NUMERICAL ANALYSIS AND MATHEMATICAL MODELLING
BANACH CENTER PUBLICATIONS, VOLUME 24
PWN-POLISH SCIENTIFIC PUBLISHERS
WARSAW 1990

UPWIND FEM FOR CONVECTION DOMINATED ELLIPTIC PROBLEMS

U. RISCH and G. LUBE

Department of Mathematics, Technical University,
Magdeburg, GDR

1. Introduction

We consider the problem

$$L_\varepsilon u = -\varepsilon \Delta u + b\nabla u + cu = f \quad \text{in } \Omega \subset R^N,\ N \geqslant 2,$$
$$u = 0 \quad \text{on } \Gamma = \partial\Omega$$

with the small parameter $0 < \varepsilon \ll 1$. The qualitative behaviour of the solution is characterized by the existence of boundary layers (narrow regions where the norms $\|u\|_{k,p}$, $k \geqslant 1$, are not bounded independently of ε). Denoting by n the unit outer normal to Ω and by Γ^- (Γ^0, Γ^+) the parts of Γ where bn is less than (equal to, greater than) zero, normally we have a boundary layer of thickness $O(\sqrt{\varepsilon}|ln\sqrt{\varepsilon}|)$ along Γ^0 and a boundary layer of thickness $O(\varepsilon|\ln\varepsilon|)$ along Γ^+. Due to these boundary layers, the application of standard finite element methods results in undesired oscillations in the numerical solution u_h unless the discretization parameter h is very small. These oscillations can spread over a region much larger than the boundary layer. So, one is interested in finite element methods which preserve the monotony behaviour and/or result in ε-independent error estimates at least in subdomains where boundary layers are excluded (local estimates). We present a short survey on the most interesting trends and some new results in this field.

2. The hybrid upwind FEM and the discontinuous Galerkin method

The weak formulation of our problem is:

Find $u \in H_0^1(\Omega)$ such that

$$\langle L_\varepsilon u, v\rangle := a(u,v) := \varepsilon(\nabla u, \nabla v) + (b\nabla u, v) + (cu, v) = \langle f, v\rangle \quad \text{for all} \quad v \in H_0^1(\Omega).$$

In this section, let $f \in L^2(\Omega)$, $b \in (W^{1,\infty}(\Omega))^2$, $c \in W^{1,\infty}(\Omega)$.

As it is known, for $c \geqslant 0$ or $c - \frac{1}{2}\operatorname{div} b \geqslant 0$ the differential operator L_ε is inversely monotone; hence e.g. follows the unique solvability of $L_\varepsilon u = f$.

For the sake of simplicity we restrict ourselves to the case $\Omega \subset \boldsymbol{R}^2$, but this restriction is not essential.

Let Ω be a polygonal convex domain. We divide Ω into triangles with angles less or equal $\pi/2$ (triangulation of "weakly acute type"). Let a triangulation $(\mathcal{T}_h)$ be regular in the usual sense and let h denote the maximal diameter of all triangles. We call such a triangulation a *general* one in order to distinguish it from a *uniform* triangulation, i.e. one in which every two adjacent triangles form a parallelogram.

Let

$$V_h = \{v_h \in C(\bar{\Omega}) : v_h|_T \in P_1(T)\}, \qquad V_{0h} = \{v_h \in V_h : v_h|_\Gamma = 0\}$$

denote spaces of piecewise linear functions.

We consider also a dual decomposition of Ω which is constructed as follows: To each node P_i there corresponds a dual polygon D_i bounded by parts Γ_{ij} of the medians (one also could choose the mid-perpendiculars, this would give no difference in the results) of adjacent triangles.

We will use the notation:

Λ_i – set of indices of the nodes adjoining to P_i,

n_i – outer unit normal to D_i.

Let $\tilde{V}_h = \{\tilde{v}_h : v_h|_{D_i} \in P_0(D_i)\}$ denote the space of piecewise constant functions. We use two interpolation operators I_h: $C(\bar{\Omega}) \to V_h$ and $\sim$: $C(\bar{\Omega}) \to \tilde{V}_h$ defined by $(I_h v)(P_i) = v(P_i)$ and $\tilde{v}(P_i) = v(P_i)$ for all nodes P_i.

