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TENSOR PRODUCTS OF CLIFFORD MODULES AND LINEAR MAXIMAL COHEN-MACAULAY MODULES ON QUADRICS

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Introduction

The purpose of this note is to make more explicit, at least for specific fields, some of the results in the paper [3] of Buchweitz–Eisenbud–Herzog. There a functor F is defined from the category $\mathcal{M}$ of $\mathbf{Z}/2\mathbf{Z}$-graded modules over the Clifford algebra C of a quadratic form f to the category $\mathcal{L}$ of linear maximal Cohen–Macaulay modules (MCM-modules) over the hypersurface ring R defined by the quadratic form f, and it is shown that this functor establishes an equivalence of categories. Using this result, it is shown in [3] that there are at most two nonisomorphic indecomposable, linear MCM-modules over R which are syzygy modules of each other, and that their rank is determined in terms of invariants of the quadratic form f. In Section 1 of this paper we briefly recall these results.

The main object of this paper is to give an explicit description of the R-representation of the unique indecomposable linear MCM-modules M and $\Omega^1(M)$ over R. Thus, if $A_1 = k[X_1, \ldots, X_n]$ and $R = A_1/f$, we want to determine the matrix of linear forms α such that

$$0 \to A_1^m \xrightarrow{\alpha} A_1^m \to M \to 0.$$

The description will be given inductively in the following sense: Suppose M is an indecomposable linear MCM-module over $k[X_1, \ldots, X_n]/f$, and N is an indecomposable linear MCM-module over $k[Y_1, \ldots, Y_m]/g$. Set $A = k[X_1, \ldots, X_n, Y_1, \ldots, Y_m]$, then $f+g$ is an element of A, and the syzygy modules $\Omega^i_{A/f+g}(M \otimes_A N)$, are linear MCM-modules over $A/f+g$.

Let W and $\Omega^1(W)$ be the indecomposable linear MCM-modules over $A/f+g$. If rank $W = \operatorname{rank} \Omega^1_{A/f+g}(M \otimes_A N)$, (which can easily be examined

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using 1.3) then, since there are at most two indecomposable linear MCM-modules over R (see [3], Prop. 3.1), W must be isomorphic, up to a shift in the grading, to $\Omega^i_{A/f+g}(M \otimes_A N)$ for some $i \in \{1, 2\}$. It is then easy to derive the presentation of $\Omega^i_{A/f+g}(M \otimes_A N)$ from the presentations of M and N.

Let $F\colon \mathcal{M} \to \mathcal{L}$ be the functor establishing the equivalence between the category of Clifford modules and the category of linear MCM-modules. It is shown in Theorem 2.2 that

$$F(M \hat{\otimes} N) \cong \Omega^2_{A/f+g}(F(M) \otimes_A F(N))(+2), \tag{1}$$

where $\hat{\otimes}$ denotes the twisted tensor product of the Clifford modules M and N as defined in § 2, and where $(+2)$ denotes a shift in the grading by $+2$. The details can be found in Section 2.

In the last section we study, motivated by (1), the twisted tensor products of Clifford modules in detail for finite fields of characteristic $\neq 2$, p-adic fields with $p \neq 2$, and the real numbers. (Other fields for which the classification of the quadratic forms is well understood can be treated similarly.) For all the fields considered we show that there is a finite set of indecomposable Clifford modules such that any other indecomposable Clifford module is a tensor product of these. More precisely, if k is any of the above fields, then there exist nondegenerate quadratic forms $f_1, \ldots, f_n$ over k with the following property: Let C_i be the Clifford algebra of f_i, and let M_i be the indecomposable Clifford module over C_i (which is unique up to isomorphisms and shifts). Then, if there is given an arbitrary nondegenerate quadratic form f over k with Clifford algebra C and Clifford module M, there exist integers $n_i \in \mathbf{N}$, and an isomorphism $\varphi\colon \hat{\bigotimes} C_i^{n_i} \to C$ of $\mathbf{Z}/2\mathbf{Z}$-graded k-algebras such that

$$_{\varphi}M \cong \hat{\bigotimes} M_i^{n_i}, \tag{2}$$

where $_{\varphi}M$ is the module M considered as $\hat{\bigotimes} C_i^{n_i}$-module via φ.

Formula (2) can be made very explicit for all the considered fields, see Propositions 3.3–3.5. As a consequence we obtain recursion formulas for the presentations of the linear MCM-modules in terms of a finite set of such presentations.

The case of finite fields (including characteristic 2) has been worked out by my student Jutta Lachfeld as part of her Diplomarbeit.

1. Review on maximal Cohen–Macaulay modules on quadrics

In this section we recall some of the results of [3] which are needed in later sections.

Let $A = k[X_1, \ldots, X_n]$ be a polynomial ring over the field k, and let $f \in A$ be a quadratic form. A graded module M over the hypersurface ring $R = A/f$ is an MCM-module if and only if it has a homogeneous free A-resolution

$$0 \to F_1 \xrightarrow{\alpha} F_0 \to M \to 0$$

M is called a *linear MCM-module* (or an *Ulrich module*) if α is a degree 1 map, which is the case if and only if $\mu(M) = e(M)$, see [2]. Here $\mu(M)$ denotes the minimal number of generators, and $e(M)$ the multiplicity of M.

We denote by $\mathscr{L}$ the full subcategory of the category of all graded R-modules, which consists of the linear MCM-modules whose generators are all of degree 0.

Now let $C := C(f)$ be the Clifford algebra of f. C is a $\mathbf{Z}/2\mathbf{Z}$-graded algebra, $C = C_0 \oplus C_1$. We denote by $\mathscr{M}$ the category of $\mathbf{Z}/2\mathbf{Z}$-graded modules $M = M_0 \oplus M_1$ with $\dim_k M < \infty$, and by $\mathscr{M}_0$ the category of C_0-modules of finite k-dimension. We call the objects in $\mathscr{M}$ *Clifford modules* for f. A $\mathbf{Z}/2\mathbf{Z}$-graded k-vector space M will be simply called a Clifford module over k if it is a Clifford module for some quadratic form f over k.

