_1) = \aleph_1$.*

THEOREM 7.3. (V = L). *There is a Δ^1_2 connective S_2 such that S_2 is not Borel but* $\mathrm{Val}(L_{\omega_1\omega}(S_2))$ *is Σ_1 over* HC. *In addition $L_{\omega_1\omega}(S_2)$ is absolute with respect to* KP$+\Sigma_1$-separation + Infinity.

The proof of Theorem 7.1 will require some familiarity with notions from recursion theory and admissibility theory. We'll recall here the specific facts which we'll need. Further explanations can be found in Shoenfield [30] and Gostanian [15].

LEMMA 7.1 (Kleene [20]). *For each $X \subseteq \omega$ there is a linear ordering*

R^X, which is recursive in X, not a well-ordering, but has no infinite descending chains hyperarithmetic in X. In addition Gandy [14] has shown that the maximal well-ordered initial segment, I_{R^X}, of R^X is isomorphic to ω_1^X.

LEMMA 7.2 (Gödel, Shepherdson, Mostowski [27]). *Every well-founded model of the Axiom of Extensionality is isomorphic (by a unique isomorphism) to a transitive set.*

The following is a version of Lemma 7.2 for admissible sets. Its proof is similar to the proof of the well-ordering isomorphism lemma of [15].

LEMMA 7.3. *If A is an admissible set and $R \in A$ is a well-founded extensional relation, then there is a transitive set $B \in A$ and a map $f \in A$ such that f is an isomorphism between R and $\in \restriction B$.*

The following lemma is a part of the folklore, but we give a proof here for the sake of completeness.

LEMMA 7.4. *If α is admissible, projectible into ω, but not recursively inaccessible, then $\alpha = \omega_1^X$ for some $X \subseteq \omega$ in L_α.*

Proof. Let β be the supremum of the admissibles less than α. Since α is not recursively inaccessible, $\beta < \alpha$. If f is the projection of α into ω and $X = \{(f(\gamma), f(\delta)) \mid \gamma < \delta < \beta\}$, then clearly $X \in L_\alpha$. Moreover, ω_1^X is admissible, greater than β, and (by Lemma 7.3) less than or equal to α. Hence $\alpha = \omega_1^X$. ■

The following definition will be used in the proof of Theorem 7.1.

DEFINITION. When α is an ordinal and f is a function from L_α into ω, we shall let $E_{\alpha,f}$ be $\{(f(x), f(y)) \mid x, y \in L_\alpha \text{ and } x \in y\}$.

Proof of Theorem 7.1. The connective S_0 which we are seeking can be taken to be $\{E_{\alpha,f} \mid \alpha$ is admissible and f is a Σ_1 projection of L_α into $\omega\}$.

To see that S_0 has the required properties we shall need the following facts.

(I) S_0 is not Σ_1^1.

(II) S_0 is Π_1^1.

(III) If α is admissible and not recursively inaccessible, then $S_0 \cap L_\alpha \in L_\alpha$.

(IV) Sat $\cap L_\alpha$ is Δ_1 over L_α for all admissible α which are not recursively inaccessible. (Sat here is $\{(\varphi, X) \mid \varphi$ is a $P_{\omega_1}(S_0)$ formula, $X \subseteq \omega$ and $\models^{S_0} \varphi[X]\}$.)

(V) For every $\beta < \aleph_1$ there is an admissible $\alpha > \beta$ such that α is not recursively inaccessible and $W \cap L_\alpha$ is not Δ_1 over L_α.

Indeed, (I) and (II) show that $L_{\omega_1\omega}(S_0)$ is a proper extension of

$L_{\omega_1\omega}$, L-reducible to $L_{\omega_1\omega}(W)$. Moreover, if $L_{\omega_1\omega}(S_0)$ were not *properly* L-reducible to $L_{\omega_1\omega}(W)$, then some $P_{\omega_1}(S_0)$ formula φ would have W as its truth set. Hence by (IV), there would be an admissible β, such that all recursively accessible $\alpha > \beta$ would be such that $W \cap L_\alpha$ is Δ_1 over L_α—a fact which would contradict (V).

Thus to complete the proof it remains to prove (I)–(V).

To prove (I) recall that if V = L is true, the set of admissibles projectible into ω is cofinal with $\aleph_1$. Hence S_0 is a set of well-founded relations of arbitrarily large countable ordinal rank, and thus not Σ^1_1 by Theorem 0.2.

To prove (II), let G be a Σ_1 formula which enumerates all one-place Σ_1 functions, i.e. G is such that for every admissible set A and every $x \in A$

$$g_x = \{(y, z) \mid y, z \in A \quad \text{and} \quad \langle A, \in\rangle \models G(\bar{x}, \bar{y}, \bar{z})\}$$

is a Σ_1 function on A, and every partial one-place Σ_1 function on A is g_x for some $x \in A$.

Recall that there is a sentence τ of the language of set theory such that for any transitive set A, $\langle A, \in\rangle \models \tau$ iff $A = L_\beta$ for some admissible β. (See e.g. Aczel and Richter [1] p. 315 for the construction of such a sentence.)

Now let $\psi(E)$ be the conjunction of the formulas of second order arithmetic expressing that

"E is well-founded",

"$\langle \mathrm{Fld}_E, E\rangle \models \tau$",

and

"$[(\exists n)\big[\big(\langle \mathrm{Fld}_E, E\rangle \models (\forall y)(\exists z \in \omega) G(\bar{n}, y, z)\big) \wedge (\forall k)\big(k \in \mathrm{Fld}_E \leftrightarrow \langle \mathrm{Fld}_E, E\rangle \models G(\bar{n}, \bar{k}, \text{'the } k\text{th integer'})\big)\big]]$".

We claim that $E \in S_0$ iff $\psi(E)$. Since it is easy to see that $\psi(E)$ is a Π^1_1 formula, we will have then shown that S_0 is Π^1_1.

To prove the claim, first note that if $E \in S_0$, then $E = E_{\alpha,f}$ for some admissible α and some Σ_1 projection f of L_α into ω. Let $x \in L_\alpha$ be such that $f = g_x$. Then since E is isomorphic to $\in \restriction L_\alpha$, it follows that E is well-founded, $\langle \mathrm{Fld}_E, E\rangle \models \tau$ and $\langle \mathrm{Fld}_E, E\rangle \models [(\forall y)(\exists z \in \omega) G(\bar{n}, y, z)]$, for $n = f(x)$. Moreover,

$$\begin{aligned} k \in \mathrm{Fld}_E \quad &\text{iff} \quad k = f(t) \text{ for some } t \in L_\alpha, \\ &\text{iff} \quad L_\alpha \models G(x, t, k), \text{ for some } t \in L_\alpha, \\ &\text{iff} \quad \langle \mathrm{Fld}_E, E\rangle \models G(n, k, \text{'the } k\text{th integer'}) \end{aligned}$$

(because $f(x) = n$, $f(t) = k$, $f(k) =$ the kth integer of $\langle \mathrm{Fld}_E, E\rangle$).

Conversely, suppose $\psi(E)$. Then E is well-founded and $\langle \mathrm{Fld}_E, E\rangle$ is a model of the Axiom of Extensionality, so by Lemma 7.2 there is a transitive set B and an isomorphism π between E and $\in \restriction B$. Thus $B = L_\alpha$

for some admissible a and $E = E_{a,\pi^{-1}}$. Then if n is as required by $\psi(E)$ and if $x = \pi(n)$, it follows that g_x is a Σ_1 projection of L_a onto ω. We shall show that $E = E_{a,g_x}$, thus completing the proof of (II).

For this, it suffices to show that $g_x = \pi^{-1}$. But, for all $k \in \mathrm{Fld}_E$, $\langle \mathrm{Fld}_E, E\rangle \models G(n, k,$ 'the kth integer'$)$, hence $\langle L_a, \in\rangle \models G(x, \pi(k), k)$, i.e. $g_x(\pi(k)) = k$.

