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Topological imbedding of Laplace distributions in Laplace hyperfunctions

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Foreword

The present paper contains the first chapter of a book planned by Bogdan Ziemian, which we started to write a few years ago.

The book written in an accessible way was intended to include results published in papers [Z1], [Z2], [P-Z], [Sz-Z2] devoted to the Laplace representation of solutions to singular differential equations as well as the last studies of Bogdan Ziemian concerning resurgent Laplace distributions and hyperfunctions [Z3].

On the day of his tragic death the first chapter of the book concerning Laplace distributions and hyperfunctions on a half-line was finished. Moreover, a preliminary version of the second chapter was ready, where n-dimensional Laplace distributions and hyperfunctions were studied in the spirit of our paper [Sz-Z2].

Under these circumstances I decided to publish the first chapter in *Dissertationes Mathematicae*; it presents in detail some passages from [Z1]. In particular, it contains the proofs of continuity of isomorphisms between relevant topological spaces, based on subtle methods of the theory of holomorphic functions and functional analysis.

I wish to thank Dr. G. Łysik for his careful reading of the manuscript, many critical remarks and discussions.

I also thank Dr. J. Kowalski for introducing important corrections and improvements to the manuscript and for carefully typing it in TEX.

Zofia Szmydt

Warszawa, October 1997

Introduction

The paper presents the foundations of the theory of Laplace distributions in one variable. Laplace distributions are investigated from the point of view of two different frameworks: functional analysis and hyperfunction theory. In the first setting the space $L'_{(\omega)}(\overline{\mathbb{R}}_+)$ of Laplace distributions is defined as the dual of the space $L_{(\omega)}(\overline{\mathbb{R}}_+)$ of Laplace test functions (see Subsection 7.1). In the second approach it is regarded as a certain quotient space of the holomorphic functions on $W \setminus \overline{\mathbb{R}}_+$, where W is a tubular neighbourhood of $\overline{\mathbb{R}}_+$ in $\mathbb{C}$.

For $\kappa \in \mathbb{R}$ let $\mathfrak{L}_\kappa(W \setminus \overline{\mathbb{R}}_+)$ (resp. $\mathfrak{L}_\kappa(W)$) be the topological space of holomorphic functions on $W \setminus \overline{\mathbb{R}}_+$ (resp. on W) having exponential growth of type $-\kappa$ as $\operatorname{Re} z \to \infty$ (see p. 42) and let $\mathfrak{L}_{(\omega)} = \varinjlim_{\kappa<\omega} \mathfrak{L}_\kappa$, where $\omega \in \mathbb{R} \cup \{\infty\}$. The quotient space

$$\mathfrak{L}_{(\omega)}(W \setminus \overline{\mathbb{R}}_+)/\mathfrak{L}_{(\omega)}(W)$$

is called the space of Laplace hyperfunctions on $\overline{\mathbb{R}}_+$ and Theorem 5.1 gives explicitly a natural topological isomorphism between this space and the space $\underset{\sim}{L}'_{(\omega)}(\overline{\mathbb{R}}_+)$ of Laplace analytic functionals (see p. 43). Since $L'_{(\omega)}(\overline{\mathbb{R}}_+) \hookrightarrow \underset{\sim}{L}'_{(\omega)}(\overline{\mathbb{R}}_+)$ (see p. 66) Theorem 5.1 yields a topological imbedding of the space of Laplace distributions in that of Laplace hyperfunctions.

To describe the image of $L'_{(\omega)}(\overline{\mathbb{R}}_+)$ under this imbedding we introduce a family $\mathfrak{L}^k_\kappa$ of subspaces of $\mathfrak{L}_\kappa$, indexed by $k \in \mathbb{N}_0$ estimating the polynomial growth of holomorphic functions near the real axis. Let

$$\mathfrak{L}^\infty_{(\omega)}(W \setminus \overline{\mathbb{R}}_+) = \varprojlim_{\kappa<\omega} \varinjlim_{k\in\mathbb{N}_0} \mathfrak{L}^k_\kappa(W \setminus \overline{\mathbb{R}}_+).$$

Theorem 7.2 gives a natural topological isomorphism

$$L'_{(\omega)}(\overline{\mathbb{R}}_+) \xrightarrow{\mathcal{J}} \mathfrak{L}^\infty_{(\omega)}(W \setminus \overline{\mathbb{R}}_+)/\mathfrak{L}_{(\omega)}(W).$$

To prove the continuity of $\mathcal{J}$ we use a refinement of the Banach–Steinhaus theorem (see Theorem 1.3) and a slight generalization of the Arzelà–Ascoli theorem to the case of functions on an unbounded interval (see Lemma 7.1). To prove the continuity of the inverse mapping $\mathcal{J}^{-1}$ we apply the method of regularization by taking primitives with simultaneous control of consecutive estimates (see Lemma 7.4 and Corollary 7.3). We use this method earlier in a simpler case, namely to prove Theorem 3.3 useful in the proof of the continuity of the mapping $\mathcal{I}$ in Theorem 4.5—the distributional version of the Köthe theorem. Note that Theorems 5.1 and 7.2 correspond to two Köthe theorems: 4.4 (cf. [Kö]) and 4.5, respectively.

The n-dimensional versions of the main theorems of Sections 4–7 were presented by B. Ziemian in [Z1, pp. 73–76] and then proved in [Sz-Z2]. We considered there only the canonical realization of hyperfunctions as sums of boundary values in wedges modelled over coordinate orthants in $\mathbb{R}^n$. That realization is convenient for applications to PDE's. Namely, solutions to a large class of constant coefficient PDE's can be represented at infinity as sums of Laplace integrals of the form $T[e^{-x\cdot z}]$, where T is a Laplace distribution whose support is related to the geometry of the complex characteristic set (see [Sz-Z1], [Z1], [Z2]).

1. Preliminaries

1.1. Terminology and notation. We employ the usual notation of set theory.

$\mathbb{R}$ denotes the set of real numbers. We consider $\mathbb{R}\cup\{\infty\}$ with the natural topology. For $x=(x_1,\ldots,x_n)\in(\mathbb{R}\cup\{\infty\})^n$ we write $\|x\|^2=x_1^2+\ldots+x_n^2$.

If $x_i<y_i$ ($x_i\le y_i$, resp.) for $i=1,\ldots,n$, we write $x<y$ ($x\le y$, resp.).

The closure of a set $A\subset\mathbb{R}^n$ is denoted by $\overline{A}$, the boundary of A by ∂A and the interior of A by $\operatorname{Int}A$.

We use the notation: $\mathbb{R}^n_+=\{x\in\mathbb{R}^n : 0<x\}$, $\mathbb{R}^n_-=\{x\in\mathbb{R}^n : x<0\}$, $I=(0,t]=\{x\in\mathbb{R}^n_+ : x\le t\}$ where $t\in\mathbb{R}^n_+$.

$\mathbb{N}$ denotes the set of positive integers, $\mathbb{N}_0$ is the set of non-negative integers, and $\mathbb{N}_0^n$ is the set of all multi-indices $\alpha=(\alpha_1,\ldots,\alpha_n)$, $\alpha_i\in\mathbb{N}_0$ for $i=1,\ldots,n$. We write $|\alpha|=\alpha_1+\ldots+\alpha_n$.

$\mathbb{C}$ denotes the set of complex numbers. For sets in $\mathbb{R}^n$ (or in $\mathbb{C}$), we write $A\Subset B$ if A is a subset of B and $\operatorname{dist}(A,\partial B)$ is strictly positive. We use the notation $B(z,r)=\{\zeta\in\mathbb{C} : |\zeta-z|<r\}$ for $z\in\mathbb{C}$, $r>0$ and write briefly $B(r)$ for $B(0,r)$.

1.2. Selected topics on complex topological vector spaces and their duals. In this section, we recall some basic notions and theorems, referring the reader to [Sz] for the proofs.

Let P be a complex vector space. We equip P with a convergence topology defined by a sequence $\{q_k\}_{k=0}^\infty$ of seminorms. Since P is a linear space it is sufficient to define convergence to zero.

We say that a sequence $\{\xi_s\}_{s=1}^\infty$ of elements of P converges to zero if

$$\lim_{s\to\infty} q_k(\xi_s)=0 \qquad (k=0,1,\ldots).$$

The space P with the convergence defined above is denoted by $\mathcal{P}_{\{q_k\}}$ or $\mathcal{P}$ for short. It is called a *multinormed space*.

It is easy to note that a necessary and sufficient condition for every convergent sequence to have a unique limit is that for every $\xi\in P$, $\xi\ne0$ there exists $k\in\mathbb{N}_0$ such that $q_k(\xi)\ne0$. In particular, this is the case if at least one of the seminorms q_k is a norm. In the following, we consider only sequences of seminorms possessing the above property.

It can be proved (see [H, §§ 3 and 4]) that $\mathcal{P}$ is a Hausdorff topological space.

The sequences $\{q_k\}$ of seminorms considered in the sequel will be *increasing* in the sense that

$$q_k(\xi) \le q_{k+1}(\xi) \quad \text{for } \xi \in P \ (k = 0, 1, 2, \ldots).$$

The theorem below characterizes continuous linear functionals on $\mathcal{P}$ with values in $\mathbb{C}$.

THEOREM 1.1. *A linear functional f on $\mathcal{P}_{\{q_k\}}$ is continuous if and only if there exist constants $C < \infty$ and $k \in \mathbb{N}_0$ such that*

$$|f(\xi)| \le C q_k(\xi) \quad \textit{for } \xi \in P.$$

We recall the Banach–Steinhaus theorem in the framework of multinormed spaces.

THEOREM 1.2. *Suppose $\mathcal{P}$ is a complete multinormed space. Let $\{f_\tau\}_{\tau\in T}$, where T is a set of indices, be a family of continuous linear functionals on $\mathcal{P}$. Assume that for every $\xi \in P$ the set $\{f_\tau(\xi) : \tau \in T\}$ is bounded in $\mathbb{C}$. Then there exist constants $C < \infty$ and $k \in \mathbb{N}_0$ such that*

$$|f_\tau(\xi)| < C q_k(\xi) \quad \textit{for } \xi \in P \ (\tau \in \mathbb{N}).$$

COROLLARY 1.1. *Let $\mathcal{P}$ be as in Theorem* 1.2. *Let $\{f_\nu\}_{\nu=1}^{\infty}$ be a sequence of continuous linear functionals on $\mathcal{P}$. Suppose that for every $\xi \in P$ the limit $f(\xi) := \lim_{\nu\to\infty} f_\nu(\xi)$ exists and is finite. Then there exist $C < \infty$ and $k \in \mathbb{N}_0$ such that*

$$|f(\xi)|, |f_\nu(\xi)| \le C q_k(\xi) \quad \textit{for } \xi \in P \ (\nu = 1, 2, \ldots).$$

We denote by $\mathcal{P}'$ the space of continuous linear functionals on $\mathcal{P}$ with the topology of pointwise convergence. It is called the *dual space* of $\mathcal{P}$. Observe that it follows from Corollary 1.1 and Theorem 1.1 that the space $\mathcal{P}'$ is complete.

Let $\{\mathcal{P}_\tau\}_{\tau\in T}$ be a family of multinormed vector spaces.

The *inductive limit* $\mathcal{S} = \varinjlim_{\tau\in T} \mathcal{P}_\tau$ of the spaces $\mathcal{P}_\tau$ is the vector space $P = \bigcup_{\tau\in T} P_\tau$ with convergence topology defined as follows: A sequence $\xi_s \in P$ $(s = 1, 2, \ldots)$ converges to zero in $\mathcal{S}$ if there exists a $\tau_0 \in T$ such that $\xi_s \in P_{\tau_0}$ $(s = 1, 2, \ldots)$ and for every P_τ such that all $\xi_s \in P_\tau$ we have $\lim_{s\to\infty} \xi_s = 0$ in $\mathcal{P}_\tau$.

The *projective limit* $\mathcal{R} = \varprojlim_{\tau\in T} \mathcal{P}_\tau$ of the spaces $\mathcal{P}_\tau$ is the vector space $R = \bigcap_{\tau\in T} P_\tau$ with the following convergence topology: A sequence $\xi_s \in R$ $(s = 1, 2, \ldots)$ tends to zero in $\mathcal{R}$ if $\xi_s \to 0$ as $s \to \infty$ in every $\mathcal{P}_\tau$, $\tau \in T$.

Note that $f \in \mathcal{R}'$ (= the dual space of $\mathcal{R}$ with the pointwise convergence topology) if and only if f is linear on R and its restriction to any $\mathcal{P}_\tau$ is in $(\mathcal{P}_\tau)'$.

The projective limit $\mathcal{R} = \varprojlim_{k\in\mathbb{N}_0} \mathcal{P}_k$ of the Banach spaces $\mathcal{P}_k$ is clearly a complete multinormed space. We shall prove the following refinement of Theorem 1.2 in the spirit of some reasoning used by R. Wawak in the particular case of $\mathcal{R} = \varprojlim_{k\in\mathbb{N}_0} D_K^k$ (cf. Subsection 2.1).

THEOREM 1.3. *Let $\mathcal{P}_k$ be a Banach space with norm q_k, $k \in \mathbb{N}_0$, such that $\mathcal{P}_{k+1} \subset \mathcal{P}_k$ and $q_k(\xi) \le q_{k+1}(\xi)$ for $\xi \in P_{k+1}$. Let $\mathcal{R} = \varprojlim_{k\in\mathbb{N}_0} \mathcal{P}_k$. Assume that $\mathcal{K}^{k+1} = \{\xi \in P_{k+1} : q_{k+1}(\xi) \le 1\}$ is precompact in $\mathcal{P}_k$. Let $f_\nu \in \mathcal{R}'$ $(\nu = 1, 2, \ldots)$ converge to zero in $\mathcal{R}'$, i.e.*

$$f_\nu(\xi) \xrightarrow[\nu\to\infty]{} 0 \quad \textit{for every } \xi \in R. \tag{1.1}$$

Then there exist $k \in \mathbb{N}_0$ *and a sequence* $\varepsilon_\nu \to 0_+$ *such that*

$$|f_\nu(\xi)| \le \varepsilon_\nu q_{k+1}(\xi) \quad \textit{for } \xi \in P_{k+1},\ \nu = 1, 2, \ldots$$

PROOF. Since $\mathcal{R}$ is a complete multinormed space, by Corollary 1.1 there exist $C < \infty$ and $k \in \mathbb{N}_0$ such that $|f_\nu(\xi)| \le C q_k(\xi)$ for $\xi \in R$, $\nu \in \mathbb{N}$. Thus by the Hahn–Banach theorem (see Theorem 1.4 below) $f_\nu \in \mathcal{P}_k'$ and

$$|f_\nu(\xi)| \le C q_k(\xi) \quad \text{for } \xi \in P_k,\ \nu \in \mathbb{N}. \tag{1.2}$$

Let $\varepsilon_\nu := \sup_{\eta \in \mathcal{K}^{k+1}} |f_\nu(\eta)|$. Then

$$|f_\nu(\xi)| \le \varepsilon_\nu q_{k+1}(\xi) \quad \text{for any } \xi \in P_{k+1}.$$

If the sequence ε_ν does not converge to zero we can find $\delta > 0$ and a subsequence ν_s so that $\varepsilon_{\nu_s} > \delta$. Take $\eta_{\nu_s} \in \mathcal{K}^{k+1}$ such that $0 < \delta/2 < |f_{\nu_s}(\eta_{\nu_s})|$. Since $\mathcal{K}^{k+1}$ is precompact in $\mathcal{P}_k$ we can select from η_{ν_s} a subsequence convergent in $\mathcal{P}_k$. For simplicity assume that $0 < \delta/2 < |f_\nu(\eta_\nu)|$ and $\eta_\nu \to \eta_0$ in $\mathcal{P}_k$ as $\nu \to \infty$. Hence $q_k(\eta_\nu - \eta_0) \to 0$, $f_\nu(\eta_0) \to 0$ and by (1.2) and (1.1) we get

$$0 < \frac{\delta}{2} < |f_\nu(\eta_\nu)| \le |f_\nu(\eta_\nu - \eta_0)| + |f_\nu(\eta_0)| \le C q_k(\eta_\nu - \eta_0) + |f_\nu(\eta_0)|,$$

which for $\nu \to \infty$ leads to the absurd $0 < \delta/2 \le 0$. ∎

The following proposition easily follows from the proof of Theorem 1.3:

PROPOSITION 1.1. *Let* $\mathcal{P}$ *be a Banach space with norm* $q_{\mathcal{P}}$ *and let* $\mathcal{Q}$ *be its subspace with underlying set* Q *and norm* $q_{\mathcal{Q}}$, *satisfying* $q_{\mathcal{P}}(\xi) \le q_{\mathcal{Q}}(\xi)$ *for* $\xi \in Q$. *Assume that the set* $\mathcal{K}_{\mathcal{Q}} = \{\xi \in Q : q_{\mathcal{Q}}(\xi) \le 1\}$ *is precompact in* $\mathcal{P}$. *Let* $f_\nu \in \mathcal{P}'$, $f_\nu \to 0$ *as* $\nu \to \infty$. *Then there exists a sequence* $\varepsilon_\nu \to 0_+$ *such that* $|f_\nu(\xi)| \le \varepsilon_\nu q_{\mathcal{Q}}(\xi)$ *for* $\xi \in Q$.

From Corollary 1.1 we get

COROLLARY 1.2. *Suppose* $\mathcal{S}$ *is the inductive limit of complete multinormed spaces* $\mathcal{P}_\tau$. *Let* $f_\nu \in \mathcal{S}'$ *and suppose the limit* $f(\xi) = \lim_{\nu\to\infty} f_\nu(\xi)$ *exists for every* $\xi \in P$ *and is finite. Then* $f \in \mathcal{S}'$.

In Section 2 we present examples of complex multinormed vector spaces and their inductive limits. Some of them are well-known from the theory of distributions.

Now consider a pair $\mathcal{Q}, \mathcal{P}$ of topological vector spaces (not necessarily multinormed) with topologies given by convergence of sequences to zero. A *topological injection* $\mathcal{Q} \hookrightarrow \mathcal{P}$ occurs if there exists a continuous imbedding $\mathcal{I} : \mathcal{Q} \to \mathcal{P}$. If the imbedding $\mathcal{I}$ is trivial we write $\mathcal{Q} \subset \mathcal{P}$. If $\mathcal{Q} \subset \mathcal{P}$ and $f \in \mathcal{P}'$ we denote by $Zf = f_{|Q}$ the restriction of f to Q.

PROPOSITION 1.2. *Suppose* $\mathcal{Q} \subset \mathcal{P}$ *and the set* Q *is dense in* $\mathcal{P}$. *Then* $Z(\mathcal{P}') \subset \mathcal{Q}'$ *and the restriction mapping* $Z : \mathcal{P}' \to \mathcal{Q}'$ *is a homeomorphism onto its image. We identify* $\mathcal{P}'$ *with* $Z(\mathcal{P}')$ *under this bijection and write* $\mathcal{P}' \subset \mathcal{Q}'$.

REMARK 1.1. Clearly if $\mathcal{Q} \subset \mathcal{P}$ and $f \in \mathcal{P}'$ then $f_{|Q} \in \mathcal{Q}'$. However, to conclude that $\mathcal{P}' \subset \mathcal{Q}'$ the density of Q in $\mathcal{P}$ is essential.

THEOREM 1.4 (The Hahn–Banach extension theorem). *Let* $\mathcal{Q} \subset \mathcal{P}$ *be topological vector spaces. Let* q *be a seminorm on* $\mathcal{P}$ *and* $f \in \mathcal{Q}'$ *be such that* $|f(\xi)| \le q(\xi)$ *for* $\xi \in Q$. *Then* *there exists a linear functional* $\widetilde{f}$ *on* $\mathcal{P}$ *which extends* f, *i.e.* $\widetilde{f}(\xi) = f(\xi)$ *for* $\xi \in Q$, *and satisfies the inequality* $|\widetilde{f}(\xi)| \le q(\xi)$ *for* $\xi \in P$.

COROLLARY 1.3. *If* $\mathcal{Q}$ *is a topological subspace of a multinormed vector space* $\mathcal{P}$ *then* $\mathcal{P}'$ *is a subspace of* $\mathcal{Q}'$ (see Proposition 1.2) *if and only if* Q *is dense in* $\mathcal{P}$. *Equivalently,* Q *is not dense in* $\mathcal{P}$ *if and only if there exists an* $f \in \mathcal{P}'$, $f \ne 0$ *such that* $f_{|Q} = 0$.

2. A review of basic facts in the theory of distributions

Let Ω be an open subset of $\mathbb{R}^n$ and $m \in \mathbb{N}_0$. Let φ be a complex-valued function defined on Ω. We say that φ is of class C^m on Ω (for short, a $C^m(\Omega)$-function) iff all the derivatives of φ up to order m exist and are continuous on Ω; φ is of class $C^\infty(\Omega)$ (or is a smooth function on Ω) iff it is of class $C^m(\Omega)$ for all $m \in \mathbb{N}_0$.

Let $k \in \mathbb{N}_0$ or $k = \infty$. We denote by $C_0^k(\Omega)$ the set of all functions in $C^k(\Omega)$ vanishing outside a compact set in Ω, and by $C_A^k(\Omega)$ the set of $C^k(\Omega)$-functions with support in A. Note that if $\varphi \in C^0(\Omega)$ and $\operatorname{supp}\varphi = A$, then A is the closure of $\operatorname{Int} A$ in Ω.