THEOREM 1.1 ([3], Th. 2.1). *The categories $\mathscr{L}$, $\mathscr{M}$ and $\mathscr{M}_0$ are all equivalent.*

A similar result holds for hypersurface rings defined by forms of higher degree, see [1], Theorem 3.9.

We briefly describe the functors which establish the equivalence of these categories: The equivalence of $\mathscr{M}_0$ and $\mathscr{M}$ is given by the so-called Atiyah–Bott–Shapiro equivalence $G: \mathscr{M}_0 \to \mathscr{M}$, where $G(M) = C \otimes_{C_0} M$. (The inverse of G is just the functor assigning to $N = N_0 \oplus N_1 \in \mathscr{M}$ the module $N_0 \in \mathscr{M}_0$.)

We now describe $F: \mathscr{M} \to \mathscr{L}$. Let $V = \bigoplus_{i=1}^n k e_i$ be an n-dimensional vector space with basis $(e_1, \ldots, e_n)$. We set $X_i := e_i^*$ for the dual elements of the e_i in the dual vector space V^*. We identify the polynomial ring A with the symmetric algebra $S(V^*)$ of V^*. The Clifford algebra of f is then just the tensor algebra $T(V)$ of V modulo the two-sided ideal generated by the elements $x \otimes x - f(x)$ for all $x \in V$. Here we have set $f(x) := f(x_1, \ldots, x_n)$ for $x = \sum x_i e_i$.

It is well-known that the natural inclusion $V \to T(V)$ induces an inclusion $V \to C$, whose image actually lies in C_1. Therefore, if we are given a Clifford module $M = M_0 \oplus M_1$, the multiplication induces k-linear maps

$$V \otimes M_0 \to M_1 \quad \text{and} \quad V \otimes M_1 \to M_0$$

whose adjoints define degree 1 maps

$$M_0 \otimes A \xrightarrow{\alpha} M_1 \otimes A \quad \text{and} \quad M_1 \otimes A \xrightarrow{\beta} M_0 \otimes A. \tag{1}$$

It turns out that (α, β) is a matrix factorization, that is,

$$f \cdot \mathrm{id} = \alpha \cdot \beta = \beta \cdot \alpha. \tag{2}$$

According to Eisenbud [4], this implies that $\operatorname{coker} \alpha$ and $\operatorname{coker} \beta$ are MCM-modules over R, and

$$\operatorname{coker} \beta \cong \Omega_R^1(\operatorname{coker} \alpha)(+1),$$

$$\operatorname{coker} \alpha \cong \Omega_R^1(\operatorname{coker} \beta)(+1).$$

It is clear that both modules are linear, and we set

(3) $$F(M) := \operatorname{coker} \alpha.$$

It follows that

(4) $$F(M(-1)) \cong \Omega_R^1(F(M)).$$

We call f *regular* if R has only an isolated singularity. It is shown in [3], Theorem 1.1 that $C_0(f)$ is semisimple if and only if f is regular. If $\operatorname{char} k \neq 2$ (which we mostly assume) then f is regular if and only if f is nondegenerate. The semisimplicity of $C_0(f)$ for a regular quadratic form f and the equivalence of the categories $\mathscr{M}_0$ and $\mathscr{M}$ is used in [3] (see proof of 3.1) to prove

THEOREM 1.2. *Up to isomorphisms and shifts in the grading there is a unique indecomposable Clifford module for f.*

As a consequence of 1.1 and 1.2 one concludes that there are at most two indecomposable linear MCM-modules over R, and they are first syzygy modules of each other. We denote this "unique" indecomposable Clifford module of f by $M(f)$.

In Proposition 3.2 of [3] the k-dimension of $M(f)$ is calculated. Notice that $\mu(F(M(f))) = 1/2 \cdot \dim_k M(f)$.

We will reformulate 3.2 of [3] to make it more accessible for our purposes, and assume from now on that $\operatorname{char} k \neq 2$ and that f is nondegenerate. We first describe the invariants of a quadratic form which determine $\dim_k M(f)$. They are the *rank*, the *discriminant* and the *Witt invariant* of f.

If $f = \sum_{i=1}^n a_i X_i^2$, $a_i \in k^*$, as we may assume, then $\operatorname{rank} f = n$, and the discriminant $\delta(f)$ of f is the class of $(-1)^{[n/2]} a_1 \cdot a_2 \cdot \ldots \cdot a_n$ in k^*/k^{*2}. To describe the Witt invariant we recall (see Scharlau [6], § 9, Thm. 2.10) that $C(f)$ is a central simple k-algebra if n is even, and $C_0(f)$ is a central simple k-algebra if n is odd. The Witt invariant is then just the class of $C(f)$ (resp. $C_0(f)$) in the Brauer group $B(k)$ if n is even (resp. if n is odd). We denote the Witt invariant of f by $c(f)$, and set

(5) $$\operatorname{inv}(f) := (\operatorname{rank} f, \delta(f), c(f)).$$

There is a unique central division k-algebra D such that

$$c(f) = [D] \quad \text{in} \quad B(k).$$

$\dim_k D$ is a square, and we set $\deg c(f) := (\dim_k D)^{1/2}$.

If $a \in B(k)$ we denote by a^* its image under the canonical map $B(k) \to B(k(\sqrt{\delta(f)}))$. Thus, if $a = [A]$, then $a^* = [A \otimes_k k(\sqrt{\delta(f)})]$.

Using these notations, we get

PROPOSITION 1.3. *Let f be a quadratic form with* $\operatorname{inv}(f) = (n, \delta, c)$, *then*

$$\dim_k M(f) = \begin{cases} 2^{(n+1)/2}\deg c, & \text{if } n \equiv 1 \bmod 2, \\ 2^{n/2}\deg c, & \text{if } n \equiv 0 \bmod 2, \text{ and } \delta = 1, \\ 2^{(n+2)/2}\deg c^*, & \text{if } n \equiv 0 \bmod 2, \text{ and } \delta \neq 1. \end{cases}$$

Proof. $C_0(f)$ is a central simple k-algebra if n is odd, and if n is even then $C_0(f) \cong A \otimes_k k(\sqrt{\delta})$, where A is a central simple k-algebra (see proof of 2.10 in [6]).

