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THE APPLICATION OF FIXED POINT THEOREMS TO GLOBAL NONLINEAR CONTROLLABILITY PROBLEMS

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§ 1. Introduction

In this paper we use fixed point theorems together with linear controllability results to obtain global controllability results for nonlinear evolution equations.

We consider a general nonlinear system

$$\dot{z} = f(z, u, t), \quad z(0) = z_0$$

where the initial state z_0 and the state $z(t)$ belong to a space Z and the control term $u(t) \in U$. We now set $z = \bar{z}+z'$, $u = \bar{u}+u'$ to make a local approximation about $(\bar{z}, \bar{u})$ to obtain a system of the form

$$\dot{z}' = A(t)z'+B(t)u'+\bar{f}(z', u', t)+f(\bar{z}, \bar{u}, t)-\dot{\bar{z}}, \quad z'(0) = z_0-\bar{z}(0).$$

Clearly if $(\bar{z}, \bar{u})$ is a solution of the original system this reduces to

$$\dot{z}' = A(t)z'+B(t)u'+\bar{f}(z', u', t), \quad z'(0) = 0.$$

However, if this is not the case, the additional term, which is a known quantity, sligtly modifies the analysis (see § 5, Remark 3). In this work we will restrict our attention to the time-invariant case with the nonlinearity dependent only on the state variable. In Section 5 we describe the modifications needed to extend the analysis to the more general system.

We will take Z to be a reflexive Banach space and assume that the system has been linearized about $(\bar{z}, \bar{u})$, $\bar{z} \in Z$, so that formally

$$\dot{z} = Az + Nz + Bu, \quad z(0) = 0. \tag{1.1}$$

B and A are linear operators and we assume A generates a strongly continuous semigroup $S(t)$ on Z. N is a nonlinear operator and the precise conditions we impose will depend on the fixed point theorem that is used. A mild solution of (1.1) is

$$z(t) = \int_0^t S(t-s)Nz(s)ds + \int_0^t S(t-s)Bu(s)ds. \tag{1.2}$$

There are many different concepts of controllability for infinite-dimensional linear systems [3]. In this paper we will restrict our attention to exact and approximate controllability. It is well known that requiring that the linear system is exactly controllable to the whole Banach space Z is very restrictive in the sense that it is impossible for most systems. So we will examine the possibility of driving the nonlinear system from the origin to a ball of radius r in a subspace V, where V is the subspace to which the linear system can be controlled. Approximate controllability requires that the system can be steered to a dense set in a ball of radius $\bar{r}$ in Z and we will examine this under the assumption that the linear system is approximately controllable to Z.

Using the implicit function theorem it is easy to obtain local results. The importance of our results is that we are able to obtain estimates for r and $\bar{r}$. In order to do this we use a variety of fixed point theorems, namely the contraction mapping theorem [6], Schauder fixed point theorem [5] and a theorem due to Bohnenblust and Karlin [2].

§ 2. Linear results

We start with some definitions relevant to the linear control system

$$z(t) = \int_0^t S(t-s)Bu(s)ds = L_t Bu. \tag{2.1}$$

Let us introduce the operator

$$G\colon L^p[0, T; U] \to Z, , \quad 1 < p < \infty$$

by

$$Gu = \int_0^T S(T-s)Bu(s)ds \tag{2.2}$$

where U is also a reflexive Banach space.

Definition 2.1. Exact controllability to a subspace V on $[0, T]$.

We say that (2.1) is *exactly controllable to a subspace V* if

$$\operatorname{Range}(G) = V.$$

DEFINITION 2.2. Approximate controllability on $[0, T]$. We say that (2.1) is *approximately controllable* if

$$\overline{\operatorname{Range}(G)} = Z.$$

In order to obtain criteria for the above controllability concepts we have the following theorem:

THEOREM 2.3. (i) $\operatorname{Range}(G) = V$ *if and only if, using a norm on V to be defined,*

$$\|G^* v^*\|_{L^q[0,T;U^*]} = \|v^*\|_{V^*}, \quad 1/q+1/p = 1, \tag{2.3}$$

(ii) $\overline{\operatorname{Range}(G)} = Z$ *if and only if*

$$\ker(G^*) = \{0\}$$

where

$$G^*\colon V^* \to L^q[0, T; U^*]$$

$$(G^* v^*)(t) = B^* S^*(t) v^*.$$

In fact we can always put a Banach space topology on $\operatorname{Range}(G)$ so that the system is exactly controllable to a subspace V and will then be approximately controllable to Z in the case V is dense in Z. To see how we construct this topology, we note that $\ker G$ is closed and thus

$$X = L^p[0, T; U]/\ker G$$

is a Banach space under the norm

$$\|[u]\|_X = \inf_{u \in [u]} \|u\|_{L^p[0,T;U]} = \inf_{G\tilde{u}=0} \|u+\tilde{u}\|_{L^p[0,T;U]}$$

where $[u]$ are the equivalence classes of u.

Now define $\tilde{G}\colon X \to Z$ by

$$\tilde{G}[u] = Gu, \quad u \in [u];$$

then $\tilde{G}$ is 1-1, and

$$\|\tilde{G}[u]\|_Z \leqslant \|G\| \, \|[u]\|_X.$$

A norm on the Range $(\tilde{G})$ is now defined by

$$\|v\|_V = \|\tilde{G}^{-1} v\|_X.$$

This norm is equivalent to the graph norm on $D(\tilde{G}^{-1}) = \operatorname{Range}(G)$, $\tilde{G}$ is bounded and since $D(\tilde{G})$ is closed (all of X), $\tilde{G}^{-1}$ is closed and so the above norm makes $\operatorname{Range}(G) = V$ a Banach space.

Moreover,

$$\|Gu\|_V = \|\tilde{G}^{-1}Gu\|_X = \|\tilde{G}^{-1}\tilde{G}[u]\|$$
$$= \|[u]\| = \inf_{u\in[u]} \|u\| \leqslant \|u\|,$$

so

$$G \in \mathscr{L}[L^p[0, T; U], V].$$

Since $L^p[0, T; U]$ is reflexive and $\ker G$ is strongly closed and hence weakly closed, the infimum in the definition of the norm on X is attained. For any $v \in V$, we can therefore choose a control $u \in L^p[0, T; U]$ such that

$$u = \tilde{G}^{-1}v.$$

Accordingly, this control minimizes $\|u\|_{L^p[0,T;U]}$ over all controls which steer the origin to V, and

$$\|u\|_{L^p[0,T;U]} = \|v\|_V. \tag{2.4}$$

Indeed, if U is strictly convex this control is unique.

There are many ways to define the norms in V which are isomorphically equivalent to the definition given by (2.4). One alternative is to use the expression (2.3) to define $\|v\|_V$. Thus, in general, there exist positive constants α, β such that

$$\alpha\|v\|_V \leqslant \|\tilde{G}^{-1}v\|_X \leqslant \beta\|v\|_V.$$

In the case where $p = 2$ and U is a Hilbert space the construction of a topology on Range(G) is easier. In fact, the operator $(GG^*)^{-1/2}$ is well defined on Z and we set

$$V = \text{Range}(GG^*)^{-1/2}$$

and

$$\|v\|_V = \|(GG^*)^{-1/2}v\|_Z.$$

In fact, $GG^*\colon V^* \to V$ is given by

$$GG^*v^* = \int_0^T S(t)BB^*S^*(t)v^*dt.$$

Then for any $v \in V$, the control $u = G^*(GG^*)^{-1}v$ steers the origin to v and minimizes the control energy $\|u\|_{L^2[0,T;U]}$:

$$\|u\|^2_{L^2[0,T;U]} = \|G^*(GG^*)^{-1}v\|^2_{L^2[0,T;U]} = \langle v, (GG^*)^{-1}v\rangle_{V,V^*}$$
$$= \|(GG^*)^{-1/2}v\|^2_Z = \|v\|^2_V.$$

§ 3. Fixed point theorems

In this section we list a number of fixed point theorems which will be used in the sequel.

THEOREM 3.1 [Contraction mapping theorem]. *Let W be a Banach space and φ: $W \to W$ satisfy a local contraction*

$$\|\varphi w - \varphi \hat{w}\| \leqslant k \|w - \hat{w}\|, \quad 0 \leqslant k < 1 \tag{3.1}$$

for any $w, \hat{w} \in D$, a subset of W. Then the iterative procedure

$$w_{i+1} = \varphi w_i \tag{3.2}$$

converges to a unique solution in D of $w = \varphi w$ if the sphere

$$S = \left\{ w \in W:\ \|w - w_1\| \leqslant \frac{k}{1-k} \|w_1 - w_0\| \right\} \tag{3.3}$$

and $w_0{}'$ lie in D.