Vector notation is also used for differentiations. Namely we write

$$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right), \quad x\frac{\partial}{\partial x} = \left(x_1\frac{\partial}{\partial x_1}, \ldots, x_n\frac{\partial}{\partial x_n}\right)$$

and if $\alpha \in \mathbb{N}_0^n$ then

$$\left(\frac{\partial}{\partial x}\right)^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}, \quad \left(x\frac{\partial}{\partial x}\right)^\alpha = \left(x_1\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(x_n\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

A non-empty set A is called *fat* if $A \subset \overline{\operatorname{Int} A}$. The support of $\varphi \in C^0(\Omega)$ with $\varphi \not\equiv 0$ is a relatively closed fat set; the support of $\varphi \in C_0^0(\Omega)$ is a fat compact set. For every fat compact set $K \subset \mathbb{R}^n$ one can find a $C^\infty(\mathbb{R}^n)$-function with support K (see [St, Ch. VI, Theorem 2]).

The proofs of the facts presented without references can be found in [Sz].

2.1. The spaces D_K and $(D_K)'$. Let K be a fat compact set in $\mathbb{R}^n$ and let $k \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^n$. We denote by D_K^k the Banach space of $C_K^k(\mathbb{R}^n)$-functions ([1]) with norm $q_k(\cdot) = \|\!| \cdot |\!\|_k$:

$$|\!\|\varphi|\!\|_k = \sum_{|\alpha| \le k} \sup_{x \in K} \left|\left(\frac{\partial}{\partial x}\right)^\alpha \varphi(x)\right| \quad \text{for } \varphi \in C_K^k.$$

We denote by D_K the space of $C_K^\infty(\mathbb{R}^n)$-functions with the topology defined by the increasing sequence $\{|\!\| \cdot |\!\|_k\}_{k \in \mathbb{N}_0}$ of norms, i.e. $D_K = \varprojlim_{k \in \mathbb{N}_0} D_K^k$. The space D_K is complete.

We denote by $(D_K)'$ ($(D_K^k)'$, resp.) the dual space of D_K (D_K^k, resp.) (see Section 1).

Let $u \in (D_K)'$. The value of u on a function $\varphi \in C_K^\infty$ is denoted by $u[\varphi]$. A linear functional u on C_K^∞ belongs to $(D_K)'$ if and only if one of the following equivalent conditions holds:

([1]) Or equivalently, of $C_K^k(\Omega)$-functions for any open neighbourhood Ω of K.

CONDITION W_1. If $\varphi_\nu \in C_K^\infty$ ($\nu = 1, 2, \ldots$) and $\lim_{\nu\to\infty} |||\varphi_\nu|||_k = 0$ ($k = 0, 1, \ldots$) then $\lim_{\nu\to\infty} u[\varphi_\nu] = 0$.

CONDITION W_2. There exist constants $C < \infty$ and $p \in \mathbb{N}_0$ such that

$$|u[\varphi]| \le C|||\varphi|||_p \quad \text{for } \varphi \in C_K^\infty.$$

$(D_K)'$ equipped with the pointwise convergence topology is complete.

PROPOSITION 2.1 (R. Wawak). *Let $u_\nu \in (D_K)'$, $u_\nu \to 0$, i.e. $u_\nu[\varphi] \to 0$ as $\nu \to \infty$ for every $\varphi \in D_K$. Then there exist $\varepsilon_\nu \to 0_+$ and $p \in \mathbb{N}_0$ such that*

$$|u_\nu[\varphi]| \le \varepsilon_\nu q_p(\varphi) \quad \textit{for } \varphi \in C_K^p.$$

This follows from Theorem 1.3 and the Arzelà theorem used to prove suitable precompactness.

2.2. The spaces $D(\Omega)$ and $D'(\Omega)$. Let $k \in \mathbb{N}_0$ and Ω be an open set in $\mathbb{R}^n$. We denote by $D(\Omega)$ ($D^k(\Omega)$, resp.) the vector space $C_0^\infty(\Omega)$ ($C_0^k(\Omega)$, resp.) equipped with the following convergence topology: a sequence $\varphi_j \in C_0^\infty(\Omega)$ ($C_0^k(\Omega)$, resp.) ($j = 1, 2, \ldots$) is said to converge to zero in Ω iff there exists a compact set $K \subset \Omega$ such that $\operatorname{supp} \varphi_j \subset K$ ($j = 1, 2, \ldots$) and for every $\alpha \in \mathbb{N}_0^n$ ($\alpha \in \mathbb{N}_0^n$, $|\alpha| \le k$, resp.) we have $\lim_{j\to\infty} (\partial/\partial x)^\alpha \varphi_j = 0$ uniformly on Ω. Thus $D(\Omega) = \varinjlim_{K\subset\Omega} D_K$, $D^k(\Omega) = \varinjlim_{K\subset\Omega} D_K^k$.

The dual space of $D(\Omega)$, denoted by $D'(\Omega)$, is called the *space of distributions on Ω*. The dual of $D^k(\Omega)$ is denoted by $(D^k(\Omega))'$. Clearly $(D^k(\Omega))' \subset D'(\Omega)$.

A linear functional u on $C_0^\infty(\Omega)$ is a distribution on Ω iff for every compact set $K \subset \Omega$ there exist constants $C(K) < \infty$ and $k(K) \in \mathbb{N}_0$ such that

$$|u[\varphi]| \le C(K)|||\varphi|||_{k(K)} \quad \text{for } \varphi \in C_K^\infty. \tag{2.1}$$

u is said to be of *finite order on Ω* iff there exists a $k \in \mathbb{N}_0$ such that $k(K) \le k$ for every $K \subset \Omega$. The smallest k satisfying this inequality is called the *order of* u on Ω.

The *support* of $u \in D'(\Omega)$ (denoted by $\operatorname{supp} u$) is the smallest relatively closed subset of Ω such that u is zero on its complement in Ω.

We denote by $D'_A(\Omega)$ the space of distributions on Ω with support in $A \subset \Omega$. Clearly the space of elements of $(D^k(\Omega))'$ supported in A, $(D^k(\Omega))'_A$, is contained in $D'_A(\Omega)$.

THEOREM 2.1 (see [Sz, Theorem 7.4]). *Let $u \in D'_K(\mathbb{R}^n)$, where K is a compact set in $\mathbb{R}^n$. Take $k \in \mathbb{N}_0$ such that the order of u is not greater than k. Then $u[\varphi] = 0$ for every function $\varphi \in C^k(\mathbb{R}^n)$ such that $(\partial/\partial x)^\alpha \varphi(x) = 0$ for $x \in K$, $|\alpha| \le k$.*

THEOREM 2.2 (see [Hö2, Theorem 2.3.10]). *Let $u \in D'_K(\mathbb{R}^n)$, where K is a connected compact set in $\mathbb{R}^n$ such that any two points $x, y \in K$ can be joined by a rectifiable curve in K of length $\le \widetilde{C}|x - y|$, $\widetilde{C} < \infty$. Then there exist $C < \infty$ and $k \in \mathbb{N}_0$ such that*

$$|u[\psi]| \le C \sum_{|\alpha|\le k} \sup_{x\in K} \left| \left(\frac{\partial}{\partial x}\right)^\alpha \psi(x) \right| \quad \textit{for } \psi \in C^k(\mathbb{R}^n).$$

The convergence in the space D' will always be understood to be the pointwise convergence. The limit of a sequence of distributions is itself a distribution.

2.3. The spaces $D(A)$ and $D'(A)$. Let $A \subset \Omega \subset \mathbb{R}^n$ be such that Ω is open in $\mathbb{R}^n$ and A is the closure of $\operatorname{Int} A$ in Ω (i.e., A is relatively closed in Ω and fat).

In applications we consider for some $t \in \mathbb{R}^n_+$ the sets $A = I = (0,t] \subset \mathbb{R}^n_+$ or $A = [0,t] \subset \mathbb{R}^n$. We can also take $A = \Omega$ open in $\mathbb{R}^n$.

LEMMA 2.1. *Let $A \subset \Omega \subset \mathbb{R}^n$, with A fat and relatively closed in Ω. Let $\widetilde{u} \in D'_A(\Omega)$. Then $\widetilde{u}[\widetilde{\varphi}] = 0$ for every $\widetilde{\varphi} \in C_0^\infty(\Omega)$ such that $\widetilde{\varphi}_{|A} = 0$.*

PROOF. Let $\widetilde{\varphi} \in C_0^\infty(\Omega)$ with $\widetilde{\varphi}_{|A} = 0$. Hence $(\partial/\partial x)^\alpha \widetilde{\varphi} = 0$ on $\operatorname{Int} A$ for every $\alpha \in \mathbb{N}_0$. Since $A \subset \overline{\operatorname{Int} A}$ and $\widetilde{\varphi} \in C^\infty(\Omega)$ we have

$$\left(\frac{\partial}{\partial x}\right)^\alpha \widetilde{\varphi} = 0 \quad \text{on } A. \tag{2.2}$$

Let $\sigma \in C_0^\infty(\Omega)$ be equal to 1 on the support of $\widetilde{\varphi}$. Then $\widetilde{\varphi} = \widetilde{\varphi}\sigma$ and hence $\widetilde{u}[\widetilde{\varphi}] = (\sigma\widetilde{u})[\widetilde{\varphi}] = v[\widetilde{\varphi}]$, where $v = \sigma\widetilde{u}$. Since A is relatively closed in Ω, the set $K := \operatorname{supp}\sigma \cap A \subset A$ is a compact subset of Ω. Thus $v \in D'_K(\Omega)$ and by (2.2), $(\partial/\partial x)^\alpha \widetilde{\varphi}(x) = 0$ for $x \in K$. By Theorem 2.1 we get $v[\widetilde{\varphi}] = 0$, i.e. $\widetilde{u}[\widetilde{\varphi}] = 0$. ∎

We denote by $C^\infty_{(0)}(A)$ ($C^\infty(A)$, resp.) the space of "extrinsically" smooth functions on A, i.e. restrictions to A of functions in $C_0^\infty(\Omega)$ ($C^\infty(\Omega)$, resp.). We equip $C^\infty_{(0)}(A)$ with an inductive limit topology as follows:

$$D(A) := \varinjlim_{K\subset\Omega} D_{K|A},$$

where K ranges over fat compact subsets of Ω and $D_{K|A}$ denotes the space of restrictions to A of elements of D_K with the topology induced from D_K.

Note that the space $D_{K|A}$ is complete.

Observe that the definition of the space $D(A)$ can also be given in the form

$$D(A) = (\varinjlim_{K\subset\Omega} D_K)_{|A} = D(\Omega)_{|A}.$$

The space $D(A)$ is independent of the choice of an open set Ω in which A is relatively closed. To prove it we may assume that $A \subset \Omega \subset \Omega'$ with A relatively closed in Ω and in Ω'. Let $\varphi \in D_K$, K a compact set in Ω'. Consider the open covering $(\Omega' \setminus A, \Omega)$ of Ω' and a partition of unity: $\psi_1 \in C_0^\infty(\Omega' \setminus A)$, $\psi_2 \in C_0^\infty(\Omega)$. Set $\widetilde{\varphi} = \varphi\cdot\psi_2$. Then $\widetilde{\varphi}_{|A} = \varphi_{|A}$, $\widetilde{\varphi} \in D_{\widetilde{K}}$ where $\widetilde{K} = K \cap \operatorname{supp}\psi_2 \subset \Omega$.

To show the topological inclusion

$$\varinjlim_{K\subset\Omega'} D_{K|A} \subset \varinjlim_{K\subset\Omega} D_{K|A}$$

take $\varphi_\nu \to 0$ in D_K and observe that $\widetilde{\varphi}_\nu := \varphi_\nu \cdot \psi_2 \to 0$ in $D_{\widetilde{K}}$.

The dual space $D'(A) := (D(A))'$ is called the *space of distributions on A*.

The convergence in $D'(A)$ will always be understood to be pointwise convergence. Hence the limit of a sequence of distributions is itself a distribution.

Note that if A is open then we take $A = \Omega$ and $D'(A)$ is the "usual" space of distributions on an open set (cf. Subsection 2.2).

To verify that a linear functional u on $C^\infty_{(0)}(A)$ is a distribution (i.e. $u \in D'(A)$) it is enough to check that for every compact set $K \subset \Omega$,

(i) for every sequence $\{\varphi_j\}_{j=1}^{\infty}$ in C_K^∞ such that $\lim_{j\to\infty}(\partial/\partial x)^\alpha \varphi_j = 0$ uniformly on Ω for $\alpha \in \mathbb{N}_0^n$, we have $u[\varphi_j|_A] \to 0$ as $j \to \infty$, or equivalently

(ii) there exist constants $C(K) < \infty$ and $k(K) \in \mathbb{N}_0$ such that

$$|u[\varphi|_A]| \le C(K) \sum_{|\alpha|\le k(K)} \sup_{x\in K} \left| \left(\frac{\partial}{\partial x}\right)^\alpha \varphi(x) \right| \quad \text{for } \varphi \in C_K^\infty.$$

We define the order of $u \in D'(A)$ (starting from (ii)) in the same way as we did for $u \in D'(\Omega)$ (starting from (2.1)).

By Lemma 2.1 we easily get

PROPOSITION 2.2. *The spaces $D'_A(\Omega)$ and $D'(A)$ are topologically isomorphic* (*written* $D'_A(\Omega) \simeq D'(A)$) *under the canonical linear isomorphism $\mathcal{L} : D'_A(\Omega) \to D'(A)$, given by*

$$D'_A(\Omega) \ni \widetilde{u} \mapsto u \in D'(A),$$

where $u[\widetilde{\varphi}|_A] = \widetilde{u}[\widetilde{\varphi}]$ for $\widetilde{\varphi} \in C_0^\infty(\Omega)$.

In view of this isomorphism we often identify u and $\widetilde{u}$.

Examples of distributions:

(a) The Dirac delta $\delta_{(y)}$ at a point $y \in A$ is defined as

$$\delta_{(y)}[\varphi] = \varphi(y) \quad \text{for } \varphi \in C_{(0)}^\infty(A).$$

(b) Locally integrable functions are imbedded in distributions by means of the identity

$$f[\varphi] = \int_A f(x)\varphi(x)\,dx \quad \text{for } \varphi \in C_{(0)}^\infty(A)$$

if $f \in L^1_{\mathrm{loc}}(A)$.

(c) We introduce a functional $\mathrm{Pf}\frac{1}{x}$ by the formula

$$\mathrm{Pf}\frac{1}{x}[\varphi] = \lim_{\varepsilon\to 0_+} \left\{ \int_{-\infty}^{-\varepsilon} \frac{\varphi(x)}{x}\,dx + \int_{\varepsilon}^{\infty} \frac{\varphi(x)}{x}\,dx \right\} \quad \text{for } \varphi \in C_0^\infty(\mathbb{R}).$$

Let $\varphi \in C_0^\infty(\mathbb{R})$ with $\varphi(x) = 0$ for $|x| \ge R$. Write $\varphi(x) = \varphi(0) + x\widetilde{\varphi}(x)$. Then

$$\left|\mathrm{Pf}\frac{1}{x}[\varphi]\right| = \left| \int_{-R}^{R} \widetilde{\varphi}(x)\,dx \right| \le 2R \sup_{|x|<R} |\varphi'(x)|,$$

which shows that $\mathrm{Pf}\frac{1}{x}$ is a distribution on $\mathbb{R}$ of order 1. Clearly for $x \ne 0$ it coincides with the function $\frac{1}{x}$.

The spaces of distributions are closed under differentiation which is defined by

$$\left(\frac{\partial}{\partial x}\right)^\alpha u[\varphi] = (-1)^{|\alpha|} u\left[\left(\frac{\partial}{\partial x}\right)^\alpha \varphi\right] \quad \text{for } u \in D'(A),\ \varphi \in D(A),$$

and under multiplication by $C^\infty(A)$-functions a defined by

$$au[\varphi] = u[a\varphi].$$

Now let us consider the important case

$$A = I = (0,t], \quad t \in \mathbb{R}_+^n,$$

and define the class $\widetilde{C}^\infty(I)$ of "intrinsically" smooth functions on I:

$$\widetilde{C}^\infty(I) = \{\varphi \in C^\infty((0,t)) : \varrho_{p,K}(\varphi) < \infty \text{ for every } p \in \mathbb{N}_0 \text{ and every compact set } K \subset (0,t]\}, \tag{2.3}$$

where

$$\varrho_{p,K}(\varphi) = \sum_{|\alpha|\le p} \sup_{x\in K\cap(0,t)} \left| \left(\frac{\partial}{\partial x}\right)^\alpha \varphi(x) \right|$$

are seminorms.

It is easy to see that $\widetilde{C}^\infty(I)$ can be defined equivalently in the following way:

$$\widetilde{C}^\infty(I) = \{\varphi \in C^\infty((0,t)) : (\partial/\partial x)^\alpha \varphi \text{ extends continuously to } (0,t] \text{ for every } \alpha \in \mathbb{N}_0^n\}.$$

Then we can also write

$$\varrho_{p,K}(\varphi) = \sum_{|\alpha|\le p} \sup_{x\in K} \left| \left(\frac{\partial}{\partial x}\right)^\alpha \varphi(x) \right|.$$

Moreover, instead of all compact sets $K \subset (0,t]$ we can take a sequence $K_j = [\delta_j, t]$ with $\delta_j \to 0_+$, exhausting I, and the seminorms ϱ_{p,K_j} define on $\widetilde{C}^\infty(I)$ an equivalent complete and locally convex topology. By the Seeley extension theorem (cf. [S])

$$\widetilde{C}^\infty(I) = C^\infty(I).$$

Now let $\widetilde{C}_0^\infty(I)$ be the space of smooth functions on I which vanish near $[0,t]\cap\partial\mathbb{R}_+^n$. We denote by $\widetilde{D}(I)$ the space $\widetilde{C}_0^\infty(I)$ endowed with the locally convex topology given by the seminorms ϱ_{p,K_j} defined above.

Theorem 2.3. *Let $t, \widetilde{t} \in \mathbb{R}_+^n$, $t < \widetilde{t}$. Then for every $0 < \varepsilon < \widetilde{t} - t$, $\varepsilon < t$ there exists a continuous linear extension map*

$$\mathcal{E}_\varepsilon : \widetilde{D}((0,t]) \to \widetilde{D}((0,\widetilde{t}])$$

such that for every $\varphi \in \widetilde{C}_0^\infty((0,t])$, $(\mathcal{E}_\varepsilon\varphi)(x) = 0$ if $t_j + \varepsilon_j < x_j < \widetilde{t}_j$ for some $1 \le j \le n$.

The proof of Theorem 2.3 can be given along the same lines as that of [Sz-Z1, Theorem 5.1] (actually it is simpler). It is based on the above-mentioned Seeley extension theorem from a half-line to the real line (cf. also [Sz-Z1, Lemma 5.1 and Exercise 5.14]).

By Theorem 2.3, $D(I) = \widetilde{D}(I)$ and hence $(D(I))' = (\widetilde{D}(I))'$. Moreover, Theorem 2.3 yields

Corollary 2.1. *Let $a, b \in \mathbb{R}^n$, $a < b$. Denote by A the polyinterval $(a,b]$, $[a,b)$, or $[a,b]$, any (fat) compact set $K_1 \times \ldots \times K_n$ with each K_j being a fat compact set in $\mathbb{R}$ (i.e. a finite sum of closed disjoint intervals with non-empty interior). Then $D(A) = \widetilde{D}(A)$ is a complete space* (cf. [Sz-Z1, Proposition 5.1]).

2.4. The spaces $D^k(K)$ and $(D^k(K))'$. In the case of distributions of finite order one does not actually need the spaces D of C^∞-functions. It is enough to consider suitable spaces of functions of a finite order of smoothness.

Let K be a fat compact set in $\mathbb{R}^n$ as in Corollary 2.1. The space $D^k(K)$ is defined analogously to $D(A)$ (see Subsection 2.3):

$$D^k(K) = \varinjlim_{H\subset\Omega} (D^k_H|_K) = \big(\varinjlim_{H\subset\Omega} D^k_H\big)|_K = D^k(\Omega)|_K,$$

where H ranges over all compact subsets of some open neighbourhood Ω of K. As in Corollary 2.1 the space

$$\widetilde{D}^k(K) = D^k(\mathbb{R}^n)|_K = D^k(K)$$

is complete and analogously to Proposition 2.2 (for $A = K$) we have

$$(D^k)'_K \simeq (D^k(K))'. \tag{2.4}$$

Moreover, it is easy to see that

$$D'(K) = \varinjlim_{k\in\mathbb{N}_0} (D^k)'_K. \tag{2.5}$$

We end this section with the following propositions:

PROPOSITION 2.3. *Let* $a, b \in \mathbb{R}^n$, $a < b$, $k \in \mathbb{N}_0$ *and define*

$$\widetilde{C}^k([a,b]) = \{\varphi \in C^k((a,b)) : (\partial/\partial x)^\alpha \varphi(x) \text{ extends continuously to } [a,b] \text{ for every } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \le k\}.$$

1° $\widetilde{C}^k([a,b])$ *equipped with the norm*

$$q_k(\varphi) = \sum_{|\alpha|\le k} \sup_{x\in[a,b]} \left|\left(\frac{\partial}{\partial x}\right)^\alpha \varphi(x)\right| \quad \text{for } \varphi \in C^k(\mathbb{R}^n)$$

is a Banach space $\widetilde{D}^k([a,b])$ *topologically equivalent to the space*

$$D^k([a,b]) = \varinjlim_{\delta>0} D^k_{[a-\delta,b+\delta]}|_{[a,b]}.$$

We write $(\widetilde{D}^k([a,b]))' = (D^k([a,b]))'$.