In the proof of 3.2 in [3] it is shown that

$$\dim_k M(f) = \begin{cases} 2^{(n+1)/2}\deg[C_0(f)], & \text{if } n \equiv 1 \bmod 2, \\ 2^{n/2}\deg[A], & \text{if } n \equiv 0 \bmod 2, \text{ and } \delta = 1, \\ 2^{(n+2)/2}\deg[A \otimes_k k\sqrt{\delta})], & \text{if } n \equiv 0 \bmod 2, \text{ and } \delta \neq 1. \end{cases}$$

Here [B] denotes the class of an algebra in $B(k)$ (respectively in $B(k(\sqrt{\delta}))$).

If $n \equiv 1 \bmod 2$, then $c(f) = [C_0(f)]$, by definition. Let $a = [A]$, so that $a^* = [A \otimes_k k(\sqrt{\delta})]$. It remains to be shown that $c = a$ if $\delta = 1$, and $c^* = a^*$ if $\delta \neq 1$.

Now $C(f) \cong A \otimes ((-1)^{m-1} a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1}, a_n)$, where $m = n/2$ (see proof of 2.10 in [6]). Here (c, d) denotes a quaternion algebra. Its class in $B(k)$ will be again denoted by (c, d).

Using the fact that $(c, d) = (-cd, d)$ in $B(k)$ the above isomorphism yields $c(f) = a \cdot ((-1)^m a_1 \cdot \ldots \cdot a_n, a_n)$ in $B(k)$, and $c(f)^* = a^* \cdot ((-1)^m a_1 \cdot \ldots \cdot a_n, a_n)^*$ in $B(k\sqrt{\delta}))$.

Now if $\delta = 1$, then $(-1)^m a_1 \cdot \ldots \cdot a_n \in k^{*2}$, and thus $((-1)^m a_1 \cdot \ldots \cdot a_n, a_n) = 1$. On the other hand, $((-1)^m a_1 \cdot \ldots \cdot a_n, a_n)^* = 1$, always, since $(-1)^m a_1 \cdot \ldots \cdot a_n \in k(\sqrt{\delta})^{*2}$.

2. Tensor products of Clifford modules

Let $A = k[X_1, \ldots, X_n]$ be a polynomial ring over a field, and let $f \in A$ be a quadratic form. As in Section 1 we consider A as the symmetric algebra $S(V^*)$ of the dual of the vector space $V = \bigoplus_{i=1}^n ke_i$, where $X_1, \ldots, X_n$ is the dual basis of $e_1, \ldots, e_n$.

Given a second quadratic form $g \in B = k[X_1, \ldots, X_m] = S(W^*)$, we define the (direct) sum $f+g$ of f and g, as the sum of $f \otimes 1$ and $1 \otimes g$ considered as elements in $A \otimes_k B \cong k[X_1, \ldots, X_n, Y_1, \ldots, Y_m]$, so that $f+g = f(X_1, \ldots, X_n) + g(Y_1, \ldots, Y_m)$.

We will identify the Clifford algebra of $f+g$ with the graded tensor product $C(f) \hat{\otimes} C(g)$ of the Clifford algebras $C(f)$ and $C(g)$ which is defined to be the graded tensor product of the underlying graded vector spaces, and whose multiplication is given by

(1) $$(a \hat{\otimes} b) \cdot (c \hat{\otimes} d) = (-1)^{(\deg b)(\deg c)} a \cdot c \hat{\otimes} b \cdot d.$$

Now $V \oplus W$ can be naturally embedded into $C(f) \hat{\otimes} C(g)$ by the map $j(x+y) = x \otimes 1 + 1 \otimes y$ for all $x \in V$ and all $y \in W$, and then $C(f) \hat{\otimes} C(g)$ is generated over k by the image of j. Using the universal properties of graded tensor products and of Clifford algebras one finds a natural isomorphism

$$\alpha\colon\ C(f) \hat{\otimes} C(g) \to C(f+g) \tag{2}$$

with $\alpha(x \otimes 1 + 1 \otimes y) = x + y$.

Similarly one defines the graded tensor product of Clifford modules: If $M = M_0 \oplus M_1$ is a Clifford module for f, $N = N_0 \oplus N_1$ a Clifford module for g, then one defines the graded tensor product $M \hat{\otimes} N$ as a Clifford module over $C(f) \hat{\otimes} C(g)$ by setting

$$(M \hat{\otimes} N)_0 = M_0 \otimes_k N_0 \oplus M_1 \otimes_k N_1 \tag{3}$$

and

$$(M \hat{\otimes} N)_1 = M_0 \otimes_k N_1 \oplus M_1 \otimes_k N_0.$$

The operation of $C(f) \hat{\otimes} C(g)$ on $M \hat{\otimes} N$ is defined by

$$(a \hat{\otimes} b) \cdot (m \hat{\otimes} n) = (-1)^{(\deg b)(\deg m)} a \cdot m \hat{\otimes} b \cdot n.$$

Using the natural isomorphism (2), we will consider henceforth $M \hat{\otimes} N$ as a $C(f+g)$-module.

For the rest of this section we will only consider fields of $\operatorname{char} k \neq 2$, and all our quadratic forms will be assumed to be nondegenerate.

Given such a quadratic form f, there is, by 1.2, up to shifts and isomorphisms a unique indecomposable Clifford module which we represent by a module denoted by $M(f)$. We also introduce the notation $M \approx N$ to indicate that M and N are Clifford modules which are isomorphic up to shifts. The question arises when

$$M(f) \hat{\otimes} M(g) \approx M(f+g)?$$

This question has a simple answer: It follows from the formulas (1) that $\dim_k M(f)$ is a power of 2, since the element $c(f)$ can be represented in the Brauer group as a product of quaternion algebras. Therefore we set

$$l(f) = \log_2 \dim_k M(f). \tag{4}$$

As a trivial consequence of 1.2 we now get

Proposition 2.1. *Let f, g be quadratic forms. The following conditions are equivalent*:

(a) $M(f) \hat{\otimes} M(g) \approx M(f+g)$,
(b) $l(f) + l(g) = l(f+g)$.

We now explain the connection with linear MCM-modules: As in the proof of 1.1 we denote by F the functor from the category $\mathcal{M}$ of Clifford modules to the category $\mathscr{L}$ of linear MCM-modules. Now let $A_1 = S(V^*)$, $A_2 = S(W^*)$ and $A = A_1 \otimes A_2$, and let $f \in A_1$ and $g \in A_2$ be quadratic forms.