To prove (III) we first note that the assumptions on a imply that

$$\{L_\beta \mid \beta < a \text{ and } \beta \text{ is admissible}\}$$

belongs to L_a. Consequently

$$\boldsymbol{S}_0^a = \{E_{\beta,f} \mid \beta < a, \beta \text{ admissible and } f \text{ is a } \Sigma_1 \text{ projection of } L_\beta \text{ into } \omega\}$$

belongs to L_a. Hence it suffices to show that $\boldsymbol{S}_0^a = \boldsymbol{S}_0 \cap L_a$.

Clearly $\boldsymbol{S}_0^a \subseteq \boldsymbol{S}_0 \cap L_a$. Conversely, if $X \in \boldsymbol{S}_0 \cap L_a$ then $X = E_\beta$, for some admissible β and some Σ_1 projection f. To show that $X \in \boldsymbol{S}_0^a$, we need only to know that $\beta < a$—something which follows immediately from Lemmas 7.2 and 7.3.

To prove (IV), first define $\mathrm{Sat}^a \subseteq L_a$ by transfinite induction on L_a as follows:

(i) $(p_n, X) \in \mathrm{Sat}^a$ iff $n \in X$,

(ii) $(\sim\varphi, X) \in \mathrm{Sat}^a$ iff $(\varphi, X) \notin \mathrm{Sat}^a$,

(iii) $(\bigwedge\Phi, X) \in \mathrm{Sat}^a$ iff $(\varphi, X) \in \mathrm{Sat}^a$ for all $\varphi \in \Phi$,

(iv) $(\mathscr{S}\varphi_n, X) \in \mathrm{Sat}^a$ iff $\{n \mid (\varphi_n, X) \in \mathrm{Sat}^a\} \in \boldsymbol{S}_0 \cap L_a$.

It is easy to check that all the inductive clauses are Δ_1 over L_a (using (III) for clause (iv)). Hence by Lemma II (iv) on p. 37 of Keisler [19], Sat^a is Δ_1 over L_a. An easy induction on the complexity of φ then shows that $\mathrm{Sat}^a = \mathrm{Sat} \cap L_a$, thus completing the proof of (IV).

To prove (V) recall that (assuming V = L) the set of admissible ordinals which are projectible into ω and not recursively inaccessible, is cofinal with $\aleph_1$. By Lemma 7.4, such ordinals are all of the form ω_X^1 such that $X \in L_{\omega_1^X}$.

Now let a be such an ordinal. Then by Lemma 7.1 there is a linear ordering $R \in L_a$ such that I_R is isomorphic to a. Since $I_R = \{n \in \mathrm{Fld}_R \mid R \restriction \{k \mid R(k, n)\} \in \boldsymbol{W}\}$, it follows that if $\boldsymbol{W} \cap L_a$ were Δ_1 over L_a, then I_R would be Δ_1 over L_a. But in this case I_R would then have to belong to L_a — something which is impossible by Lemma 7.3. ∎

We remark that by reasoning similar to that of (IV), we could actually show that $\boldsymbol{S}_0 \cap L_a$ is provably Δ_1 (in KP$+\Sigma_1$-separation $+$ Infinity), for all admissible a. This can then be used to show that $L_{\omega_1\omega}(\boldsymbol{S}_0)$ is absolute (in the sense of Barwise [3]) with respect to KP$+\Sigma_1$-separation $+$ Infinity.

After completion of this paper, the second author showed in [35] that, in the constructible universe, the $\boldsymbol{\Pi}_1^1$ connectives $\boldsymbol{S}$, such that $L_{\omega_1\omega}(\boldsymbol{S})$

is a proper extension of $L_{\omega_1\omega}$ strictly weaker than $L_{\omega_1\omega}(\boldsymbol{W})$, are densely ordered by $\leqslant_L$; also, there are $2^{\aleph_0}$ such connectives, mutually incomparable in $\leqslant_L$. In particular, there is no $\leqslant_L$-minimal non-Borel Π^1_1 connective. Actually, all of the above results hold for Solovay degrees rather than just L-degrees.

The main result of the second part of this section is the construction (contained in the proof of Theorem 7.3) of a Δ^1_2-connective $\boldsymbol{S}_2$, such that $L_{\omega_1\omega}(\boldsymbol{S}_2)$ is a proper extension of $L_{\omega_1\omega}$, but yet $\mathrm{Val}(L_{\omega_1\omega}(\boldsymbol{S}_2))$ is Σ_1 over HC. The proof of this will be a refinement of the proof of a weaker result (Theorem 7.2) which needs only the Continuum Hypothesis. Since both proofs will be presented using forcing-like constructions, we shall first pause to develop the necessary machinery.

We let F^* be the set of all formulas of $P_{\omega_1}(\boldsymbol{S})$ (with $\bigvee$ and $\mathscr{S}$ as well as $\bigwedge$ and $\not\mathscr{S}$) in which the negation symbol occurs only in front of propositional variables. For each $\varphi \in F^*$, we let φ^* denote the formula of F^* obtained from $\sim\varphi$ by pushing the negation symbol all the way inside. φ^* is thus logically equivalent to $\sim\varphi$.

We shall call (A_1, A_2) a *condition* if A_1 and A_2 are countable sets of reals such that $A_1 \cap A_2 = \emptyset$. We let P be the set of all conditions. If (B_1, B_2) and (A_1, A_2) are conditions, we say that (B_1, B_2) is *stronger* than (A_1, A_2) (in symbols, $(B_1, B_2) < (A_1, A_2)$) iff $A_1 \subseteq B_1$ and $A_2 \subseteq B_2$.

The relation $\Vdash$, between conditions, formulas from F^*, and reals, is defined inductively as follows:

$$
\begin{array}{lll}
(A_1, A_2) \Vdash p_n[X] & \text{iff} & n \in X,\\
(A_1, A_2) \Vdash \sim p_n[X] & \text{iff} & n \notin X,\\
(A_1, A_2) \Vdash \bigwedge\varphi_n[X] & \text{iff} & (A_1, A_2) \Vdash \varphi_n[X] \text{ for all } n \in \omega,\\
(A_1, A_2) \Vdash \bigvee\varphi_n[X] & \text{iff} & (A_1, A_2) \Vdash \varphi_n[X] \text{ for some } n \in \omega,\\
(A_1, A_2) \Vdash \mathscr{S}\varphi_n[X] & \text{iff} & \text{for all } n \in \omega, \text{ either } (A_1, A_2) \Vdash \varphi_n[X] \text{ or}\\
(A_1, A_2) \Vdash \varphi_n^*[X], & \text{and} & \{n \in \omega \mid (A_1, A_2) \Vdash \varphi_n[X]\} \in A_1,\\
(A_1, A_2) \Vdash \not\mathscr{S}\varphi_n[X] & \text{iff} & \text{for all } n \in \omega, \text{ either } (A_1, A_2) \Vdash \varphi_n[X] \text{ or}\\
(A_1, A_2) \Vdash \varphi_n^*[X] & \text{and} & \{n \in \omega \mid (A_1, A_2) \Vdash \varphi_n[X]\} \in A_2.
\end{array}
$$

$\Vdash$ has some of the usual properties of forcing as shown by

Lemma 7.5. *For all $\varphi \in F^*$ and all $X \in P(\omega)$,*

(a) *$(A_1, A_2) \Vdash \varphi[X]$ and $(A_1, A_2) \Vdash \varphi^*[X]$ is impossible.*

(b) *If $(A_1, A_2) \Vdash \varphi[X]$ and $(B_1, B_2) < (A_1, A_2)$, then $(B_1, B_2) \Vdash \varphi[X]$.*

(c) *For every $(A_1, A_2) \in P$ there is a $(B_1, B_2) \in P$ such that $(B_1, B_2) < (A_1, A_2)$ and $(B_1, B_2) \Vdash \varphi[X]$ or $(B_1, B_2) \Vdash \varphi^*[X]$.*

All three parts are proven rather easily by induction on the complexity of φ. We give details only for the case of φ of the form $\mathscr{S}\varphi_n$ in part (c).