2° *Let* $u_\nu \in (D^k([a,b]))'$, $\nu = 1, 2, \ldots$, $u_\nu \to 0$. *Then there exists a sequence* $\varepsilon_\nu \to 0_+$ *such that*

$$|u_\nu[\varphi]| \le \varepsilon_\nu q_k[\varphi] \quad \text{for } \varphi \in C^{k+1}(\mathbb{R}^n).$$

For the proof let $\mathcal{K}^{k+1} = \{\eta \in C^{k+1}([a,b]) : q_{k+1}(\eta) \le 1\}$. Show that $\mathcal{K}^{k+1}$ is precompact in $D^k([a,b])$ and apply Proposition 1.1.

PROPOSITION 2.4. *Let* K *be a compact set in* $\mathbb{R}^n$ *as in Corollary* 2.1. *If* $u_\nu \in D'(K)$ $(\nu = 1, 2, \ldots)$ *and* $\lim_{\nu\to\infty} u_\nu = 0$ *then there exists a sequence* $\varepsilon_\nu \to 0_+$ *and* $p \in \mathbb{N}_0$ *such that* $|u_\nu[\varphi]| \le \varepsilon_\nu q_p(\varphi)$ *for* $\varphi \in C^p(\mathbb{R}^n)$.

3. Selected topics in the theory of holomorphic functions of one variable

3.1. Basic notions and theorems. In this subsection we present (without proofs) a collection of theorems and notions of the theory of holomorphic functions of one variable which will be used in the sequel.

The Cauchy theorem. *Let U be a bounded domain in $\mathbb{C}$ whose boundary $\gamma = \partial U$ is a positively oriented rectifiable curve* (1). *Suppose that F is holomorphic on U and continuous on $\overline{U}$. Then*

$$\int_\gamma F(\zeta)\,d\zeta = 0, \quad F(z) = \frac{1}{2\pi i}\int_\gamma \frac{F(\zeta)}{\zeta - z}\,d\zeta \quad \textit{for } z \in U.$$

The maximum modulus principle. *Let U be a bounded open set in $\mathbb{C}$ and suppose F is holomorphic on U and continuous on $\overline{U}$. Then*

$$\max\{|F(z)| : z \in \overline{U}\} = \max\{|F(z)| : z \in \partial U\}.$$

Next we recall the following singularity free version of the Mittag-Leffler theorem (see [Hö1]).

The Mittag-Leffler theorem. *Let $\{U_\tau\}_{\tau\in\mathcal{T}}$ be a family of open sets in $\mathbb{C}$ and let $\varphi_{\tau,\delta} \in \mathcal{O}(U_\tau \cap U_\delta)$ for $U_\tau \cap U_\delta \neq \emptyset$, $\tau, \delta \in \mathcal{T}$, be a family of functions such that*

$$\varphi_{\tau,\delta} = -\varphi_{\delta,\tau},$$
$$\varphi_{\tau,\delta} + \varphi_{\delta,\gamma} + \varphi_{\gamma,\tau} = 0 \quad \textit{on } U_\tau \cap U_\delta \cap U_\gamma \neq \emptyset \textit{ for } \tau, \delta, \gamma \in \mathcal{T}$$

(*such a family of functions is called a* cocycle). *Then there exist functions $\varphi_\tau \in \mathcal{O}(U_\tau)$ such that*

$$\varphi_{\tau,\delta} = \varphi_\delta - \varphi_\tau \textit{ on } U_\tau \cap U_\delta \neq \emptyset \textit{ for } \tau, \delta \in \mathcal{T}.$$

For two sets the above theorem reduces to

Corollary 3.1. *Let U_0 and U_1 be open sets in $\mathbb{C}$ such that $U_0 \cap U_1 \neq \emptyset$. For each $F \in \mathcal{O}(U_0 \cap U_1)$ there exist $F_j \in \mathcal{O}(U_j)$, $j = 0, 1$, such that*

$$F(z) = F_1(z) - F_0(z) \quad \textit{for } z \in U_0 \cap U_1. \tag{3.1}$$

The corollary can be verbalized as follows: after modification by a suitable (non-unique) function on U_0 (on U_1) every holomorphic function on $U_0 \cap U_1$ extends holomorphically to U_1 (to U_0).

The Vitali theorem (see [N]). *If a sequence $\{F_n\}$ of uniformly bounded holomorphic functions in a domain U is convergent on a set E having a condensation point $z_0 \in U$, then F_n converges locally uniformly in U.*

3.2. The spaces $A(K)$ and $A'(K)$. By the *germ* of an analytic function at a point $a \in \mathbb{R}$ (resp. at a compact set $K \subset \mathbb{R}$) we understand the equivalence class of analytic functions which coincide in a neighbourhood of a (resp. of K) in $\mathbb{R}$. Analogously we define the germs of holomorphic functions at $z \in \mathbb{C}$ or at a compact set $K \subset \mathbb{C}$. In that case we consider open neighbourhoods in $\mathbb{C}$. We equip the set of analytic (holomorphic) germs with a natural vector space structure. The space thus obtained is denoted by $A(K)$.

Let V be a bounded open set in $\mathbb{C}$. We denote by $H(\overline{V})$ the space of continuous functions in $\overline{V}$ which are holomorphic in V. We equip $H(\overline{V})$ with the maximum norm:

$$\|F\|_V = \sup_{z\in V} |F(z)|.$$

(1) Note that γ need not be connected.

It is easy to see that $H(\overline{V})$ is complete and thus it is a Banach space. As usual we denote the dual space by $H'(\overline{V})$.

Now we equip the linear space $A(K)$ with the inductive limit topology:

$$A(K) = \varinjlim_{V\supset K} H(\overline{V}),$$

where V ranges over open bounded sets in $\mathbb{C}$ containing K. Note that we identify $F_1 \in H(\overline{V}_1)$ with $F_2 \in H(\overline{V}_2)$ if $F_1 = F_2$ in an open neighbourhood of K. We also note that $\varphi_\nu \to 0$ in $A(K)$ as $\nu \to \infty$ if and only if there exists $V \supset K$ such that $\sup_{z\in V} |\varphi_\nu(z)| \to 0$.

The dual space of $H(\overline{V})$ is denoted by $H'(\overline{V})$. Let a sequence $u_\nu \in H'(\overline{V})$ be such that $\lim_{\nu\to\infty} u_\nu[\varphi]$ exists and is finite for every $\varphi \in H(\overline{V})$. Denote the limit by $u[\varphi]$. Since $H(\overline{V})$ is a Banach space, by the Banach–Steinhaus theorem there exists $C < \infty$ such that

$$|u_\nu[\varphi]|, |u[\varphi]| \le C \sup_{z\in V} |\varphi(z)| \quad \text{for } \nu \in \mathbb{N},\ \varphi \in H(\overline{V})$$

and therefore $u \in H'(\overline{V})$.

The dual space of $A(K)$ is denoted by $A'(K)$ and its elements are called *analytic functionals in* K. It follows that a linear functional u on $A(K)$ is an analytic functional if and only if for every open bounded set V with $K \subset V \subset \mathbb{C}$ the restriction $u|_{H(\overline{V})}$ belongs to $H'(\overline{V})$, i.e. there exists a constant $C_V < \infty$ such that

$$|u[\varphi]| < C_V \sup_V |\varphi| \quad \text{for } \varphi \in H(\overline{V}). \tag{3.2}$$

We say that a sequence $u_\nu \in A'(K)$ converges if $u_\nu[\varphi]$ converges to a finite limit for every $\varphi \in A(K)$. We then define $u[\varphi] := \lim_{\nu\to\infty} u_\nu[\varphi]$ for $\varphi \in A(K)$ and write $u = \lim_{\nu\to\infty} u_\nu$. The above definitions and remarks yield

Corollary 3.2. *Let $u_\nu \in A'(K)$ be a convergent sequence, and $u = \lim_{\nu\to\infty} u_\nu$. Then for every bounded set $V \supset K$ there exists a constant C_V such that*

$$|u_\nu[\varphi]|, |u[\varphi]| \le C_V \sup_{\zeta\in V} |\varphi(\zeta)| \quad \textit{for } \nu \in \mathbb{N},\ \varphi \in H(\overline{V})$$

and therefore $u \in A'(K)$.

For a given analytic functional $u \in A'(K)$, the set K is called the *carrier* of u.

Now assume that $K \subset \mathbb{R}$ is a non-empty compact set. Let $u \in A'(K)$. In this case there exists a closed non-empty set $K_{\min} \subset K$ such that u extends from $A(K)$ to $A(K_{\min})$ as a functional in $A'(K_{\min})$ but no proper subset of $K_{\min}$ has this property. It is natural to call $K_{\min}$ the *support* of u and we have $u \in A'(\operatorname{supp} u)$.

In the same way we define the support of $u \in A'(K)$, where K is a non-empty compact subset of a half-line $\mathring{\zeta} + \overline{\mathbb{R}}_+$ ($\mathring{\zeta} \in \mathbb{C}$).

If a single point $\mathring{z} \in \mathbb{C}$ is a carrier of an analytic functional u then $\{\mathring{z}\} = \operatorname{supp} u$. Such an example is given in point 2° below.

Proposition 3.1. *Let K be a fat compact set in $\mathbb{R}$. The set $A(K)$ is dense in $D^k(K)$ for any $k \in \mathbb{N}_0$.*

Proof. Let $\psi \in D^k(K)$, $\widetilde{a} = \min_{x\in K} x$ and $\widetilde{b} = \max_{x\in K} x$. There exists $\widetilde{\psi} \in C_0^k(\mathbb{R})$ such that $\psi = \widetilde{\psi}|_K$ and $\operatorname{supp}\widetilde{\psi} \subset [a,b]$, where $a < \widetilde{a}$, $b > \widetilde{b}$. By the Weierstrass approximation theorem there exists a sequence $\{w_j\}$ of polynomials such that

$$w_j \xrightarrow[j\to\infty]{} \left(\frac{d}{dx}\right)^k \widetilde{\psi} \quad \text{uniformly on } [a,b]. \tag{3.3}$$

Define

$$Jf(x) = \int_a^x f(t)\,dt \quad \text{for } x \in [a,b] \text{ and } f \in C^0([a,b]).$$

Observe that (since $(d/dx)^j \widetilde{\psi}(a) = 0$, $j = 0,1,\ldots,k$)

$$J^p\bigg(\left(\frac{d}{dx}\right)^k \widetilde{\psi}\bigg)(x) = \left(\frac{d}{dx}\right)^{k-p} \widetilde{\psi}(x) = \left(\frac{d}{dx}\right)^{k-p} \psi(x) \quad \text{for } x \in K,\ p = 1,\ldots,k$$

and we have

$$\left| J^p w_j(x) - \left(\frac{d}{dx}\right)^{k-p} \widetilde{\psi}(x)\right| \le (b-a)^p \sup_{t\in[a,b]} \left| w_j(t) - \left(\frac{d}{dt}\right)^k \widetilde{\psi}(t)\right| \tag{3.4}$$

for $p = 0,1,\ldots,k$ and $x \in [a,b]$.

Let $v_j = J^k w_j$ for $j = 1,2,\ldots$ It follows from (3.3) and (3.4) that for $l = 0,1,\ldots,k$,

$$\left(\frac{d}{dx}\right)^l (v_j - \widetilde{\psi}) = J^{k-l} w_j - \left(\frac{d}{dx}\right)^l \widetilde{\psi} \xrightarrow[j\to\infty]{} 0$$

uniformly on $[a,b]$ and hence $v_j \to \psi$ in $D^k(K)$. ∎

Proposition 3.2. *If $\varphi_j \in A(K)$ and $\varphi_j \to 0$ in $A(K)$ as $j \to \infty$ then $\varphi_j \to 0$ in $D^k(K)$ for any $k \in \mathbb{N}_0$.*

Proof. There exists a complex neighbourhood V of K such that $\varphi_j \in \mathcal{O}(V)$ and $\sup_V |\varphi_j(z)| \to 0$. Choose a simple positively oriented rectifiable curve γ in V encircling K once. Let $\varrho = \operatorname{dist}(K,\gamma)$. We have $\varrho > 0$ and for z in the domain bounded by γ and containing K,

$$\varphi_j^{(l)}(z) = \frac{l!}{2\pi i}\int_\gamma \frac{\varphi_j(\zeta)}{(\zeta - z)^{l+1}}\,d\zeta, \quad l = 0,1,2,\ldots$$

Hence for $l = 0,1,2,\ldots,$

$$|\varphi_j^{(l)}(z)| \le \frac{l!}{2\pi}|\gamma|\frac{1}{\varrho^{l+1}}\sup_V |\varphi_j(\zeta)|$$

for z in K. Thus $\varphi_j \to 0$ in $D^k(K)$. ∎

Propositions 3.1 and 3.2 yield

Corollary 3.3. $(D^k(K))' \hookrightarrow A'(K)$ *for any $k \in \mathbb{N}_0$.*

Remark 3.1. Proposition 3.1 extends to compact sets $K \subset \mathbb{C} \simeq \mathbb{R}^2$. For the proof it suffices to apply the Stone–Weierstrass theorem instead of the Weierstrass theorem. Hence also Corollary 3.3 holds for $K \subset \mathbb{C}$, and moreover

$$\varinjlim_{k\in\mathbb{N}_0} \big(D^k(K)\big)' \hookrightarrow A'(K).$$

Thus (2.4) and (2.5) yield $D'(K) \hookrightarrow A'(K)$.

We now make two more observations.

1° Let K be a compact set in $\mathbb{C}$. A sequence $\{F_n\}$ in $A(K)$ converges in $A(K)$ iff for some open set V with $K \subset V \subset \mathbb{C}$, it converges uniformly on $\overline{V}$, and $F_n \in H(\overline{V})$ for $n = 1, 2, \ldots$ Consequently, the sequences $\{F_n^{(p)}\}$ are uniformly convergent on $\overline{V}$ $(p = 1, 2, \ldots)$.

2° The formal series

$$u[\varphi] = \sum_{j=0}^{\infty} a_j \frac{\varphi^{(j)}(0)}{j!} \quad \text{for } \varphi \in A(\{0\}),\ a_j \in \mathbb{C}$$

defines $u \in A'(\{0\})$ if and only if for every $\varepsilon > 0$ there exists $0 < C_\varepsilon < \infty$ such that (2)

$$|a_j| \le C_\varepsilon \varepsilon^j,\ j = 0, 1, 2, \ldots \tag{3.5}$$

For sufficiency take arbitrary $\delta > 0$, $V = (-\delta, \delta)$ and $\varphi \in H(\overline{V})$. Apply the Cauchy inequalities and (3.5) with $\varepsilon < \delta$. For necessity apply (3.2) with $V = (-\varepsilon, \varepsilon)$ and $\varphi_k(z) = z^k$ $(k = 0, 1, 2, \ldots)$.

3.3. Boundary values of holomorphic functions of one variable. An open simply connected set $V \subset \mathbb{C}$ is called a *complex neighbourhood* of a set $E \subset \mathbb{R}$ if E is a relatively closed subset of V. In the case when E is open in $\mathbb{R}$ we denote it by Ω (we then have $V \cap \mathbb{R} = \Omega$). If E is closed in $\mathbb{R}$ it is denoted by K.

Let Ω be an open set in $\mathbb{R}$ and V its complex neighbourhood and let $F \in \mathcal{O}(V \setminus \Omega)$. We are interested in conditions the function F should satisfy near Ω in order to extend holomorphically to V. The simplest such condition is given by the classical Painlevé theorem which states that a function F holomorphic on $V \setminus \Omega$ and continuous on V extends holomorphically to V. The Painlevé theorem does not extend the set where the function F is defined but only asserts that the singularities of F on Ω are removable. The theorem follows easily from the classical Morera theorem. The Painlevé theorem admits a generalization to the case where the set $\Omega \subset \mathbb{R}$ is replaced by a rectifiable curve: more precisely, we have

THEOREM 3.1 (see [R]). *Let V be a domain in $\mathbb{C}$. Suppose V is divided into two disjoint domains V_1 and V_2 by a rectifiable curve contained in V except for its end points. Suppose that $F \in \mathcal{O}(V_1 \cup V_2)$ and is continuous in the closure of V. Then F is holomorphic in V.*

The continuity of F on Ω can be formulated as the existence and equality of the limits

$$\lim_{\substack{z \to x \\ \operatorname{Im} z > 0}} F(z) = \lim_{\substack{z \to x \\ \operatorname{Im} z < 0}} F(z) \quad \text{for } x \in \Omega,$$

or in other words as equality of the boundary values of F from above and below.

We now proceed to a generalization of the notion of boundary values. In what follows V is an open set in $\mathbb{C}$, $V \cap \mathbb{R} = \Omega$, and F is a holomorphic function on $V \setminus \Omega$.

DEFINITION 3.1. Let $F \in \mathcal{O}(V \setminus \Omega)$. Assume that the limits of the integral means

$$b_\pm F[\varphi] = F(\,\cdot\, \pm i0)[\varphi] = \lim_{\varepsilon \to 0_+} \int_\Omega F(\alpha \pm i\varepsilon)\varphi(\alpha)\, d\alpha \tag{3.6}$$

(2) Note that (3.5) implies $\lim_{j\to\infty} \sqrt[j]{|a_j|} = 0$.

exist for every $\varphi \in C_0^\infty(\Omega)$. Then the Banach–Steinhaus theorem implies that $b_\pm F \in D'(\Omega)$. They are called the *distributional boundary values of F from above* (+) and *below* (−), respectively. Their difference, called the *jump of F across* $\mathbb{R}$ and denoted by

$$bF = b_+F - b_-F,$$

is also a distribution in $D'(\Omega)$.

Now we formulate a condition under which F has the distributional boundary values $b_\pm F$.

THE CONDITION OF POLYNOMIAL GROWTH. We say that a function $F \in \mathcal{O}(V \setminus \Omega)$, where $V \cap \mathbb{R} = \Omega$ and V is open in $\mathbb{C}$, is *of polynomial growth* (near the real axis) if for every compact set $K \subset \Omega$ there exist constants $N = N(K) \in \mathbb{N}_0$, $C = C(K)$ and $\varepsilon = \varepsilon(K) > 0$ such that

$$|F(\alpha + i\beta)| \le C|\beta|^{-N} \quad \text{for } \alpha \in K,\ 0 < |\beta| < \varepsilon. \tag{3.7}$$

THEOREM 3.2. *Suppose that $F \in \mathcal{O}(V \setminus \Omega)$ is of polynomial growth. Then the distributional boundary values* (3.6) *exist and $bF = b_+F - b_-F \in D'(\Omega)$.*

Let $F_\nu \in \mathcal{O}(V \setminus \Omega)$ ($\nu = 1, 2, \ldots$) be a sequence of functions such that for every compact set $K \subset \Omega$ there exist $N = N(K) \in \mathbb{N}_0$, $\varepsilon = \varepsilon(K) > 0$ (independent of ν) and $C_\nu = C_\nu(K) \to 0$ as $\nu \to \infty$ such that

$$|F_\nu(\alpha + i\beta)| \le C_\nu|\beta|^{-N} \quad \textit{for } \alpha \in K,\ 0 < |\beta| < \varepsilon. \tag{3.7ν}$$

Then the boundary values

$$F_\nu(\,\cdot\, \pm i0)[\varphi] = \lim_{\varepsilon \to 0_+} \int_\Omega F_\nu(\alpha \pm i\varepsilon)\varphi(\alpha)\, d\alpha \quad \textit{for } \varphi \in C_0^\infty(\Omega) \tag{3.6ν}$$

exist, they are distributions and moreover $\lim_{\nu\to\infty} F_\nu(\,\cdot\, \pm i0) = 0$ *in $D'(\Omega)$. Hence*

$$bF_\nu = b_+F_\nu - b_-F_\nu = F_\nu(\,\cdot\, + i0) - F_\nu(\,\cdot\, - i0) \xrightarrow[\nu\to\infty]{} 0 \quad \textit{in } D'(\Omega).$$

PROOF. We may assume that Ω is an open interval. Let K be a fixed closed subinterval of Ω. Choose $0 < \eta < 1$ so that

$$A := \{z = \alpha + i\beta \in \mathbb{C} : \alpha \in K,\ 0 < \beta \le \eta\} \subset V_+ := V \cap \{\operatorname{Im} z > 0\}$$

and (3.7) holds for $\alpha + i\beta \in A$. Fix $\mathring{z} = \mathring{\alpha} + i\eta \in A$ and let γ_z be an arbitrary rectifiable curve joining $\mathring{z}$ to $z = \alpha + i\beta \in V_+$ and contained in V_+. Define

$$J_+F(z) = \int_{\gamma_z} F(\theta)\, d\theta, \quad z \in V_+.$$

Clearly $(d/dz)J_+F(z) = F(z)$ for $z \in V_+$. To get estimates of $J_+F(z)$ for $z \in A$ we take for γ_z the union of two segments $[\mathring{z}, \alpha + i\eta]$ and $[\alpha + i\eta, \alpha + i\beta]$ (lying in A). Then

$$J_+F(z) = \int_{\mathring{\alpha}}^{\alpha} F(t + i\eta)\, dt + i\int_\eta^\beta F(\alpha + is)\, ds. \tag{3.8}$$

Thus if $N = 0$ in (3.7) we have $|J_+F(\alpha + i\beta'') - J_+F(\alpha + i\beta')| \le C|\beta' - \beta''|$, hence the limit $\lim_{\beta\to 0_+} J_+F(\alpha + i\beta)$ exists uniformly for $\alpha \in K$.