THEOREM 2.2. *Let M be a Clifford module for f, N a Clifford module for g, then*

$$F(M \hat{\otimes} N) \cong \Omega^2_{A/f+g}(F(M) \otimes_A F(N))(+2).$$

(The shift by 2 in this formula is needed since $F(M \hat{\otimes} N)$ has all its generators in degree 0, while the second syzygy-module of $F(M) \otimes_A F(N)$ has all its generators in degree 2.)

The proof of this theorem will follow from the next lemma which is an immediate consequence of the definitions (1), (2) in Section 1, and (3) of Section 2.

LEMMA 2.3. *Let (α, β) be the matrix factorization of f associated with the Clifford module M, and let (γ, δ) be the matrix factorization of g associated with the Clifford module N. Then (η, σ) is the matrix factorization of $f+g$ associated with $M \hat{\otimes} N$, where*

$$\eta = \begin{pmatrix} \mathrm{id}_{M_0 \otimes A} \otimes \gamma, & \alpha \otimes \mathrm{id}_{N_0 \otimes A} \\ \beta \otimes \mathrm{id}_{N_1 \otimes A}, & -\mathrm{id}_{M_1 \otimes A} \otimes \delta \end{pmatrix}$$

and

$$\sigma = \begin{pmatrix} \mathrm{id}_{M_0 \otimes A} \otimes \delta, & \alpha \otimes \mathrm{id}_{N_1 \otimes A} \\ \beta \otimes \mathrm{id}_{N_0 \otimes A}, & -\mathrm{id}_{M_1 \otimes A} \otimes \gamma \end{pmatrix}.$$

In terms of matrix factorizations this lemma says that, whenever the conditions of 2.1 are satisfied, and (α, β) (resp. (γ, δ)) is the matrix factorization of minimal size for f (resp. for g), then (η, σ), as defined in the Lemma, is the minimal matrix factorization for $f+g$.

Proof of 2.2. We use the notations of 2.3. According to § 1, (1), (2), and (3) the modules $F(M)$ and $F(N)$ have the presentations

$$0 \to M_0 \otimes_k S(V^*) \xrightarrow{\alpha} M_1 \otimes_k S(V^*) \to F(M) \to 0$$

and

$$0 \to N_0 \otimes_k S(W^*) \xrightarrow{\gamma} N_1 \otimes_k S(W^*) \to F(N) \to 0.$$

We may consider $F(M)$ and $F(N)$ via the canonical epimorphisms $A = S(V^*) \otimes_k S(W^*) \to S(V^*)$ and $A \to S(W^*)$ as A-modules. The tensor product $F(M) \otimes_A F(N)$ has then the following free A-resolution

$$0 \to F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \to F(M) \otimes_A F(N) \to 0$$

with

$$\varphi_1 = \begin{pmatrix} \alpha \otimes \mathrm{id}_{N_1 \otimes A} \\ -\mathrm{id}_{M_1 \otimes A} \otimes \gamma \end{pmatrix} \quad \text{and} \quad \varphi_2 = (\mathrm{id}_{M_0 \otimes A} \otimes \gamma, \alpha \otimes \mathrm{id}_{N_0 \otimes A}).$$

Since $f+g$ lies in the annihilator of $F(M)\otimes_A F(N)$ one obtains, according to Eisenbud [4], the $A/f+g$-resolution of $F(M)\otimes F(N)$ as the associated total complex of the following double complex

$$\begin{array}{ccccccc} 0 \to F_2\otimes \bar{A} & \xrightarrow{\bar\varphi_2} & F_1\otimes \bar{A} & \xrightarrow{\bar\varphi_1} & F_0\otimes \bar{A} \to 0 \\ \bar\psi_2 \uparrow & & \bar\psi_1 \uparrow & & \\ \dots \to F_1\otimes \bar{A} & \xrightarrow[\bar\varphi_1]{} & F_0\otimes A & & \\ \uparrow & & & & \\ \dots \to F_0\otimes \bar{A}. & & & & \end{array} \tag{5}$$

Here $\bar{A} = A/f+g$. $\bar\varphi_i = \varphi_i\otimes\bar{A}$, and $\bar\psi_i = \psi_i\otimes\bar{A}$ with maps $\psi_1\colon F_0\to F_1$ and $\psi_2\colon F_1\to F_2$ such that $\varphi_1\circ\psi_1 = \psi_1\circ\varphi_1 = (f+g)\cdot\mathrm{id}_{F_0}$ and $\varphi_2\circ\psi_2 = \psi_2\circ\varphi_2 = (f+g)\cdot\mathrm{id}_{F_2}$.

It is clear that we can choose

$$\psi_1 = (\beta\otimes\mathrm{id}_{N_1\otimes A},\ -\mathrm{id}_{M_1\otimes A}\otimes\delta) \quad\text{and}\quad \psi_2 = \begin{pmatrix}\mathrm{id}_{M_0\otimes A}\otimes\delta\\ \beta\otimes\mathrm{id}_{N_0\otimes A}\end{pmatrix}.$$

Now $\Omega^2_{A/f+g}(F(M)\otimes_A F(N)) \cong \mathrm{Im}\begin{pmatrix}\bar\varphi_2\\ \bar\psi_1\end{pmatrix}$, and it follows from the complex (5) that this module shifted by 2 has the A-free presentation

$$0\to G_1 \xrightarrow{\sigma} G_0 \to \Omega^2_{A/f+g}(F(M)\otimes_A F(N))(2)\to 0$$

with

$$\sigma = (\psi_1, \varphi_1) = \begin{pmatrix}\mathrm{id}_{M_0\otimes A}\otimes\delta, & \alpha\otimes\mathrm{id}_{N_1\otimes A}\\ \beta\otimes\mathrm{id}_{N_0\otimes A}, & -\mathrm{id}_{M_1\otimes A}\otimes\gamma\end{pmatrix}.$$

Now 2.3 and formulas (3) and (4) of § 1 imply the assertion.