For given $(A_1, A_2) \in P$ and $X \subseteq \omega$, construct a sequence of conditions $(A_{1,i}, A_{2,i})$ such that

$$A_{1,0} = A_1, \quad A_{2,0} = A_2,$$

$$(A_{1,i+1}, A_{2,i+1}) < (A_{1,i}, A_{2,i})$$

and

$$(A_{1,i+1}, A_{2,i+1}) \Vdash \varphi_i[X] \quad \text{or} \quad (A_{1,i+1}, A_{2,i+1}) \Vdash \varphi_i^*[X].$$

Let $A_1' = \bigcup_{i\in\omega} A_{1,i}$ and $A_2' = \bigcup_{i\in\omega} A_{2,i}$. Then clearly $(A_1', A_2') \in P$ and, by part (b), $(A_1', A_2') \Vdash \varphi_i[X]$ or $(A_1', A_2') \Vdash \varphi_i^*[X]$ for all $i \in \omega$.

Now let $Z = \{i \in \omega \mid (A_1', A_2') \Vdash \varphi_i[X]\}$. If $Z \in A_1'$ or $Z \in A_2'$, let $(B_1, B_2) = (A_1', A_2')$. Otherwise let $B_1 = A_1' \cup \{Z\}$ and $B_2 = A_2'$. In either case $(B_1, B_2) \Vdash \mathscr{S}\varphi_i[X]$ or $(B_1, B_2) \Vdash \mathscr{\not S}\varphi_i^*[X]$. ∎

Another easy induction on the complexity of φ proves the following definability lemma.

LEMMA 7.6. *For every* $\varphi \in F^*$, $\{(A_1, A_2, X) \mid (A_1, A_2) \Vdash \varphi[X]\}$ *is Borel.* (*Here of course it is understood that countable sets of reals are coded in some standard way by a real, for example each real A may code* $\{(A)_n \mid n \in \omega\}$.)

Given a connective $S \subseteq P(\omega)$, we say that $(A_1, A_2) \in P$ is an *S-condition* if $A_1 \subseteq S$ and $A_2 \subseteq P(\omega) - S$.

The next lemma connects the notion of forcing with the notion of truth.

LEMMA 7.7. *For all* $\varphi \in F^*$ *and all* $X \subseteq \omega$, $\models^S \varphi[X]$ *iff there is an S-condition* (A_1, A_2) *such that* $(A_1, A_2) \Vdash \varphi[X]$.

By induction of the complexity of φ. The only non-trivial case is when φ is of the form $\mathscr{S}\varphi_n$ (and similarly $\mathscr{\not S}\varphi_n$).

The inductive assumption yields S-conditions $(A_{1,n}, A_{2,n})$ such that

$$\models^S \varphi_n[X] \quad \text{iff} \quad (A_{1,n}, A_{2,n}) \Vdash \varphi_n[X],$$

$$\models^S \varphi_n^*[X] \quad \text{iff} \quad (A_{1,n}, A_{2,n}) \Vdash \varphi_n^*[X].$$

Hence if $\models^S \varphi[X]$, then $\{n \mid \models^S \varphi_n[X]\} = Z \in S$, so that $(A_1, A_2) = \left(\bigcup_{n\in\omega} A_{1,n} \cup \{Z\}, \bigcup_{n\in\omega} A_{2,n}\right)$ is an S-condition and $(A_1, A_2) \Vdash \mathscr{S}\varphi_n[X]$.

Conversely if $\models^S \varphi[X]$ is not the case, then $Z \in P(\omega) - S$ so that $(A_1, A_2) = \left(\bigcup_{n\in\omega} A_{1,n}, \bigcup_{n\in\omega} A_{2,n} \cup \{Z\}\right)$ is an S-condition and $(A_1, A_2) \Vdash \mathscr{\not S}\varphi_n^*[X]$. Since $\mathscr{\not S}\varphi_n^*$ is $(\mathscr{S}\varphi_n)^*$, it follows, from Lemma 7.5 (a), that $(A_1, A_2) \Vdash \mathscr{S}\varphi_n[X]$ fails. ∎

We can now commence with the proofs of the main results.

Proof of Theorem 7.2. Let $\langle \varphi_\alpha \mid \alpha < \aleph_1 \rangle$ be some listing of all formulas of F^* and let $\langle (B_{1,\alpha}, B_{2,\alpha}) \mid \alpha < \aleph_1 \rangle$ be some listing of all conditions. (Here is where the assumption $2^{\aleph_0} = \aleph_1$ is used.)

We shall construct a sequence $(A_{1,\alpha}, A_{2,\alpha})$ of conditions by transfinite induction as follows:

$$A_{1,0} = A_{2,0} = \emptyset,$$

$$A_{1,\beta} = \bigcup_{\nu<\beta} A_{1,\nu}, \quad A_{2,\beta} = \bigcup_{\nu<\beta} A_{2,\nu} \quad \text{for limit ordinals } \beta,$$

$(A_{1,\alpha+1}, A_{2,\alpha+1})$ is the first condition (B_1, B_2) in the above listing such that $(B_1, B_2) < (A_{1,\alpha}, A_{2,\alpha})$ and there is a real X for which $(B_1, B_2) \Vdash \varphi_\alpha[X]$ if such (B_1, B_2) exists; otherwise $(A_{1,\alpha+1}, A_{2,\alpha+1}) = (A_{1,\alpha}, A_{2,\alpha})$.

We let $S_1 = \bigcup_{\alpha<\aleph_1} A_{1,\alpha}$. It is clear that $(A_{1,\alpha}, A_{2,\alpha})$ is an S_1-condition for each $\alpha < \aleph_1$. We now show that S_1 has the required properties.

First we show that S_1 is not Borel. To see this let T be Borel and let φ_T be an P_{ω_1} formula of F^* such that

$$X \in T \quad \text{iff} \quad \models \varphi_T[X].$$

Pick α_T such that

$$\varphi_{\alpha_T} = (\mathscr{S} p_n \wedge \varphi_T^*) \vee (\mathscr{S} \sim p_n \wedge \varphi_T).$$

(Note that φ_{α_T} expresses the fact that "$S \neq T$".)

It is easy to verify that for all $(B_1, B_2) \in P$,

$$\text{(I)} \qquad \begin{aligned} (B_1, B_2) &\Vdash \mathscr{S} p_n[X] && \text{iff} \quad X \in B_1, \\ (B_1, B_2) &\Vdash \mathscr{S} \sim p_n[X] && \text{iff} \quad X \in B_2, \\ (B_1, B_2) &\Vdash \varphi_T && \text{iff} \quad X \in T, \\ (B_1, B_2) &\Vdash \varphi_T^* && \text{iff} \quad X \notin T. \end{aligned}$$

Using (I) it is easy to show the existence of a condition (B_1, B_2) such that $(B_1, B_2) < (A_{1,\alpha_T}, A_{2,\alpha_T})$ and $(B_1, B_2) \Vdash \varphi_{\alpha_T}[X]$, for some $X \subseteq \omega$. Hence the α_T+1st step in the inductive construction insures that $(A_{1,\alpha_T+1}, A_{2,\alpha_T+1}) \Vdash \varphi_{\alpha_T}[X]$ for some $X \subseteq \omega$. Hence either $(A_{1,\alpha_T+1}, A_{2,\alpha_T+1}) \Vdash (\mathscr{S} p_n \wedge \varphi_T^*)[X]$ or $(A_{1,\alpha_T+1}, A_{2,\alpha_T+1}) \Vdash (\mathscr{S} \sim p_n \wedge \varphi_T)[X]$, so that by (I) again, it follows that either $X \in S_1$ and $X \notin T$, or $X \notin S_1$ and $X \in T$. Thus S_1 cannot be Borel.

To complete the proof we must show that $h(S_1) = \aleph_1$. Hence assume that $h(S_1) > \aleph_1$. Then by the proof of Theorem 6.1, $W \in \Sigma(B_S)$. Thus if we can show that $W \neq \Sigma_\varphi = \{X \subseteq \omega \mid \models^{S_1} \varphi[(X, Y)] \text{ for some } Y \subseteq \omega\}$, for any $\varphi \in F^*$, the result follows.