If $N > 0$ from (3.7) and the fact that F is bounded on $K + i\eta$ we get

$$|J_+F(z)| \le \frac{C|K|}{\eta^N} + C\int_\beta^\eta t^{-N}\,dt \quad \text{for } z \in A.$$

Hence there exists a constant C_1 independent of C and depending on N, η and $|K|$ such that for $\alpha + i\beta \in A$,

$$|J_+F(\alpha+i\beta)| \le \begin{cases} CC_1\beta^{-N+1} & \text{if } N > 1,\\ CC_1|\ln\beta| & \text{if } N = 1.\end{cases}$$

Iterating the above N times we find $J_+^N F \in \mathcal{O}(V_+)$, $|J_+^N F(\alpha+i\beta)| \le CC_N|\ln\beta|$ on A with C_N independent of C and by (3.8) (with $J_+^N F$ instead of F) we get

$$|J_+^{N+1}F(\alpha+i\beta'') - J_+^{N+1}F(\alpha+i\beta')| \le CC_N|\beta'\ln\beta' - \beta''\ln\beta'' - \beta' + \beta''|,$$

which is arbitrarily small if β', β'' are sufficiently small. Thus $\lim_{\varepsilon\to 0_+} J_+^{N+1}F(\alpha+i\varepsilon) =: F_+^{N+1}(\alpha)$ exists uniformly on K and $(d^{N+1}/dz^{N+1})J_+^{N+1}F = F$ on V_+.

Hence F_+^{N+1} is continuous on K. Thus $J_+^{N+1}F$ extends to a continuous function on $\overline{A} = \{z \in \mathbb{C} : \alpha \in K,\ 0 \le \beta \le \eta\}$.

Similarly by (3.8) we get

$$|J_+^{N+1}F(\alpha+i\beta)| \le CC_N|K|\cdot|\ln\eta| + CC_N(\eta + \eta|\ln\eta|) = CC_{N+1}$$

with C_{N+1} independent of C. Hence $|F_+^{N+1}(\alpha)| \le CC_{N+1} < \infty$ for $\alpha \in K$.

Let $\varphi \in C_K^\infty(\Omega)$. Integrating by parts we get by the Lebesgue theorem

$$\lim_{\varepsilon\to 0_+}\int_\Omega F(\alpha+i\varepsilon)\varphi(\alpha)\,d\alpha = \lim_{\varepsilon\to 0_+}(-1)^{N+1}\int_\Omega J_+^{N+1}F(\alpha+i\varepsilon)\left(\frac{d}{d\alpha}\right)^{N+1}\varphi(\alpha)\,d\alpha$$
$$= (-1)^{N+1}\int_\Omega F_+^{N+1}(\alpha)\left(\frac{d}{d\alpha}\right)^{N+1}\varphi(\alpha)\,d\alpha =: F(\,\cdot\,+i0)[\varphi].$$

By symmetry one defines J_-F and $J_-^{N+1}F$ on $V_- = V \cap \{\operatorname{Im} z < 0\}$ and one proves the existence of the limit $F(\,\cdot\,-i0)$. Thus

$$(3.9)\qquad F(\,\cdot\,\pm i0)[\varphi] = (-1)^{N+1}\int_\Omega F_\pm^{N+1}(\alpha)\left(\frac{d}{d\alpha}\right)^{N+1}\varphi(\alpha)\,d\alpha \quad \text{for } \varphi \in C_K^\infty(\Omega).$$

Hence $F(\,\cdot\,\pm i0)$ is a linear functional on $C_K^\infty(\Omega)$ and

$$(3.10)\qquad |F(\,\cdot\,\pm i0)[\varphi]| \le CC_{N+1}|K|\sup_{\alpha\in K}\left|\left(\frac{d}{d\alpha}\right)^{N+1}\varphi(\alpha)\right| \quad \text{for } \varphi \in C_K^\infty(\Omega)$$

since $\sup_{\alpha\in K}|F_\pm^{N+1}(\alpha)| \le CC_{N+1}$.

Thus, by (2.1), $F(\,\cdot\,\pm i0)$ is a distribution, since K was an arbitrary closed subinterval of Ω.

Considering now the sequence $\{F_\nu\}$ we arrive as above at the relations (3.6ν) with $F_\nu(\,\cdot\,\pm i0) \in D'(\Omega)$. Moreover, by (3.7$\nu$) we get for $\varphi \in C_K^\infty$ the estimates (3.10) with F_ν replaced F and C_ν replaced C. The assumption $C_\nu \to 0$ then yields $F_\nu(\,\cdot\,\pm i0)[\varphi] \to 0$.

Remark 3.2. Note the method of regularization by taking primitives, used in the proof of Theorem 3.2. In the sequel, variants of this method will be used in Lemmas 7.4 and 7.5.

We shall prove later on that every distribution on Ω can be represented as the jump of a function F holomorphic on a set $V \setminus \Omega$, where V is a complex neighbourhood of Ω, of polynomial growth near Ω (see Theorem 4.6′).

Remark 3.3. Via the translation $z \mapsto \mathring{z} + z$, the result of Theorem 3.2 extends to the case where $\mathbb{R}$ is replaced by the line $\mathring{z} + \mathbb{R}$, with an arbitrary fixed $\mathring{z} \in \mathbb{C}$.

Now we are going to prove a generalization of the Painlevé theorem assuming equality of the distributional boundary values $b_+(F) = b_-(F)$. We start with

Proposition 3.3. *Let Ω be open in $\mathbb{R}$, V open in $\mathbb{C}$ and $V \cap \mathbb{R} = \Omega$. Let $F \in \mathcal{O}(V \setminus \Omega)$ admit distributional boundary values on Ω: $F(\cdot + i0)$ and $F(\cdot - i0)$. Then F is of polynomial growth* (*near the real axis*).

We postpone the proof of Proposition 3.3 to Subsection 4.5. It will use some tools of hyperfunction theory.

Theorem 3.2 and Proposition 3.3 imply

Corollary 3.4. *Let $F \in \mathcal{O}(V \setminus \Omega)$. Then the following conditions are equivalent*:

(i) *F is of polynomial growth* (*near the real axis*).

(ii) *F has distributional boundary values from above and below on Ω.*

Theorem 3.3 (the distributional Painlevé theorem). *Retain the assumptions of Proposition* 3.3 *and suppose that*

$$bF = F(\cdot + i0) - F(\cdot - i0) = 0. \tag{3.11}$$

Then F extends to a holomorphic function on V.

Proof. It follows from formula (3.9) in the proof of Theorem 3.2 and the definition of the derivative of a distribution that

$$F(\cdot \pm i0) = \left(\frac{d}{d\alpha}\right)^{N+1} F_\pm^{N+1}.$$

Hence $(d/d\alpha)^{N+1}(F_+^{N+1} - F_-^{N+1}) = 0$ by (3.11), which leads to the conclusion that $F_+^{N+1} - F_-^{N+1}$ is a polynomial (denote it by w) of degree at most N. Consequently, the function

$$F^{N+1}(z) = \begin{cases} J_+^{N+1}F(z) & \text{for } z \in V_+, \\ F_+^{N+1}(z) & \text{for } z \in \Omega, \\ J_-^{N+1}F(z) + w(z) & \text{for } z \in V_-, \end{cases}$$

is continuous on V. Now it follows from the classical Painlevé theorem that it is holomorphic on V. Hence $(d/dz)^{N+1}F^{N+1} \in \mathcal{O}(V)$ and coincides with F on $V_+ \cup V_-$, giving the desired extension. ∎

Example 3.1. With $\mathrm{Pf}\frac{1}{x}$ being the distribution of example (c) in Subsection 2.3 and $\delta_0 = \delta_{(0)}$ as in (a) we have

$$\lim_{\varepsilon\to 0_+} \frac{1}{x \pm i\varepsilon} = \mp i\pi\delta_0 + \mathrm{Pf}\frac{1}{x}$$

and hence

$$\lim_{\varepsilon\to 0_+} \left(\frac{1}{x+i\varepsilon} - \frac{1}{x-i\varepsilon}\right) = -2\pi i\delta_0, \quad \text{i.e.} \quad \frac{1}{2\pi i} b\left(\frac{-1}{z}\right) = \delta_0.$$

The proof is standard in the theory of distributions.

We end this section with the following statements which will be applied in Section 4:

3° Let $L \subset \mathbb{R}$ be closed. If $F \in \mathcal{O}(\mathbb{C} \setminus L)$ is of polynomial growth then $bF \in D'_L$.

4° If $F \in \mathcal{O}(\mathbb{C}\setminus\mathbb{R})$ admits distributional boundary values on $\mathbb{R}$ from above and below then $b_\pm(dF/dz)$ both exist and $b_\pm(dF/dz) = (d/dx)b_\pm F$.

5° Let Y be the Heaviside function. Then $b\big(-\frac{1}{2\pi i}\operatorname{Ln}(-z)\big) = Y$, where Ln denotes the principal branch of the logarithm.

4. Hyperfunctions in one variable

4.1. Definitions and basic properties of hyperfunctions. Throughout this section we denote by Ω an open set in $\mathbb{R}$ and by V a complex neighbourhood of Ω (see the beginning of Subsection 3.3).

Definition 4.1. Let $F \in \mathcal{O}(V \setminus \Omega)$. The equivalence class $[F]$ of F in $\mathcal{O}(V \setminus \Omega)$ modulo $\mathcal{O}(V)$ (i.e. the set $F + \mathcal{O}(V)$) is called a *hyperfunction* on Ω. The function F is called a *defining function* for the hyperfunction $f = [F]$.

Actually, one should write $[F]_{\mathrm{mod}\,\mathcal{O}(V)}$ instead of $[F]$ (and we do so in some cases). The notation $[F]$ suggests that the equivalence class of F is independent of the choice of V containing Ω as a closed subset; this is proved in Theorem 4.1 below.

Clearly the defining function is not uniquely determined by f: one may always add an arbitrary function holomorphic on V. A defining function for the zero hyperfunction is an arbitrary holomorphic function on V.

The space of hyperfunctions on an open set $\Omega \subset \mathbb{R}$ is denoted by $B(\Omega)$, i.e.

$$B(\Omega) = \mathcal{O}(V \setminus \Omega)/\mathcal{O}(V) \ (^1).$$

It is a $\mathbb{C}$-linear space: for arbitrary $\lambda, \mu \in \mathbb{C}$, $f = [F]$, $g = [G] \in B(\Omega)$, $F, G \in \mathcal{O}(V \setminus \Omega)$ we define $\lambda f + \mu g = [\lambda F + \mu G]$.

We now prove that the space $B(\Omega)$ is independent of the choice of a complex neighbourhood V of Ω such that Ω is relatively closed in V. More precisely, we have

Theorem 4.1. *If W and V are complex neighbourhoods of $\Omega \subset \mathbb{R}$ then the quotient space $\mathcal{O}(V \setminus \Omega)/\mathcal{O}(V)$ is canonically isomorphic to $\mathcal{O}(W \setminus \Omega)/\mathcal{O}(W)$.*

(1) We regard $\mathcal{O}(V)$ as a vector subspace of $\mathcal{O}(V \setminus \Omega)$ under the restriction operation (cf. Remark 4.1).

Proof. We may assume $W \subset V$ (otherwise we take $W \cap V \subset V$ and $W \cap V \subset W$). Let $B_V(\Omega) = \mathcal{O}(V \setminus \Omega)/\mathcal{O}(V)$ and $B_W(\Omega) = \mathcal{O}(W \setminus \Omega)/\mathcal{O}(W)$. The restriction mappings

$$\mathcal{O}(V \setminus \Omega) \to \mathcal{O}(W \setminus \Omega) \quad \text{and} \quad \mathcal{O}(V) \to \mathcal{O}(W)$$

induce a linear mapping

$$I : B_V(\Omega) \to B_W(\Omega)$$

defined for $f = [F]_{\mathrm{mod}\,\mathcal{O}(V)}$, where $F \in \mathcal{O}(V \setminus \Omega)$, by

$$[F]_{\mathrm{mod}\,\mathcal{O}(V)} \mapsto [F_{|W\setminus\Omega}]_{\mathrm{mod}\,\mathcal{O}(W)}.$$

The map I is injective. Indeed, if $f \in B_V(\Omega)$ with defining function $F \in \mathcal{O}(V \setminus \Omega)$ and $If = 0$ in $B_W(\Omega)$ (i.e. $F_{|W\setminus\Omega}$ extends holomorphically to W), then F extends holomorphically to V, i.e. $f = 0$ in $B_V(\Omega)$.

To prove that I is surjective take $f \in B_W(\Omega)$, $f = [F]$, $F \in \mathcal{O}(W \setminus \Omega)$. We apply Corollary 3.1 to the function F and the open sets $U_0 = W$ and $U_1 = V \setminus \Omega$. It follows that there exist $F_1 \in \mathcal{O}(V \setminus \Omega)$ and $F_0 \in \mathcal{O}(W)$ such that (3.1) holds for $z \in W \cap (V \setminus \Omega) = W \setminus \Omega$, i.e. $F(z) + F_0(z)$ extends holomorphically from the set $W \setminus \Omega$ to a function $F_1 \in \mathcal{O}(V \setminus \Omega)$. Putting $f_1 = [F_1] \in B_V(\Omega)$ we infer that $If_1 = [F_{1|W\setminus\Omega}]_{\mathrm{mod}\,\mathcal{O}(W)} = [F + F_0]_{\mathrm{mod}\,\mathcal{O}(W)} = [F]_{\mathrm{mod}\,\mathcal{O}(W)} = f$ since $F_0 \in \mathcal{O}(W)$. ∎

The independence of the definition of a hyperfunction from the choice of a complex neighbourhood plays an important role for restriction, gluing and extension of hyperfunctions. We describe these operations below.

Let $\Omega' \subset \Omega$ be an open subset of Ω. To define the *restriction mapping*

$$B(\Omega) \ni f \mapsto f_{|\Omega'} \in B(\Omega')$$

we fix a complex neighbourhood V of Ω, represent $B(\Omega) = \mathcal{O}(V \setminus \Omega)/\mathcal{O}(V)$ and take $V' = (V\setminus\Omega)\cup\Omega'$, $B(\Omega') = \mathcal{O}(V'\setminus\Omega')/\mathcal{O}(V')$. If $f \in B(\Omega)$ then $f = [F]_{\mathrm{mod}\,\mathcal{O}(V)}$ for some $F \in \mathcal{O}(V \setminus \Omega)$. Since we also have $F \in \mathcal{O}(V' \setminus \Omega')$ and V' is a complex neighbourhood of Ω' we define $f|_{\Omega'} = [F]_{\mathrm{mod}\,\mathcal{O}(V')}$.

The above definitions imply

Lemma 4.1. *Let $\Omega' \subset \Omega \subset \mathbb{R}$ be open sets and let V be a complex neighbourhood of Ω. Take $f = [F]_{\mathrm{mod}\,\mathcal{O}(V)}$ where $F \in \mathcal{O}(V \setminus \Omega)$. Then F extends holomorphically from $V \setminus \Omega$ to $V' = (V \setminus \Omega) \cup \Omega'$ if and only if $f|_{\Omega'} = 0$.*

Example 4.1. The Dirac delta hyperfunction $\delta_{(0)} \in B(\Omega)$, $0 \in \Omega$, defined by $\delta_{(0)} = -\frac{1}{2\pi i}[\frac{1}{z}]_{\mathrm{mod}\,\mathcal{O}(V)}$, where V is any complex neighbourhood of Ω, equals zero on $\Omega \setminus \{0\}$. Since Ω is an arbitrary open (in $\mathbb{R}$) set containing 0 and V its arbitrary complex neighbourhood, we write simply

$$\delta_{(0)} = -\frac{1}{2\pi i}\left[\frac{1}{z}\right].$$

The $\delta_{(0)}$ hyperfunction, denoted for simplicity by δ, will serve to illustrate different concepts and results discussed later on. The corresponding examples will be denoted by $4.1^{I}, 4.1^{II}$.

Now, we pass to the *gluing* of hyperfunctions.

THEOREM 4.2. *Let* $\{\Omega_j\}_{j\in J}$, $\Omega_j \subset \Omega$, *be an open covering of* Ω.

(i) *If* $f \in B(\Omega)$ *and* $f_{|\Omega_j} = 0$ *for all* j *then* $f = 0$ *in* $B(\Omega)$.

(ii) *If* $f_j \in B(\Omega_j)$, $j \in J$, *satisfy the condition*

$$f_{j|\Omega_j\cap\Omega_k} = f_{k|\Omega_j\cap\Omega_k} \quad \textit{for } j,k \textit{ such that } \Omega_j \cap \Omega_k \neq \emptyset$$

(*such a family of functions is called* compatible) *then there exists* $f \in B(\Omega)$ *such that* $f_{|\Omega_j} = f_j$ *for* $j \in J$.

PROOF. (i) Let V be a complex neighbourhood of Ω, $f = [F]_{\mathrm{mod}\,\mathcal{O}(V)}$, $V_j = (V \setminus \Omega) \cup \Omega_j$, $0 = f_{|\Omega_j} = [F]_{\mathrm{mod}\,\mathcal{O}(V_j)}$. Hence $F \in \mathcal{O}(V_j)$ for every j and since $\bigcup_j V_j = V$ it follows that $F \in \mathcal{O}(V)$ and consequently $[F] = 0$.

(ii) The proof will be based on the Mittag-Leffler theorem (see Subsection 3.1). Let $f_j = [F_j]$ where $F_j \in \mathcal{O}(V_j \setminus \Omega_j)$ and V_j are complex neighbourhoods of Ω_j. Put $V = \bigcup_j V_j$ and observe that V is a complex neighbourhood of Ω. Let $G_{ij} = F_j - F_i$. Since $f_{j|\Omega_j\cap\Omega_i} = f_{i|\Omega_j\cap\Omega_i}$ we see that $G_{ij} \in \mathcal{O}(V_i \cap V_j)$. It is clear that the G_{ij} form a cocycle. Hence the Mittag-Leffler theorem yields the existence of $G_j \in \mathcal{O}(V_j)$ such that $G_{ij} = G_j - G_i$ on $V_i \cap V_j$. We thus have

$$F_i - G_i = F_j - G_j \quad \text{on } (V_i \setminus \Omega_i) \cap (V_j \setminus \Omega_j). \tag{4.1}$$

Let

$$F(z) = F_j(z) - G_j(z) \quad \text{for } z \in V_j \setminus \Omega_j \text{ and all } j.$$

The definition is correct due to property (4.1) and leads to a function $F \in \mathcal{O}(V \setminus \Omega)$. Let $f = [F]$. We have $f_{|\Omega_j} = [F_{|V_j\setminus\Omega_j}] = [F_j - G_j] = [F_j] = f_j$ since $G \in \mathcal{O}(V_j)$. ■

Property (ii) in Theorem 4.2 is called the *localization property*. It constitutes together with property (i) the fundamental requirement to be met by objects which generalize the notion of a function. The property (i) allows us to define the *support* of a hyperfunction to be the minimal closed subset of Ω such that f vanishes on its complement (in Ω).

EXAMPLE 4.1^I. The support of Dirac's delta considered in Example 4.1 equals $\{0\}$.

In contrast to distributions and analytic functions, hyperfunctions have the following unrestricted extendability property (cf. Example 4.2):

THEOREM 4.3. *Let* Ω' *be a proper open subset of an open set* $\Omega \subset \mathbb{R}$. *Then every hyperfunction* $f \in B(\Omega')$ *extends to a hyperfunction* $\widetilde{f} \in B(\Omega)$ *with support in* $\overline{\Omega'}$. *Such an extension is unique up to a hyperfunction with support in* $\partial\Omega'$.

PROOF. Choose a complex neighbourhood V of Ω. Then $V_1 = V \setminus \partial\Omega'$ is a complex neighbourhood of Ω' and $V_1 \setminus \Omega' = V \setminus \overline{\Omega'}$. Moreover, $V_2 = V \setminus \overline{\Omega'}$ is a complex neighbourhood of $\Omega' \setminus \overline{\Omega'}$.

