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STABILIZATION OF SOLUTIONS OF A REACTION-DIFFUSION EQUATION

MAREK FILA

Department of Mathematical Analysis, Comenius University
Bratislava, Czechoslovakia

1. Introduction

Consider the reaction-diffusion equation

$$(1)\qquad u_t = u_{xx}+f(u), \quad |x| < L,\ t > 0,$$

with the Dirichlet boundary conditions

$$(2)\qquad u(\pm L, t) = 0, \quad t > 0,$$

where $f\colon \mathbf{R}\to\mathbf{R}$ is C^2, $f(u) = 0$ for $u \leqslant 0$, $f(u)$ is positive and nondecreasing for $u > 0$, $\lim_{u\to\infty}(f(u)/u) = 0$. We suppose further that $\varphi(u) = uf(u)-u^2 f'(u)$ has a zero at a point $r_0 > 0$, is negative for $0 < u < r_0$ and nondecreasing for $u > r_0$.

As an example of a function satisfying these assumptions (which has also a physical meaning) we can take $f(u) = e^{-1/u}$ for $u > 0$.

Let us write $D = (-L, L)$, $C_0^1(\bar{D}) = \{v\in C^1(\bar{D})\colon v(\pm L) = 0\}$. It is known that the solution u of the equation (1) with the boundary conditions (2) and an initial condition

$$(3)\qquad u(x, 0) = u_0(x), \quad x\in\bar{D},$$

where $u_0\in C_0^1(\bar{D})$, exists for all $t > 0$, and $u(\cdot, t)$ belongs to $C_0^1(\bar{D})\cap C^2(D)$. From the assumption $\lim_{u\to\infty}(f(u)/u) = 0$ it follows that u is bounded in $D\times(0, \infty)$ in the sup-norm by a constant depending on f, u_0, L ([1], [2]). Applying the results of Matano ([3]) we then deduce that for every u_0 the solution u of the problem (1)–(3) tends (as $t\to\infty$) in $C_0^1(\bar{D})\cap C^2(D)$ to a stationary solution v, i.e. to a solution of the problem

$$(4)\qquad v''+f(v) = 0, \quad |x| < L,$$

$$(5)\qquad v(\pm L) = 0.$$

In Section 2 we show via the time-map that there exists a positive number L_0 such that for $0 < L < L_0$ the stationary problem (4), (5) has only trivial solution, for $L = L_0$ a global bifurcation occurs and a nontrivial solution suddenly appears, while for every $L > L_0$ there are precisely two nontrivial solutions $v_1, v_2, v_1 < v_2$ in D. ($v \equiv 0$ is a solution for every $L > 0$.)

In Section 3 we prove by comparison with sub- and supersolutions of the stationary problem that for $L > L_0$

$$\lim_{t\to\infty} u(t, u_0) = v_2 \quad \text{if } u_0 > v_1 \text{ in } D,$$

$$\lim_{t\to\infty} u(t, u_0) = 0 \quad \text{if } u_0 < v_1 \text{ in } D$$

($u(t, u_0)$ is the solution of (1)–(3) regarded as a function of x depending on parameters t, u_0.) Using this method one can also determine in some cases to which stationary solution $u(t, u_0)$ converges if u_0 crosses v_1.

2. Global bifurcation of stationary solutions

Each nontrivial stationary solution is positive for $|x| < L$ and achieves its positive maximum m at a point $x_0 \in (-L, L)$. Multiplying the equation (4) by v' and integrating from x_0 to x we obtain

$$\tag{6} \tfrac{1}{2}v'^2 + F(v) = F(m), \qquad F(s) = \int_0^s f(u)\,du.$$

(F is increasing on $(0, \infty)$.) From (6) we conclude that $v(x) = v(-x)$, $x_0 = 0$ and two nontrivial solutions cannot intersect in $(-L, L)$. The equation (6) yields

$$\frac{1}{\sqrt{2}} \int_{v(x)}^{m} \frac{dv}{\sqrt{F(m) - F(v)}} = |x|.$$

The only singularity of the integrand at $v = m$ is integrable.