THEOREM 3.2 [Schauder fixed point theorem]. *Every continuous operator which maps a closed convex subset of a Banach space into a precompact subset has a fixed point.*

THEOREM 3.3 [Bohnenblust and Karlin theorem].

STRONG VERSION. *Let S be a convex closed set in a Banach space W. To each $\omega \in S$ associate a nonempty subset $\varphi(\omega) \subset S$. If*

(a) *$\varphi(\omega)$ is convex for each $\omega \in S$,*

(b) *$\omega_n \to \omega$, $p_n \to p$, $p_n \in \varphi(\omega_n)$ implies $p \in \varphi(\omega)$,*

(c) *$\bigcup_{\omega \in S} \varphi(\omega)$ is contained in some sequentially compact set,*

then there is a point $\omega_0 \in S$ such that $\omega_0 \in \varphi(\omega_0)$.

WEAK VERSION. *Let W be a Banach space with W^* weakly separable and S a convex weakly closed set in W. To each $\omega \in S$ associate a nonempty subset $\varphi(\omega) \subset S$ such that*

(a) *$\varphi(\omega)$ is convex for each $\omega \in S$,*

(b) *$\omega_n \xrightarrow{\text{weakly}} \omega$, $p_n \xrightarrow{\text{weakly}} p$, $p_n \in \varphi(\omega_n)$ implies $p \in \varphi(\omega)$,*

(c) *$\bigcup_{\omega \in S} \varphi(\omega)$ is contained in some weakly sequentially compact set,*

then there exists an $\omega_0 \in S$ such that $\omega_0 \in \varphi(\omega_0)$.

§ 4. Nonlinear controllability

We will employ two different methods of analysis. In the first, assuming the state of the system is known, we will construct a control which drives the origin to any point in a ball in V. When this control is played over the interval $[0, T]$ the state of the system is defined implicitly and we will use fixed point Theorems 3.1, 3.2 to ensure a solution. The second method does not construct a particular control but considers the totality of controls which steer the system to the required end point. Thus multi-valued maps are obtained and we use Theorems 3.3 to obtain a fixed point.

4.1. Method 1. Let V be the subspace to which the linearized system can be steered and consider

$$z(t) = \int_0^t S(t-s)Nz(s)ds + \int_0^t S(t-s)Bu(s)ds \tag{4.1}$$

or

$$z(t) = L_t Nz + L_t Bu. \tag{4.2}$$

We will impose conditions on N so that (4.1), (4.2) are well defined. Since $\text{Range}(G) = V$, we can define the operator $\tilde{G}^{-1}$ as in § 2. Now consider the control

$$\begin{aligned} u(s) &= \tilde{G}^{-1}\Big[v - \int_0^T S(T-p)Nz(p)dp\Big](s) \\ &= \tilde{G}^{-1}[v - L_T Nz](s) \end{aligned} \tag{4.3}$$

which we will show steers the nonlinear system to v at T.

Substituting this control into the right-hand side of (4.2) we obtain the operator

$$(\varphi z)(t) = L_t Nz + L_t B\tilde{G}^{-1}[v - L_T Nz]. \tag{4.4}$$

Moreover,

$$(\varphi z)(T) = L_T Nz + L_T B\tilde{G}^{-1}[v - L_T Nz].$$

But $L_T B = G$ and so

$$(\varphi z)(T) = L_T Nz + G\tilde{G}^{-1}[v - L_T Nz] = v.$$

So if we can impose conditions on the various operators to obtain a fixed point for φ, then for control (4.3), system (4.1) is well defined with $z(T) = v$. In the following we do not attempt to optimize the conditions on the various operators and often make crude estimates. Our intention is to illustrate the main ideas.

(a) *Application of contraction mapping theorem*

In this section we will apply Theorem 3.1 with $W = L^r[0, T; \underline{Z}]$ where $\underline{Z}$ is a Banach space and $r \geqslant 1$, and the operator will be defined by (4.4). For simplicity, we assume that $w_0 = 0$ so that

$$w_1 = L_t B\tilde{G}^{-1}v.$$

Most nonlinearities map $\underline{Z}$ into a larger space and we take account of this in the following theorem (see [4]).

THEOREM 4.1. *Let a nonlinear system be described by*

$$\dot{z} = Az + Nz + Bu, \quad z(0) = 0$$

and consider its mild solution

$$z(t) = L_t Nz + L_t Bu.$$

Assume that

(a) *the nonlinear operator maps $\underline{Z}$ to a larger space $\bar{Z}$ such that*

$$N\colon L^r[0, T; \underline{Z}] \to L^s[0, T; \bar{Z}]$$

is continuous (where $r, s \geqslant 1$) and satisfies a Lipschitz-type condition

$$\|Nz - N\hat{z}\|_{L^s[0,T;\bar{Z}]} \leqslant k(\|z\|, \|\hat{z}\|)\|z - \hat{z}\| \tag{4.5}$$

where the norms on the right-hand side are computed in $L^r[0, T; \underline{Z}]$. The function $k(\cdot, \cdot)\colon R^+ \times R^+ \to R^+$ is continuous, symmetric and $k(0, 0) = 0$;

(b) *the semigroup $S(t)$ generated by A satisfies*

$$S(t) \in \mathscr{L}(\bar{Z}, \underline{Z}) \cap \mathscr{L}(\bar{Z}, V) \quad \text{for} \quad t > 0,$$

$$\|S(t)z\|_{\underline{Z}} \leqslant \bar{g}(t)\|z\|_{\bar{Z}}, \quad \|\bar{g}\|_{L^q[0,T]} = c, \tag{4.6}$$

$$\|S(t)z\|_V \leqslant \bar{\bar{g}}(t)\|z\|_{\bar{Z}}, \quad \|\bar{\bar{g}}\|_{L^\omega[0,T]} = d \tag{4.7}$$

where c and d are constants dependent on T and where q, ω are positive real numbers defined by

$$1/r = 1/q + 1/s - 1, \quad 1/s + 1/\omega = 1$$

and

$$\Big\| \int_0^{\cdot} S(\cdot - s)Bu(s)ds \Big\|_{L^r[0,T;\underline{Z}]} \leqslant R\|u\|_{L^p[0,T;U]} \tag{4.8}$$

where R is a constant dependent on T;

(c) *a is chosen so that the condition*

$$(Rd + c) \sup_{0 \leqslant \theta_1, \theta_2 \leqslant a} k(\theta_1, \theta_2) = K < 1 \tag{4.9}$$

is satisfied.

Then the state of the system described by (4.2) *can be steered from the origin to any final state, v, satisfying*

$$\|v\|_V \leqslant \frac{1 - K}{R} a \tag{4.10}$$

in the time interval $[0, T]$.

Proof. Let the set D of Theorem 3.1 be the ball of radius a in $L^r[0, T; \underline{Z}]$; then

$$\begin{aligned}\|\varphi z - \varphi \hat{z}\|_{L^r[0,T;\underline{Z}]} &\leq \|\bar{g}\|_{L^q[0,T]} \|Nz - N\hat{z}\|_{L^s[0,T;\bar{Z}]} + R\|\tilde{G}^{-1} L_T(Nz - N\hat{z})\|_X \\ &\leq \|\bar{g}\|_{L^q[0,T]} \|Nz - N\hat{z}\|_{L^s[0,T;\bar{Z}]} + R\|\bar{\bar{g}}\|_{L^\infty[0,T]} \|Nz - N\hat{z}\|_{L^s[0,T;\bar{Z}]} \\ &\leq (R\|\bar{\bar{g}}\|_{L^\infty[0,T]} + \|\bar{g}\|_{L^q[0,T]}) k(\|z\|_{L^r[0,T;\underline{Z}]}, \|\hat{z}\|_{L^r[0,T;\underline{Z}]}) \cdot \\ &\qquad \cdot \|z - \hat{z}\|_{L^r[0,T;\underline{Z}]}\end{aligned}$$

by (4.7). So φ is a contraction on D by (4.8).