Let $B(\Omega') \ni f = [F]_{\mathrm{mod}\,\mathcal{O}(V_1)}$, $F \in \mathcal{O}(V_1 \setminus \Omega')$. Since $V \setminus \Omega \subset V \setminus \overline{\Omega'} = V_1 \setminus \Omega'$ the following definition is correct: $B(\Omega') \ni \widetilde{f} = [F]_{\mathrm{mod}\,\mathcal{O}(V)}$. Moreover, $B(\Omega') \ni \widetilde{f}_{|\Omega'} = [F]_{\mathrm{mod}\,\mathcal{O}(V_1)} = f$, $B(\Omega \setminus \overline{\Omega'}) = \widetilde{f}_{|\Omega\setminus\overline{\Omega'}} = [F]_{\mathrm{mod}\,\mathcal{O}(V_2)} = 0$ (since $F \in \mathcal{O}(V_2)$), and hence $\operatorname{supp}\widetilde{f} \subset \overline{\Omega'}$. ■

To end this subsection we distinguish an important subspace of $B(\Omega)$, namely the subspace of hyperfunctions with support in a compact set K, denoted by $B_K(\Omega)$. Below we observe that $B_K(\Omega)$ may be represented in the following special way:

PROPOSITION 4.1. *Fix a compact set $K \subset \mathbb{R}$. Then for every open (in $\mathbb{R}$) set Ω containing K and any complex neighbourhood U of Ω we have the isomorphism*

$$B_K(\Omega) \simeq \mathcal{O}(U \setminus K)/\mathcal{O}(U). \tag{4.2}$$

Moreover, if Ω_1, Ω_2 are two open sets in $\mathbb{R}$ containing K, then

$$B_K(\Omega_1) \simeq B_K(\Omega_2).$$

Therefore we shall denote the space in (4.2) *by B_K suppressing Ω.*

PROOF. Take $f \in B_K(\Omega)$. Then $f = 0$ on $\Omega \setminus K$. Represent f as $f = [F]_{\mathrm{mod}\,\mathcal{O}(U)}$ with $F \in \mathcal{O}(U \setminus \Omega)$ and observe by Lemma 4.1 that F extends holomorphically to $\widetilde{F} \in \mathcal{O}(U \setminus K)$. Clearly the mapping $\mathcal{J} : B_K(\Omega) \ni f = [F]_{\mathrm{mod}\,\mathcal{O}(U)} \mapsto [\widetilde{F}]_{\mathrm{mod}\,\mathcal{O}(U)} \in \mathcal{O}(U \setminus K)/\mathcal{O}(U)$ is an injection. To see that it is surjective take $\widetilde{F} \in \mathcal{O}(U \setminus K)$, and put $F = \widetilde{F}_{|U \setminus \Omega}$ and $f = [F]_{\mathrm{mod}\,\mathcal{O}(U)} \in B(\Omega)$. Since F extends holomorphically to $U \setminus K = (U \setminus \Omega) \cup (\Omega \setminus K)$, by Lemma 4.1 we have $f_{|\Omega \setminus K} = 0$ and hence $f \in B_K(\Omega)$ and $\mathcal{J} f = [\widetilde{F}]_{\mathrm{mod}\,\mathcal{O}(U)}$. ∎

PROPOSITION 4.2. *The space of functions $F \in \mathcal{O}(U \setminus K)$ holomorphically extendable to U is closed in the space $\mathcal{O}(U \setminus K)$ with the locally uniform convergence topology. Hence the quotient space $\mathcal{O}(U \setminus K)/\mathcal{O}(U)$ is a Hausdorff topological space.*

PROOF. Let $\widetilde{F}_\nu \in \mathcal{O}(U)$, $F_\nu = \widetilde{F}_{\nu|U \setminus K}$ ($\nu = 1, 2, \ldots$), and suppose that $F_\nu \to F$ as $\nu \to \infty$ locally uniformly on $U \setminus K$. Clearly $F \in \mathcal{O}(U \setminus K)$ and by the maximum modulus principle (see Subsection 3.1) the sequence $\widetilde{F}_\nu$ converges locally uniformly on U. Thus F extends holomorphically on U. ∎

Another proof of Proposition 4.2 follows from Lemma 4.2 below.

REMARK 4.1. In the case of the space $B(\Omega) = \mathcal{O}(V \setminus \Omega)/\mathcal{O}(V)$ (V an open set in $\mathbb{C}$, $\Omega = V \cap \mathbb{R}$) the space $\mathcal{O}(V)$ is not closed in $\mathcal{O}(V \setminus \Omega)$; hence the quotient topology in $B(\Omega)$ has pathological properties and therefore is not interesting.

4.2. Imbedding of analytic functions in hyperfunctions: $A(\Omega) \hookrightarrow B(\Omega)$. Let Ω be an open set in $\mathbb{R}$. We denote by $A(\Omega)$ the space of analytic functions on Ω. Since every analytic function on Ω extends to a complex neighbourhood U of Ω we may regard $A(\Omega)$ as

$$A(\Omega) = \varinjlim_{U \supset \Omega} \mathcal{O}(U).$$

Let $\varphi \in A(\Omega)$ and let $\Phi \in \mathcal{O}(U)$ be such that $\Phi_{|\Omega} = \varphi$, where U is a complex neighbourhood of Ω. Set $U_+ = U \cap \{\operatorname{Im} z > 0\}$, $U_- = U \cap \{\operatorname{Im} z < 0\}$ and

$$\Phi_+(z) = \begin{cases} \Phi(z) & \text{for } z \in U_+, \\ 0 & \text{for } z \in U_-. \end{cases}$$

We define the mapping

$$A(\Omega) \ni \varphi \mapsto [\Phi_+] \in B(\Omega) \tag{4.3}$$

and say that the hyperfunction $[\Phi_+]$ *coincides with the analytic function* φ. Observe that by Theorem 4.1 the mapping (4.3) is well defined (i.e. it is independent of the choice of U). Also it is easy to see that (4.3) is injective and coincides with

$$A(\Omega) \ni \varphi \mapsto -[\Phi_-] \in B(\Omega),$$

where

$$\Phi_-(z) = \begin{cases} 0 & \text{for } z \in U_+, \\ \Phi(z) & \text{for } z \in U_-. \end{cases}$$

We now describe "intrinsically" the image of $A(\Omega)$ in the space of hyperfunctions:

PROPOSITION 4.3. *A hyperfunction* $f = [F] \in B(\Omega)$ *is an analytic function in* Ω *if and only if both* $F_{|\{\operatorname{Im} z>0\}}$ *and* $F_{|\{\operatorname{Im} z<0\}}$ *extend holomorphically to a complex neighbourhood of* Ω.

PROOF. Let $\widetilde{F}_+$ and $\widetilde{F}_-$ be holomorphic extensions of $F_{|\{\operatorname{Im} z>0\}}$ and $F_{|\{\operatorname{Im} z<0\}}$ respectively to a complex neighbourhood of Ω. Clearly $[\widetilde{F}_-] = 0$ so that $f = [F - \widetilde{F}_-]$ and we have $F - \widetilde{F}_- = 0$ on $\{\operatorname{Im} z < 0\}$. Set $\varphi = (\widetilde{F}_+ - \widetilde{F}_-)_{|\Omega}$. Then φ is the analytic function which is mapped onto f by the mapping (4.3). The converse implication is obvious. ∎

EXAMPLE 4.2. Consider the analytic function $\varphi(x) = e^{-1/x}$ for $x \in \mathbb{R} \setminus \{0\}$. Clearly φ does not extend to an analytic function on $\mathbb{R}$ and even considered as a distribution in $D'(\mathbb{R} \setminus \{0\})$ it does not extend to an element of $D'(\mathbb{R})$ (cf. [Sz, Exercise 3.7]). On the other hand, the hyperfunction $f = [\Phi_+]$ with defining function

$$\Phi_+(z) = \begin{cases} e^{-1/z} & \text{for } \operatorname{Im} z > 0, \\ 0 & \text{for } \operatorname{Im} z < 0, \end{cases}$$

belongs to $B(\mathbb{R})$. Its restriction to $\mathbb{R} \setminus \{0\}$ coincides with the analytic function φ. In other words, the analytic function φ, regarded as a hyperfunction, extends to the hyperfunction f on $\mathbb{R}$.

4.3. Elementary operations on hyperfunctions. $B(\Omega)$ is naturally equipped with the structure of a $\mathbb{C}$-linear space. Now we introduce in it the following operations:

Multiplication by analytic functions. Let $\varphi \in A(\Omega)$, $f \in B(\Omega)$. Letting V be a complex neighbourhood of Ω such that φ extends to a $\Phi \in \mathcal{O}(V)$ and representing $f = [F]$ with $F \in \mathcal{O}(V \setminus \Omega)$ we define

$$\varphi f = [\Phi F] \in B(\Omega).$$

Further on we shall use the same symbol φ for an analytic function $\varphi \in A(\Omega)$ and its extension to a complex neighbourhood of Ω.

Differentiation. If $f = [F] \in B(\Omega)$ we define

$$\frac{d}{d\alpha} f = \left[\frac{d}{dz} F\right] \in B(\Omega).$$

Note that this is compatible with the standard differentiation of analytic functions under the natural imbedding $A(\Omega) \hookrightarrow B(\Omega)$.

4.4. The Köthe theorem. Let K be a compact set in $\mathbb{R}$ and $f = [F] \in B_K$. Denote by $\mathcal{I}f$ the functional defined by

$$A(K) \ni \varphi \mapsto (\mathcal{I}f)[\varphi] := -\int_\gamma F(z)\varphi(z)\,dz, \tag{4.4}$$

where γ is a Jordan curve encircling K in the anticlockwise direction and contained in the sets of holomorphy of F and of (a holomorphic continuation of) φ. Clearly $\mathcal{I}f$ is a well defined linear and continuous functional on $A(K)$, which means that $\mathcal{I}f[\varphi_\nu] \to 0$ as $\nu \to \infty$ if $\varphi_\nu \to 0$ in the inductive limit topology of $A(K)$ (Subsection 3.2). Thus $\mathcal{I}$ is a mapping

$$\mathcal{I} : B_K \ni f \mapsto \mathcal{I}f \in A'(K). \tag{4.5}$$

We aim at proving that $\mathcal{I}$ is a topological isomorphism. We begin with

Lemma 4.2. *Let $K \subset \mathbb{R}$ be a compact set and U a domain in $\mathbb{C}$ containing K. Let $G \in \mathcal{O}(U \setminus K)$. Then $G \in \mathcal{O}(U)$ if and only if*

$$\int_\gamma G(z)\varphi(z)\,dz = 0 \quad \textit{for } \varphi \in A(K), \tag{4.6}$$

where γ encircles K once in the anticlockwise direction and lies in the set where both G and φ are holomorphic.

Proof. The "only if" part is obvious. To prove the "if" part assume that $G \in \mathcal{O}(U \setminus K)$. Take a two-connected domain P_{γ^-,γ^+} bounded by the inner curve γ^- and the outer curve γ^+ encircling K, with both γ^- and γ^+ being Jordan curves. Observe that for $\zeta \in P_{\gamma^-,\gamma^+}$ the function $\varphi_\zeta(x) = \frac{1}{2\pi i}\frac{1}{x-\zeta}$ is in $A(K)$ and hence by the Cauchy theorem and by (4.6) we get

$$G(\zeta) = \frac{1}{2\pi i}\int_{\gamma^+} \frac{G(z)}{z-\zeta}\,dz.$$

Thus G is holomorphic on the domain U_{γ^+} bounded by γ^+ and hence on U, since $G \in \mathcal{O}(U \setminus K)$. ∎

Lemma 4.3. *Let $K \subset \mathbb{R}$ be a compact set and U a domain in $\mathbb{C}$ containing K. Let $[F]_{\mathrm{mod}\,\mathcal{O}(U)} = f \in B_K$, $F \in \mathcal{O}(U \setminus K)$. For $z \in \mathbb{C} \setminus K$ define*

$$F^{\mathrm{s}}(z) = -\frac{1}{2\pi i}\int_{\widetilde{\gamma}} \frac{F(\zeta)}{\zeta - z}\,d\zeta, \tag{4.7}$$

where $\widetilde{\gamma} \subset U\setminus K$ is a Jordan curve which encircles K in the anticlockwise direction leaving the point z on the right. Then $F^{\mathrm{s}} \in \mathcal{O}(\mathbb{C}\setminus K)$ and $F - F^{\mathrm{s}} \in \mathcal{O}(U)$, i.e. $[F^{\mathrm{s}}]_{\mathrm{mod}\,\mathcal{O}(U)} = f$.

The function F^{s} is called the *standard defining function* for f and (by the Liouville theorem) it is uniquely determined by the conditions: $F^{\mathrm{s}} \in \mathcal{O}(\mathbb{C} \setminus K)$ and $F^{\mathrm{s}}(z) \to 0$ as $z \to \infty$.

Proof. Clearly $F^{\mathrm{s}} \in \mathcal{O}(\mathbb{C} \setminus K)$ and hence the function $G(z) := F(z) - F^{\mathrm{s}}(z)$ for $z \in U \setminus K$ is holomorphic on $U \setminus K$. Take a function $\varphi \in A(K)$ and a Jordan curve $\gamma \subset U$ encircling K and contained in the set of holomorphy of φ. We show that (4.6) holds.

To this end take a Jordan curve $\widetilde{\gamma}$ encircling K and contained in the domain V_γ bounded by γ and observe that by the Cauchy theorem and (4.7),

$$\int_\gamma F(z)\varphi(z)\,dz = \int_{\widetilde{\gamma}} F(\zeta)\varphi(\zeta)\,d\zeta = \frac{1}{2\pi i}\int_{\widetilde{\gamma}} F(\zeta)\bigg(\int_\gamma \frac{\varphi(z)}{z-\zeta}\,dz\bigg)\,d\zeta$$
$$= -\frac{1}{2\pi i}\int_\gamma \varphi(z)\bigg(\int_{\widetilde{\gamma}} \frac{F(\zeta)}{\zeta - z}\,d\zeta\bigg)\,dz = \int_\gamma F^{\mathrm{s}}(z)\varphi(z)\,dz,$$

which proves (4.6). Thus by Lemma 4.2, $F - F^{\mathrm{s}} \in \mathcal{O}(U)$. ■

Now we introduce a sequence topology (compatible with the quotient topology) in the space $B_K \simeq \mathcal{O}(U \setminus K)/\mathcal{O}(U)$, U open in $\mathbb{C}$, $U \supset K$. Let $f_\nu \in B_K$, $f_\nu = [F_\nu]$, where $F_\nu \in \mathcal{O}(U \setminus K)$ are arbitrary defining functions for f_ν $(\nu = 1, 2, \ldots)$.

Definition 4.2. We say that the sequence f_ν of hyperfunctions with support in K is *convergent* if there exist defining functions F_ν such that the sequence F_ν is locally uniformly convergent on $U \setminus K$. Clearly $F := \lim_{\nu\to\infty} F_\nu \in \mathcal{O}(U \setminus K)$. We set

$$\lim_{\nu\to\infty} f_\nu = f := [F].$$

Theorem 4.4 (Köthe [Kö], cf. also [Ko1]). *For every compact set $K \subset \mathbb{R}$ there exists a natural topological isomorphism*

$$B_K \simeq A'(K)$$

given by (4.5), *where $\mathcal{I}f$ is defined in* (4.4). *The inverse mapping is*

$$A'(K) \ni g \mapsto \mathcal{J}g = [\mathcal{C}g], \tag{4.8}$$

where

$$\mathcal{C}g(z) = \frac{-1}{2\pi i}\, g\bigg[\frac{1}{z-\alpha}\bigg] \quad \text{for } z \in \mathbb{C} \setminus K \tag{4.9}$$

is the Cauchy transform of g.

Proof. We prove that $\mathcal{J}$ is inverse to $\mathcal{I}$. Since for every $z \in \mathbb{C} \setminus K$ the function $\varphi_z(\alpha) = \frac{-1}{2\pi i}\frac{1}{z-\alpha}$ is in $A(K)$, formula (4.9) defines a function holomorphic on $\mathbb{C} \setminus K$. Thus by Proposition 4.1 we get $[\mathcal{C}g] \in B_K$. It remains to prove that $\mathcal{I} \circ \mathcal{J} = \mathrm{Id}$ in $A'(K)$ and $\mathcal{J} \circ \mathcal{I} = \mathrm{Id}$ in B_K.

To check the first identity fix $g \in A'(K)$. Let $\varphi \in A(K)$. We have to show that $\mathcal{I} \circ \mathcal{J}g[\varphi] = g[\varphi]$. Choosing the curve γ appropriately we get by (4.4), (4.8) and the Cauchy integral formula,

$$\mathcal{I} \circ \mathcal{J}g[\varphi] = -\int_\gamma (\mathcal{C}g)(z)\varphi(z)\,dz = \frac{1}{2\pi i}\int_\gamma g\bigg[\frac{1}{z-\alpha}\bigg]\varphi(z)\,dz$$
$$= g\bigg[\frac{1}{2\pi i}\int_\gamma \frac{\varphi(z)}{z-\alpha}\,dz\bigg] = g[\varphi].$$

The interchange of integration and the action of the linear functional g can be justified by considering Riemann sums and using the linearity and continuity of g.

Now, suppose $f \in B_K$ and let $g = \mathcal{I} f$ (defined by (4.4)). Then by (4.9) and (4.7),

$$\mathcal{C}g(z) = -\frac{1}{2\pi i} g\left[\frac{1}{z - \cdot}\right] = \frac{1}{2\pi i} \int_\gamma F(\zeta) \frac{1}{z-\zeta}\, d\zeta = F^{\mathrm{s}}(z) \quad \text{for } z \in \mathbb{C} \setminus K \tag{4.10}$$

and by Lemma 4.3,

$$\mathcal{J} \circ \mathcal{I} f = \mathcal{J} g = [\mathcal{C} g] = [F^{\mathrm{s}}] = f,$$

which ends the proof that $\mathcal{J} = \mathcal{I}^{-1}$.

Now we prove that the algebraic isomorphism $\mathcal{I}$ is a topological isomorphism.

Let $B_K \ni f_\nu \to 0$ as $\nu \to \infty$. Then there exist defining functions F_ν for the hyperfunctions f_ν such that F_ν is locally uniformly convergent to F on $U \setminus K$ and $[F] = 0$, i.e. $F \in \mathcal{O}(U)$. Hence by (4.4) we get for any $\varphi \in A(K)$ the following estimates:

$$\begin{aligned} |\mathcal{I} f_\nu[\varphi]| &= \Big| \int_\gamma \varphi(z) F_\nu(z)\, dz \Big| = \Big| \int_\gamma \varphi(z)(F_\nu(z) - F(z))\, dz \Big| \\ &\le \int_\gamma |\varphi(z)|\, |dz| \cdot \sup_{z\in\gamma} |F_\nu(z) - F(z)| \end{aligned}$$

and therefore $\mathcal{I} f_\nu \to 0$ in $A'(K)$.

To prove the converse assume that $g_\nu \to 0$ in $A'(K)$, i.e. $g_\nu[\varphi] \to 0$ for every $\varphi \in A(K)$. Since $\mathcal{J} g_\nu = [\mathcal{C} g_\nu]$, it suffices to show that $\mathcal{C} g_\nu \to 0$ locally uniformly on $U \setminus K$. Observe that $\varphi_z(\zeta) = \frac{-1}{2\pi i} \frac{1}{z-\zeta} \in A(K)$ for $z \in \mathbb{C} \setminus K$ and hence by (4.9) and the assumption on g_ν we get the pointwise convergence: $\mathcal{C} g_\nu(z) = g_\nu[\varphi_z] \to 0$ for $z \in \mathbb{C} \setminus K$.

Next we prove that this convergence is locally uniform on $\mathbb{C} \setminus K$. Take an arbitrary compact set $\Lambda \subset \mathbb{C} \setminus K$ and choose an open bounded set V such that $K \subset V$ and $\operatorname{dist}(V, \Lambda) = \varrho > 0$. Clearly $\varphi_z \in H(\overline{V})$ for $z \in \Lambda$ and by Corollary 3.2 we get

$$|\mathcal{C} g_\nu(z)| \le C_V \sup_{\zeta \in V} |\varphi_z(\zeta)| \le C_V \frac{1}{2\pi\varrho} \quad \text{for } z \in \Lambda,\ \nu \in \mathbb{N}.$$

Hence $\mathcal{C} g_\nu$ is a convergent sequence of holomorphic functions, uniformly bounded on Λ. By the Vitali theorem it is locally uniformly convergent on $\mathbb{C} \setminus K$ since $\Lambda \subset \mathbb{C} \setminus K$ was arbitrary. ■

Theorem 4.4 yields

COROLLARY 4.1. *A sequence of hyperfunctions $f_\nu \in B_K$ is convergent if and only if the sequence of standard defining functions $F^{\mathrm{s}}_\nu = \mathcal{C}(\mathcal{I} f_\nu)$ is locally uniformly convergent on $\mathbb{C} \setminus K$ (with $\mathcal{I}$, $\mathcal{C}$ defined by* (4.4), (4.10)).

Owing to Corollary 4.1 we can define convergence of series:

DEFINITION 4.3. A series $\sum_{j=0}^\infty f_j$ of hyperfunctions $f_j \in B_K$ $(j \in \mathbb{N}_0)$ is *convergent* if the corresponding series $\sum_{j=0}^\infty F^{\mathrm{s}}_j$ of standard defining functions is convergent locally uniformly on $\mathbb{C} \setminus K$. We set

$$\sum_{j=0}^\infty f_j = \Big[\sum_{j=0}^\infty F^{\mathrm{s}}_j\Big]. \tag{4.11}$$

Clearly the following proposition holds.