To do so let $\psi \in F^*$ be such that for all $X, Y, Z \subseteq \omega$,

$$\models^{S_1} \psi[(X, Y, Z)] \quad \text{iff} \quad \models^{S_1} \varphi[(X, Y)] \wedge (\forall m \in Z)(\exists n \in Z)(\langle n, m \rangle \in X)$$

(i.e. Z is an infinite descending chain through the subset of ω^2 coded by X). The formula ψ can be chosen to be

$$\varphi[p_i/p_{2i}]\wedge\psi'$$

where ψ' is

$$\bigwedge_{m\in\omega}\bigvee_{n\in\omega}\left(\sim p_{2m+1}\vee(p_{2n+1}\wedge p_{4(n,m)})\right)$$

and $\varphi[p_i/p_{2i}]$ is obtained from φ by replacing each p_i by p_{2i}.

Now pick a such that $\varphi_a=\psi$. Then either (1) or (2) must occur at stage $a+1$:

(1) there is a real (X, Y, Z) such that

$$(A_{1,a+1}, A_{2,a+1})\ \Vdash \varphi_a[(X, Y, Z)];$$

(2) for all $(B_1, B_2)<(A_{1,a}, A_{2,a})$ and for all reals X, Y, Z

$$(B_1, B_2)\ \Vdash \varphi_a[(X, Y, Z)] \text{ fails.}$$

Since $\varphi_a=\psi$ and since $(A_{1,a+1}, A_{2,a+1})$ is an S_1-condition, it follows, in case that (1) holds, that $\models^{S_1}\psi[(X, Y, Z)]$, so that $\models^{S_1}\varphi[(X, Y)]$ and Z witnesses the fact that X does not code a well-ordering of ω^2. Hence $X\in\Sigma_\varphi-W$.

In the case that (2) holds consider

$$C_\varphi=\{X\subseteq\omega \mid \text{there exist } Y\in P(\omega) \text{ and } (B_1, B_2)\in P \text{ such that } (B_1, B_2)<(A_{1,a}, A_{2,a}) \text{ and } (B_1, B_2)\ \Vdash \varphi[(X, Y)]\},$$

which by Lemma 7.6 is Σ_1^1. Take $X\in C_\varphi$ and pick $Y\in P(\omega)$ and $(B_1, B_2)\in P$ witnessing the fact that $X\in C_\varphi$. Then if Z were a descending chain in the subset of ω^2 coded by X, we would have $\models^{S_1}\psi'[(X, Y, Z)]$ and hence $(B_1, B_2)\ \Vdash \psi'[(X, Y, Z)]$. Thus $(B_1, B_2)\ \Vdash \psi[(X, Y, Z)]$ which contradicts the assumption on (B_1, B_2) in (2). Hence $C_\varphi\subseteq W$ (and thus by Theorem 0.2 $C_\varphi \subsetneq W$).

We conclude the argument by showing that $\Sigma_\varphi\subseteq C_\varphi$. Hence suppose that $X\in\Sigma_\varphi$. Then $\models^{S_1}\varphi[(X, Y)]$ for some $Y\in P(\omega)$. Thus, by Lemma 7.7, there is an S_1-condition (A_1', A_2') such that $(A_1', A_2')\ \Vdash \varphi[(X, Y)]$. Since $(A_{1,a}, A_{2,a})$ is also an S_1-condition, it follows that $(B_1, B_2)=(A_{1,a}\cup A_1', A_{2,a}\cup A_2')$ is an S_1-condition. Since (B_1, B_2) witnesses the fact that $X\in C_\varphi$, the proof is now complete. ■

It turns out that the connective S_1 of Theorem 7.2 is such that $\mathrm{Val}(L_{\omega_1\omega}(S_1))$ will be Σ_1 over HC if the listings of formulas and conditions used in the construction are chosen in some kind of definable manner. The content of Theorem 7.3 is precisely that this can be done, assuming $V=L$.

Proof of Theorem 7.3. To show that $\mathrm{Val}(L_{\omega_1\omega}(S_1))$ is Σ_1 over HC, it suffices, by Lemma 6.1, to show that $\mathrm{Val}(P_{\omega_1}(S_1))$ is Σ_1 over HC. The main idea of the proof is in noticing that the validity of φ_a is decided at stage $a+1$ in the construction of S_1. More precisely, for $\varphi \in F^*$, the statement

$(*)$ $\quad \varphi \in \mathrm{Val}(P_{\omega_1}(S_1))$

is equivalent to

$(**)$ $\quad$ There is no $a < \omega_1$ such that $\varphi^* = \varphi_a$ and $(A_{1,a+1}, A_{2,a+1}) \Vdash \varphi_a[X]$, for some $X \subseteq \omega$.

Indeed if $\varphi \in \mathrm{Val}(P_{\omega_1}(S_1))$, then $\models^{S_1}\varphi[X]$ for all X, so by Lemma 7.7, the S_1-condition $(A_{1,a+1}, A_{2,a+1})$ cannot force $\varphi^*[X]$, for any X.

Conversely, if $\varphi \notin \mathrm{Val}(P_{\omega_1}(S_1))$, then $\models^{S_1}\varphi^*[X]$, for some X. But in this case there would be an S_1-condition $(B_1, B_2) < (A_{1,a}, A_{2,a})$ such that $(B_1, B_2) \Vdash \varphi^*[X]$. Since $\varphi^* = \varphi_a$, it follows that $(A_{1,a+1}, A_{2,a+1}) \Vdash \varphi^*[X]$.

Hence we shall prove the result by proving $(**)$. $\mathrm{V} = \mathrm{L}$ enters by providing a definable well-ordering, $<_L$, of the universe such that

(D1) $\{(x, y) \in \mathrm{HC} \mid x <_L y\}$ is Δ_1 over HC and has order type $\aleph_1$, and

(D2) If ψ is equivalent to both a Π_1 and a Σ_1 formula, then $\{x \in \mathrm{HC} \mid (\forall y)(y <_L z \to \psi(x, y))\}$ and $\{x \in \mathrm{HC} \mid (\exists y)(y <_L z \wedge \psi(x, y))\}$ are both Δ_1 over HC (see Devlin [8]).

Now let φ_v be the vth formula of F^* with respect to $<_L$ and let $(A_{1,v}, A_{2,v})$ be the vth condition with respect to $<_L$.

We will show that if S_1 is defined (as in the proof of Theorem 7.2) *from the above choice of listings*, then $(**)$ can be written in Σ_1 form over HC. Thus $\mathrm{Val}(L_{\omega_1\omega}(S_1))$ will be Σ_1 over HC.

To do this we must analyze the logical complexity of the conditions used in the definitions of $\Vdash$ and S_1. We will use the method of coding HC sets by reals as explained for example in Barwise et al. [5]. The main features of this coding which we shall need are

(B1) $T \subseteq \omega$ codes an element of HC iff T is a well-founded tree, and

(B2) If $A \subseteq \mathrm{HC}$, then A is Σ_1 over HC iff $\{T \subseteq \omega \mid T$ codes an element of $A\}$ is Σ^1_2.

In some cases (B2) can be improved.

(B3) $\quad \{T \subseteq \omega \mid T$ is a code$\}$ is Π^1_1,
$\{T \subseteq \omega \mid T$ codes a real$\}$ is Δ^1_1, and
$\{T \subseteq \omega \mid T$ codes a forcing condition$\}$ is Δ^1_1.

The crucial step in this part of the proof is in showing that the forcing relation, $\Vdash$, is Δ^1_1 in codes (for formulas). More precisely, there are Σ^1_1 and Π^1_1 formulas, ψ_1 and ψ_2, respectively, of second order arithmetic such that for all $T_{B_1}, T_{B_2}, T_X, T_\varphi \subseteq \omega$, if T_{B_1} codes B_1, T_{B_2} codes B_2, T_X codes X and T_φ codes φ (when $\varphi \in F^*$), then

(B_1, B_2) is a condition and X is a real such that

$$(B_1, B_2) \Vdash \varphi[X] \quad \text{iff } \psi_1(T_{B_1}, T_{B_2}, T_X, T_\varphi),$$

(B_1, B_2) is a condition and X is a real such that

$$(B_1, B_2) \Vdash \varphi[X] \text{ fails} \quad \text{iff } \psi_2(T_{B_1}, T_{B_2}, T_X, T_\varphi).$$

This is clear when one notices that the inductive clauses in the definition of $\Vdash$ are expressible by arithmetical formulas (in codes).