3. Generating sets of Clifford modules

Suppose f and g are equivalent quadratic forms; then their Clifford algebras are isomorphic. Let $\varphi\colon C(f)\to C(g)$ be such an isomorphism, and let $M(f)$ (respectively $M(g)$) be the indecomposable Clifford modules of f (respectively of g). Then by the uniqueness of these modules we have $M(f)\approx M(g)$, where in this section we use the symbol "$\approx$" to indicate that there is an isomorphism φ: $C(f)\to C(g)$ such that the modules $M(f)$ and ${}_\varphi M(g)$ (which is $M(g)$ considered as a $C(f)$-module via φ) are isomorphic up to a shift in the grading. We therefore may talk about the Clifford module of the equivalence class of a quadratic form f.

We say that a field k has the *property C* if the quadratic forms over k are classified by the rank, the discriminant and the Witt invariant. Such fields are for instance the p-adic fields, the finite fields and the nonreal algebraic number fields. A general characterization of such fields is given in Theorem 14.5 in [6].

Now suppose that k is a field with the property C. If f is a quadratic form over k with $\operatorname{inv}(f) = (n, \delta, c)$ (see § 1, (5)), then we write $M(n, \delta, c)$ for $M(f)$, since $M(f)$ depends only on $\operatorname{inv}(f)$.

We call a set $\mathcal{L} = \{M_1, \ldots, M_n\}$ of indecomposable Clifford modules over k a *generating set* of Clifford modules for k if for any other indecomposable Clifford module M over k there exist integers m_i such that

$$M \approx M_1^{m_1} \hat{\otimes} M_2^{m_2} \hat{\otimes} \ldots \hat{\otimes} M_n^{m_n}.$$

For certain fields with the property C we want to describe such generating sets. To demonstrate the method we consider a trivial example: Assume k is algebraically closed, then the rank determines the equivalence class of a quadratic form, and thus for each $n \in \mathbf{N}$ we have an indecomposable Clifford module $M(n)$ belonging to the class of quadratic forms of rank n. It follows from 1.3 that

$$\dim_k M(n) = \begin{cases} 2^{(n+1)/2}, & \text{if } n \equiv 1 \bmod 2, \\ 2^{n/2}, & \text{if } n \equiv 0 \bmod 2. \end{cases}$$

Applying 2.1 we find that

$$M(n) \approx \begin{cases} \hat{\otimes}^{n/2} M(2), & \text{if } n \equiv 0 \bmod 2. \\ \hat{\otimes}^{[n/2]} M(2) \hat{\otimes} M(1), & \text{if } n \equiv 1 \bmod 2, \end{cases}$$

Notice that $M(2)$ and $M(1) \hat{\otimes} M(1)$ are not isomorphic, since $\dim_k M(2) = 2$, while $\dim_k M(1) \otimes M(1) = 4$. Thus we see that in this simple case $\{M(1), M(2)\}$ is a generating set of the Clifford modules over k.

The above formulas imply that

$$M(n+2) \approx M(n) \otimes M(2) \quad \text{for all } n \geqslant 1.$$

In terms of matrix factorizations this isomorphism says that if (α, β) is the indecomposable matrix factorization of f, then

$$\left[\begin{bmatrix} X, & \alpha \\ \beta, & -Y \end{bmatrix}, \begin{bmatrix} Y, & \alpha \\ \beta, & -X \end{bmatrix}\right]$$

is the indecomposable matrix factorization of $f + XY$, which is Knörrer's periodicity theorem [5] in the graded case.

We now assume that k is a field of characteristic $\neq 2$. The subset $S(k)$ of elements $(n, \delta, c) \in \mathbf{N} \times k^*/k^{*2} \times B(k)$ for which there exists a quadratic form with these invariants. It is clear that $S(k) = \dot{\bigcup}_{n \geqslant 1} S_n(k)$, where $S_n(k)$ is the set of elements of $S(k)$ whose first component is n. Let $s_1 \in S_n(k)$, $s_2 \in S_m(k)$, and choose quadratic forms f and g such that $s_1 = \operatorname{inv}(f)$ and $s_2 = \operatorname{inv}(g)$. We define the product $s_1 \cdot s_2 \in S_{n+m}(k)$ by setting $s_1 \cdot s_2 = \operatorname{inv}(f+g)$.

This definition does not depend on the choice of f and g, and thus $S(k)$, together with this multiplication, is a graded, associative semigroup. The multiplication can be made explicit without referring to quadratic forms. The result is compiled in the next

LEMMA 3.1 ([6], page 81).

$$\cdot(n, \delta, c)(n', \delta', c') = \begin{cases} (n+n', \delta\delta', cc'(\delta, \delta')), & n \equiv 0 \bmod 2 \text{ and } n' \equiv 0 \bmod 2, \\ (n+n', \delta\delta', cc'(\delta, -\delta')), & n \equiv 0 \bmod 2 \text{ and } n' \equiv 1 \bmod 2, \\ (n+n', \delta\delta', cc'(-\delta, \delta')), & n \equiv 1 \bmod 2 \text{ and } n' \equiv 0 \bmod 2, \\ (n+n', \delta\delta', cc'(\delta, \delta')), & n \equiv 1 \bmod 2 \text{ and } n' \equiv 1 \bmod 2, \end{cases}$$

(Here and in the whole section we denote the class of a quaternion algebra (a, b) in $B(k)$ again by (a, b).)

The hyperbolic form $X_1^2 - X_2^2$ has the invariants (2, 1, 1). Thus (2, 1, 1) always belongs to $S(k)$. We let $(2, 1, 1)S(k)$ be the "ideal" $\{(2, 1, 1)(n, \delta, c) \mid (n, \delta, c) \in S(k)\}$. There is the following simple general result.

PROPOSITION 3.2. *If k has the property C then*

$$\{M(2, 1, 1)\} \cup \{M(n, \delta, c) \mid (n, \delta, c) \in S(k) \setminus (2, 1, 1)S(k)\}$$

is a generating set of Clifford modules.