Proposition 4.4. *Let* $f_j \in B_K$, $g_j = \mathcal{I} f_j$ $(j = 0, 1, 2, \ldots)$ (cf. (4.4)) *and denote by* F_j^{s} *the standard defining function of* f_j. *Then the following three conditions are equivalent*:

(w) $\lim_{N\to\infty} \sum_{j=0}^{N} g_j[\varphi] = g[\varphi]$ *for every* $\varphi \in A(K)$.

(w′) $\sum_{j=0}^{\infty} F_j^{\mathrm{s}}$ *is locally uniformly convergent on* $U \setminus K$.

(w″) *There exist defining functions* G_j *for* f_j, $j = 0, 1, 2, \ldots$, *such that* $\sum_{j=0}^{\infty} G_j$ *is locally uniformly convergent on* $U \setminus K$.

If one of the above conditions holds then $\sum_{j=0}^{\infty} f_j = [\mathcal{C}g]$.

Remark 4.2. The Köthe theorem can be used to define the support of an analytic functional $g \in A'(K)$, K a compact set in $\mathbb{R}$, as the support of the corresponding hyperfunction $\mathcal{I}^{-1}g$. Note that this definition agrees with the definition given in Subsection 3.2.

Remark 4.3. Theorem 4.4 extends easily to compact sets $K \subset \mathbb{C}$ with B_K replaced by $\mathcal{O}(\mathbb{C} \setminus K)/\mathcal{O}(\mathbb{C})$ and more generally to compact sets on the universal covering space of $\mathbb{C} \setminus \{0\}$ (cf. Subsection 6.1).

The following proposition gives an example of such an extension.

Proposition 4.5. *Let* $G(z, w)$ *be a function holomorphic on* $\mathbb{C} \times \mathbb{C} \setminus \{(z, w) : z = w\}$. *Suppose that the function* $\mathbb{C} \ni z \mapsto G(z, w)$ *has a simple pole at* $z = w$ *with residue* 1 *and the function* $\mathbb{C} \ni w \mapsto G(z, w)$ *has a simple pole at* $w = z$ *with residue* -1. *Let* K *be any compact set in* $\mathbb{C}$. *Then there exists a natural topological isomorphism*

$$\mathcal{O}(\mathbb{C} \setminus K)/\mathcal{O}(\mathbb{C}) \simeq A'(K)$$

given by

$$\mathcal{O}(\mathbb{C} \setminus K)/\mathcal{O}(\mathbb{C}) \ni [F] \mapsto g \in A'(K),$$

where $F \in \mathcal{O}(\mathbb{C} \setminus K)$, $g[\varphi] = -\int_\gamma \varphi(z)F(z)\,dz$ *for* $\varphi \in A(K)$ *and a closed curve* γ *encircling* K *in the anticlockwise direction and contained in the set of holomorphy of* φ. *The inverse mapping is* $A'(K) \ni g \mapsto [\mathcal{C}_G g]$, *where* $\mathcal{C}_G g(z) = -\frac{1}{2\pi i} g[G(z, \cdot)]$ *is holomorphic on* $\mathbb{C} \setminus K$.

Outline of the proof. Let $\widetilde{F}(z) = -\frac{1}{2\pi i}\int_\gamma F(w)G(z, w)\,dw$ for $z \in \mathbb{C} \setminus K$ and a curve γ encircling once K in the anticlockwise direction and leaving z on the right. Prove that $F(z) - \widetilde{F}(z)$ extends holomorphically over K (cf. the proof of Lemma 4.3). ∎

4.5. Imbedding $D'_K \hookrightarrow B_K$, K compact in $\mathbb{R}$. We begin with a remark.

Remark 4.4. Every $u \in D'_K$ (K a compact set in $\mathbb{R}$) can be considered as an element of $A'(K)$, i.e. there exists a natural imbedding $D'_K \hookrightarrow A'(K)$.

Indeed, by Proposition 2.1 we consider u as an element of $(D(K))'$ and actually of $(D^k(K))'$ with some $k \in \mathbb{N}_0$ since u is of finite order. Hence by Corollary 3.3, $u \in A'(K)$.

Now by Theorem 4.4 we get the imbedding:

$$D'_K \ni u \mapsto [\mathcal{C}u] \in B_K \quad \text{where} \quad (\mathcal{C}u)(z) = -\frac{1}{2\pi i} u\left[\frac{1}{z - \alpha}\right]. \tag{4.12}$$

EXAMPLE 4.1^{II}. Recall that the Dirac $\delta_{(0)}$ distribution belongs to $D'_{\{0\}}$ and hence by Remark 4.4 to $A'(\{0\})$. Hence by (4.12),

$$(\mathcal{C}\delta_{(0)})(z) = -\frac{1}{2\pi i}\delta_{(0)}\left[\frac{1}{z-\alpha}\right] = -\frac{1}{2\pi i}\frac{1}{z}$$

and by the isomorphism of Theorem 4.4 the hyperfunction $\left[-\frac{1}{2\pi i}\frac{1}{z}\right]$ corresponds to the Dirac $\delta_{(0)}$ distribution. Combining this with Example 3.1 we can write

$$b\left(-\frac{1}{2\pi i}\frac{1}{z}\right) = \delta_{(0)} \simeq \left[-\frac{1}{2\pi i}\frac{1}{z}\right],$$

where b denotes the difference of boundary values in the distributional sense. This justifies the name $\delta_{(0)}$-hyperfunction given to $\left[-\frac{1}{2\pi i}\frac{1}{z}\right]$ in Example 4.1. We write briefly δ for $\delta_{(0)}$.

Note the following properties of the function $\mathcal{C}u$ with $u \in D'_K$ (which are evident for $u = \delta$ in Example 4.1^{II}):

1° $\mathcal{C}u \in \mathcal{O}(\mathbb{C}\setminus K)$ tends to zero as $z \to \infty$ and is of polynomial growth near the real axis (see Subsection 3.3).

2° $\mathcal{C}u$ has distributional boundary values from above and below. Their difference equals u:

$$b(\mathcal{C}u) = u. \tag{4.13}$$

PROOF. 1° Let $\varepsilon > 0$ and $\kappa \in C_0^\infty(\mathbb{R})$, $\kappa = 1$ on $K_{\varepsilon/2}$, $\kappa = 0$ on $\mathbb{R}\setminus K_\varepsilon$ (where $K_\varepsilon = \{\alpha \in \mathbb{R} : \operatorname{dist}(\alpha, K) \le \varepsilon\}$). Since u has support in K it is of finite order k and consequently

$$\begin{aligned}|(\mathcal{C}u)(z)| &= \frac{1}{2\pi}\left|u\left[\frac{1}{z-\alpha}\right]\right| = \frac{1}{2\pi}\left|u\left[\kappa(\alpha)\frac{1}{z-\alpha}\right]\right| \\ &\le C\sum_{j=0}^{k}\sup_{\alpha\in K_\varepsilon}\left|\frac{d^j}{d\alpha^j}\left(\kappa(\alpha)\frac{1}{z-\alpha}\right)\right| \le C_1\sum_{j=0}^{k}\sup_{\alpha\in K_\varepsilon}\left|\frac{d^j}{d\alpha^j}\frac{1}{z-\alpha}\right|.\end{aligned}$$

Hence our assertion follows.

2° The existence of the boundary values follows from 1° by Theorem 3.2. To prove (4.13) take $\varphi \in C_0^\infty(\mathbb{R})$ and observe that

$$b(\mathcal{C}u)[\varphi] = -\lim_{\varepsilon\to 0_+}\frac{1}{2\pi i}\int_{\mathbb{R}} u\left[\frac{1}{a+i\varepsilon-\alpha} - \frac{1}{a-i\varepsilon-\alpha}\right]\varphi(a)\,da \overset{(2)}{=} \lim_{\varepsilon\to 0_+} u[\sigma_\varepsilon] = u[\varphi]$$

where

$$\begin{aligned}\sigma_\varepsilon(\alpha) &= -\frac{1}{2\pi i}\int_{\mathbb{R}}\left(\frac{1}{a+i\varepsilon-\alpha} - \frac{1}{a-i\varepsilon-\alpha}\right)\varphi(a)\,da \\ &= -\frac{1}{2\pi i}\int_{\mathbb{R}}\left(\frac{1}{x+i\varepsilon} - \frac{1}{x-i\varepsilon}\right)\varphi(x+\alpha)\,dx\end{aligned}$$

and $\lim_{\varepsilon\to 0}\sigma_\varepsilon = \varphi$ locally uniformly on $\mathbb{R}$ with all derivatives. ■

Formula (4.13) implies at once

(2) By passing to Riemann sums of the integral over $\operatorname{supp}\varphi$.

Corollary 4.2. *Every $u \in D'_K$ can be considered as a jump* (*in the sense made precise in Theorem* 3.2) *of the function $\mathcal{C}u \in \mathcal{O}(\mathbb{C} \setminus K)$ of polynomial growth near the real axis. By* (4.12) *we can write*

$$b(\mathcal{C}u) = u \simeq [\mathcal{C}u] \tag{4.14}$$

where $\simeq$ is the isomorphism from the Köthe theorem.

Corollary 4.3. *Let $K \subset \mathbb{R}$ be a compact set and suppose that $F \in \mathcal{O}(\mathbb{C} \setminus K)$ is of polynomial growth near the real axis. Then*

$$bF[\varphi] = -\int_\gamma F(z)\varphi(z)\,dz \quad \textit{for } \varphi \in A(K) \tag{4.15}$$

where γ is a continuous curve encircling K once in the anticlockwise direction and included in the intersection of $\mathbb{C} \setminus K$ with the set to which φ extends holomorphically.

Proof. By point 3° of Subsection 3.3, $u := b(F) \in D'_K$. Hence by Corollary 4.2, $u = b(\mathcal{C}u)$ and by Theorem 3.3, $F - \mathcal{C}u$ extends to an entire function. Thus $[F] = [\mathcal{C}u]$. From Remark 4.4, $u \in A'_K$ and so the Köthe theorem yields $\mathcal{I}^{-1}u = [\mathcal{C}u] \in B_K$. Therefore $u[\varphi] = \mathcal{I}(\mathcal{I}^{-1}u)[\varphi] = \mathcal{I}([F])[\varphi] = -\int_\gamma F(z)\varphi(z)\,dz$ for $\varphi \in A(K)$ and so (4.15) follows since $u = b(F)$. ■

Now we prove Proposition 3.3 using some facts presented in this section.

Proof of Proposition 3.3. Let $u := bF \in D'(\Omega)$. Take an arbitrary compact set $K \subset \Omega$. Choose $\delta > 0$ small enough that $K_{2\delta} = \{x \in \Omega : \operatorname{dist}(x, K) < 2\delta\} \Subset \Omega$ and take $\chi \in C_0^\infty(\Omega)$ with $\chi = 1$ on K_δ and $\chi = 0$ outside $K_{2\delta}$. Let $v = \chi u$ and $\widetilde{K} = \operatorname{supp} v$. Then $\widetilde{K}$ is compact, $\widetilde{K} \subset \Omega$ and $v \in D'_{\widetilde{K}}$. Thus by 1° the function $G := \mathcal{C}v$ holomorphic on $\mathbb{C} \setminus \widetilde{K}$ is of polynomial growth near the real axis. By 2°, $bG = v$. Hence $H := F - G \in \mathcal{O}(V \setminus \Omega)$, $bH = bF - bG = u - v$ and since $u = v$ on K_δ, $bH = 0$ on K_δ. Let $h = [H]_{\mathrm{mod}\,\mathcal{O}(V)}$. Then $h_{|K_\delta} = 0$ and by Lemma 4.1, H extends holomorphically to $(V \setminus \Omega) \cup K_\delta$. Hence H is bounded on some complex neighbourhood of K_δ and since $F = H + G$ and G is of polynomial growth near the real axis, inequality (3.8) holds with some $N = N(K)$, $C = C(K)$ and $\varepsilon = \varepsilon(K)$. This ends the proof since the compact set $K \subset \Omega$ was arbitrary. ■

Now we sketch a longer but more natural proof of Corollary 4.3 which does not refer to the Köthe theorem.

Elementary proof of Corollary 4.3. Take $K = [x_1, x_2]$, $\varphi \in A(K)$ and V open in $\mathbb{C}$ such that $\varphi \in \mathcal{O}(V)$ and $K \subset V$. Fix $\eta > 0$ and $\varrho > 0$ such that the rectangle $P_\eta = [x_1 - \varrho, x_2 + \varrho] + i[-\eta, \eta]$ is contained in V. Let γ_b be the boundary of a rectangle P_b $(0 < b < \eta)$ encircling K in the anticlockwise direction and observe that $\int_{\gamma_b} F(z)\varphi(z)\,dz$ does not depend on b. By point 3° of Subsection 3.3, $bF \in D'_K$ and hence $bF[\varphi] = bF[\widetilde{\varphi}]$ where $\widetilde{\varphi} \in C_0^\infty(\mathbb{R})$, $(\widetilde{\varphi} - \varphi)_{|[x_1-\varrho, x_2+\varrho]} = 0$ and $\widetilde{\varphi}(x) = 0$ for $x \notin [x_1 - 2\varrho, x_2 + 2\varrho]$. Observe that

$$bF[\varphi] = \lim_{b \to 0_+} \int_{x_1-\varrho}^{x_2+\varrho} F(x + ib)\varphi(x)\,dx - \lim_{b \to 0_+} \int_{x_1-\varrho}^{x_2+\varrho} F(x - ib)\varphi(x)\,dx$$

and

$$\lim_{b\to 0}\int_{-b}^{b} F(x_1-\varrho+iy)\varphi(x_1-\varrho+iy)\,dy = 0 = \lim_{b\to 0}\int_{-b}^{b} F(x_2+\varrho+iy)\varphi(x_2+\varrho+iy)\,dy.$$

Hence to end the proof it suffices to show that

$$\lim_{b\to 0}\int_{x_1-\varrho}^{x_2+\varrho} F(x\pm ib)(\varphi(x)-\varphi(x\pm ib))\,dx = 0.$$

This is clear if F is bounded near the real line. Otherwise one proceeds analogously to the proof of Theorem 3.2 (cf. the proof of Lemma 7.5). ■

4.6. The distributional version of the Köthe theorem. Let K be a compact set in $\mathbb{R}$, U a bounded domain in $\mathbb{C}$ containing K and let $k\in\mathbb{N}_0$. We define the following topological spaces.

The space $\mathcal{O}^k(U\setminus K)$ of holomorphic functions on $U\setminus K$ is defined by

$$\mathcal{O}^k(U\setminus K) := \{H\in\mathcal{O}(U\setminus K) : q_{k,V}(H)<\infty \text{ for every compact set } V\subset U\},$$

where $q_{k,V}(H) := \sup_{z\in V} ||\mathrm{Im}\, z|^k H(z)|$ are seminorms defining the topology in $\mathcal{O}^k(U\setminus K)$.

Thus $H_\nu\to 0$ in $\mathcal{O}^k(U\setminus K)$ as $\nu\to\infty$ if and only if $q_{k,V}(H_\nu)\to 0$ as $\nu\to\infty$ for every compact set $V\subset U$.

Define

$$\mathcal{O}^\infty(U\setminus K) := \varinjlim_{k\in\mathbb{N}_0} \mathcal{O}^k(U\setminus K) \quad\text{and}\quad B_K^\infty := \mathcal{O}^\infty(U\setminus K)/\mathcal{O}(U).$$

Thus $f\in B_K^\infty$ if and only if there exist $k\in\mathbb{N}_0$ and $H\in\mathcal{O}^k(U\setminus K)$ such that $f=[H]_{\mathrm{mod}\,\mathcal{O}(U)}$.

PROPOSITION 4.6. *Let K be a compact set in $\mathbb{R}$, and U open in $\mathbb{C}$ with $K\subset U$. Let $H\in\mathcal{O}^\infty(U\setminus K)$. Then there exists $k\in\mathbb{N}$ such that*

$$\sup_{z\in V}(\mathrm{dist}(z,K))^k|H(z)|<\infty \quad \textit{for every compact set } V\subset U.$$

PROOF. By 3° of Subsection 3.3, $u := bH\in D'_K$ since H is of polynomial growth. Observe that $\mathcal{C}u - H\in\mathcal{O}(U)$ and that by Theorem 2.2 there exist $C<\infty$ and $k\in\mathbb{N}_0$ such that

$$|\mathcal{C}u(z)|\le C\frac{1}{(\mathrm{dist}(z,K))^k}.$$

These facts yield the desired assertion. ■

We say that a sequence of hyperfunctions $f_\nu\in B_K^\infty$ converges to zero if there exist $k\in\mathbb{N}_0$ and defining functions F_ν such that $f_\nu=[F_\nu]$ and $F_\nu\to 0$ in $\mathcal{O}^k(U\setminus K)$ as $\nu\to\infty$.

THEOREM 4.5. *There exists a natural topological isomorphism*

$$B_K^\infty \simeq D'(K)$$

compatible with the isomorphism of Theorem 4.4.

Proof. Let $u \in D'(K)$. By (2.5) and (2.4) there exists $k \in \mathbb{N}_0$ such that $u \in (D^k(K))'$ and by Corollary 3.3, $u \in A'(K)$. By Theorem 4.4, $\mathcal{I}^{-1}u = [\mathcal{C}u]_{\operatorname{mod}\mathcal{O}(U)}$, where the function $\mathcal{C}u$ defined by (4.9) belongs to $\mathcal{O}^{k+1}(U \setminus K)$. Hence $\mathcal{I}^{-1}u \in B_K^\infty$.

If $u_\nu \in D'(K)$ $(\nu = 1, 2, \ldots)$ and $\lim_{\nu\to\infty} u_\nu = 0$ in $D'(K)$ then by Proposition 2.4 there exist $p \in \mathbb{N}_0$ and a sequence $\varepsilon_\nu \to 0_+$ such that

$$|u_\nu[\varphi]| \le \varepsilon_\nu q_p(\varphi) \quad \text{for } \varphi \in C^p(\mathbb{R}), \quad \text{where} \quad q_p(\varphi) = \sum_{j\le p} \sup_{x\in K} \left|\left(\frac{d}{dx}\right)^j \varphi(x)\right|.$$

Then there exists $C < \infty$ such that

$$|\,|\operatorname{Im} z|^{p+1}\mathcal{C}u_\nu(z)| \le \frac{1}{2\pi}\left||\operatorname{Im} z|^{p+1} u_\nu\left[\frac{1}{z-\alpha}\right]\right| \le C\varepsilon_\nu$$

and hence $\mathcal{C}u_\nu \to 0$ in $\mathcal{O}^{p+1}(U \setminus K)$. Thus $\mathcal{I}^{-1} : D'(K) \to B_K^\infty$ is continuous.

If $f \in B_K^\infty$ then there exist $k \in \mathbb{N}_0$ and $F \in \mathcal{O}^k(U \setminus K)$ such that $f = [F]$. By point 3° of Subsection 3.3 and Proposition 2.2, $bF \in D'_K \simeq D'(K)$, and by (4.4) and Corollary 4.3, $bF = \mathcal{I}f$. Thus $\mathcal{I}f \in D'(K)$.

To prove the continuity of $\mathcal{I}$ observe that if $f_\nu \to 0$ in B_K^∞ as $\nu \to \infty$ then there exist $k \in \mathbb{N}_0$ and defining functions $F_\nu \in \mathcal{O}^k(U \setminus K)$ such that $F_\nu \to 0$ in $\mathcal{O}^k(U \setminus K)$. By Theorem 3.2, $bF_\nu \to 0$. ∎

4.7. Imbedding $D'(\Omega) \hookrightarrow B(\Omega)$, Ω open in $\mathbb{R}$. The imbedding $D'_K \hookrightarrow B_K$ of Subsection 4.5 is now extended in a compatible way to an imbedding of $D'(\Omega)$ into $B(\Omega)$ for every Ω open in $\mathbb{R}$. This is done thanks to the localization property shared by both $D'(\Omega)$ and $B(\Omega)$ (cf. Subsection 4.1). We start with the following general method which works in any dimension.

(I) Fix $\mathring{x} \in \Omega$ and take an open bounded set Ω_1 such that $\mathring{x} \in \Omega_1 \Subset \Omega$. Let $f \in D'(\Omega)$ and $\chi \in C_0^\infty(\Omega)$ with $\chi = 1$ on Ω_1. Clearly χf is a distribution with a compact support K_χ. Denote by f^χ the hyperfunction corresponding to χf under the imbedding $D'_{K_\chi} \hookrightarrow B_{K_\chi}$ from Subsection 4.5. Clearly $f^\chi \in B_{K_\chi}$. We claim that the restriction $f^\chi|_{\Omega_1}$ does not depend on the choice of χ such that $\chi = 1$ on Ω_1. In fact, let $\widetilde{\chi} \in C_0^\infty(\Omega)$ be another such function. Then the distribution $(\chi - \widetilde{\chi})f$ has compact support contained in $(K_\chi \cup K_{\widetilde{\chi}}) \setminus \Omega_1$ and the corresponding hyperfunction $f^\chi - f^{\widetilde{\chi}}$ is in $B^\infty_{(K_\chi \cup K_{\widetilde{\chi}})\setminus\Omega_1}$. Hence $f^\chi|_{\Omega_1} = f^{\widetilde{\chi}}|_{\Omega_1}$.