Now the sphere S (3.3) is

$$\|z - L_{(\cdot)} B\tilde{G}^{-1} v\|_{L^r[0,T;\underline{Z}]} \leq \frac{k}{1-k} \|L_{(\cdot)} B\tilde{G}^{-1} v\|$$

and S will certainly lie in D if

$$\left(1 + \frac{k}{1-k}\right) \|L_{(\cdot)} B\tilde{G}^{-1} v\|_{L^r[0,T;\underline{Z}]} \leq a.$$

This will be the case if

$$\frac{1}{1-k} R\|v\|_V \leq a$$

or

$$\|v\|_V \leq \frac{a(1-k)}{R}$$

which is guaranteed by (4.10). Thus by Theorem 3.1 there is a unique fixed point of φ in a ball of radius a contained in $L^r[0, T; Z]$, whenever v satisfies (4.10). Hence we obtain exact controllability of system (4.2) from the origin to such a final state v in the interval $[0, T]$.

Remarks. 1. The novelty in the above formulation is the condition (4.5) which allows for a large class of nonlinear operators to be considered.

2. The ball in V determined by (4.10) depends on the constants c, d, R which are all dependent on the time T. Hence the size of the ball will vary for different intervals $[0, T]$.

3. If we assume

(a)
$$S(t) \in \mathscr{L}(\bar{Z}, \hat{Z}), \quad t > 0$$
$$\|S(t)\bar{z}\|_{\hat{Z}} \leq \hat{g}(t)\|\bar{z}\|_{\bar{Z}}, \quad \hat{g} \in L^{\hat{p}}[0, T], \ \hat{p} \geq \omega;$$

(b)
$$L_{(\cdot)} Bu \in C[0, T; \hat{Z}] \quad \forall u \in L^p[0, T; U],$$

then the unique fixed point solution also lies in $C[0,T;\hat{Z}]$. Particularly important are the cases $\hat{Z}=Z$, $\hat{Z}=V$.

(b) *Application of Schauder fixed point theorem*

In this section we consider the Banach space $C[0,T;Z]$ and let S be the closed convex set

$$\|z\|_{C[0,T;Z]}\leqslant a. \tag{4.11}$$

THEOREM 4.2. *Let a nonlinear system be described by*

$$z=Az+Nz+Bu,\quad z(0)=0$$

and consider its mild solution

$$z(t)=L_tNz+L_tBu.$$

Assume that

(a) *the nonlinear operator maps Z to a larger space $\bar{Z}$ such that*

$$N\colon C[0,T;Z]\to L^s[0,T;\bar{Z}],\quad s\geqslant 1$$

is continuous and satisfies

$$\|Nz\|_{L^s[0,T;\bar{Z}]}\leqslant \varrho(\|z\|)\,\|z\| \tag{4.12}$$

where the norms on the right-hand side are computed in $C[0,T;Z]$. The function $\varrho(\cdot)\colon R^+\to R^+$ is continuous and $\varrho(\theta)\to 0$ as $\theta\to 0$;

(b) *the semigroup $S(t)$ generated by A satisfies*

$$S(t)\in\mathscr{L}(\bar{Z},Z)\cap\mathscr{L}(\bar{Z},V)\quad \textit{for}\quad t>0,$$

$$\|S(t)z\|_Z\leqslant \bar{g}(t)\,\|z\|_{\bar{Z}},\quad \|\bar{g}\|_{L^\omega[0,T]}=c, \tag{4.13}$$

$$\|S(t)z\|_V\leqslant \bar{\bar{g}}(t)\,\|z\|_{\bar{Z}},\quad \|\bar{\bar{g}}\|_{L^\omega[0,T]}=d \tag{4.14}$$

where c and d are constants dependent on T and where ω is a positive real number defined by

$$1/s+1\omega=1,$$

and

$$\Big\|\int_0^{\cdot} S(\cdot-s)Bu(s)\,ds\Big\|_{C[0,T;Z]}\leqslant R\,\|u\|_{L^p[0,T;U]} \tag{4.15}$$

where R is a constant dependent on T;

(c) *the following compactness conditions are satisfied:*

the operator $L_tB\colon L^p[0,t;U]\to Z$ is compact for each $t\in[0,T]$, *and*

the operator $L_tN\colon C[0,t;Z]\to Z$ *is compact for each* $t\in[0,T]$;

(d) *a is chosen so that the condition*

$$(Rd+c)\sup_{\theta\leqslant a}\varrho(\theta)=K<1 \tag{4.16}$$

is satisfied.

Then the state of the system described by (4.2) *can be steered from the origin to any final state, v, satisfying*

$$\|v\|_V\leqslant\frac{1-K}{R}a \tag{4.17}$$

in the time interval $[0,T]$.

Proof. Since

$$(\varphi z)(t)=L_tNz+L_tB\tilde{G}^{-1}[v-L_TNz]$$

we have

$$\begin{aligned}\|\varphi z\|_{C[0,T;Z]}&\leqslant\|\bar{g}\|_{L^\infty[0,T]}\|Nz\|_{L^p[0,T;\bar{Z}]}+R\|v\|_V+R\|L_TNz\|_V\\&\leqslant(\|\bar{g}\|_{L^\infty[0,T]}+R\|\bar{\bar{g}}\|_{L^\infty[0,T]})\sup_{\theta\leqslant a}\varrho(\theta)a+R\|v\|_V\\&\leqslant ka+a(1-k)\leqslant a\end{aligned}$$

by (4.14) and (4.15). So φ maps S into S.

In order to prove φ is continuous we compute

$$\|\varphi(z+h)-\varphi(z)\|_{C[0,T;Z]}\leqslant(\|\bar{g}\|_{L^\infty[0,T]}+R\|\bar{\bar{g}}\|_{L^\infty[0,T]})\|N(z+h)-Nz\|_{L^p[0,T;\bar{Z}]}$$

and so by assumption (b) φ is continuous.

Finally we have to show that φ maps S into a precompact subset of S. In order to do this we use condition (c) together with the Arzelà–Ascoli theorem. Let us consider the operator $L_tN\colon C[0,t;Z]\to Z$. Now

$$\begin{aligned}\|L_tNz-L_{t_0}Nz\|_Z&\leqslant\Big\|(S(t-t_0)-I)\int_0^{t_0}S(t_0-s)Nz(s)ds\Big\|_Z+\\&\qquad+\Big\|\int_{t_0}^{t}S(t-s)Nz(s)ds\Big\|_Z\\&\leqslant\|(S(t-t_0)-I)L_{t_0}Nz\|+\|\bar{g}\|_{L^\infty[t_0,t]}\|Nz\|_{L^p[t_0,t;\bar{Z}]}.\end{aligned}$$

But the map $(t,z)\to S(t)z$ is continuous and hence uniformly continuous on $[0,T]\times\Omega$ for some compact set Ω in Z. Moreover, $L_{t_0}Nz$ is a precompact set for $\|z\|_{C[0,T;Z]}\leqslant a$, so that

$$\|(S(t-t_0)-I)L_{t_0}Nz\|\to0\quad\text{as}\quad t\downarrow t_0$$

uniformly on $\|z\|_{C[0,T;Z]} \leqslant a$. Also $\|\bar{g}\|_{L^\infty[t_0,t]} \to 0$ as $t \downarrow t_0$. Hence $L_t Nz$ is equicontinuous from the right on $\|z\|_{C[0,T;Z]} \leqslant a$.

To show equicontinuity from the left we take $t > \varepsilon > h > 0$; then

$$\|L_t Nz - L_{t-h} Nz\|_Z \leqslant \Big\| \int_0^{t-\varepsilon} S(t-s)Nz(s)ds - \int_0^{t-\varepsilon} S(t-h-s)Nz(s)ds \Big\|_Z +$$

$$+ \Big\| \int_{t-\varepsilon}^{t} S(t-s)Nz(s)ds \Big\|_Z + \Big\| \int_{t-\varepsilon}^{t-h} S(t-h-s)Nz(s)ds \Big\|_Z$$

$$\leqslant \big\| \big(S(\varepsilon) - S(\varepsilon - h)\big) L_{t-\varepsilon} Nz \big\|_Z + 2\, \|\bar{g}\|_{L^{\bar{p}}[0,\varepsilon]} \|Nz\|_{L^p[0,T;\bar{Z}]}.$$

Then by first letting $h \downarrow 0$ and then $\varepsilon \downarrow 0$ and using (4.12) and the precompactness of $L_{t-\varepsilon} Nz$, we see that $L_t Nz$ is equicontinuous on the left.