Proof. By 3.1 we have $(2, 1, 1)(n, \delta, c) = (n+2, \delta, c)$ for all $(n, \delta, c) \in S(k)$. Similarly as in § 2, (5) we set

$$l(n, \delta, c) = \log_2 \dim_k M(n, \delta, c) \tag{1}$$

Now 1.3 implies that $l(2, 1, 1) = 1$, and that $l(n+2, \delta, c) = 1 + l(n, \delta, c) = l(2, 1, 1) + l(n, \delta, c)$, so that by 2.1: $M(n+2, \delta, c) \approx M(2, 1, 1) \hat{\otimes} M(n, \delta, c)$. Since any $(n, \delta, c) \in S(k)$ can be written as $(n, \delta, c) = (2, 1, 1)^r(n', \delta', c')$ with $(n', \delta', c') \notin (2, 1, 1)S(k)$ we get $M(n, \delta, c) \approx \hat{\otimes}^r M(2, 1, 1) \hat{\otimes} M(n', \delta', c')$, as required.

We now consider more specific fields.

1. Finite fields

Let $k = F_q$ be the Galois field with $q = p^m$ elements, where p is a prime number $\neq 2$, and where $m \geqslant 1$. In the following the distinction $-1 \in k^{*2}$ or $-1 \notin k^{*2}$ will be necessary. Notice that $-1 \in k^{*2}$ if $m > 1$ or $p \equiv 1 \bmod 4$, and $-1 \notin k^{*2}$ if $p \equiv 3 \bmod 4$ (and $m = 1$). In any case k^*/k^{*2} is a group of order two. We denote its generator by u, which may be chosen to be -1 if $p \equiv 3 \bmod 4$.

By the theorem of Wedderburn we have $B(k) = \{1\}$, and as k has property C, rank and discriminant alone classify the quadratic forms. Thus we simply write (n, δ) instead of (n, δ, c). (This should not be confused with the symbol (a, b) for quaternion algebras. It should be clear from the context what is meant.)

PROPOSITION 3.3. *Assume k is a finite field of characteristic $\neq 2$ Then*
(a) $\{M(1, 1), M(1, u), M(2, 1)\}$ *is a generating set for the Clifford modules.*
(b)

$$M(n, 1) \approx \begin{cases} \hat{\otimes}^{n/2} M(2, 1), & \textit{if } n \equiv 0 \bmod 2, \\ \hat{\otimes}^{[n/2]} M(2, 1) \hat{\otimes} M(1, 1), & \textit{if } n \equiv 1 \bmod 2, \end{cases}$$

and

$$M(n, u) \approx \begin{cases} \hat{\otimes}^{[(n-1)/2]} M(2, 1) \hat{\otimes} M(1, 1) \hat{\otimes} M(1, -u), & \textit{if } n \equiv 0 \bmod 2, \\ \hat{\otimes}^{[n/2]} M(2, 1) \hat{\otimes} M(1, -1), & \textit{if } n \equiv 1 \bmod 2. \end{cases}$$

Notice that

$$M(1, -u) \approx \begin{cases} M(1, u), & \text{if } -1 \in k^{*2}, \\ M(1, 1), & \text{if } -1 \notin k^{*2}. \end{cases}$$

and

$$M(1, -1) \approx \begin{cases} M(1, 1), & \text{if } -1 \in k^{*2}, \\ M(1, u), & \text{if } -1 \notin k^{*2}. \end{cases}$$

Proof of 3.3. (a) $(1, 1)$ $(1, u) \in S_1(k)$, and $(1, \delta)$ $(n, \delta') = (n+1, (-1)^n \delta\delta')$. Since $(-1)^n \delta\delta' \equiv 1$ or $u \bmod k^{*2}$ for suitable δ, δ', it follows that $S_1(k)$ generates the semigroup $S(k)$. Therefore $S(k)/(2, 1)S(k) = S_1(k)$, and thus (a) follows from 3.2.

(b) We have seen in the proof of 3.2 that $M(n+2, \delta) \approx M(2, 1) \hat{\otimes} M(n, \delta)$. Thus if $n \equiv 0 \bmod 2$, one proves by induction that $M(n, \delta) \approx \hat{\otimes}^{[(n-1)/2]} M(2, 1) \hat{\otimes} M(2, \delta)$. Since $l(1, \delta) = 1$ for $\delta = 1, u$, and $l(2, u) = 2$, and since $(2, u) = (1, 1)(1, -u)$, it follows from 2.1 that $M(2, u) \approx M(1, 1) \hat{\otimes} M(1, -u)$. Thus we have proved (b) in the case that $n \equiv 0 \bmod 2$. The case $n \equiv 1 \bmod 2$ is treated similarly.

The following example demonstrates how one uses 3.3 to find explicitly the indecomposable matrix factorization of a quadratic form. Let k be a finite field with $-1 \notin k^{*2}$. We want to factorize

$$f = X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2.$$

Since $\delta = 1$, 2.8 implies that $M(f) \approx M(2, 1) \hat{\otimes} M(2, 1) \hat{\otimes} M(1, 1)$. To write down the factorization we first have to pass to the equivalent form

$$g = Y_1 Y_2 + Y_3 Y_4 + Y_5^2.$$

Using 2.3, the factorizations corresponding to the modules in terms of the Y_i are given as follows:

$M(2, 1)$: $((Y_1), (Y_2))$,

$M(2, 1) \hat{\otimes} M(2, 1)$:

$$(\alpha, \beta) = \left[\begin{bmatrix} Y_3, & Y_1 \\ Y_2, & -Y_4 \end{bmatrix}, \begin{bmatrix} Y_4, & Y_1 \\ Y_2, & -Y_3 \end{bmatrix} \right],$$

$M(2,1)\hat{\otimes} M(2,1)\hat{\otimes} M(1,1)$:

$$\left[\begin{bmatrix} Y_5, & \alpha \\ \beta, & -Y_5 \end{bmatrix}, \begin{bmatrix} Y_5, & \alpha \\ \beta, & -Y_5 \end{bmatrix}\right].$$

Thus

$$(Y_1 Y_2 + Y_3 Y_4 + Y_5^2)\cdot E_4 = \begin{bmatrix} Y_5, & 0 & Y_3, & Y_1 \\ 0, & Y_5, & Y_2, & -Y_4 \\ Y_4, & Y_1, & -Y_5, & 0 \\ Y_2, & -Y_3, & 0, & -Y_5 \end{bmatrix}^2.$$