The hybrid upwind FEM is described as follows:

Find $u_h \in V_{0h}$ such that

$$a_h(u_h, v_h) = \varepsilon(\nabla u_h, \nabla v_h) + b_h(u_h, v_h) + (\tilde{c}\tilde{u}_h, \tilde{v}) = (f, \tilde{v}_h), \qquad v_h \in V_{0h},$$

where

$$b_h(u_h, v_h) = \sum v_h(P_i) \sum \beta_{ij}(\lambda_{ij} - 1)(u_h(P_i) - u_h(P_j))$$

and

$$\beta_{ij} = \int_{\Gamma_{ij}} b n_i \, d\Gamma_{ij}, \qquad \lambda_{ij} = \tfrac{1}{2}(1 + \operatorname{sgn} \beta_{ij}).$$

According to this choice of the λ_{ij} we can state that the part B_h of the Galerkin matrix which is descended from the convection term is of non-negative type (that means, B_h is off-diagonal non-positive and the sum of the elements in each row of B_h is non-negative).

The weakly acute type of triangulation is needed to ensure that the non-negative type of the Galerkin matrix is not disturbed by the diffusion term. Moreover, one may easily prove that for $c \geqslant 0$ (or $c - \frac{1}{2}\operatorname{div} b \geqslant \alpha_0 > 0$ and sufficiently small h) the Galerkin matrix becomes an M-matrix. Thus, by this rather simple method the preservation of inverse monotony is guaranteed (and in this fact lies the importance of the hybrid upwind FEM). Regarding the localization of boundary layers, the hybrid upwind FEM is not as successful as some of the methods described in the following sections, but nevertheless local estimates have been proven:

THEOREM 1. *If* $\sigma := \max(\varepsilon, h)$ *sufficiently small,*

$$|b| \geqslant b_0 > 0, \quad |bn| \geqslant b_1 > 0 \quad on\ \Gamma^-,$$

no characteristic of the reduced equation starts from Γ^0, *and* $u \in W^{2,2}(\Omega)$ *then there exist constants* C_1, C_2 *independent of* ε, h *such that for all* $\Omega' \subset \Omega$ *with*

$$\operatorname{dist}(\Omega', \Gamma^0) \geqslant C_1 \sqrt{\sigma} |\ln \sigma|, \quad \operatorname{dist}(\Omega', \Gamma^+) \geqslant C_1 \sigma |\ln \sigma|$$

and with a uniform triangulation of $\Omega \backslash \Omega'$, *we have*

$$\|u_h - I_h u\|_{0,2,\Omega'} \leqslant C_2(\sigma + \sqrt{h}).$$

If we additionally assume $c \geqslant 0$, $u \in W^{2,\infty}(\Omega)$,

$$\operatorname{dist}(\Omega', \Gamma^0) \geqslant C_1 \sqrt{\sigma} |\ln \sigma \ln h|, \quad \operatorname{dist}(\Omega', \Gamma^+) \geqslant C_1 \sigma |\ln \sigma \ln h|.$$

then $\|u_h - I_h u\|_{0,\infty,\Omega'} \leqslant C_2 h^{1/2-\varkappa}$, *with an arbitrary* $\varkappa > 0$. *Moreover, if the whole domain* Ω *is triangulated uniformly, we get*

$$\|u_h - I_h u\|_{0,2,\Omega'} \leqslant C_2 \sigma \quad and \quad \|u_h - I_h u\|_{0,\infty,\Omega'} \leqslant C_2 \sigma^{1-\varkappa}.$$

For a more detailed presentation of the results and a sketch of the proof see [11]; some weaker results but with all details of the proof can be found in [10].