Finally,

$$\|L_{(\cdot)} Nz\|_{C[0,T;Z]} \leqslant \|\bar{g}\|_{L^\infty[0,T]} \|Nz\|_{L^p[0,T;\bar{Z}]}$$
$$\leqslant \|\bar{g}\|_{L^\infty[0,T]} \sup_{\theta \leqslant a} \varrho(\theta) a$$

by (4.12). So $L_t Nz$ is uniformly bounded and hence we may apply the Arzelà–Ascoli theorem to conclude that $L_t Nz$ is precompact. Similarly it can be shown that $L_t B$ is compact from $L^p[0, T; U] \to C[0, T; Z]$ and hence φ maps S into a precompact set in S. Thus there is a fixed point in $\|z\|_{C[0,T;Z]} \leqslant a$, and so the nonlinear system is exactly controllable to a ball of radius $a(1-K)/R$ in V.

Remark. In applications the compactness requirements in (c) may arise in two different ways. Either the operators N and B are compact or the semigroup $S(t)$ smooths the space Z yielding, for example, that the map $L_t B$ is bounded from $L^p[0, T; U]$ into $\bar{V}$ where the injection i: $\bar{V} \to Z$ is compact. In order to verify this assumption, use will be made of Rellich's Theorem [1] which states that if Ω is an open bounded set in R^n, then the injections

$$H^m(\Omega) \to H^{m-1}(\Omega),$$
$$H_0^m(\Omega) \to H_0^{m-1}(\Omega)$$

are compact.

4.2. Method 2. (a) *Application of the Bohnenblust–Karlin Theorem* (*Strong version*). In the application of the Contraction and Schauder fixed point theorems we used the controllability of the linear system to construct a control which drove the system to any $v \in V$. In general there will be many controls which perform this task, so if we do not specify the control we are led to the application of a set-valued fixed point theorem. Hence we illustrate this approach with an application to approximate controllability. Later we will make some remarks indicating the modifications required if instead we consider controllability to a subspace.

We consider the system (4.2), namely

$$z(t) = \int_0^t S(t-s)Nz(s)ds + \int_0^t S(t-s)Bu(s)ds$$

which we write as

$$z = F(z, u)$$

where $F\colon C[0,T;Z]\times L^p[0,T;U]\to C[0,T;Z]$. We will assume that the control action is bounded, so that $u\in\mathscr{U}_M$, where

$$\mathscr{U}_M = \{u\in L^p[0,T;U]\colon \|u\|_{L^p[0,T;U]}\leqslant M\}.$$

Given any $z_1\in Z$ and $\varepsilon>0$ define the set

(4.18) $\quad S = \{z\in C[0,T;Z]\colon \|z\|_{C[0,T;Z]}\leqslant a,\ z(0) = 0,\ \|z(T)-z_1\|_Z\leqslant\varepsilon\}.$

Clearly S is closed, convex and bounded.

Now define the set-valued map φ on S with values in the subsets of S,

(4.19) $\quad \varphi(b) = \{z\in S\colon \exists u\in\mathscr{U}_M \text{ such that } z = F(b,u) \text{ with } b\in S\}.$

The objective is to prove the existence of a fixed point of φ, that is to show there exists $z\in S$ such that $z\in\varphi(z)$. This will imply there exists a $\hat{u}\in\mathscr{U}_M$ such that

$$z = F(z,\hat{u})$$

and $z\in C[0,T;Z]$ satisfies $\|z(T)-z_1\|_Z\leqslant\varepsilon$. If this can be done for any ε and any z_1 in a ball of radius r in Z, then (4.2) is said to be *approximately controllable to an r-ball*. We have the following theorem:

THEOREM 4.3. *Consider the mild solution*

$$z(t) = L_tNz+L_tBu$$

of the nonlinear system

$$\dot{z} = Az+Nz+Bu, \qquad z(0) = 0.$$

Assume that

(a) *the nonlinear operator maps Z to a larger space $\bar{Z}$ such that*

$$N\colon C[0,T;Z]\to L^s[0,T;\bar{Z}] \qquad (s\geqslant 1)$$

and satisfies

(4.20) $$\|Nz\|_{L^s[0,T;\bar{Z}]}\leqslant\varrho(\|z\|)\|z\|$$

where the norms on the right-hand side are computed in $C[0,T;Z]$. The function $\varrho(\cdot)\colon R^+\to R^+$ is continuous and $\varrho(\theta)\to 0$ as $\theta\to 0$;

(b) *the semigroup* $S(t)$ *generated by* A *satisfies*

$$S(t) \in \mathscr{L}(\bar{Z}, Z) \quad \textit{for} \quad t > 0$$

$$\|S(t)z\|_Z \leqslant \bar{g}(t)\|z\|_{\bar{Z}}, \qquad \|\bar{g}\|_{L^\omega[0,T]} = c < \infty \tag{4.21}$$

where c *is a constant dependent on* T *and* ω *is a positive real number defined by*

$$1/s + 1/\omega = 1$$

and

$$\Big\| \int_0^\cdot S(\cdot - s)Bu(s)\,ds \Big\|_{C[0,T;Z]} \leqslant R\|u\|_{L^p[0,T;U]} \tag{4.22}$$

where the constant R *depends on* T;

(c) *the following compactness conditions are satisfied:*

the operator $L_tB\colon L^p[0, t; U] \to Z$ *is compact for each* $t \in [0, T]$

and

the operator $L_tN\colon C[0, t; Z] \to Z$ *is compact for each* $t \in [0, T]$.

(d) a *is chosen so that the condition*

$$\frac{(R + l)c}{l} \sup_{\theta \leqslant a} \varrho(\theta) = K < 1 \tag{4.23}$$

is satisfied, where the closure of the range of G *(defined by (2.2)) contains the ball of radius* l *when the controls are restricted to lie in* $\mathscr{U}_1$.

Then the system described by (4.2) *is approximately controllable from the origin to any final state,* z_1, *satisfying*

$$\|z_1\|_Z \leqslant \frac{a(1 - K)l}{R} = r \tag{4.24}$$

in the time interval $[0, T]$.

Proof. First we show that $\varphi(b)$ is nonempty for each $b \in S$ and $\varphi(b)$ is a subset of S. If $b \in C[0, T; Z]$ and $z = F(b, u)$ then it is easy to show that $z \in C[0, T; Z]$. Moreover, if $u \in \mathscr{U}_M$ we have

$$\begin{aligned}\|z(t)\|_Z &\leqslant \|\bar{g}\|_{L^\omega[0,T]}\|Nz\|_{L^s[0,T;\bar{Z}]} + RM \\ &\leqslant \|\bar{g}\|_{L^\omega[0,T]} \sup_{\theta \leqslant a} \varrho(\theta)a + RM.\end{aligned}$$

If we set

$$M = \frac{a}{R}\big(1 - \|\bar{g}\|_{L^\omega[0,T]} \sup_{\theta \leqslant a} \varrho(\theta)\big)$$

which is positive by condition (d), then

$$\|z\|_{C[0,T;Z]} \leq a.$$

By linearity the closure of the range of G when the controls are restricted to lie in $\mathscr{U}_M$ contains the ball of radius Ml. Moreover,

$$\begin{aligned}\Big\|z_1 - \int_0^T S(T-s)Nb(s)\,ds\Big\|_T &\leq \|z_1\| + \|\bar{g}\|_{L^\infty[0,T]} \|Nb\|_{L^p[0,T;\bar{Z}]} \\ &\leq \|z_1\| + \|\bar{g}\|_{L^\infty[0,T]} \sup_{\theta \leq a} \varrho(\theta) a \\ &\leq r + a - RM \leq Ml.\end{aligned}$$

Hence there exists a control $\bar{u} \in \mathscr{U}_M$ such that if

$$z = F(b, \bar{u})$$

then

$$\|z(T) - z_1\|_Z \leq \varepsilon$$

for any small $\varepsilon > 0$, and so $\varphi(b)$ is nonempty.

Now we show that $\varphi(b)$ is convex. Let $z_1, z_2 \in \varphi(b) \subset S$, then for $0 \leq \lambda \leq 1$, $\lambda z_1 + (1-\lambda) z_2 \in S$ since S is convex and there exists u_i, $i = 1, 2$ such that

$$z_i(t) = L_t Nb + L_t Bu_i \quad i = 1, 2.$$

Hence

$$\lambda z_1(t) + (1-\lambda) z_2(t) = L_t Nb + L_t B\big(\lambda u_1 + (1-\lambda) u_2\big).$$

But

$$u = \lambda u_1 + (1-\lambda) u_2 \in \mathscr{U}_M$$

since $\mathscr{U}_M$ is convex, so $\varphi(b)$ is convex.