Now to see that (**) can be expressed in Σ_1 form over HC let us consider the formula $\xi = (\exists\varphi)(\exists A_1)(\exists A_2)(\exists X)(\exists D)(\exists \leqslant)[\chi \wedge (7)]$ where χ is the conjunction of the set theoretic formulas expressing

(1) φ, A_1, A_2, X are functions with domain D,

(2) D is well-ordered by $\leqslant$,

(3) $(\forall \nu \in D)\varphi_\nu$ is the $<_L$-first element of F^* greater than (in the sense of $<_L$) all φ_a for $a < \nu$,

(4) if ν_0 is the first (in the sense of $\leqslant$) element of D, then $A_{1,\nu_0} = A_{2,\nu_0} = \emptyset$,

(5) $(\forall \nu \in D)$ if ν is a limit in the sense of $\leqslant$, then $A_{1,\nu} = \bigcup_{a<\nu} A_{1,a}$ and $A_{2,\nu} = \bigcup_{a<\nu} A_{2,a}$,

(6) $(\forall \nu \in D)$ if ν^+ is the $\leqslant$-successor of ν, then either

(6a) $(A_{1,\nu^+}, A_{2,\nu^+})$ is a condition such that $(A_{1,\nu^+}, A_{2,\nu^+}) \Vdash \varphi_\nu[X]$, and for all conditions (B_1, B_2) less than $(A_{1,\nu^+}, A_{2,\nu^+})$ (in the sense of $<_L$) and stronger than $(A_{1,\nu}, A_{2,\nu})$ it is not the case that $(B_1, B_2) \Vdash \varphi[Y]$ for any $Y \subseteq \omega$, or

(6b) $A_{1,\nu^+} = A_{1,\nu}$, $A_{2,\nu^+} = A_{2,\nu}$ and for any condition (B_1, B_2) stronger than $(A_{1,\nu}, A_{2,\nu})$ it is not the case that $(B_1, B_2) \Vdash \varphi_\nu[Y]$ for any $Y \subseteq \omega$,

and (7) is the formula expressing the fact that

for some $a \in D$, $[\varphi^* = \varphi_a$

and it is not the case that $(A_{1,a}, A_{2,a}) \Vdash \varphi_a[Y]$ for any $Y \subseteq \omega]$.

Since it is easy to see that (**) is equivalent to ξ, it thus suffices to show that (1)–(7) are all expressible in Δ_1 form over HC.

Clearly (1)–(5) are all Δ_1 over HC (use (D1) and (D2) for (3)). Since the previous discussion concerning the logical complexity of $\Vdash$ shows that when (B_1, B_2) is a condition and $\varphi_\nu \in F^*$,

$$(\forall Y \subseteq \omega)(B_1, B_2) \Vdash \varphi_\nu[Y]$$

is Π_1^1 (in codes) and since the quantification over all conditions (B_1, B_2) in (6a) is bounded into an initial segment of $<_L$, it follows, from (B1) and (D2), that (6a) is Δ_1 over HC. (6b) is Δ_1 over HC because it is Π_1^1 (in codes for formulas from F^*). Finally the bracketted part of (7) is Π^1_1 (in codes for formulas from F^*) so that (7) is also Δ_1 over HC. Hence we have shown that $\mathrm{Val}\big(L_{\omega_1\omega}(\boldsymbol{S}_1)\big)$ is Σ_1 over HC.

To complete the proof we must still show that $\boldsymbol{S}_1$ is Δ^1_2, and that $L_{\omega_1\omega}(\boldsymbol{S}_1)$ is absolute (in the sense of Barwise [3]) with respect to KP+ $+\Sigma_1$-separation + Infinity. Both facts however follow quite routinely once it is shown that $\boldsymbol{S}_1$ is provably Δ_1 in KP$+\Sigma_1$-separation + Infinity. Moreover, since $\boldsymbol{S}_1$ will be provably Δ_1 in KP$+\Sigma_1$-separation + Infinity if $\mathrm{Val}\big(P_{\omega_1}(\boldsymbol{S}_1)\big)$ is Δ_1 in KP$+\Sigma_1$-separation + Infinity, we need only prove this last-mentioned fact.

To do this, first note that (**), besides being expressible in Σ_1 form, is also expressible (over HC) by the Π_1 formula ξ', where ξ' is

$$(\forall\varphi, A_1, A_2, X, D, \leqslant)\big[\chi\to\big\{(\forall a\in D)\big(\varphi^* = \varphi_a\to(\forall Y\subseteq\omega)(A_{1,a}, A_{2,a}) \\ \Vdash \varphi_a[Y] \text{ fails}\big)\big\}\big].$$

(Hence $\mathrm{Val}\big(P_{\omega_1}(\boldsymbol{S}_1)\big)$ is actually Δ_1 over HC.) Then let $R(x)$ be a Σ_1 formula defining F^* over V. Since every Π^1_1 set of reals is provably Δ_1 in KP$+\Sigma_1$-separation + Infinity, it follows easily, by examining ξ and ξ', that

$$(\forall\varphi)[R(\varphi)\to\xi(\varphi)\leftrightarrow\xi'(\varphi)]$$

is a theorem of KP$+\Sigma_1$-separation + Infinity. Thus $\mathrm{Val}\big(L_{\omega_1\omega}(\boldsymbol{S}_1)\big)$ is provably Δ_1 in KP$+\Sigma_1$-separation + Infinity. ∎

The three theorems proven here are of course not the final word on the behavior of propositional extensions of $L_{\omega_1\omega}$ in the constructible universe. Many questions remain unanswered. For example, are there minimal extensions of $L_{\omega_1\omega}$ in L? Are there extensions having the Beth property, or at least the propositional Beth property? These are but a few of the possibilities for further work in this area.

8. The Souslin connective

Because of the intimate relationship between $L_{\omega_1\omega}$ and the Borel sets (see Lemma 3.1 for one such relationship as well as Scott [29] for a discussion of some others), it is perhaps most natural to attempt to extend $L_{\omega_1\omega}$ by a propositional operation somehow connected with the Σ^1_1 (or Π^1_1) sets. Indeed this was the point of view we shared when we began the research into the topic of this paper. It lead us to the idea of formulating the classical operation $\mathscr{A}$ in a propositional setting and resulted in a connective which we called the *Souslin connective* and a corresponding logic which we called *Souslin logic.*

After working out many of the properties of this Souslin logic we learned that it had first been considered by Nerode a few years earlier and subsequently, studied, from various differing points of view, by Campbell [7], Ellentuck [10] and Halpern [17]. In this section we wish to briefly describe some of the properties of this logic—particularly those properties connected with the general theme of the present paper.

In order to define the Souslin connective we must first introduce some notations. We let Seq be the set of all finite sequences of natural numbers. By using one of the usual codings of finite sequences of natural numbers by natural numbers, we can thus identify Seq with ω. Given a subset X of Seq, and a function f, from ω into ω, we say that f is a *branch through* X, if for all $n \in \omega$, $f \restriction n \in X$.