In the case $\varepsilon = 0$ there is a simple possibility to generalize the hybrid upwind FEM to higher order approximations, namely the discontinuous Galerkin method:

Let the finite space V_h consist of all functions v_h which are on each element polynomials of degree $\leqslant k$ (they need not be continuous); let

$$V_{0h} := \{v_h \in V_h,\ v_h = 0 \text{ on } \partial\Omega\}$$

For $u_h, v_h \in V_h$, let $(b\nabla u_h, v_h)$ be approximated by

$$b_h(u_h, v_h) := (b\nabla u_h, v_h) + \sum_{T \in T_h} \int_{\partial T^-} |bn|(u_h^+ - u_h^-)\, v_h\, ds$$

where T^- is the part of T on which $bn \leqslant 0$ and $u_h^{+,-}(x) = \lim_{t \to +0} u_h(x \pm tb)$ if the limit exists and $u_h^{+,-} = 0$ elsewhere.

The discrete problem now consists in finding $u_h \in V_{0h}$ such that $b_h(u_h, v_h) + (cu_h, v_h) = (f, v_h)$ for all $v_h \in V_{0h}$.

Obviously, if we choose as our triangulation the dual decomposition of Ω defined for the hybrid upwind FEM, the discontinuous Galerkin method for $k = 0$ leads to the same discretization as the hybrid upwind FEM. Regarding some properties and error estimates for the discontinuous Galerkin method see e.g. [3].

3. The streamline diffusion FEM

The streamline diffusion FEM (SDFEM) of Hughes–Brooks [1] preserves the localization properties of the hybrid upwind FEM (Th. 1) but in general not inverse monotony. Other features of SDFEM are additional control on the streamline derivative $\partial u_h/\partial b$ and high-order error estimates. Let $(\mathcal{T}_h)$ be a regular triangulation of Ω and let

$$V_h = \{v_h \in C(\bar{\Omega}) : v_h|_T \in P_k(T)\}, \qquad V_{0h} = \{v_h \in V_h : v_h|_\Gamma = 0\}$$

be spaces of piecewise polynomials of degree $k \geqslant 1$. With the perturbed bilinearform on $V_h \times V_h$

$$a_\delta(u_h, v_h) := a(u_h, v_h) + \sum_{T \in \mathcal{T}_h} \delta(L_\varepsilon u_h,\, b\cdot\nabla v_h)_T,$$

the SDFEM reads:

Find $u_h \in V_{0h}$ such that

$$a_\delta(u_h, v_h) = (f, v_h + \delta b\cdot\nabla v_h) \qquad \forall v_h \in V_{0h}.$$

Global convergence results are stated in

THEOREM 2. *Let* $c - \frac{1}{2}\operatorname{div} b \geqslant \alpha_0 > 0$ *and* $\varepsilon\delta \leqslant Ch^2$. *Then*

$$\lim_{h\to 0} |u - u_h|_{1,2,\Omega} = 0$$

for fixed $\varepsilon > 0$ *and* $u \in W^{1,2}(\Omega)$. *If additionally* $u \in W^{l+1,2}(\Omega)$ *and* $\delta = Ch$, $1 \leqslant l \leqslant k$, *then there exist a constant C independent on* ε, h *such that for* $\varepsilon \leqslant Ch$

$$\|u - u_h\|_{0,2,\Omega} + \sqrt{h}\|b\cdot\nabla(u - u_h)\|_{0,2,\Omega} + \sqrt{\varepsilon}|u - u_h|_{1,2,\Omega} \leqslant Ch^{l+1/2} u_{l+1,2\Omega}.$$

Furthermore, we have local error estimates:

THEOREM 3. *Let the assumptions of Theorem* 1 *be fulfilled and let the constants* $C_1 = C_1(l)$, $C_2 = C_2(l)$ *be sufficiently large. If* $u \in W^{l+1,2}(\Omega')$ *then*

$$\|u - u_h\|_{0,2,\Omega''} + \sqrt{h}\|b\cdot\nabla(u - u_h)\|_{0,2,\Omega''} + \sqrt{\varepsilon}|u - u_h|_{1,2,\Omega''} \leqslant Ch^{l+1/2}, \qquad 1 \leqslant l \leqslant k$$

holds for all $\Omega'' \subset \Omega'$, *with C independent on* ε, h. *If additionally* $u \in W^{l+1,\infty}(\Omega')$, *then*

$$\|u - u_h\|_{0,\infty,\Omega''} \leqslant Ch^{l-1/2}.$$

For a proof of the error estimates in integral norms, see [9]. The $W^{1,2}$-convergence result and the (non-optimal) local L_∞-estimates can be found in [6]. Note that the results of Theorems 2, 3 remain valid in case of $\varepsilon = 0$ (with the relaxed condition $0 \leqslant l \leqslant k$).