We can prove that $L_{(\cdot)}Nz\colon C[0, T; Z] \to C[0, T; Z]$ is continuous and $L_{(\cdot)}Bu\colon L^p[0, T; U] \to C[0, T; Z]$ is compact as in the proof of Theorem 4.2. Now let $b_n \to b$, $z_n \to z$, where $z_n = \varphi(b_n)$; then there exists a sequence $\{u_n\} \subset \mathscr{U}_M$ such that

$$z_n = F(b_n, u_n).$$

Since $\mathscr{U}_M$ is bounded, there exists a subsequence converging weakly to a point $u \in L^p[0, T; U]$. But since $\mathscr{U}_M$ is convex and strongly closed it is weakly closed and hence $u \in \mathscr{U}_M$. So

$$z_n(t) = L_t Nb_n + L_t Bu_n.$$

Now using the compactness of $L_{(\cdot)}B$, the continuity of $L_{(\cdot)}N$ and the closedness of S we have

$$z(t) = L_t Nb + L_t Bu$$

or

$$z = \varphi(b).$$

So we have satisfied condition (b) of Theorem 3.3 and all we need to show is that $\bigcup_{b\in S} \varphi(b)$ is contained in some sequentially compact set. But this follows from the boundedness of S and $\mathscr{U}_M$ and the compactness of the operators $L_t N$ and $L_t B$.

Thus φ has a fixed point and the nonlinear system is approximately controllable to an r-ball.

Remarks. 1. If we replace condition (d) by

$$1 > K = \left(\|\bar{g}\|_{L^\infty[0,T]} + \frac{R}{l}\|\bar{\bar{g}}\|_{L^\infty[0,T]}\right)\sup_{\theta\leqslant a} \varrho(\theta) \tag{4.25}$$

where we assume $S(t) \in \mathscr{L}(\bar{Z}, V), t > 0,$

$$\|S(t)z\|_V \leqslant \bar{\bar{g}}(t)\|z\|_{\bar{Z}}, \qquad \bar{\bar{g}} \in L^{\bar{\bar{p}}}[0, T] \tag{4.26}$$

with $\bar{\bar{p}} \geqslant \omega$ and V the subspace to which the linearized system is exactly controllable, then the nonlinear system is exactly controllable to a ball of radius r in V. The proof is straightforward and carried out by setting

$$S = \{z \in C[0, T; Z];\ \|z\|_{C[0,T;Z]} \leqslant a,\ z(0) = 0,\ z(T) = z_1\}.$$

(b) *Application of the Bohnenblust–Karlin Theorem (Weak version)*

In the application of the strong version we imposed compactness conditions on B, N and/or the semigroup $S(t)$ in order to obtain a compact set in $C[0, T; Z]$. We can also obtain compactness by considering a reflexive Banach space (for example $L^r[0, T; V]$) and then working with the weak topology when every bounded set is weakly precompact.

We again consider the system (4.2)

$$z(t) = \int_0^t S(t-s)Nz(s)\,ds + \int_0^t S(t-s)Bu(s)\,ds$$

and write

$$z = F(z, u)$$

but now

$$F\colon L^r[0, T; V] \times L^p[0, T; U] \to L^r[0, T; V]$$

where U, V are assumed to be reflexive Banach spaces and $\infty > r, p > 1$.

We again define

$$\mathscr{U}_M = \{u \in L^p[0, T; U]: \|u\|_{L^p[0,T;U]} \leqslant M\}$$

and assume that the linear system is exactly controllable to V where the range of G contains a ball of radius l when the controls are restricted to $\mathscr{U}_1$.

We now define a space X as the space of equivalence classes of r-integrable measurable functions with

$$\|z\|_X = [\|z(0)\|^r_{\underline{Z}} + \|z(T)\|^r_{\underline{Z}} + \|z\|^r_{L^r[0,T;\underline{Z}]}]^{1/r}$$

where $\underline{Z}$ is a reflexive Banach space. This norm makes X a reflexive Banach space.

Let

$$S = \{z \in X: z(0) = 0,\ z(T) = z_1,\ \|z\|_{L^r[0,T;\underline{Z}]} \leqslant a\} \tag{4.27}$$

where $z_1 \in V$. Then define a map φ from S to the subsets of S by

$$\varphi(b) = \{z \in S: \exists u \in \mathscr{U}_M \text{ such that } z = F(b, u) \text{ with } b \in S\}. \tag{4.28}$$

We now apply the weak version to obtain a fixed point of φ.

THEOREM 4.4. *Let a nonlinear system be described by*

$$z = Az + Nz + Bu, \quad z(0) = 0$$

and consider its mild solution

$$z(t) = L_t Nz + L_t Bu.$$

Assume that

(a) *the nonlinear operator maps $\underline{Z}$ to a larger space $\bar{Z}$ such that*

$$N: L^r[0, T; \underline{Z}] \to L^s[0, T; \bar{Z}] \quad (r, z \geqslant 1)$$

is weakly continuous and satisfies

$$\|Nz\|_{L^s[0,T;\bar{Z}]} \leqslant \varrho(\|z\|)\,\|z\| \tag{4.29}$$

where the norms on the right-hand side are computed in $L^r[0, T; \underline{Z}]$. The function $\varrho(\cdot): R^+ \to R^+$ is continuous and $\varrho(\theta) \to 0$ as $\theta \to 0$;

(b) *the semigroup $S(t)$ generated by A satisfies*

$$S(t) \in \mathscr{L}(\bar{Z}, \underline{Z}) \cap \mathscr{L}(\bar{Z}, V) \quad \textit{for} \quad t > 0$$

$$\|S(t)z\|_{\underline{Z}} \leqslant \bar{g}(t)\,\|z\|_{\bar{Z}}, \quad \|\bar{g}\|_{L^q[0,T]} = c < \infty \tag{4.30}$$

$$\|S(t)z\|_V \leqslant \bar{\bar{g}}(t)\,\|z\|_{\bar{Z}}, \quad \|\bar{\bar{g}}\|_{L^\omega[0,T]} = d < \infty \tag{4.31}$$

where c and d are T-dependent constants and where q, ω are positive real *numbers defined by*

$$1/r = 1/q + 1/s - 1, \quad 1/\omega + 1/s = 1$$

and

$$(4.32) \qquad \left\| \int_0^{\cdot} S(\cdot - s) Bu(s)\, ds \right\|_{L^r[0,T;\underline{Z}]} \leqslant R \|u\|_{L^p[0,T;U]}$$

where constant R is T-dependent;

(c) *a is chosen so that the condition*

$$\left(\frac{Rd}{l} + c\right) \sup_{\theta < a} \varrho(\theta) = K < 1$$

is satisfied.

Then the state of the system described by (4.2) *can be steered from the origin to any final state,* z_1, *satisfying*

$$(4.33) \qquad \|z_1\|_V \leqslant \frac{a(1-K)l}{R} = r$$

in the time interval $[0, T]$.

Proof. First we show that S is weakly closed in X. Let z_n converge weakly to z in X with $\{z_n\} \subset S$. Thus

$$\langle f_1, z_n(0) \rangle_{\underline{Z}^*,\underline{Z}} + \langle f_2, z_n(T) \rangle_{\underline{Z}^*,\underline{Z}} + \langle f_3, z_n \rangle_{L^{r'}[0,T;\underline{Z}^*], L^r[0,T;\underline{Z}]}$$

converges to

$$\langle f_1, z(0) \rangle_{\underline{Z}^*,\underline{Z}} + \langle f_2, z(T) \rangle_{\underline{Z}^*,\underline{Z}} + \langle f_3, z \rangle_{L^{r'}[0,T;\underline{Z}^*], L^r[0,T;\underline{Z}]}$$

for all $(f_1, f_2, f_3) \in \underline{Z}^* \times \underline{Z}^* \times L^{r'}[0, T; \underline{Z}^*]$ with $1/r + 1/r' = 1$. Take $f_2 = f_3 = 0$; then

$$\langle f_1, z_n(0) \rangle_{\underline{Z}^*,\underline{Z}} \to \langle f_1, z(0) \rangle_{\underline{Z}^*,\underline{Z}} \qquad \forall f_1 \in \underline{Z}^*.$$

But $z_n \in S$, and so $z_n(0) = 0$; hence

$$\langle f_1, z(0) \rangle_{\underline{Z}^*,\underline{Z}} = 0 \qquad \forall f_1 \in \underline{Z}^*,$$

thus $z(0) = 0$. Now take $f_3 = 0$, then a similar argument yields

$$\langle f_2, z_1 - z(T) \rangle_{\underline{Z}^*,\underline{Z}} = 0 \qquad \forall f_2 \in \underline{Z}^*$$

or

$$z(T) = z_1.$$

Since z_n converges weakly to z,

$$\begin{aligned}\|z\|_X^r &\leqslant \liminf \|z_n\|_X^r \\ &= \liminf\left(\|z_n(0)\|_{\underline{Z}}^r + \|z_n(T)\|_{\underline{Z}}^r + \|z_n\|^r_{L^r[0,T;\underline{Z}]}\right).\end{aligned}$$