Let us define the *time-map* T: $\boldsymbol{R}_+ \to \boldsymbol{R}_+$ by the formula

$$T(m) = \frac{1}{\sqrt{2}} \int_0^m \frac{dv}{\sqrt{F(m) - F(v)}}.$$

The image of every m is the number $T(m) = L$ for which the solution v of the equation (4) with initial data $v(0) = m$, $v'(0) = 0$ solves the problem (4), (5). We shall show that the graph of the time-map is as depicted in Figure 1.

LEMMA 1. $\lim_{m\to\infty} T(m) = \lim_{m\to\alpha} T(m) = \infty$.

Proof. The substitution $y = v/m$ leads to the expression

$$T(m) = \frac{m}{\sqrt{2}} \int_0^1 \frac{dy}{\sqrt{F(m) - F(my)}}.$$

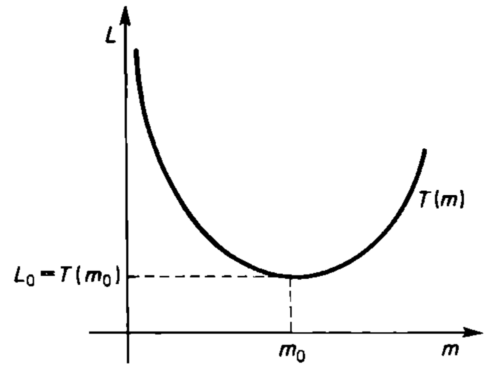

Fig. 1

We have

$$F(m)-F(my) = \int_{my}^{m} f(t)\,dt \leqslant m(1-y)f(m)$$

and so $T(m) \geqslant \sqrt{2m/f(m)}$. According to the properties of f ($f'(0)=0$, $\lim_{u\to\infty}(u/f(u)) = \infty$) the assertion holds.

The existence of a global minimum $L_0 = T(m_0) > 0$ is an immediate corollary of Lemma 1. We shall show that m_0 is a unique singular point of T in the same way as Smoller and Wasserman in [5]. These authors studied the time-map for cubic functions f.

Lemma 2. *T has exactly one singular point.*

Proof. It suffices to prove that T has at most one singular point. To this end we will show that

$$T''(m) > 0 \quad \text{if} \quad T'(m) = 0.$$

We have ([4], [5])

$$2\sqrt{2}\,T'(m) = \int_0^m [g(m)-g(v)]\,[F(m)-F(v)]^{-3/2}\,dv,$$

where $g(u) = 2F(u) - uf(u)$. But $g'(u) = \varphi(u)/u$ for $u > 0$, where φ is defined in Section 1. Thus g decreases on $(0, r_0)$, attains its minimum at r_0 and $g(m)-g(v) < 0$ for $0 \leqslant v < m \leqslant r_0$. Hence $T'(m) < 0$ for $0 < m \leqslant r_0$. For every m such that $T'(m) = 0$ the following inequality holds ([4], [5]):

$$2\sqrt{2}\,T''(m) \geqslant \frac{1}{m^2}\int_0^m [mg'(m)-vg'(v)]\,[F(m)-F(v)]^{-3/2}\,dv.$$

Writing $ug'(u) = \varphi(u)$ gives for $m > r_0$

$$mg'(m) - vg'(v) \geqslant 0 \quad \text{if} \quad 0 \leqslant v < m.$$

(Equality can occur only for some $v \geqslant r_0$.) Thus $T''(m) > 0$.

The following theorem reveals the significance of Lemmas 1, 2.

THEOREM 3. (i) *For* $L < L_0$ *no nontrivial solution of* (4), (5) *exists.*

(ii) *For* $L = L_0$ *there is one nontrivial solution.*

(iii) *For every* $L > L_0$ *there are exactly two nontrivial solutions.*

3. Stability of stationary solutions

If $L < L_0$, then for every u_0 the solution of the problem (1)–(3) tends to zero in $C_0^1(\bar{D}) \cap C^2(D)$ as $t \to \infty$, since we already know that a solution of (1)–(3) converges to some stationary solution.

For $L > L_0$ denote by $v_1(x, L)$ the lower stationary solution and by $v_2(x, L)$ the upper stationary solution on $[-L, L]$.