Although the question of optimal local L_∞-estimates seems to be open for SDFEM, in a recent paper [4] an improved estimate was given in the special case $k = 1$ for a modified SDFEM. The diffusion term $\varepsilon(\nabla u_h, \nabla v_h)$ is weighted in the crosswind direction b^T for small ε.

With

$$\tilde{a}_\delta(u_h, v_h) := a_\delta(u_h, v_h) + (\bar{\varepsilon} - \varepsilon)\left(\frac{b^T \nabla u_h}{|b^T|}, \frac{b^T \nabla v_h}{|b^T|}\right),$$

$$\bar{\varepsilon} := \begin{cases} \varepsilon & \text{if } \varepsilon \geqslant h^{3/2}, \\ h^{3/2} & \text{if } \varepsilon < h^{3/2}, \end{cases}$$

this method reads:

Find $\tilde{u}_h \in V_{0h}$ such that

$$\tilde{a}_\delta(u_h, v_h) = (f, v_h + \delta\, b \cdot \nabla v_h) \qquad \forall v_h \in V_{0h}.$$

Under similar assumptions to those of Theorem 3, we have in the special case $b = (1,0)^T$ the estimate

$$\|u - u_h\|_{0,\infty,\Omega''} \leqslant C h^{5/4} |\ln h|.$$

Furthermore, the thickness of a numerical boundary layer at Γ^0 or of interior layers along a streamline is shown to be of order $h^{3/4}|\ln h|$ rather than $\sqrt{h}\,|\ln h|$. Unfortunately, the estimation technique of [4], which is essentially based on the crosswind correction of the diffusion term, does not allow analogous results for the original SDFEM.

4. Asymptotically fitted FEM's

In order to achieve information on convergence properties in the boundary layers, global error estimates which are valid uniformly with respect to ε (say, for $\varepsilon \leqslant Ch$) are desirable. For the modified SDFEM, under similar assumptions to those of Theorem 3 and for simple boundary and interior layer structures with $\varepsilon\|\Delta u\|_{0,1,\Omega} + |u|_{1,2,\Omega} \leqslant C$, we have

$$\|u - u_h\|_{0,1,\Omega} < C\sqrt{h}|\ln^{5/2} h| \qquad \text{if } \varepsilon \leqslant h^{3/2}.$$

A well-known way to derive uniformly in ε converging methods is "asymptotical fitting". Recently, Schieweck [12] proposed an exponential – fitted Galerkin FEM with adding some boundary layer – like test functions to V_{0h}. In case of $b_1, b_2 > 0$ on $\Omega = (0,1) \times (0,1)$ and $k = 1$ he derived the global estimate

$$\|u - u_h\|_{0,2,\Omega} + \sqrt{\varepsilon}|u - u_h|_{1,2,\Omega} < C\left(\sqrt{\varepsilon} + \sqrt{h} + (\varepsilon/h)^m\right)$$

with $m > 0$, arbitrary. Unfortunately, this method is too complicated in the case of a complex geometry for $\Omega \subset \boldsymbol{R}^N, N \geqslant 2$.