Thus

$$\|z(T)\|_{\underline{Z}}^r + \|z\|^r_{L^r[0,T;\underline{Z}]} \leqslant \|z_1\|_{\underline{Z}}^r + \liminf \|z_n\|^r_{L^r[0,T;\underline{Z}]}.$$

Hence $\|z\|^r_{L^r[0,T;\underline{Z}]} \leqslant a$ and so $z \in S$, thus S is weakly closed. The proof that S is convex is the same as in Theorem 4.3 and to show that $\varphi(b)$ is nonempty and lies in S we have

$$\begin{aligned}\|z\|_{L^r[0,T;\underline{Z}]} &\leqslant \|\bar{g}\|_{L^q[0,T]} \|Nz\|_{L^s[0,T;\bar{Z}]} + RM \\ &\leqslant \|\bar{g}\|_{L^q[0,T]} \sup_{\theta \leqslant a} \varrho(\theta) a + RM.\end{aligned}$$

Now let $M = \dfrac{a}{R}\left(1 - \|\bar{g}\|_{L^q[0,T]} \sup\limits_{\theta \leqslant a} \varrho(\theta)\right)$; then

$$\|z\|_{L^r[0,T;\underline{Z}]} \leqslant a.$$

Also

$$\begin{aligned}\left\|z_1 - \int_0^T S(T-s)Nb(s)ds\right\|_V &\leqslant \|z_1\|_V + \|\bar{g}\|_{L^\infty[0,T]} \sup_{\theta \leqslant a} \varrho(\theta) a \\ &\leqslant \frac{a(1-K)d}{R} + \|\bar{g}\|_{L^\infty[0,T]} \sup_{\theta \leqslant a} \varrho(\theta) a \leqslant Ml\end{aligned}$$

by condition (c). So there exists a control $\bar{u} \in \mathscr{U}_M$ such that if

$$z = F(b, \bar{u})$$

then $z(T) = z_1$, and so $\varphi(b)$ is nonempty.

Since S is bounded in X and $\bigcup\limits_{b \in S} \varphi(b) \subset S$ with X reflexive, it follows that $\bigcup\limits_{b \in S} \varphi(b)$ is contained is some weakly sequentially compact set. So all we have to prove is that if $b_n \to b$ weakly in X, $z_n \to z$ weakly in X where $z_n \in \varphi(b_n)$ then $z \in \varphi(b)$. To do this we use condition (a) where by the weak continuity of N we mean that if $z_n \to z$ weakly in $L^r[0, T; \underline{Z}]$, then $Nz_n \to Nz$ weakly in $L^s[0, T; \bar{Z}]$. It follows that N is weakly closed and our main problem is to extend this to prove that $L_t Nz$ is weakly closed from X to X. We prove this in the following lemma.

LEMMA. *If N is weakly closed from X to $L^s[0, T; \bar{Z}]$ then $L_t Nz$ is weakly closed from X to X. (Here an operator is weakly closed iff $z_n \to z$ weakly, $Nz_n \to y$ weakly, imply $Nz = y$.)*

Proof. Assume $Nz_n \to y$ weakly, that is

$$\langle h, Nz_n\rangle_{L^{s'}[0,T;\bar{Z}^*],L^s[0,T;\bar{Z}]}$$

converges to $\langle h, y\rangle_{L^{s'}[0,T;\bar{Z}^*],L^s[0,T;\bar{Z}]}$ where $1/s'+1/s = 1$.

Let h be given by

$$h(t) = \int_t^T S^*(s-t)f(s)ds + S^*(T-t)g = h_1(t)+h_2(t)$$

where $f \in L^{r'}[0, T; \underline{Z}^*]$, $g \in \underline{Z}^*$. Such an h is allowable since

$$\|h_1\|_{L^{s'}[0,T;\bar{Z}^*]} \leqslant \|S^*(t)\|_{L^q[0,T]}\|f\|_{L^{r'}[0,T;\underline{Z}^*]}$$

and

$$1/s' = 1-1/s = 1/q-1/r = 1/q+1/r'-1,$$

$$\|h_2\|_{L^{s'}[0,T;\bar{Z}^*]} \leqslant \|S^*(\cdot)\|_{L^{s'}[0,T]}\|g\|_{\underline{Z}^*}.$$

Now

$$\begin{aligned}
\langle h, y\rangle_{L^{s'}[0,T;\bar{Z}^*],L^s[0,T;\bar{Z}]} &= \int_0^T \Big\langle \int_t^T S^*(s-t)f(s)ds + S^*(T-t)g, y(t)\Big\rangle_{\bar{Z}^*,\bar{Z}} dt \\
&= \int_0^T\int_t^T \langle f(s), S(s-t)y(t)\rangle_{\underline{Z}^*,\underline{Z}} ds\,dt + \int_0^T \langle g, S(T-t)y(t)\rangle_{\underline{Z}^*,\underline{Z}} dt \\
&= \int_0^T\int_0^s \langle f(s), S(s-t)y(t)\rangle_{\underline{Z}^*,\underline{Z}} ds\,dt + \int_0^T \langle g, S(T-t)y(t)\rangle_{\underline{Z}^*,\underline{Z}} dt \\
&= \int_0^T \langle f(s), L_s y\rangle_{\underline{Z}^*,\underline{Z}} ds + \langle g, L_T y\rangle_{\underline{Z}^*,\underline{Z}} \\
&= \langle f(\cdot), L_{(\cdot)}y\rangle_{L^{r'}[0,T;\underline{Z}^*],L^r[0,T;\underline{Z}]} + \langle g, L_T y\rangle_{\underline{Z}^*,\underline{Z}}.
\end{aligned}$$

Thus if

$$\langle h, Nz_n\rangle_{L^{s'}[0,T;\bar{Z}^*],L^s[0,T;\bar{Z}]} \to \langle h, y\rangle_{L^{s'}[0,T;\bar{Z}^*],L^s[0,T;\bar{Z}]}$$

then

$$\langle f(\cdot), L_{(\cdot)}Nz_n\rangle_{L^{r'}[0,T;\underline{Z}^*],L^r[0,T;\underline{Z}]} + \langle g, L_T Nz_n\rangle_{\underline{Z}^*,\underline{Z}} \to$$
$$\to \langle f(\cdot), L_{(\cdot)}y\rangle_{L^{r'}[0,T;\underline{Z}^*],L^r[0,T;\underline{Z}]} + \langle g, L_T y\rangle_{\underline{Z}^*,\underline{Z}}.$$

Thus if $Nz_n \to y$ weakly in $L^s[0, T; \bar{Z}]$ then $L_t Nz_n \to L_t y$ weakly in X.

Now if $z_n \to z$ weakly then $\{z_n\}$ is a bounded sequence and since N maps bounded sets into bounded sets, it follows that $\{Nz_n\}$ is a bounded sequence in $L^s[0, T; \bar{Z}]$. Hence there exists a subsequence $\{Nz_{n_k}\}$ such that $z_{n_k} \to z$ weakly and $Nz_{n_k} \to y$ weakly. But since N is weakly closed,

$Nz = y$. Now since $Nz_{n_k} \to y$ weakly in $L^s[0, T; \bar{Z}]$, $L_t N z_{n_k} \to L_t y$ weakly in X and $L_t y = L_t N z$. That is, $L_t N$ is weakly closed on X and the Lemma is proved.

Now since $z_n \in \varphi(b_n)$ there exists $\{u_n\} \subset \mathscr{U}_M$ such that

$$z_n = F(b_n, u_n).$$

Moreover, since $\mathscr{U}_M$ is bounded in the reflexive Banach space $L^p[0, T; U]$, there exists a subsequence $\{u_{n_k}\}$ such that $u_{n_k} \to u$ weakly in $L^p[0, T; U]$ and $u \in \mathscr{U}_M$ since $\mathscr{U}_M$ is strongly closed and convex, and hence weakly closed. Thus relabelling

$$b_n \to b, \quad z_n \to z, \quad u_n \to u$$

and

$$z_n(t) = L_t N b_n + L_t B u_n.$$

Now since $L_{(\cdot)} B \in \mathscr{L}(L^p[0, T; U], X)$, it is easy to show that $L_{(\cdot)} B$ is weakly continuous and hence

$$L_t N b_n = z_n(t) - L_t B u_n \to z(t) - L_t B u \text{ weakly}.$$

But using the Lemma we must have

$$L_t N b = z(t) - L_t B u$$

or

$$z = \varphi(b).$$

So all the conditions for the weak version of the theorem are satisfied and hence there is a fixed point for φ.