(II) To prove that the local imbeddings (constructed above) are compatible take $\Omega_i \Subset \Omega$ and $\chi_i \in C_0^\infty(\Omega)$ with $\chi_i = 1$ on Ω_i $(i = 1, 2)$, and assume that $\Omega_{1,2} := \Omega_1 \cap \Omega_2 \ne \emptyset$, $\chi_{1,2} \in C_0^\infty(\Omega)$ and $\chi_{1,2} = 1$ on $\Omega_{1,2}$. Then by (I),

$$f^{\chi_1}|_{\Omega_{1,2}} = f^{\chi_{1,2}}|_{\Omega_{1,2}} = f^{\chi_2}|_{\Omega_{1,2}}$$

since $\chi_1 = \chi_2 = \chi_{1,2} = 1$ on $\Omega_{1,2}$.

Now, we present a more effective method of imbedding $D'(\mathbb{R}) \hookrightarrow B(\mathbb{R})$ by constructing explicitly the pertinent defining functions. This is done by restricting distributions on $\mathbb{R}$ to compact sets and summing the corresponding Cauchy transforms à la Mittag-Leffler. We begin with the following theorem (see [B]):

THEOREM 4.6. *Every* $u \in D'(\mathbb{R})$ *is the jump of a holomorphic function with polynomial growth near the real axis* (see Subsection 3.3). *More precisely*, *if* $L = \operatorname{supp} u$ (L *closed in* $\mathbb{R}$ *but not necessarily bounded*) *then there is* $F \in \mathcal{O}(\mathbb{C} \setminus L)$ *of polynomial growth near the real axis such that for every* $\varphi \in C_0^\infty(\mathbb{R})$ *the limits* $\lim_{\varepsilon\to 0_+} \int_{-\infty}^{\infty} F(x \pm i\varepsilon)\varphi(x)\,dx$ *exist and* $b(F) = u$, *i.e.*

$$\lim_{\varepsilon\to 0_+} \int_{-\infty}^{\infty} (F(x+i\varepsilon) - F(x-i\varepsilon))\varphi(x)\,dx = u[\varphi] \quad \textit{for } \varphi \in C_0^\infty(\mathbb{R}). \tag{4.16}$$

The function F *satisfying* (4.16) *and belonging to* $\mathcal{O}(\mathbb{C} \setminus L)$ *is unique up to an entire function.*

PROOF. It suffices to show that there exists $F \in \mathcal{O}(\mathbb{C} \setminus L)$ of polynomial growth near the real axis and satisfying (4.16). Indeed, it then follows by Theorem 3.2 that $\lim_{\varepsilon\to 0_+} \int_{-\infty}^{\infty} F(x \pm i\varepsilon)\varphi(x)\,dx$ exists for $\varphi \in C_0^\infty(\mathbb{R})$ and by Theorem 3.3 we get the uniqueness assertion.

A function $F \in \mathcal{O}(\mathbb{C} \setminus L)$ satisfying (4.16) will be constructed from Corollary 4.2 by applying a C_0^∞ partition of unity $\{\chi_\nu\}$ subordinate to the open, locally finite covering $\mathbb{R} = \bigcup_{\nu=-\infty}^{\infty} \Omega_\nu$, where $\Omega_\nu = \{x : |x - \nu| < 1\}$, $\nu = 0, \pm 1, \pm 2, \ldots$ Take $\chi_\nu \in C_0^\infty(\Omega_\nu)$ ($\nu = 0, \pm 1, \pm 2, \ldots$) with $\sum_{\nu=-\infty}^{\infty} \chi_\nu = 1$ and put $u_\nu = \chi_\nu u$. Then

$$u[\varphi] = \lim_{n\to\infty} \sum_{\nu=-n}^{n} u_\nu[\varphi] \quad \text{for } \varphi \in C_0^\infty(\mathbb{R}).$$

By Corollary 4.2 the functions $\mathcal{C}u_\nu$ are of polynomial growth near the real axis and

$$F_\nu := \mathcal{C}u_\nu \in \mathcal{O}(\mathbb{C} \setminus L), \quad b(F_\nu) = u_\nu \quad \text{for } \nu = 0, \pm 1, \pm 2, \ldots \tag{4.17}$$

Now we modify F_ν to get $\widetilde{F}_\nu$ such that the sequence $\{\sum_{\nu=-n}^{n} \widetilde{F}_\nu\}_n$ converges as $n \to \infty$ to a function F with the desired properties. Let $B(r) = \{z \in \mathbb{C} : |z| < r\}$. Observe that $\operatorname{supp} u_\nu \cap \overline{B(|\nu| - 1)} = \emptyset$ for $|\nu| \ge 2$, hence $bF_\nu = u_\nu = 0$ on $\overline{B(|\nu| - 1)}$ and consequently F_ν is holomorphic on $\overline{B(|\nu| - 1)}$. Write its power series expansion at zero:

$$F_\nu(z) = \sum_{\mu=0}^{\infty} a_\mu^\nu z^\mu, \quad |\nu| \ge 2,$$

and take n_ν large enough so that with $h_\nu(z) = \sum_{\mu=0}^{n_\nu} a_\mu^\nu z^\mu$,

$$|F_\nu(z) - h_\nu(z)| < 2^{-|\nu|} \quad \text{for } z \in \overline{B(|\nu| - 1)}, \ |\nu| \ge 2.$$

Hence for every compact set $H \subset \mathbb{C}$ the series $\sum_{|\nu| \ge |\nu_0|} (F_\nu(z) - h_\nu(z))$ is uniformly convergent on H if $|\nu_0|$ is so large that $H \subset \overline{B(|\nu_0| - 1)}$. Let

$$F(z) = F_0(z) + F_1(z) + F_{-1}(z) + \sum_{|\nu| \ge 2} (F_\nu(z) - h_\nu(z)). \tag{4.18}$$

Since h_ν are entire functions and $F_\nu \in \mathcal{O}(\mathbb{C} \setminus L)$ are of polynomial growth near the real axis it follows that $F \in \mathcal{O}(\mathbb{C} \setminus L)$ and is also of polynomial growth.

It remains to prove formula (4.16). Fix $\varphi \in C_0^\infty(\mathbb{R})$ and let $H = \operatorname{supp}\varphi \subset B(N-1)$. Then $H \cap \operatorname{supp}\chi_\nu = \emptyset$ for $|\nu| \geq N$, hence $\varphi = \sum_{|\nu|\leq N-1} \varphi\chi_\nu$ and by (4.17) we get

$$(4.19)\quad u[\varphi] = \sum_{|\nu|\leq N-1} u_\nu[\varphi] = \sum_{|\nu|\leq N-1} \lim_{\varepsilon\to 0_+} \int_{-\infty}^{\infty} (F_\nu(x+i\varepsilon) - F_\nu(x-i\varepsilon))\varphi(x)\,dx$$
$$= \lim_{\varepsilon\to 0_+} \int_{-\infty}^{\infty} \Big(\sum_{|\nu|\leq N-1} F_\nu(x+i\varepsilon) - \sum_{|\nu|\leq N-1} F_\nu(x-i\varepsilon)\Big)\varphi(x)\,dx.$$

Observe that for $|\nu| \geq N$ the functions F_ν are analytic on $H = \operatorname{supp}\varphi$, hence, since h_ν are entire, the function

$$\lambda(z) := \sum_{|\nu|\geq N} (F_\nu(z) - h_\nu(z)) - \sum_{2\leq|\nu|\leq N-1} h_\nu(z)$$

is analytic on H. Thus

$$\lim_{\varepsilon\to 0_+} \int_{-\infty}^{\infty} (\lambda(x+i\varepsilon) - \lambda(x-i\varepsilon))\varphi(x)\,dx = 0,$$

which together with (4.19) yields (4.16). ■

Remark 4.5. If L in Theorem 4.6 is a compact subset of $[-N+1, N-1]$ then for F_ν, h_ν introduced in the proof we have $F_\nu = 0$ and $h_\nu = 0$ if $\nu \geq N$ and the sum (4.18) is finite. Hence

$$(4.20)\qquad F(z) = \sum_{\nu=-N+1}^{N-1} F_\nu(z) + \sum_{2\leq|\nu|\leq N-1} h_\nu(z) = \sum_{\nu=-N+1}^{N-1} (\mathcal{C}u_\nu)(z) - h(z)$$
$$= \mathcal{C}\Big(\sum_{\nu=-N+1}^{N-1} u_\nu\Big)(z) - h(z) = (\mathcal{C}u)(z) - h(z),$$

where $h(z) = \sum_{2\leq|\nu|\leq N-1} h_\nu(z) \in \mathcal{O}(\mathbb{C})$.

Note the following important consequences of Theorem 4.6.

Corollary 4.4 (Imbedding $D'(\mathbb{R}) \hookrightarrow B(\mathbb{R})$). *The mapping $\mathcal{J} : D'(\mathbb{R}) \ni u \mapsto [F] \in B(\mathbb{R})$, where F is the function from Theorem* 4.6, *is well defined since F is unique up to entire functions* (*hence $[F]$ is unique*). *It defines a natural imbedding $D'_L \hookrightarrow B_L$* ($L = \operatorname{supp} u$ *not necessarily bounded*) *compatible with the imbeddings $D'_K \hookrightarrow B_K$ of Subsection* 4.5 *for K compact.*

Indeed, it follows from (4.20) that for $u \in D'_K(\mathbb{R})$ with K compact, $\mathcal{J}u = [F] = [\mathcal{C}u]$ in agreement with (4.14). Hence $\mathcal{J}$ is an injection.

Corollary 4.5. *A hyperfunction $[F]$ is a distribution on $\mathbb{R}$ if and only if F is of polynomial growth near the real axis* (see Theorem 3.2).

Remark 4.6. The uniqueness assertion of Theorem 4.6 ensures that the hyperfunction $[F]$ with defining function F satisfying (4.16) is unique. The method of constructing F satisfying (4.16), and in particular the choice of the partition of unity, are unimportant.

Remark 4.7. Note that the distributional derivative and the derivative in the sense of hyperfunctions are compatible under the imbedding $D'(\mathbb{R}) \hookrightarrow B(\mathbb{R})$ (cf. point 4° of Subsection 3.3).

Remark 4.8. The natural imbedding $L^1_{\mathrm{loc}} \hookrightarrow D'$ and Corollary 4.4 lead to the imbedding $L^1_{\mathrm{loc}} \hookrightarrow B$, but in order to construct a defining function for the hyperfunction corresponding to a given function in L^1_{loc} with unbounded support, we must in general find the sum of the series (4.18). In that case we can take a partition of unity: $1 = \sum_{-\infty<k<\infty} \chi_{[k,k+1]}$ almost everywhere, where $\chi_{[k,k+1]}$ is the characteristic function of the interval $[k, k+1]$ (see Remark 4.6).

Example 4.3. Let Y be the Heaviside function. Clearly $Y \in L^1_{\mathrm{loc}}$. It follows by point 5° of Subsection 3.3 and Corollary 4.4 that the hyperfunction $\left[-\frac{1}{2\pi i}\,\mathrm{Ln}(-z)\right]$, where Ln denotes the principal branch of the logarithm, corresponds to Y under the imbedding $L^1_{\mathrm{loc}} \hookrightarrow B$.

Example 4.4. Let p be a measurable function on $\mathbb{R}$ such that $|p(t)| \le Ce^{M|t|}$ for $t \in \mathbb{R}$ and some $C \in \mathbb{R}_+$ and $M \ge 0$. Then for $\varrho > M$ the function

$$F(z) = \frac{1}{2\pi i}\int_{-\infty}^{0} \frac{p(t)}{e^{-\varrho(t-z)}(t-z)}\,dt + \frac{1}{2\pi i}\int_{0}^{\infty} \frac{p(t)}{e^{\varrho(t-z)}(t-z)}\,dt \quad \text{for } z \in \mathbb{C}\setminus\mathbb{R}$$

is a defining function for p considered as a distribution (hyperfunction) on $\mathbb{R}$.

To prove this observe first that the function F is well defined and holomorphic on $\mathbb{C}\setminus\mathbb{R}$ since $|t-z| \ge |\mathrm{Im}\, z| > 0$ if $t \in \mathbb{R}$, $z \in \mathbb{C}\setminus\mathbb{R}$ and $\int_0^\infty e^{(M-\varrho)t}\,dt < \infty$, $\int_{-\infty}^0 e^{(\varrho-M)t}\,dt < \infty$. Fix $R > 0$ and let $\Omega = (-R, R)$. It suffices to show that the jump of F across Ω equals p (restricted to Ω). Let

$$J_+^R(z) = \frac{1}{2\pi i}\int_0^R \frac{p(t)}{e^{\varrho(t-z)}(t-z)}\,dt, \quad J_-^R(z) = \frac{1}{2\pi i}\int_{-R}^0 \frac{p(t)}{e^{-\varrho(t-z)}(t-z)}\,dt$$

and observe that there exists an entire function G_1 such that

$$J_+^R(z) = \frac{1}{2\pi i}\int_0^R \frac{p(t)}{t-z}\,dt + \frac{1}{2\pi i}\int_0^R G_1(t-z)p(t)\,dt$$

and clearly the second summand on the right-hand side is also an entire function. Observe that $\frac{1}{2\pi i}\int_0^R \frac{p(t)}{t-z}\,dt$ is a defining function for $p\chi_{[0,R]}$. Thus $b(J_+^R(z)) = p\chi_{[0,R]}$ and similarly $b(J_-^R(z)) = p\chi_{[-R,0]}$. Since $F(z) - J_+^R(z) - J_-^R(z)$ is holomorphic in $\mathbb{C}\setminus((-\infty,-R]\cup[R,\infty))$, the jump of F across $\Omega = (-R, R)$ equals $p\chi_\Omega$.

Now consider an open set Ω in $\mathbb{R}$ and a set $L \subset \Omega$, relatively closed in Ω. Note the following slight generalization of Theorem 4.6 for this case.

Theorem 4.6′. *Let $u \in D'_L(\Omega)$. Then there exists a function $F \in \mathcal{O}(\mathbb{C}\setminus L)$ (of polynomial growth) such that $u = bF$, where*

$$u[\varphi] = bF[\varphi] = \lim_{\varepsilon\to 0_+}\int_\Omega (F(x+i\varepsilon) - F(x-i\varepsilon))\varphi(x)\,dx \quad \text{for } \varphi \in C_0^\infty(\Omega). \tag{4.21}$$

Corollary 4.6. *A linear functional u on $C_0^\infty(\Omega)$ is a distribution on Ω with support in L if and only if there exists a function $F \in \mathcal{O}(\mathbb{C}\setminus L)$ such that (4.21) holds.*

Proof. The "only if" part follows by Theorem 4.6′. For the "if" part see Subsection 3.3: Proposition 3.3 and point 3°. ∎

4.8. Hyperfunctional boundary values of holomorphic functions. With every hyperfunction $f = [F] \in B(\mathbb{R})$ with defining function $F \in \mathcal{O}(\mathbb{C} \setminus \mathbb{R})$ we associate two hyperfunctions $[F_+]$ and $[F_-]$ with defining functions

$$F_+(z) = \begin{cases} F(z) & \text{for } \operatorname{Im} z > 0, \\ 0 & \text{for } \operatorname{Im} z < 0, \end{cases} \qquad F_-(z) = \begin{cases} 0 & \text{for } \operatorname{Im} z > 0, \\ F(z) & \text{for } \operatorname{Im} z < 0. \end{cases}$$

Suppose first that F is of polynomial growth near the real axis. Then by Theorem 3.2 there exist distributional boundary values of F from above and below denoted by $F(\,\cdot \pm i0)$. The mapping $\mathcal{J}$ from Corollary 4.4 associates with the distributions $F(\,\cdot + i0)$ and $F(\,\cdot - i0)$ the hyperfunctions $[F_+]$ and $[-F_-]$ respectively. This motivates, also in the case of an arbitrary function F holomorphic in $\mathbb{C} \setminus \mathbb{R}$, the name of hyperfunctional boundary values of F from above and below given to the hyperfunctions $[F_+]$, $[-F_-]$, denoted also by $F(\,\cdot + i0)$ and $F(\,\cdot - i0)$ respectively. Their difference will be called the *jump* of F across $\mathbb{R}$ and denoted by

$$bF = F(\,\cdot + i0) - F(\,\cdot - i0) = b_+F - b_-F.$$

Clearly we have $bF = [F]$. Hence equality of hyperfunctional boundary values of F from above and below (i.e. $bF = 0$) yields $[F] = 0$ and thus F extends holomorphically across the real line. This is the most general version of the Painlevé Theorem 3.3 and, at the same time, a trivial one.

4.9. Hyperfunctions supported by a single point. For simplicity we deal with hyperfunctions supported by the origin.

Example 4.5. Let

$$f_n = 2\pi i \frac{1}{n!} \frac{1}{(n+1)!} \delta^{(n)} \in B_{\{0\}}, \quad n = 0, 1, 2, \ldots$$

By Example 4.1^{II} from Subsection 4.5 and from Subsection 4.3 it follows that

$$\delta^{(n)} = \left[\frac{(-1)^{n+1} n!}{2\pi i} \frac{1}{z^{n+1}} \right], \quad n = 0, 1, \ldots$$

Hence $f_n = [F_n]$, where $F_n = \frac{(-1)^{n+1}}{(n+1)!} \frac{1}{z^{n+1}}$ is the standard defining function of the hyperfunction f_n. Note that locally uniformly on $\mathbb{C} \setminus \{0\}$,

$$\sum_{n=0}^{\infty} F_n = \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} \frac{1}{z^j} = e^{-1/z} - 1.$$

Hence by (4.11),

$$\sum_{n=0}^{\infty} f_n = \Big[\sum_{n=0}^{\infty} F_n \Big] = [e^{-1/z} - 1] = [e^{-1/z}].$$

Here $F(z) = e^{-1/z} - 1$ is the standard defining function for $\sum_{n=0}^{\infty} f_n$ since it tends to zero as $|z| \to \infty$. Clearly $F \in \mathcal{O}(\mathbb{C} \setminus \{0\})$ and $\sum_{n=0}^{\infty} f_n \in B_{\{0\}}$.

Theorem 4.7. *The set $B_{\{0\}}$ coincides with the set of series*

$$f = 2\pi i \sum_{n=0}^{\infty} a_n \frac{(-1)^{n+1}}{n!}\delta^{(n)}, \quad a_n \in \mathbb{C},\ n \in \mathbb{N}_0,\ \lim_{n\to\infty} \sqrt[n]{|a_n|} = 0. \tag{4.22}$$

The standard defining function of f equals $F(z) = \sum_{n=0}^{\infty} a_n/z^{n+1}$.

Proof. Let $f \in B_{\{0\}}$. By Proposition 4.1, $f = [G]$ with $G \in \mathcal{O}(\mathbb{C}\setminus\{0\})$. Let $G(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ be the Laurent expansion of G at zero. Putting $a_n = c_{-(n+1)}$ for $n = 0, 1, 2, \ldots$ we see that $F(z) = \sum_{n=0}^{\infty} a_n/z^{n+1}$ is the standard defining function for f: we have $f = [F]$ and $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 0$. On the other hand, $F_n(z) = a_n/z^{n+1}$ is the standard defining function for $2\pi i a_n(-1)^{n+1}\frac{1}{n!}\delta^{(n)}$ $(n = 0, 1, \ldots)$. Thus (4.22) follows from Definition 4.3 since $\sum_{n=0}^{\infty} F_n$ is locally uniformly convergent on $\mathbb{C}\setminus\{0\}$.

Conversely, suppose that (4.22) holds. Then the series $\sum_{n=0}^{\infty} a_n/z^{n+1}$ is locally uniformly convergent on $\mathbb{C}\setminus\{0\}$ and by Definition 4.3,

$$2\pi i \sum_{n=0}^{\infty} a_n(-1)^{n+1}\frac{1}{n!}\delta^{(n)} = \Big[\sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}}\Big],$$

i.e. f is a hyperfunction with defining function $F(z) = \sum_{n=0}^{\infty} a_n/z^{n+1} \in \mathcal{O}(\mathbb{C}\setminus\{0\})$. Thus $f \in B_{\{0\}}$. ■

Remark 4.9. The series in (4.22) makes sense as an analytic functional on $A(\{0\})$ (see also point 2° of Subsection 3.2).

4.10. Substitution in a hyperfunction and in an analytic functional (distribution). Let $\tau : \Omega_1 \to \Omega$ be an invertible analytic mapping of an open set $\Omega_1 \subset \mathbb{R}$ onto $\Omega \subset \mathbb{R}$. Given a hyperfunction $f \in B(\Omega)$ we define the composite $f \circ \tau$ as follows. First since $\tau(\xi)$ is analytic it extends to a holomorphic mapping $\tau(\zeta)$ of a complex neighbourhood V_1 of Ω_1 onto a complex neighbourhood V of Ω. We represent f by means of a defining function $F \in \mathcal{O}(V\setminus\Omega)$ and define $f\circ\tau$ to be the element of $\mathcal{O}(V_1\setminus\Omega_1)/\mathcal{O}(V_1)$ determined by the function $\operatorname{sgn}(J\tau)\cdot F\circ\tau(\zeta)$, where $J\tau$ denotes the Jacobian of the mapping τ. Note that the appearence of the sign of $J\tau$ is necessary since (the complexification of) τ may interchange the upper and lower parts of the sets V_1 and V.