Another way consists in asymptotical-fitting of boundary values. The method is based on the observation that any unrefined mesh cannot resolve for $\varepsilon < h$ the downstream layer at Γ^+, and for $\varepsilon < h^{3/2}$, the characteristic layers along Γ^0, respectively. The idea is to replace step by step the sharp layers by more smooth layers. More precisely, let

$$V_h^\varepsilon := \begin{cases} \{v_h \in V_h : v_h|_{\Gamma^- \cup \Gamma^0} = 0\} & \text{if } Ch \geqslant \varepsilon \geqslant h^\varkappa \\ \{v_h \in V_h : v_h|_{\Gamma^-} = 0\} & \text{if } h^\varkappa > \varepsilon \end{cases}$$

Then the asymptotically fitted streamline diffusion FEM (ASDFEM): Find $\hat{u}_h \in V_h^\varepsilon$ such that

$$a_\delta(\hat{u}_h, v_h) = (f, v_h + \delta b \nabla v_h) \qquad \forall v_h \in V_h^\varepsilon,$$

is an approximation of the solution $\hat{u}$ of (*) with

$$(*) \qquad \begin{cases} L_\varepsilon \hat{u} = f \quad \text{in } \Omega \\ \hat{u} = u^+ \quad \text{if } Ch \geqslant \varepsilon \geqslant h^\varkappa \quad \text{with } u^+|_{\Gamma^- \cup \Gamma^0} = 0, \left.\dfrac{\partial u^+}{\partial n}\right|_{\Gamma^+} = 0 \\ \hat{u} = u^0 \quad \text{if } h^\varkappa > \varepsilon \quad \text{with } u^0|_{\Gamma^-} = 0, \left.\dfrac{\partial u^0}{\partial n}\right|_{\Gamma^+ \cup \Gamma^0} = 0. \end{cases}$$

A global error estimate is given by

THEOREM 4. *Let* $c - \frac{1}{2}\operatorname{div} b \geqslant \alpha_0 > 0$ *and* $\varepsilon\delta \leqslant Ch^2$. *Furthermore assume that*

$$\|u - u^+\|_{0,2} \leqslant C\sqrt{\varepsilon}, \qquad |u^+|_{2,2} \leqslant C\varepsilon^{-3/4},$$
$$\|u - u^0\|_{0,2} \leqslant C\varepsilon^{1/4}, \qquad |u^0|_{2,2} < C\varepsilon^{-1/2},$$

and that $u_0 \in W^{r,2}(\Omega)$, *where* u_0 *is the solution of*

$$L_0 u_0 = b \cdot \nabla u_0 + c u_0 = f, \qquad u_0|_\Gamma = 0 \textit{ with } r \geqslant 1.$$

With $\varkappa = 3/2$, *we have for* ASDFEM *the estimate*

$$\|u - u_h\|_{0,2,\Omega} \leqslant C \min\{\varepsilon^{1/2} + h^{3/2}\varepsilon^{-3/4};\ \varepsilon^{1/4} + h^{3/2}\varepsilon^{-1/2};\ \varepsilon^{1/4} + h^{\bar{r}-1/2} + \varepsilon h^{-2}\},$$
$$\bar{r} = \min\{r; k+1\}.$$

For $\varepsilon \to 0$ (more precisely, if $\varepsilon \leqslant h^{\max(\bar{r}+3/2, 4\bar{r}-2)}$, the higher-order estimate

$$\|u - u_h\|_{0,2} \leqslant C h^{\min(r,k+1)-1/2}$$

is valid.

The localization results of Theorem 3 remain valid for ASDFEM. Hence, ASDFEM results in improved convergence properties in the boundary layers at $\Gamma^+ \cup \Gamma^0$. This theoretical superiority of ASDFEM to SDFEM is reflected by numerical calculations ([6]).

A survey on asymptotically fitted Galerkin and streamline diffusion methods is given in [7].

5. Streamline diffusion FEM with discontinuity-capturing

Despite the success of SDFEM, it does not exclude overshooting and undershooting (sometimes even restricted oscillations) about sharp layers. Although ASDFEM reduces such instabilities in boundary layers, this is not the case for interior layers ("shocks").

Linear and monotonicity-preserving methods (cf. Sect. 2) which are a candidate to preclude interior oscillations are at most first-order accurate and overdiffusive. Mizukami and Hughes [8] proposed a nonlinear method satisfying the maximum principle and based on triangles and piecewise linear elements. Unfortunately, extensions to higher-order-element methods are not apparent.