The Souslin connective, $\boldsymbol{A}$ is now taken to be

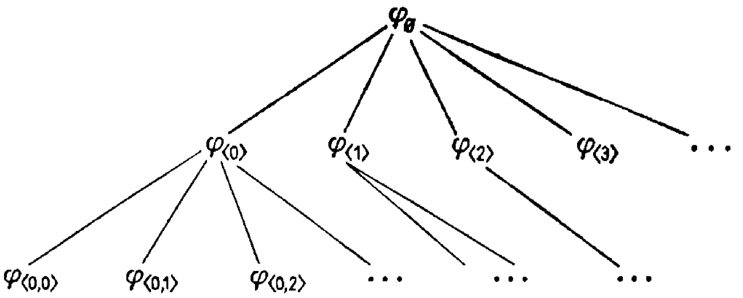
$$\{X \subseteq \mathrm{Seq} \mid \text{there is a branch through } X\}.$$

By the coding mentioned above, $\boldsymbol{A}$ is of course a subset of $P(\omega)$. Moreover, if for each $n \in \omega$, $A_n \subseteq P(\omega)$, then

$$X \in \boldsymbol{A} A_n \quad \text{iff} \quad \text{there is a branch through } \{n \mid \in \mathrm{Seq} \mid X \in A_n\}.$$

Hence the operation $\boldsymbol{A}$ is nothing more than the usual operation $\mathscr{A}$ of descriptive set theory. The elements of the corresponding boolean algebra $\boldsymbol{B}_A$, are what is known in the classical literature as the *ensembles criblés*, or simply the C-sets.

It is somewhat instructive to point out that the Souslin connective can be regarded, as we originally did, as a connective which operates on a tree of formulas of the form



The resulting formula, $\underset{\tau\in\mathrm{Seq}}{\mathscr{S}} \varphi_\tau$, is then satisfied in a structure $\mathfrak{A}$, by an assignment x, if there is an infinite path through the tree, all of whose nodes are satisfied in $\mathfrak{A}$ by x.

Since $\boldsymbol{A}$ is quite obviously Σ^1_1, the results of Section 4 concerning connectives with covering pairs of cardinality $\aleph_1$ apply to $L_{\omega_1\omega}(\boldsymbol{A})$. Thus, for example, we can conclude that $\mathrm{Craig}(L_{\omega_1\omega}(\boldsymbol{A}), L_{\omega_2\omega})$ holds.

The well-known classical connections between the use of the operation $\mathscr{A}$ and the use of sieves can be expressed in our terminology as follows

THEOREM 8.1. $\boldsymbol{B}_A = \boldsymbol{B}_W$.

Proof. By Lemma 3.2 it suffices to show

(a) $\boldsymbol{A} \in \boldsymbol{B}_{\boldsymbol{W}}$, and

(b) $\boldsymbol{W} \in \boldsymbol{B}_{\boldsymbol{A}}$.

To prove (a) we let

$$\varphi_n = \begin{cases} p_{(n)_0} \wedge p_{(n)_1} & \text{if} \quad (n)_0 <_{\mathrm{KB}} (n)_1, \\ p_0 \wedge \sim p_0 & \text{otherwise}, \end{cases}$$

where $<_{\mathrm{KB}}$ is the Kleene–Brouwer ordering of Seq (see Kleene [20]). Then, since a subset X of Seq has no branch iff $<_{\mathrm{KB}} \cap X$ is well-ordered, it follows that

$$X \in \boldsymbol{A} \quad \text{iff} \quad \models^{\boldsymbol{W}} \sim \mathscr{S} \varphi_n[X].$$

To prove (b) we define for each $n \in \omega$, a finitary propositional formula ψ_n, such that if n codes the finite sequence $\langle t_0, \ldots, t_k \rangle$ then

$$\psi_n = p_{(t_1,t_0)} \wedge p_{(t_2,t_1)} \wedge \qquad \wedge p_{(t_k t_{k-1})}.$$

Then clearly

$$X \in \boldsymbol{W} \quad \text{iff} \quad \models^{\boldsymbol{A}} \sim \mathscr{S} \psi_n[X]$$

so that (b) follows. ■

Theorem 8.1 thus shows that the Souslin logic is really the same as $L_{\omega_1\omega}(\boldsymbol{W})$. Hence from Section 6 we can conclude that

(1) $\mathrm{Val}(L_{\omega_1\omega}(\boldsymbol{A}))$ is not Σ_1 over HC, and

(2) Weak Beth $(L_{\omega_1\omega}(\boldsymbol{A}), L_{\omega_1\omega}(\mathbf{Full}))$ fails.

Thus the Souslin extension—which, both from the point of view of its motivation as well as from the perspective of Theorems 3.4 and 3.5, is the most natural propositional extension of $L_{\omega_1\omega}$—already fails to have strong model-theoretic properties. However, since the Σ_1^1 sets have more manageable properties than the C-sets (e.g. the pre-well-orderings of the former type are all of countable length), it is not unreasonable to expect that the part of the Souslin logic which corresponds to the Σ_1^1 sets has nicer model-theoretic properties than the whole Souslin logic. This indeed turns out to be true as we shall show in the rest of this section.

We begin with some more terminology. Given a connective $\boldsymbol{S}$, we say that a formula φ of $L_{\omega_1\omega}(\boldsymbol{S})$ is *positive* if $\mathscr{S}$ does not occur in φ and $\sim$ occurs only in front of atomic formulas. The set of positive formulas of $L_{\omega_1\omega}(\boldsymbol{S})$ constitutes the positive $\boldsymbol{S}$-logic, denoted by $p\text{-}L_{\omega_1\omega}(\boldsymbol{S})$. Thus $p\text{-}L_{\omega_1\omega}(\boldsymbol{A})$ corresponds to the Σ_1^1 sets and $p\text{-}L_{\omega_1\omega}(\boldsymbol{\forall})$ corresponds to the Π_1^1 sets.

DEFINITION. We write Craig$^+$ $(p\text{-}L_{\omega_1\omega}(\boldsymbol{S}))$ to mean that for any two sentences, φ and ψ, of $p\text{-}L_{\omega_1\omega}(\boldsymbol{S})$, such that $\varphi \wedge \psi$ has no model, there is an $L_{\omega_1\omega}$ sentence θ all of whose non-logical symbols are common to both φ and ψ and such that $\models^{\boldsymbol{S}} \varphi \rightarrow \theta$ and $\models^{\boldsymbol{S}} \psi \rightarrow \sim \theta$.

The following theorem summarizes the main positive model-theoretic facts concerning $p\text{-}L_{\omega_1\omega}(\mathbf{A})$ and $p\text{-}L_{\omega_1\omega}(\mathbf{V})$.

THEOREM 8.2. (a) $\mathrm{Val}\big(p\text{-}L_{\omega_1\omega}(\mathbf{V})\big)$ *is* Σ_1 *over* HC.

(b) $\mathrm{Craig}^+\big(p\text{-}L_{\omega_1\omega}(\mathbf{A})\big)$.

Proof. (a). Since Lemma 6.1 extends to positive parts, it suffices to show that $\mathrm{Val}\big(p\text{-}P_{\omega_1}(\mathbf{V})\big)$ is Σ_1 over HC, where $p\text{-}P_{\omega_1}(\mathbf{V})$ is the propositional part of $p\text{-}L_{\omega_1\omega}(\mathbf{V})$. To do so, define $R(\varphi, X)$ iff $\models^A \varphi[X]$, for φ from $p\text{-}P_{\omega_1}(\mathbf{V})$.

It is easy to see that $R(\varphi, X)$ holds iff

$(\forall f)$ [if f is a function defined on the set of all subformulas of φ, with values in $\{0, 1\}$ such that

$f(p_i) = 1$ iff $i \in X$,

$f(\sim p_i) = 1$ iff $i \notin X$,

$f(\bigvee \varphi_n) = 1$ iff $f(\varphi_n) = 1$ for some $n \in \omega$,

$f(\bigwedge \varphi_n) = 1$ iff $f(\varphi_n) = 1$ for all $n \in \omega$, and

if $f(\mathscr{S}\varphi_n) = 0$, then there is a branch through $\{n \mid f(\varphi_n) = 1\}$

then $f(\varphi) = 1$].

Then by coding $P_{\omega_1}(\mathbf{V})$ formulas (which are HC sets) by reals, in the usual way, it is easy to see that R, viewed as a relation between codes of formulas and realizations, is Π^1_1. Thus $\mathrm{Val}\big(p\text{-}P_{\omega_1}(\mathbf{V})\big)$ must be Δ_1 over HC by (B2) of Section 7.