§ 5. Remarks

1. The results in this work have been obtained for the time-invariant nonlinear system (1.1) but they can be extended to the case where the operators A, B and N are time-dependent. A mild solution of the system now has the form

$$z(t) = \int_0^t U(t, s) N(s) z(s) ds + \int_0^t U(t, s) B(s) u(s) ds$$

where $U(t, s)$ is the mild evolution operator associated with $A(t)$ (see [1]). The controllability definitions in § 2 can be directly extended and the fixed point theorems can be applied as before with a few modifications. In particular, the smoothing property of $U(t, s)$ now leads to conditions of the form

$$\|U(t, s)z\|_{\underline{Z}} \leqslant g(t-s)\|z\|_{\bar{Z}}, \qquad g \in L^p[0, T]$$

when $U(t, s) \in \mathscr{L}(\bar{Z}, \underline{Z})$, $t > s$.

2. A more general nonlinear system is one where the nonlinearity depends on both the state and control variable. The analysis can be adapted to deal with this case, where we have the system

$$\dot{z} = Az + Bu + f(z, u), \qquad z(0) = 0$$

with a mild solution

$$z(t) = \int_0^t S(t-s)f\big(z(s), u(s)\big)ds + \int_0^t S(t-s)Bu(s)ds.$$

Using the techniques in Method 1 of Section 4 we must obtain a fixed point (z, u) of the operator

$$\varphi\begin{bmatrix} z \\ u \end{bmatrix}(t) = \begin{cases} L_t f(z, u) + L_t B\tilde{G}^{-1}[v - L_T f(z, u)], \\ \tilde{G}^{-1}[v - L_T f(z, u)]\,(t). \end{cases}$$

We can adapt Theorem 4.1 directly to obtain a unique fixed point of φ in the space $L^r[0, T; \underline{Z}] \times L^p[0, T; U]$. In the application of the Schauder fixed point theorem we require a fixed point of φ in $C[0, T; Z] \times \times L^p[0, T; U]$. This presents some extra difficulty since we can no longer employ the Arzelà–Ascoli theorem to ensure the compactness condition for the second component. However, by examining the dual mapping $(\tilde{G}^{-1})^*$ this compactness condition can often be verified.

Theorems 4.3 and 4.4 and also be extended with suitable modifications to the conditions on $U(t, s)$ and the nonlinearity.

3. In the linearization process we assumed that $(\bar{z}, \bar{u})$ was a solution of the nonlinear system. This simplifies the analysis but the results follow when $(\bar{z}, \bar{u})$ does not satisfy the original differential equation. In the general case we have

$$\dot{z} = Az + Bu + Nz + w, \qquad z(0) = z_0 - \bar{z}(0)$$

where w is known, and a mild solution is

$$\dot{z}(t) = L_t(Nz + w) + L_t Bu + S(t)\big(z_0 - \bar{z}(0)\big).$$

If we choose $\bar{z}$ such that $\bar{z}(0) = z_0$, the control generated using Method 1 has the form

$$u(t) = \tilde{G}^{-1}[v - L_T(Nz + w)]\,(t)$$

and so we need a fixed point of the operator

$$(\varphi z)(t) = L_t(Nz+w) + L_t B\tilde{G}^{-1}[v - L_T(Nz+w)].$$

The appearance of the known quantity w makes only minor changes in the application of the fixed point theorems to obtain controllability results.

§ 6. Examples

In this section we consider two very simple examples.

EXAMPLE 6.1. We consider a diffusion equation with a nonlinearity and apply Theorem 4.1. Let

$$z_t = z_{xx} + Nz + u, \qquad z(0,t) = z(1,t) = 0.$$

If $Az = z_{xx}$, $z \in D(A)$ where $D(A) = H^2(0,1) \cap H_0^1(0,1)$ then A generates a strongly continuous semigroup $S(t)$ on $L^2[0,1]$ given by

$$S(t)z = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} \varphi_n \langle \varphi_n, z \rangle$$

where $\varphi_n(x) = \sqrt{2}\sin n\pi x$. If $u \in L^2[0,T; L^2[0,1]]$ it can be shown that the linearized system is exactly controllable to $H_0^1(0,1)$ ([3]).

In Theorem 4.1 let $Nz = z^2$ where $Z = L^2[0,1]$, $\bar{Z} = H^{1/2}(0,1)$, $\underline{Z} = H^{1/2}(0,1)$ and $V = H_0^1(0,1)$. Let $r = s = \infty$, $q = \omega = 1$. Then

$$\|S(t)z\|_{H^{1/2}(0,1)} \leqslant c_1 \|z\|_{H^{1/2}(0,1)}$$

$$\|S(t)z\|_{H_0^1(0,1)} \leqslant \frac{c_2}{t^{1/4}} \|z\|_{H^{1/2}(0,1)}, \qquad \frac{c_2}{t^{1/4}} \in L^1[0,T].$$

Thus in (4.9) $c = c_1 T$ and $d = \frac{4}{3} c_2 T^{3/4}$. Moreover, (4.5) is satisfied with $k(\theta_1, \theta_2) = c_3(\theta_1 + \theta_2)$.

Therefore

$$K = 2c_3 a\left(\tfrac{4}{3} c_2 T^{3/4} R + c_1 T\right)$$

where R is also dependent on T. We now write K in the form $K = \beta(T)a$, where $\beta(T)$ is a T-dependent constant. Hence if we have

$$\|v\|_V = \|u\|_{L^2[0,T;L^2[0,1]]} \leqslant \frac{a}{R(T)}\left(1 - \beta(T)a\right)$$

in the interval $[0,T]$, then the state of the system can be steered to v. Furthermore, since the norm in V is equivalent to the norm in $H_0^1(0,1)$, the nonlinear system is exactly controllable to a ball in $H_0^1(0,1)$ of radius

$$\gamma \frac{a}{R(T)}\left(1 - \beta(T)a\right)$$

for constant $\gamma > 0$. For each value of T we can optimize the size of this region. The constraint $K < 1$ implies that $a < 1/\beta(T)$ and for $a = 1/2\beta(T)$, $K = 1/2$ we have the maximal result

$$\|v\|_{H_0^1(0,1)} \leqslant \frac{\gamma}{4R(T)\beta(T)}.$$

EXAMPLE 6.2. To illustrate the application of Theorems 4.2, 4.3 and 4.4, we consider a controlled hyperbolic system with nonlinearity,

$$w_{tt} = w_{xx} + \bar{N}w + bu,$$

$$w(0, t) = w(1, t) = 0, \quad w(x, 0) = w_t(x, 0) = 0.$$

We define $\bar{A}w = -w_{xx}$, $w \in D(\bar{A})$ with $D(\bar{A}) = H^2(0, 1) \cap H_0^1(0, 1)$. The system is written in the form

$$\dot{z}(t) = Az(t) + Nz(t) + Bu$$

where

$$z = \begin{bmatrix} w \\ w_t \end{bmatrix}, \quad A = \begin{bmatrix} 0 & I \\ -\bar{A} & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 0 \\ \bar{N} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ b \end{bmatrix} \quad \text{and}$$

$$D(A) = D(\bar{A}) \times D(\bar{A}^{1/2}).$$

Then A generates a strongly continuous semigroup $S(t)$ on $D(\bar{A}^{1/2}) \times \times L^2[0, 1]$ given by

$$S(t)\begin{bmatrix} z' \\ z'' \end{bmatrix} = \begin{bmatrix} \sum_{n=1}^{\infty} \left\{ \cos n\pi t \langle z', \varphi_n \rangle + \frac{1}{n\pi} \sin n\pi t \langle z'', \varphi_n \rangle \right\} \varphi_n \\ \sum_{n=1}^{\infty} \{ -n\pi \sin n\pi t \langle z', \varphi_n \rangle + \cos n\pi t \langle z'', \varphi_n \rangle \} \varphi_n \end{bmatrix}$$

where $\varphi_n(x) = \sqrt{2} \sin n\pi x$. Now let $b = \sum_{n=1}^{\infty} b_n \varphi_n$ where $b_n = \langle b, \varphi_n \rangle$ and assume that $\liminf_{n\to\infty} n|b_n| > 0$ and $b_n \neq 0$, $n = 0, 1, 2, \ldots$ (e.g. $b_n = 1/n$). Then, if $u \in L^2[0, T]$ it can be shown that the linearized system is exactly controllable to $D(\bar{A}) \times D(\bar{A}^{1/2})$ for $T \geqslant 2$ (see [8]).