Recently, Hughes et al. [2] developed an extension of the SDFEM including a discontinuity-capturing term. Leb $b_{\|}$ denote the projection of b onto ∇u_h, that is

$$b_{\|} := \begin{cases} \dfrac{(b\cdot\nabla u_h)}{|\nabla u_h|^2}\nabla u_h & \text{if } \nabla u_h \neq 0 \\ 0, & \text{if } \nabla u_h = 0. \end{cases}$$

Note that $b_{\|}\cdot\nabla u_h = b\cdot\nabla u_h$.

With test functions of the form

$$\tilde{v}_h = v_h + \delta_1 b\cdot\nabla v_h + \delta_2 b_{\|}\cdot\nabla v_h$$

the proposed method reads:

$$\begin{aligned} \bar{a}_\delta(u_h, v_h) &:= a(u_h, v_h) + \sum_{T\in\mathcal{T}_h} (L_\varepsilon u_h, \delta_1 b\cdot\nabla v_h + \delta_2 b_{\|}\cdot\nabla v_h)_T \\ &= (f, v_h + \delta_1 b\cdot\nabla v_h + \delta_2 b_{\|}\cdot\nabla v_h). \end{aligned}$$

The additional term engenders control on gradients in the direction ∇u_h and increases the robustness of SDFEM, as it is known from numerical experiments ([8], [5]). A rigorous mathematical analysis of the method is in general an open problem. A first step toward this problem is paper [5] concerning e.g. Cauchy's problem for Burger's equation in one space dimension. The authors proposed the shock capturing modification to allow a real break through in the practical use of streamline methods for compressible flow.

References

[1] T. J. R. Hughes and A. Brooks, *A multidimensional upwind scheme with no crosswind diffusion*, in AMD v. 34: Finite element methods for convection dominated flows (T. J. R. Hughes ed.), ASME, New York 1979, 19–35.

[2] T. J. R. Hughes, M. Mallet and A. Mizukami, *A new finite element formulation for computational fluid dynamics: II.* Beyond SUPG, Comp. Meth. Appl. Mech. Engrg. 54 (1986), 341–355.

[3] C. Johnson and J. Pitkäranta, *An analysis of the discontinuous Galerkin method for a scalar hyperbolic equation*, Math. Comp. 46 (1986) 173, 1–26.

[4] C. Johnson, A. H. Schatz and L. B. Wahlbin, *Crosswind smear and pointwise errors in streamline diffusion finite element methods*, Math. Comp. 49 (1987), 25–38.

[5] C. Johnson and A. Szepessy, *On the convergence of streamline diffusion finite element methods for hyperbolic conservation laws*, in: AMD v. 78: Numerical methods for compressible flows – Finite difference, element and volume techniques, ASME, New York 1986, 75–91.

[6] G. Lube, *An asymptotically fitted finite element method for convection dominated convection-reaction-diffusion problems*, Preprint Techn. Universität Magdeburg, DDR 1988.

[7] –, *Uniform in ε discretization error estimates for convection dominated convection-diffusion problems*, RAIRO, M^2AN 22 (1988), 517–538.

[8] A. Mizukami and T. J. R. Hughes, *A Petrov-Galerkin finite element method for convection-dominated flows — an accurate upwinding technique for satisfying the maximum principle*, Comput. Methods Appl. Mech. Engrg. 50 (1985), 181–193.

[9] U. Nävert, *A finite element method for convection-diffusion problems*, Ph. D. Thesis, Chalmers Univ. of Technology, Göteborg, Sweden 1982.

[10] U. Risch, *An upwind* FEM *for singularly perturbed elliptic problems and local estimates in the L_∞-norm*, to appear in RAIRO, M^2AN.

[11] –, *Error estimates for the hybrid upwind* FEM *for convection dominated problems*, Preprint Math-3, Magdeburg 1987.

[12] F. Schieweck, *Eine asymptotisch angepaßte Finite-Element-Methode für singulär gestörte elliptische Randwertaufgaben*, Dissertation, Techn. Hochschule Magdeburg, DDR 1986.

Presented to the Semester
Numerical Analysis and Mathematical Modelling
February 25 – May 29, 1987