(b) This will be proven by a simple reduction to interpolation for $L_{\omega_1\omega}$. We will give the details only in the case that the languages of φ and ψ are finite.

We begin by showing that for each sentence φ of $p\text{-}L_{\omega_1\omega}(\mathbf{A})$, there is a sentence φ^+ of $L_{\omega_1\omega}$ (with an additional unary relation symbol $\bar{Q}$, an additional binary relation symbol $\bar{I}$, additional constant symbols $\bar{c}_n$ for all $n \in \omega$, and an additional unary relation symbol $\bar{B}_\psi$ for each subformula ψ of φ, of the form $\psi = \mathscr{S}\psi_n$) such that for all $\mathfrak{A}$

$\mathfrak{A} \models^A \varphi$ iff there are $Q \subseteq |\mathfrak{A}|$, $I \subseteq |\mathfrak{A}|^2$,

$c_n \in |\mathfrak{A}|$, $B_\psi \subseteq |\mathfrak{A}|$ such that $\langle \mathfrak{A}, Q, I, c_n, B_\psi \rangle \models \varphi^+$.

To do this we first define φ' inductively as follows:

φ' is φ, if φ is atomic or negated atomic,

φ' is $\bigwedge \varphi'_n \big(\bigvee \varphi'_n, (\forall v)\psi', (\exists v)\psi'\big)$, if φ is $\bigwedge \varphi_n \big(\bigvee \varphi_n, (\forall v)\psi, (\exists v)\psi\big)$,

φ' is $(\exists v_0)\bar{B}_\varphi(v_0) \wedge (\forall v_0)\big[\bar{B}_\varphi(v_0) \to (\exists v_1)\big(\bar{B}_\varphi(v_1) \wedge \bar{I}(v_0, v_1)\big] \wedge \bigwedge_{n\in\omega} \big(\bar{B}_\varphi(\bar{c}_n) \to \to \varphi'_n\big)$, if φ is $\mathscr{S}\varphi_n$.

Clearly, φ' is a sentence of $L_{\omega_1\omega}$ whenever φ is a sentence of $p\text{-}L_{\omega_1\omega}(\mathbf{A})$.

We next let φ'' be the conjunction of

$\bar{c}_i \neq \bar{c}_j \ (i \neq j)$,

$(\forall v_0)\big(\bar{Q}(v_0) \leftrightarrow \bigvee_{n\in\omega} v_0 = \bar{c}_n\big)$,

$(\forall v_0)(\forall v_1)[\bar{I}(v_0, v_1) \to \bar{Q}(v_0) \wedge \bar{Q}(v_1)]$,

$\bar{I}(\bar{c}_i, \bar{c}_j)$ (whenever the sequence coded by i is a proper initial segment of the sequence coded by j),

and φ'.

Finally, let φ^+ be $\varphi'' \vee \bigvee$(diagram of B), where the last disjunction is taken over all finite models B of φ. It is clear that φ^+ works.

Now suppose that $\varphi \wedge \psi$ has no model. Let φ^+, ψ^+ be the sentences constructed as above, and such that the additional symbols used in φ^+ do not occur in ψ^+ and conversely. Then $\varphi^+ \wedge \psi^+$ is an $L_{\omega_1\omega}$ sentence without any models. By interpolation for $L_{\omega_1\omega}$, there is a sentence θ in the language common to both φ^+ and ψ^+ (which is, of course, the same as the language common to both φ and ψ) such that $\models \varphi^+ \to \theta$ and $\models \psi^+ \to \ \sim \theta$. But then clearly $\models \varphi \to \theta$ and $\models \psi \to \ \sim \theta$, which is what we wished to show.

If the language of φ or of ψ is infinite, there might be uncountably many nonisomorphic finite models, so that φ^+ or ψ^+ might not be a sentence of $L_{\omega_1\omega}$. In this case we can still prove the theorem—but this time one must use the many sorted interpolation theorem of Feferman [11]. We omit the details. ∎

Our next theorem shows that the properties dual to those in the previous theorem fail—something which might be expected, bearing in mind the failure of separation for the Π^1_1 sets.

THEOREM 8.3. (a) $\mathrm{Val}\big(p\text{-}L_{\omega_1\omega}(A)\big)$ *is not* Σ_1 *over* HC.

(b) $\mathrm{Craig}^+\big(p\text{-}L_{\omega_1\omega}(V)\big)$ *fails.*

Proof. (a) The proof of Theorem 6.1 can be used here with some minor changes. It will suffice to show that $\mathrm{Val}\big(p\text{-}P_{\omega_1}(A)\big)$ is not Σ_1 over HC.

We let Σ_V denote the set of all projections of truth sets of formulas of $p\text{-}P_{\omega_1}(V)$. Part (b) of the proof of Theorem 8.1 actually shows that $W \in \Sigma_V$. From this it follows that Σ_V contains all Σ^1_2 sets.

Now let T be a Σ^1_2 non Π^1_2 element of Σ_V, and let φ be a $p\text{-}P_{\omega_1}(V)$ formula such that

$$X \in T \quad \text{iff} \quad \models^V \varphi[(X, Y)] \text{ for some } Y.$$

Defining φ_X, as before, we shall have

$$X \in T \quad \text{iff} \quad \models^V \sim\varphi_X \text{ fails}.$$

Then if φ^*_X is obtained from $\sim\varphi_X$ by pushing negation signs all the way down to atomic constituents, it follows that φ^*_X is a formula of $p\text{-}P_{\omega_1}(A)$. Moreover, the function mapping X to φ^*_X is clearly Δ_1 over HC.

Hence, since

$$X \in T \quad \text{iff} \quad \varphi_x^{\bullet} \notin \mathrm{Val}(p\text{-}P_{\omega_1}(A)),$$

it follows that if $\mathrm{Val}(p\text{-}P_{\omega_1}(A))$ were Σ_1 over HC, T would be Π_1 over HC, and therefore Π_2^1 (by Theorem 0.1), contradicting the choice of T.

(b) Using the fact that the formula (of Lemma 6.2) of $L_{\omega_1\omega}(W)$ which pins down $\aleph_1$ is actually a formula of $p\text{-}L_{\omega_1\omega}(V)$ it is easy to show, by following the proof of Theorem 6.2, that $p\text{-}L_{\omega_1\omega}(A)$ fails to have the weak Beth property. The failure of Craig$^+$ can then be derived in the usual way. ∎

We close this section by mentioning that independently of us, and at about the same time, Halpern in [17] studied the countable admissible fragments of $p\text{-}L_{\omega_1\omega}(A)$ and $p\text{-}L_{\omega_1\omega}(V)$. His main result was the formulation of a uniform axiomatic system, with deduction rules using only *countably* many premises, such that $p\text{-}L_A(A)$ is complete with respect to this system over all countable admissible sets A. He then used his main result to derive analogues of parts (a) and (b) of our Theorem 8.2 for all the countable admissible fragments of the respective logics in question. Furthermore, he conjectured that $\mathrm{Val}(p\text{-}L_A(A))$ is not Σ_1 over any countable admissible A. Our Theorem 8.3 (a) of course verifies the corresponding conjecture for $\mathrm{Val}(p\text{-}L_{\omega_1\omega}(A))$. From the proof, however, it is easy to see that Halpern's conjecture is true for many countable admissibles.

9. Concluding remarks

In this final section we would like to make some general observations, based on our results, about the possibility of extending $L_{\omega_1\omega}$ by propositional operations. Our basic contention here is that, *among all the propositional extensions of $L_{\omega_1\omega}$, only $L_{\omega_1\omega}$ itself warrants any interest from a traditional model-theoretic point of view.* This, however, does not preclude the possibility of having interesting and useful extensions of $L_{\omega_1\omega}$ from the point of view of applications to mathematics (a point which we will amplify later).

To support our basic contention we must first agree on what properties make a logic model-theoretically interesting. Without generating much controversy it is probably safe to list the suitable generalizations of such properties as

1. compactness,
2. interpolation and Beth definability, and
3. recursive enumerability of the set of valid sentences,

as candidates.