By a *subsolution* (*supersolution*) of the problem (4), (5) we mean a function y_1 (y_2) satisfying the inequalities

$$y'' + f(y) \geqslant 0 \quad (\leqslant 0) \quad \text{in } (-L, L),$$

$$y(\pm L) \leqslant 0 \quad (\geqslant 0).$$

The following theorem is well known, it is an easy consequence of the maximum principle for parabolic equations.

THEOREM 4. *If* y_1 (y_2) *is subsolution* (*supersolution*) *of the stationary problem and*

$$y_1(x) \leqslant u_0(x) \quad (\leqslant y_2(x)) \quad \text{in } \bar{D},$$

then

$$y_1(x) \leqslant u(x, t, u_0) \quad (\leqslant y_2(x)) \quad \text{in } \bar{D} \times (0, \infty),$$

where $u(x, t, u_0)$ *is the solution of* (1)–(3).

This assertion yields

THEOREM 5. *If* $L > L_0$ *and* $u_0 \leqslant y_2$, *where* y_2 *is a supersolution of* (4), (5) *such that there exists a point* $x_0 \in (-L, L)$ *at which* $y_2(x_0) < v_1(x_0, L)$, *then*

$$\lim_{t \to \infty} u(x, t, u_0) = 0 \quad \text{in } C_0^1(\bar{D}) \cap C^2(D).$$

(*If* $u_0 \geqslant y_1$, *where* y_1 *is a subsolution which is at some point greater than* $v_1(x, L)$, *then* $u(x, t, u_0) \to v_2(x, L)$.)

COROLLARY 6. *If* $L > L_0$, *then*

(i) $u(x, t, u_0) \to v_2(x, L)$ *if* $u_0(x) > v_1(x, L)$ *in* $(-L, L)$.

(ii) $u(x, t, u_0) \to 0$ *if* $u_0(x) < v_1(x, L)$ *in* $(-L, L)$.

Proof. If $u_0(x) > v_1(x, L)$ in $(-L, L)$ then we can choose $L' < L$ such that $u_0(x) > v_1(x, L')$. $v_1(x, L')$ is obviously a subsolution of (4), (5) on $[-L, L]$, and from the graph of T we see that

$$v_1(0, L') = \max_{x \in [-L, L]} v_1(x, L') > \max_{x \in [-L, L]} v_1(x, L) = v_1(0, L).$$

Analogously if $u_0(x) < v_1(x, L)$ in $(-L, L)$ we can choose $L'' > L$ such that $u_0(x) < v_1(x, L'')$, and $v_1(x, L'')$ is a supersolution on $[-L, L]$, $v_1(0, L'') < v_1(0, L)$.

An example of a function $u_0(x)$ intersecting $v_1(x, L)$ such that $u(x, t, u_0) \to v_2(x, L)$ is drawn in Fig. 2.

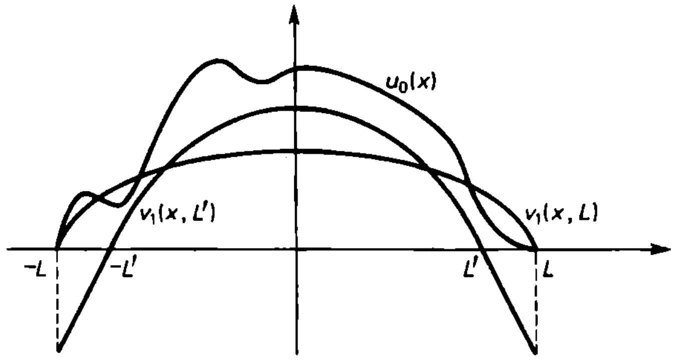

Fig. 2

For completeness we remark that for $L = L_0$ the following holds:

(i) If $u_0(x) > v(x, L_0)$ in $(-L_0, L_0)$ ($v(x, L_0)$ is the nontrivial solution of (4), (5)), then $u(x, t, u_0) \to v(x, L_0)$.

(ii) If $u_0 < v$, then $u(x, t, u_0) \to 0$.

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