To get the factorization of f itself we have to express the Y_i in the matrix factorization by the X_i. Since k is finite there exist $a, b \in k$ such that $-1 = a^2 + b^2$, and we can express the Y_i as follows:

$$\begin{aligned} Y_1 &= 2(X_1 - bX_2 + aX_3), \\ Y_2 &= X_1 + X_2 + bX_3, \\ Y_3 &= X_1 + (a-b)X_2 + (a+b)X_3 - X_4, \\ Y_4 &= X_1 + (a-b)X_2 + (a+b)X_3 - X_4 \quad \text{and} \quad Y_5 = X_5. \end{aligned}$$

2. p-adic fields

As a second example we consider the p-adic fields $k = \mathbf{Q}_p$, where p is a prime number $\neq 2$. For general properties of this field we refer to Scharlau [6], Serre [7], and Vigneras [8]. We shall need the following facts:

(1) $k^*/k^{*2} = \{1, u, p, up\} \bmod k^{*2}$, where $u \in \hat{\mathbf{Z}}_p$ such that $u \bmod p$ is a nonsquare in F_p. If $p \equiv 1 \bmod 4$, one may choose $u = -1$.

(2) There is a unique quaternion algebra Q over k which splits for every quadratic extension of k. The class $[Q]$ of Q in $B(k)$ generates a subgroup of order 2 which we identify with $\{\pm 1\}$. Hence we may write

(1) $$(a,b) = \begin{cases} 1, & \text{if } (a,b) \text{ splits,} \\ -1, & \text{if } (a,b) \text{ is a division algebra.} \end{cases}$$

One gets the following table

(2)

(a,b)	1	u	p	up
1	1	1	1	1
u	1	1	-1	-1
p	1	-1	ε	$-\varepsilon$
up	1	-1	$-\varepsilon$	ε

where the entries are (a, b), and where

$$\varepsilon = \begin{cases} 1, & \text{if } -1 \in k^{*2}, \\ -1, & \text{if } -1 \notin k^{*2}. \end{cases}$$

(3) Rank, discriminant and Witt invariant classify the quadratic forms

over k. If (n, δ, c) are the invariants of a quadratic form over k, then with the identifications in (2) we have $c = \pm 1$.

LEMMA 3.4: *The number* $l(n, \delta, c) = \log_2 \dim_k M(n, \delta, c)$ *is given by*

$$l(n, \delta, c) = \begin{cases} (n-c)/2+1, & \text{if } n \equiv 1 \bmod 2, \\ (n+1-c)/2, & \text{if } n \equiv 0 \bmod 2 \text{ and } \delta = 1, \\ (n+2)/2, & \text{if } n \equiv 0 \bmod 2 \text{ and } \delta \neq 1. \end{cases}$$

Proof. We apply the formulas 1.3 to our situation. Identifying c with ± 1 as we did above we get

$$\deg c = \begin{cases} 1, & \text{if } c = 1, \\ 2, & \text{if } c = -1. \end{cases}$$

Using that $c^* = 1$ (by (2) above) the assertions follow immediately from 1.3.

PROPOSITION 3.5. *Let* $p \neq 2$ *be a prime number and let* $k = \mathbf{Q}_p$.

(a) $\{M(2, 1, 1),\ M(4, 1, -1)\} \cup \{M(1, \delta, 1) | \delta \in \{1, u, p, up\} \bmod k^{*2}\}$ *is a generating set of Clifford modules.*

(b) *For all* $(n, \delta, c) \in S(k)$ *we have*

$$M(n, \delta, c) \approx M(n', \delta', c') \hat{\otimes} (\hat{\otimes}^m M(2, 1, 1))$$

for a suitable m, where $(n', \delta', c') \in S(k) \backslash (2, 1, 1) S(k)$.

(c) $S(k) \backslash (2, 1, 1) S(k)$

$$= S_1(k) \cup S_2(k) \cup \{(3, \delta, -1) | \delta \in k^*/k^{*2}\} \cup \{(4, 1, -1)\},$$

and

$$S_1(k) = \{(1, \delta, 1) | \delta \in k^*/k^{*2}\},$$
$$S_2(k) = \{(2, \delta, c) | \delta \in k^*/k^{*2},\ c = \pm 1\} \backslash \{(2, 1, -1)\}.$$

(d) *The modules in the set* $S(k) \backslash (2, 1, 1) S(k)$ *are expressed by the generating Clifford modules as follows*:

$$M(2, \delta, 1) \approx M(1, 1, 1) \hat{\otimes} M(1, -\delta, 1) \text{ for all } \delta \in k^*/k^{*2}.$$

$$M(2, u, -1) \approx \begin{cases} M(1, p, 1) \hat{\otimes} M(1, up, 1), & \text{if } -1 \in k^{*2}, \\ M(1, p, 1) \hat{\otimes} M(1, p, 1), & \text{if } -1 \notin k^{*2}, \end{cases}$$

$$M(2, p, -1) \approx \begin{cases} M(1, u, 1) \hat{\otimes} M(1, up, 1), & \text{if } -1 \in k^{*2}, \\ M(1, u, 1) \hat{\otimes} M(1, p, 1), & \text{if } -1 \notin k^{*2}, \end{cases}$$

$$M(2, up, -2) \approx \begin{cases} M(1, u, 1) \hat{\otimes} M(1, p, 1), & \text{if } -1 \in k^{*2}, \\ M(1, u, 1) \hat{\otimes} M(1, up, 1), & \text{if } -1 \notin k^{*2}, \end{cases}$$

and

$$M(3, 1, -1) \approx M(1, u, 1) \hat{\otimes} M(2, u, 1),$$
$$M(3, u, -1) \approx M(1, 1, 1) \hat{\otimes} M(2, u, -1),$$

$$M(3, p, -1) \approx \begin{cases} M(1, 1, 1) \hat{\otimes} M(2, p, -1), & \textit{if } -1 \in k^{*2}, \\ M(1, 1, 1) \hat{\otimes} M(2, p, 1), & \textit{if } -1 \notin k^{*2}, \end{cases}$$

$$M(3, up, -1) \approx \begin{cases} M(1, 1, 1) \hat{\otimes} M(2, up, -1), & \textit{if } -1 \in k^{*2}, \\ M(1, 1, 1) \hat{\otimes} M(2, up, 1), & \textit{if } -1 \notin k^{*2}, \end{cases}$$

Sketch of the proof. The proof is similar to the one of 3.3, therefore we only give an outline of the proof. In the first step one verifies that the semigroup $S(k)$ is generated by $S_1(k)$. For the proof of this fact one uses Lemma 3.1 and table 2 for the quaternion algebras. Moreover it is needed that $(2, 1, -1) \notin S_2(k)$, see Serre [7], IV, § 3, Proposition 6. As a side result of the calculations in the proof of this first step one obtains assertion (c) of the Proposition. The statement (b) follows from the proof of 3.2.