Of course, for $L_{\omega_1\omega}(S)$ compactness is too much to expect since even $L_{\omega_1\omega}$ fails to have Σ_1-compactness. Beth definability on the other hand, which expresses the fact that a logic is closed under implicit definitions by its own formulas, is certainly something one would want to have. Similarly, the recursive enumerability of the set of valid formulas is also desirable. The precise meaning of this requirement however, is somewhat less transparent.

In the present paper we have exclusively used Σ_1-definability as our notion of recursive enumerability. Over countable transitive structures, Σ_1-definability is equivalent to any reasonable notion of recursive enumerability and thus is not problematic. However, for the case of HC, one might wish to argue (as has been done for example by Kunen [22] and Kreisel [21]) that s.i.i.d (see Kunen [22] for the definition) rather than Σ_1-definability is the proper notion of recursive enumerability. As we have no quarrels with this position, we would readily agree with anyone objecting to the possible interest of our Theorem 6.1 on this basis. However, there are at least two things which should be said in favor of our decision to study $\mathrm{Val}(L_{\omega_1\omega}(S))$ in terms of Σ_1-definability.

Firstly, since the notion of s.i.i.d over HC is so vast, it seems that any analysis of $\mathrm{Val}(L_{\omega_1\omega}(S))$ in terms of s.i.i.d would become completely unmanageable. For example, it is quite easy to see that $\mathrm{Val}(L_{\omega_1\omega}(S))$ is s.i.i.d whenever S is Π^2_1. Hence any attempt to fatten up $L_{\omega_1\omega}$ with propositional connectives in the hope of obtaining a complete s.i.i.d validity predicate seems somewhat futile.

Second, and more important, is the following observation concerning fragments of $L_{\omega_1\omega}(S)$ of the form $L_A(S)$, with A transitive:

If S is a connective for which there is a theory T, in the language of set theory, which

(i) has arbitrarily large countable models, and

(ii) is such that $\mathrm{Val}(L_A(S))$ is *uniformly* Σ_1 over each A which is a model of T (uniformly here means that there is a single formula without parameters which works for all A),

then $\mathrm{Val}(L_{\omega_1\omega}(S))$ is Σ_1 over HC.

Hence the assertion that $\mathrm{Val}(L_{\omega_1\omega}(S))$ is not Σ_1 over HC rules out the possibility of a tower of countable fragments each of which is complete (and possibly Σ_1-compact). Since it has been argued that the existence of such a tower for $L_{\omega_1\omega}$ is precisely the reason why $L_{\omega_1\omega}$ has such nice model-theoretic properties, it would be desirable to find connectives S such that $L_{\omega_1\omega}(S)$ has the same property.

We next turn to a discussion of the major consequences of our negative results in Section 6. Here we have Theorems 6.2 and 6.1, which, respectively, rule out properties (2) and (3) for $L_{\omega_1\omega}(S)$ whenever $h(S) > \aleph_1$ Thus if we knew that $h(S)$ is always greater than $\aleph_1$, we would have.

very strong evidence for the privileged character of $L_{\omega_1\omega}$. Unfortunately, this is only true under false assumptions such as AD.

However, two lines of argument still remain open. First, although we showed in Section 7, Theorem 2, that the existence of connectives with Σ_1 validity is consistent with ZFC, it is still possible (as we have conjectured at the end of Section 6) that the Beth definability property always fails. If this turns out to be true, the evidence for the special status of $L_{\omega_1\omega}$ will greatly increase.

Second, one can argue—from assumptions that are more likely to be true than false—that those connectives, S, which are such that $\mathrm{Val}(L_{\omega_1\omega}(S))$ is Σ_1, or that $\mathrm{Craig}(L_{\omega_1\omega}(S))$ holds, should be rendered inadmissable on the grounds that they are totally uninteresting.

To be more precise let us assume that the notion of *ordinal definable from a real* (OD from a real) is known (see Myhill–Scott [28] for the definition). Since it seems quite reasonable that a class of structures should be of interest to mathematicians (or even logicians) only if it is definable in set theory (perhaps from some real or ordinal parameters), the notion of a class of structures OD from a real seems to capture the concept of an interesting class completely.

Hence suppose that $L_{\omega_1\omega}(S)$ has an elementary class OD from a real, which is not elementary in $L_{\omega_1\omega}$. Then quite obviously, the set of all structures in this class with domain ω must also be OD from a real, so that by Lemma 3.4, $\boldsymbol{B}_S$ would contain a non-Borel set $\boldsymbol{T}$ which is OD from a real. Then by assuming that all games which are OD from a real are determined—an axiom suggested by Martin and Solovay—one could quite easily show, using the arguments of Section 3, that $\boldsymbol{W} \leqslant_L \boldsymbol{T}$ and $\boldsymbol{T} \leqslant_L S$. Thus from Lemma 6.2 it would follow that $h(S) > \aleph_1$. Hence if $L_{\omega_1\omega}(S)$ were capable of expressing something interesting, then either $L_{\omega_1\omega}$ could express the same thing, or both Σ_1-validity and Beth definability for $L_{\omega_1\omega}(S)$ would fail.

Similarly, even if the universe is less playful, the idea of the above considerations would still apply. For example, if only projective determinacy holds (i.e. the assumption that all projective games are determined) we would still be able to show that if $L_{\omega_1\omega}(S)$ possesses an elementary class which is elementary in *second order* $L_{\omega_1\omega}$, but not elementary in $L_{\omega_1\omega}$, then $h(S) > \aleph_1$, so that again Σ_1-validity (over HC) and Beth definability fail.

Finally, without any assumption of determinacy we could still show that, if $L_{\omega_1\omega}(S)$ contains an elementary class which is either PC or CPC in $L_{\omega_1\omega}$, but not elementary in $L_{\omega_1\omega}$, then again $h(S) > \aleph_1$, with the usual consequences. Since it seems that all extensions of $L_{\omega_1\omega}$ with any real mathematical interest would almost always have this last mentioned property, the search for both mathematically and model-theoretically interesting propositional extensions of $L_{\omega_1\omega}$ does not seem to be very fruitful.

By this, however, we do not mean to suggest that the study of extensions of $L_{\omega_1\omega}$ is a useless endeavor. After all, there are many logics with even worse model-theoretic properties than the ones we have been considering, which have been quite successfully applied to problems of mathematical interest. For example, Vaught in [32] and [33] has made excellent use of the game logic in working out some descriptive set theoretic aspects of the $L_{\omega_1\omega}$ space of models. In addition, many people have achieved some quite interesting results by applying $L_{\varkappa\lambda}$ back and forth techniques (for $\lambda > \omega$) to problems related to algebra (see Dickman [9] for a comprehensive account.) We may even offer the second half of our Theorem 5.1 as a modest example of a purely set-theoretic result proved with the aid of the logic $L_{\omega_1\omega}(S)$. Thus it is probably true, as Kreisel has suggested, that the future of large infinitary logics in general, and the kind of logics we consider here in particular, lies in the formulation and solution of questions which are quite different from those presently considered by model theorists. Indeed the truly relevant questions will probably be those arising out of possible mathematical applications.

Note added in proof: The results contained in this paper were obtained in the winter of 1974 and the spring of 1975 and presented to the meeting of the Association for Symbolic Logic in Clermont–Ferrand in July 1975 (see [37]). Several related results appeared since that time. Harrington [39] constructed connectives that have the Craig property, and thus disproved our conjecture on p. 34. His construction assumes only the Axiom of Choice and produces connectives with other nice properties, for example that of S_1 in Theorem 7.2. Harrington [38] showed that the existence of a single Π_1^1 connective that yields a proper extension of $L_{\omega_1\omega}$ strictly weaker than $L_{\omega_1\omega}[W]$ (our Theorem 7.1) is equivalent to $(\forall X)(X^{\#}$ exists) (see also Theorem 3.5). Finally, Hrbacek and Simpson [36] proved the existence of $2^{\aleph_0}$ mutually incomparable such connectives in all generic extensions of L.

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