The type of nonlinearity to be considered depends on the theorem to be employed. In Theorems 4.2 and 4.3, N is chosen so that the operator $L_t N$ is compact from $C[0, t; Z]$ to Z for all $t \in [0, T]$. Theorem 4.4 does not require such a condition but N is chosen to be a weakly continuous operator. Certainly, in the particular case when N is a compact operator the requirement for Theorem 4.4 is satisfied. The rest of this example is devoted to the application of Theorems 4.2 and 4.3 in the case where $\bar{N}w = w^3$.

(i) *Application of Theorem* 4.2

Let $Z = D(\bar{A}^{1/2}) \times L^2[0,1]$ and $V = \bar{Z} = D(\bar{A}) \times D(\bar{A}^{1/2})$. Then conditions (4.13) and (4.14) are satisfied with $\bar{g}(t) = \bar{\bar{g}}(t) = 1$. Let $\bar{N}z = \begin{bmatrix} 0 \\ w^2 \end{bmatrix}$ where

$$N\colon C[0, T; D(\bar{A}^{1/2}) \times L^2[0,1]] \to L^2[0, T; D(\bar{A}) \times D(\bar{A}^{1/2})]$$

and therefore (4.12) is satisfied with $\varrho(\theta) = C\theta$ for some constant C.

Now consider

$$L_t Bu = \int_0^t S(t-s) \begin{bmatrix} 0 \\ b \end{bmatrix} u(s)\,ds;$$

then

$$L_t Bu = \begin{bmatrix} \sum_{n=1}^{\infty} \left\{ \frac{b_n}{n\pi} \int_0^t \sin n\pi(t-s) u(s)\,ds \right\} \varphi_n \\ \sum_{n=1}^{\infty} \left\{ b_n \int_0^t \cos n\pi(t-s) u(s)\,ds \right\} \varphi_n \end{bmatrix} = \begin{bmatrix} \sum_{n=1}^{\infty} \eta_n \varphi_n \\ \sum_{n=1}^{\infty} \zeta_n \varphi_n \end{bmatrix}.$$

Suppose $T = 2$; then the Fourier coefficients of u in $L^2[0,2]$ are $\{\int_0^2 \sin n\pi s\, u(s)\,ds\}$, $n = 0, 1, 2, \ldots$ Now let

$$u_1(s) = \begin{cases} u(t-s), & s \in [0, t], \\ 0, & s \in (t, 2] \end{cases}$$

and take $b_n = 1/n\pi$, then

$$\eta_n = \frac{1}{(n\pi)^2} \int_0^2 \sin n\pi s\, u(s)\,ds.$$

Since $u_1 \in L^2[0,2]$, we have

$$\sum_{n=1}^{\infty} \{(n\pi)^2 \eta_n\}^2 \leqslant \|u_1\|^2_{L^2[0,2]} \leqslant \|u\|^2_{L^2[0,2]}.$$

However, if $z \in D(\bar{A})$ then

$$\|z\|_{D(\bar{A})} = \sum_{n=1}^{\infty} \{(n\pi)^2 \langle z, \varphi_n \rangle\}^2.$$

Furthermore,

$$\sum_{n=1}^{\infty} \{n\pi \zeta_n\}^2 \leqslant \|u\|^2_{L^2[0,2]}.$$

Hence

$$\|L_t Bu\|^2_{D(\bar{A})\times D(\bar{A}^{1/2})} \leqslant 2\,\|u\|^2_{L^2[0,2]}$$

for $t \in [0, 2]$. Thus for each $t \in [0, 2]$,

$$\|L_t Bu\|_{D(\bar{A})\times D(\bar{A}^{1/2})} \leqslant \sqrt{2}\,\|u\|_{L^2[0,2]}.$$

Clearly $L_t Bu$ is bounded in $D(\bar{A})\times D(\bar{A}^{1/2})$ if u is bounded in $L^2[0,2]$ and so $L_t B$ is a compact operator from $L^2[0,t]$ to $D(\bar{A}^{1/2})\times L^2[0,1]$ for all $t \in [0,2]$.

Now for all $t \in [0,2]$

$$\|L_t Nz\|_{E(\bar{A})\times D(\bar{A}^{1/2})} \leqslant \int_0^t \left\|\begin{bmatrix} 0 \\ \bar{N}w(\cdot, s)\end{bmatrix}\right\|_{D(\bar{A})\times D(\bar{A}^{1/2})} ds = \int_0^t \|w^2(\cdot, s)\|_{D(\bar{A}^{1/2})} ds$$
$$\leqslant \int_0^t \left\|\begin{bmatrix} w(\cdot, s) \\ 0\end{bmatrix}\right\|^2_{D(\bar{A}^{1/2})\times L^2[0,1]} ds.$$

Hence $L_t Nz$ is bounded in $D(\bar{A})\times D(\bar{A}^{1/2})$ if $z(t)$ is bounded in $C[0,2; D(\bar{A})\times D(\bar{A}^{1/2})]$ and so $L_t N$ is a compact operator from $C[0,2; D(\bar{A}^{1/2})\times L^2[0,1]]$ to $D(\bar{A}^{1/2})\times L^2[0,1]$ for all $t \in [0,2]$.

Finally, if

$$\|v\|_{D(\bar{A})\times D(\bar{A}^{1/2})} \leqslant \beta_1(1-\beta_2 a)a$$

where β_1, β_2 are constants, then all the conditions of Theorem 4.2 are satisfied and hence there exists a fixed point of (4.4) in $\|z\|_{C[0,T;D(A^{1/2})\times L^2[0,1]]} \leqslant a$. Thus for $T = 2$, the nonlinear system is exactly controllable to v.

(ii) *Application of Theorem* 4.3

In Theorem 4.3 we considered the approximate controllability of the nonlinear system. In this example, if $u \in L^2[0,T]$ the linearized system is approximately controllable to $D(\bar{A}^{1/2})\times L^2[0,1]$ for time $T \geqslant 2$ (see [8]). As before we take $Z = D(\bar{A}^{1/2})\times L^2[0,1]$, $\bar{Z} = D(\bar{A})\times D(\bar{A}^{1/2})$ and $Nz = \begin{bmatrix} 0 \\ w^2 \end{bmatrix}$. Then condition (4.21) is satisfied with $\bar{g}(t) = 1$ and (4.20) holds with $\varrho(\theta) = C\theta$.

In the same way as above we show that

$$L_t B\colon L^2[0,t] \to D(\bar{A}^{1/2})\times L^2[0,1]$$

is compact for each $t \in [0,T]$ and

$$L_t N\colon C[0,t;\ D(\bar{A}^{1/2})\times L^2[0,1]] \to D(\bar{A}^{1/2})\times L^2[0,1]$$

is compact for each $t \in [0,T]$. Then φ, as defined by (4.19), has a fixed point provided

$$\|z_1\|_{D(\bar{A}^{1/2})\times L^2[0,1]} \leqslant \gamma_1(1-\gamma_2 a)a$$

for constants γ_1, γ_2. Consequently, the nonlinear system can be approximately controlled to a ball in $D(\bar{A}^{1/2}) \times L^2[0,1]$ for $T \geqslant 2$.

References

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, New York 1975.

[2] H. F. Bohnenblust and S. Karlin, *On a Theorem of Ville*, in: *Contributions to the Theory of Games*, vol. 1, Princeton University Press, 1960.

[3] R. F. Curtain and A. J. Pritchard, *Infinite Dimensional Linear Systems Theory*, Springer-Verlag, 1978.

[4] A. Ichikawa and A. J. Pritchard, *Existence, uniqueness and stability of nonlinear evolution equations*, J. Math. Anal. Appl. 68 (1979), 454–476; Control Theory Centre Report No. 65, University of Warwick.

[5] L. W. Kantorovich and G. P. Akilov, *Functional Analysis in Normed Spaces*, Macmillan, New York 1964.

[6] R. H. Martin, *Nonlinear Operators and Differential Equations in Banach Spaces*, John Wiley, New York 1976.

[7] A. J. Pritchard and J. Zabczyk, *Stability and stabilizability of infinite dimensional systems*, SIAM Rev. 23 (1981), 25–52. Control Theory Centre Report No. 70, University of Warwick.

[8] D. L. Russell, *Non-harmonic Fourier series in the control theory of distributed parameter systems*, J. Math. Anal. Appl. 18 (1967), 542–559.