To prove (a) and (d) notice that

$$M(n, \delta, c) \hat{\otimes} M(n', \delta', c') \approx M(n'', \delta'', c'')$$

if and only if $(n, \delta, c)\ (n', \delta', c') = (n'', \delta'', c'')$, and $l(n, \delta, c) + l(n', \delta', c') = l(n'', \delta'', c'')$, where l is defined as in 3.4. Using 3.2, 3.4 and table 2 one checks (a) and (d) after some calculations.

To give an idea of the sort of calculations which are needed in the proof of (a) we show that $M(4, 1, -1)$ cannot be omitted in the list of the generating modules. $(4, 1, -1)$ can be decomposed in $S(k)$ in the following ways:

$$(4, 1, -1) = (2, \delta, 1)(2, \delta, -1),\ \delta \neq 1 \text{ and } (4, 1, -1) = (1, \delta, 1)(3, -\delta, -1).$$

According to 3.4 we have $l(4, 1, -1) = 3$, and $l(2, \delta, \pm 1) = 2$, so that $l(4, 1, -1) \neq l(2, \delta, 1) + l(2, \delta, -1)$, and thus $M(4, 1, -1)$ and $M(2, \delta, 1) \hat{\otimes} M(2, \delta, -1)$ are not isomorphic. Similarly $M(4, 1, -1)$ cannot be isomorphic to $M(1, \delta, 1) \hat{\otimes} M(3, -\delta, -1)$, since $l(3, -\delta, -1) = 3$.

3. The real numbers

As a last example we consider the real quadratic forms. Any quadratic form f over $\mathbf{R}$ is equivalent to a form $\sum_{i=1}^{r} X_i^2 - \sum_{i=r+1}^{r+s} X_i^2$. The pair (r, s) is called the *signature* of f. The signature classifies the quadratic forms over $\mathbf{R}$. Notice however that $\operatorname{inv}(f)$ does not classify the real quadratic forms. For example if $f = \sum_{i=1}^{4} X_4^2$, then $\operatorname{inv}(f) = \operatorname{inv}(-f) = (4, 1, -1)$. However $\operatorname{sign} f = (4, 0)$, while $\operatorname{sign}(-f) = (0, 4)$. (Here, as for $k = \mathbf{Q}_p$, we may identify $c(f)$ with ± 1, since there is a unique nonsplit quaternion algebra, namely, $(-1, -1)$ over $\mathbf{R}$.)

More generally it is clear that if $\operatorname{sign} f = (r, s)$, then $\operatorname{inv}(f) = (1, \varepsilon, 1)^{|r-s|} (2, 1, 1)^{\min(r,s)}$, where

$$\varepsilon = \begin{cases} 1, & \text{if } r \geqslant s, \\ -1, & \text{if } r < s. \end{cases}$$

As there are $n+1$ pairs (r, s) with $r+s = n$, there are exactly $n+1$ nonequivalent quadratic forms of rank n, while $\mathrm{inv}(f) = (n, \pm 1, \pm 1)$ takes at most 4 values.

We thus denote the unique indecomposable module of a quadratic form with signature (r, s) by $M(r, s)$. Moreover, as $\dim_k M(r, s)$ is a power of 2 we set similarly as before $l(r, s) = \log_2 \dim_k M(r, s)$.

PROPOSITION 3.5. (a) $l(r, s) = l(s, r)$.

(b) *For* $l(r, 0)$ *we have the following table*:

r	1	2	3	4	5	6	7	8
$l(r, 0)$	1	2	3	3	4	4	4	4

(c) *The Atiyah, Bott and Shapiro periodicity modulo* 8:

$$M(r, s)\hat{\otimes} M(8, 0) \approx M(r+8, s),$$
$$M(r, s)\hat{\otimes} M(0, 8) \approx M(r, s+8).$$

(d) $M(1, 0)$, $M(0, 1)$, $M(4, 0)$, $M(0, 4)$, $M(6, 0)$, $M(0, 6)$, $M(7, 0)$, $M(0, 7)$, $M(8, 0)$, $M(0, 8)$ *and* $M(1, 1)$ *are the generating Clifford modules for* $\mathbf{R}$.

Sketch of the proof. To prove (a) let k be an arbitrary field of characteristic $\neq 2$, and let f be a quadratic form over k. For all $x \in k^*$ one has $\dim_k M(f) = \dim_k M(xf)$. In fact, suppose (α, β) is the matrix factorization belonging to $M(f)$, then $(x\alpha, \beta)$ is a matrix factorization of xf. The corresponding Clifford module N clearly satisfies $\dim_k M(f) = \dim_k N \geqslant \dim_k M(xf)$. By symmetry we get the other inequality as well. In our particular case, if $\mathrm{sign}(f) = (r, s)$ then $\mathrm{sign}(-f) = (s, r)$, and so $l(r, s) = l(s, r)$.

The table (b) is computed using 3.1 and 3.4. For the proof of (c) let f be a quadratic form with signature (r, s), and let $f_8 = \sum_{i=1}^{8} X_i^2$. Then $f+f_8$ has signature $(r+8, s)$, and we get

$$\mathrm{inv}(f+f_8) = \mathrm{inv}(f)\cdot \mathrm{inv}(f_8) = (n, \delta, \pm 1)\cdot(8, 1, 1) = (n+8, \delta, \pm 1).$$

Thus it follows that $l(r, s)+l(8, 0) = l(r, s)+4 = l(r+8, s)$, which proves the first isomorphism. Similarly the second isomorphism follows. Finally, (d) is a consequence of (a), (b) and (c).

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