Example 4.6. Let $\tau(\xi) = e^{-\xi}$ for $\xi \in \mathbb{R}$, $r \ge 1$ and $\Omega_1 = (-\ln r, \infty)$, $\Omega = (0, r)$. Then $\tau : \Omega_1 \to \Omega$ is an invertible analytic mapping of Ω_1 onto Ω. Let V be an open sector bounded by two segments $z = \varrho e^{\pm i\theta}$, $0 < \varrho \le r$, and the arc $z = re^{i\varphi}$, $-\theta \le \varphi \le \theta$, $\theta > 0$. Clearly V is a complex neighbourhood of Ω. The boundary of $V_1 = \tau^{-1}(V)$ consists of two half-lines $\zeta = \alpha \pm i\theta$, $-\ln r \le \alpha < \infty$, and the segment $\zeta = -\ln r + i\beta$, $-\theta \le \beta \le \theta$. Note that τ^{-1} preserves the orientation of the boundaries and transforms the positively oriented segment $(0, r)$ onto the negatively oriented half-line $(-\ln r, \infty)$. Given a hyperfunction $f \in B(\Omega)$ with $f = F + \mathcal{O}(V)$, $F \in \mathcal{O}(V\setminus\Omega)$, we define $f\circ\tau = -F\circ\tau + \mathcal{O}(V_1)$ since $\operatorname{sgn}(J\tau) = -1$.

Now let $f \in B_K(\Omega)$ where K is a compact subset of Ω. By the Köthe Theorem 4.4 the space B_K is isomorphic to $A'(K)$ under the isomorphism $\mathcal{I} : B_K \ni [F] = f \mapsto \mathcal{I}f \in A'(K)$

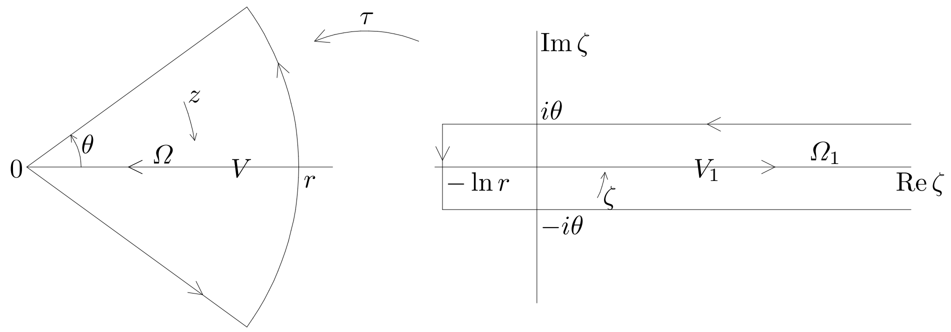

Fig. 1

defined by

$$\mathcal{I}f[\varphi] = -\int_\gamma \varphi(z)F(z)\,dz \quad \text{for } \varphi \in A(K),$$

where γ is an anticlockwise oriented curve encircling K. We now prove that the substitution $f \circ \tau$ in the hyperfunction $f \in B_K$ is compatible with the substitution in the corresponding analytic functional $\mathcal{I}f \in A'(K)$, defined to be the mapping

$$A'(K) \ni u \mapsto u \circ \tau \in A'(K_1)$$

where ([3])

$$(u \circ \tau)[\psi] = u[\psi \circ \tau^{-1}(z)(\tau^{-1})'(z)\operatorname{sgn}(\tau^{-1})'(x)] \quad \text{for } \psi \in A(K_1) \text{ and } K_1 = \tau^{-1}(K).$$

To prove this it suffices, in view of the Köthe theorem applied to the hyperfunctions $f \in B_K$ and $f \circ \tau \in B_{K_1}$, to show that

$$\mathcal{I}(f \circ \tau) = (\mathcal{I}f) \circ \tau \quad \text{in } A'(K_1).$$

Take $\psi \in A(K_1)$ and an anticlockwise oriented curve γ_1 encircling K_1. Let $f = F + \mathcal{O}(V)$. It follows from the definition of $f \circ \tau$ that

$$f \circ \tau = \operatorname{sgn}\tau'(\xi)(F \circ \tau)(\zeta) + \mathcal{O}(V_1)$$

and by applying twice the Köthe theorem we get for $\psi \in A(K_1)$,

$$\begin{aligned}
\mathcal{I}(f \circ \tau)[\psi] &= -\int_{\gamma_1} \operatorname{sgn}\tau'(\xi)(F \circ \tau)(\zeta)\psi(\zeta)\,d\zeta \\
&= -\int_{\tau(\gamma_1)} F(z)(\psi \circ \tau^{-1})(z)\operatorname{sgn}\tau'(\xi)(\tau^{-1})'(z)\,dz \\
&= -\int_{\tau(\gamma_1)} F(z)(\psi \circ \tau^{-1})(z)\operatorname{sgn}(\tau^{-1})'(x)(\tau^{-1})'(z)\,dz \\
&= \mathcal{I}f[(\psi \circ \tau^{-1})(z)\operatorname{sgn}(\tau^{-1})'(x)(\tau^{-1})'(z)] = ((\mathcal{I}f) \circ \tau)[\psi].
\end{aligned}$$

([3]) Observe that this formula agrees with the usual formula for the substitution in a distribution if u is a distribution on Ω_1.

We have the following commutative diagram:

$$\begin{array}{ccc}
B_K(\Omega) \ni f & \xrightarrow[\text{isomorphism}]{\text{Köthe}} & \mathcal{I}f \in A'(K) \\
\Big\downarrow{\scriptstyle\tau} & & \Big\downarrow{\scriptstyle\tau} \\
B_{K_1}(\Omega_1) \ni f \circ \tau & \xrightarrow[\text{isomorphism}]{\text{Köthe}} & \mathcal{I}(f \circ \tau) = (\mathcal{I}f) \circ \tau \in A'(K_1).
\end{array}$$

5. Laplace hyperfunctions and Laplace analytic functionals in one variable

Laplace hyperfunctions considered in this section on $\overline{\mathbb{R}}_+$ can be regarded as a special case of Fourier hyperfunctions (cf. [K]). To define them we need some special sets in $\mathbb{C}$.

Tubular sets. We say that W is a *tubular neighbourhood* of $\overline{\mathbb{R}}_+$ in $\mathbb{C}$, and write $\overline{\mathbb{R}}_+ \Subset W$, if W is open in $\mathbb{C}$ and

$$(\overline{\mathbb{R}}_+)_\varepsilon \subset W \subset (\overline{\mathbb{R}}_+)_M \quad \text{for some } 0 < \varepsilon < M < \infty, \tag{5.1}$$

where (1) $(\overline{\mathbb{R}}_+)_\varrho = \{z \in \mathbb{C} : \operatorname{dist}(z, \overline{\mathbb{R}}_+) \le \varrho\}$ for any $\varrho > 0$. We say that $\widetilde{W}$ is a *proper subset* of W, denoted by $\widetilde{W} \Subset W$, if $\operatorname{dist}(\widetilde{W}, \partial W) > 0$.

In a similar way we define tubular neighbourhoods of the half-lines $\mathbb{L}_+ = \zeta + \overline{\mathbb{R}}_+$ and $\mathbb{L}_- = \zeta - \overline{\mathbb{R}}_+$ and of the line $\mathbb{L} = \zeta + \mathbb{R}$ where $\zeta \in \mathbb{C}$. A set W is called *tubular* if W contains a tubular neighbourhood of a half-line $\mathbb{L}_+$ or $\mathbb{L}_-$.

Next we define certain spaces of holomorphic functions on W of exponential growth at infinity. Let W be a tubular neighbourhood of $\overline{\mathbb{R}}_+$ in $\mathbb{C}$. Let $\kappa \in \mathbb{R}$. We introduce the topological space

$$\mathfrak{L}_\kappa(W) = \{H \in \mathcal{O}(W) : \sup_{\zeta \in \widetilde{W}} |e^{\kappa\zeta} H(\zeta)| < \infty \text{ for every proper subset } \widetilde{W} \Subset W\}$$

with the topology given by the seminorms $\sup_{\zeta \in \widetilde{W}} |e^{\kappa\zeta} H(\zeta)|$.

Now if $\omega \in \mathbb{R} \cup \{\infty\}$ we define

$$\mathfrak{L}_{(\omega)}(W) = \varprojlim_{\kappa < \omega} \mathfrak{L}_\kappa(W),$$

so that $H_\nu \to 0$ in $\mathfrak{L}_{(\omega)}(W)$ as $\nu \to \infty$ if $\sup_{\zeta \in \widetilde{W}} |e^{\kappa\zeta} H_\nu(\zeta)| \to 0$ as $\nu \to \infty$ for every $\kappa < \omega$ and $\widetilde{W} \Subset W$. In an analogous way we define the space $\mathfrak{L}_{(\omega)}(W \setminus \overline{\mathbb{R}}_+)$.

REMARK 5.1. Let $H \in \mathfrak{L}_\kappa(W)$ and let W' be a proper subset $W' \Subset W$. Then $H \in \mathcal{O}(W') \cap C^0(\overline{W'})$. Observe that

$$\mathfrak{L}_\kappa(W') = \{H \in \mathcal{O}(W') \cap C^0(\overline{W'}) : \sup_{\zeta \in W'} |e^{\kappa\zeta} H(\zeta)| < \infty\}$$

(with the natural topology) is a Banach space (which could also be denoted by $\mathfrak{L}_\kappa(\overline{W'})$).

---

(1) Note that (5.1) is equivalent to the condition $0 < \operatorname{dist}(\partial W, \overline{\mathbb{R}}_+) < \infty$ and $\overline{\mathbb{R}}_+ \subset W$. Here $\operatorname{dist}(z, \zeta) = |z - \zeta|$, but for the use in Section 6 it will be convenient to take $\operatorname{dist}(z, \zeta) = \max(|\operatorname{Re}(z - \zeta)|, |\operatorname{Im}(z - \zeta)|)$.

Next we can write

$$\mathfrak{L}_{(\omega)}(W) = \varprojlim_{\kappa<\omega} \varprojlim_{W' \Subset W} \mathfrak{L}_{\kappa}(W').$$

REMARK 5.2. Let W be a tubular neighbourhood of $\overline{\mathbb{R}}_+$ in $\mathbb{C}$ and let $\kappa \in \mathbb{R}$. Denote by $\mathfrak{L}^*_{\kappa}(W)$ the linear space

$$\mathfrak{L}^*_{\kappa}(W) = \{H \in \mathcal{O}(W) : \sup_{\zeta \in \widetilde{W}} |e^{\kappa\zeta} H(\zeta)| < \infty \text{ for every proper tubular subset } \widetilde{W} \Subset W\},$$

equipped with a convergence topology defined as follows: $H_\nu \to 0$ in $\mathfrak{L}^*_{\kappa}(W)$ as $\nu \to \infty$ if for every $\widetilde{W} \Subset W$ there exists $C < \infty$ such that $\sup_{\zeta \in \widetilde{W}} |e^{\kappa\zeta} H_\nu(\zeta)| < C$ for $\nu \in \mathbb{N}$ and the sequence $|e^{\kappa\zeta} H_\nu(\zeta)|$ converges to zero locally uniformly on W.

Note that for any $\omega \in \mathbb{R} \cup \{\infty\}$, $\mathfrak{L}_{(\omega)}(W) = \varprojlim_{\kappa<\omega} \mathfrak{L}_{\kappa}(W) = \varprojlim_{\kappa<\omega} \mathfrak{L}^*_{\kappa}(W)$.

Now we introduce the spaces $\underset{\sim}{L}_a(W)$, $a \in \mathbb{R}$:

$$\underset{\sim}{L}_a(W) = \{\sigma \in \mathcal{O}(W) : \varrho_{a,V}(\sigma) < \infty \text{ for every proper tubular } V \Subset W\}$$

with the topology defined by the family of seminorms

$$\varrho_{a,V}(\sigma) = \sup_{\zeta \in V} |e^{-a\zeta}\sigma(\zeta)| \quad \text{for any } V \Subset W,$$

and set

$$\underset{\sim}{L}_a(\overline{\mathbb{R}}_+) := \varinjlim_{W \Supset \overline{\mathbb{R}}_+} \underset{\sim}{L}_a(W),$$

where W ranges over tubular neighbourhoods of $\overline{\mathbb{R}}_+$. We put $\underset{\sim}{L}_{-\infty}(\overline{\mathbb{R}}_+) = \varprojlim_{a\in\mathbb{R}} \underset{\sim}{L}_a(\overline{\mathbb{R}}_+)$ and

$$\underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}_+) := \varinjlim_{a<\omega} \underset{\sim}{L}_a(\overline{\mathbb{R}}_+) \quad \text{for any } \omega \in \mathbb{R}.$$

We also have

$$\underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}_+) = \varinjlim_{W \Supset \overline{\mathbb{R}}_+} \underset{\sim}{L}_{(\omega)}(W) = \varinjlim_{W \Supset \overline{\mathbb{R}}_+} \varinjlim_{a<\omega} \underset{\sim}{L}_a(W).$$

Note that

1° the space $\underset{\sim}{L}_a(\overline{\mathbb{R}}_+)$ can be defined equivalently as follows: $\underset{\sim}{L}_a(\overline{\mathbb{R}}_+) = \varinjlim_{V \Supset \overline{\mathbb{R}}_+} \underset{\sim}{L}_a(\overline{V})$, where

$$\underset{\sim}{L}_a(\overline{V}) = \{\sigma \in \mathcal{O}(V) \cap C^0(\overline{V}) : \varrho_{a,V}(\sigma) < \infty\}$$

is a Banach space.

The space $\underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}_+)$ is complete. $\sigma \in \underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}_+)$ if and only if there exist $a < \omega$ and a tubular neighbourhood W of $\overline{\mathbb{R}}_+$ such that $\sigma \in \mathcal{O}(W)$ and $\varrho_{a,V}(\sigma) < \infty$ for every $V \Subset W$.

The dual space of $\underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}_+)$ is called the space of *Laplace analytic functionals* and is denoted by $\underset{\sim}{L}'_{(\omega)}(\overline{\mathbb{R}}_+)$ or sometimes by $\underset{\sim}{L}'_{(\omega)}(\overline{\mathbb{R}}_+ \cup \{\infty\})$ to stress the fact that there exist Laplace analytic functionals with carrier at infinity, i.e. whose carrier is contained in $[r, \infty)$ for any $r > 0$ (cf. [M-Y] and Example 6.1).

2° $T \in \underset{\sim}{L}'_{(\omega)}(\overline{\mathbb{R}}_+)$ if and only if for every $a < \omega$, T is a linear functional on $\underset{\sim}{L}_a(\overline{\mathbb{R}}_+)$ and for every tubular neighbourhood V of $\overline{\mathbb{R}}_+$ there exists $C_{a,V}$ such that

$$|T[\sigma]| \le C_{a,V}\,\varrho_{a,V}(\sigma) \quad \text{for } \sigma \in \underset{\sim}{L}_a(W),$$

where W is an arbitrary open tubular set such that $V \Subset W$.

Another equivalent definition of Laplace analytic functionals carried by $\overline{\mathbb{R}}_+ \cup \{\infty\}$ is obtained by extending the Köthe theorem in Subsection 4.4 to the case of a non-compact carrier (see Theorem 5.1 below). The basic idea of the proof remains the same, but we must keep track of the growth order of pertinent functions at infinity. For that purpose we replace the standard Cauchy convolution kernel $\mathbb{C} \setminus \{0\} \ni w \mapsto K(w) := -1/w$ by a modified convolution kernel $\mathbb{C} \setminus \{0\} \ni w \mapsto -e^{-w^2}/w$ which differs from K by the entire function $\frac{1}{w}(1 - e^{-w^2})$. In the sequel the function

$$\Lambda(\zeta, w) = -\frac{e^{-(w-\zeta)^2}}{w-\zeta}$$

will play the fundamental role. We call it the *modified Cauchy kernel*.

We denote by γ (sometimes γ^+, γ^-, $\widetilde{\gamma}$) a regular curve in $W \setminus \overline{\mathbb{R}}_+$ encircling $\overline{\mathbb{R}}_+$ once in the anticlockwise direction with $\operatorname{dist}(\gamma, \overline{\mathbb{R}}_+) > 0$. Let $G \in \mathfrak{L}_{(\omega)}(W \setminus \overline{\mathbb{R}}_+)$ and $\sigma \in \underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}_+)$. By the definition of these functions, for every tubular set V on which both of them are defined, there exist $a < \kappa < \omega$ and $C < \infty$ such that

$$|G(z)\sigma(z)| \le Ce^{-(\kappa-a)\operatorname{Re} z}. \tag{5.2}$$

When considering the integral $\int_\gamma G(z)\sigma(z)\,dz$ we shall always assume that both G and σ are holomorphic on γ.

LEMMA 5.1. *Fix a tubular neighbourhood $W \Supset \overline{\mathbb{R}}_+$ and a tubular set $V^1 \Subset W \setminus \overline{\mathbb{R}}_+$. Choose a tubular set V^2 with $\overline{\mathbb{R}}_+ \Subset V^2 \Subset W$ such that $\operatorname{dist}(V^1, V^2) > 0$ and let $a \in \mathbb{R}$. Then there exists $C < \infty$ such that*

$$\sup_{w\in V^2}\sup_{\zeta\in V^1} |e^{a(\zeta-w)}\Lambda(\zeta, w)| < C < \infty. \tag{5.3}$$

In particular, for fixed $\zeta \in W \setminus \overline{\mathbb{R}}_+$ and $\mathbb{R}_+ \Subset V \Subset W$ with $\operatorname{dist}(\zeta, V) > 0$ we have $\sup_{w\in V} |e^{-aw}\Lambda(\zeta, w)| < \infty$, which means that $\Lambda(\zeta, \cdot) \in \underset{\sim}{L}_a(V)$ and hence ($a \in \mathbb{R}$ being arbitrary) $\Lambda(\zeta, \cdot) \in \underset{\sim}{L}_{(\omega)}(V)$ for every $\omega \in \mathbb{R} \cup \{\infty\}$ and $\Lambda(\zeta, \cdot) \in \underset{\sim}{L}_{-\infty}(V)$.

This follows from the estimates

$$\begin{aligned}\sup_{w\in V^2}\sup_{\zeta\in V^1} |e^{a(\zeta-w)}\Lambda(\zeta, w)| &\le \widetilde{C}\sup_{w\in V^2}\sup_{\zeta\in V^1} e^{\operatorname{Re}(\zeta-w)(a-\operatorname{Re}(\zeta-w))}\\ &\le \widetilde{C}\sup_{\xi\in\mathbb{R}} e^{\xi(a-\xi)} \le C < \infty,\end{aligned}$$

which are true since $\operatorname{Im}(\zeta - w)$ is bounded for any $\zeta, w \in W \subset (\overline{\mathbb{R}}_+)_M$.

Lemma 5.1 implies the correctness of the following definition.

DEFINITION 5.1. Let $\omega \in \mathbb{R} \cup \{\infty\}$ and $T \in \underset{\sim}{L}'_{(\omega)}(\overline{\mathbb{R}}_+)$. The function

$$(\mathcal{C}_\Lambda T)(\zeta) = -\frac{1}{2\pi i}T[\Lambda(\zeta, \cdot)] \quad \text{for } \zeta \in \mathbb{C} \setminus \overline{\mathbb{R}}_+$$

is called the *modified Cauchy transform of T*.

PROPOSITION 5.1. *Let* $\omega \in \mathbb{R} \cup \{\infty\}$, $T \in \underset{\sim}{L}'_{(\omega)}(\overline{\mathbb{R}}_+)$ *and let* W *be a tubular neighbourhood of* $\overline{\mathbb{R}}_+$. *Then* $\mathcal{C}_\Lambda T \in \mathfrak{L}_{(\omega)}(W \setminus \overline{\mathbb{R}}_+)$.

PROOF. Take $a < \omega$ and let V^1, V^2 be as in Lemma 5.1. Let $\overline{\mathbb{R}}_+ \Subset V \Subset V^2$. Then $T \in \underset{\sim}{L}'_a(V^2)$, $\Lambda(\zeta, \cdot) \in \underset{\sim}{L}_a(V^2)$ for $\zeta \in V^1 \Subset W \setminus \overline{\mathbb{R}}_+$ and $\Psi := \mathcal{C}_\Lambda T$ is well defined for $\zeta \in V^1$. By Definition 5.1, point 2° and (5.3) we get, with some $\widetilde{C} < \infty$,

$$\begin{aligned}\sup_{\zeta \in V^1} |e^{a\zeta}\Psi(\zeta)| &\le \frac{1}{2\pi} C_{a,V} \sup_{\zeta \in V^1} |e^{a\zeta} \sup_{w \in V} |e^{-aw}\Lambda(\zeta, w)|| \\ &\le \frac{1}{2\pi} C_{a,V} \sup_{\zeta \in V^1} \sup_{w \in V} |e^{a(\zeta - w)}\Lambda(\zeta, w)| \le \widetilde{C} < \infty.\end{aligned}$$

Recall that V^1 was an arbitrary tubular set $\Subset W \setminus \overline{\mathbb{R}}_+$ and $a < \omega$ was also arbitrary. To prove that $\Psi \in \mathfrak{L}_{(\omega)}(W \setminus \overline{\mathbb{R}}_+)$ it suffices to show that it is holomorphic on $W \setminus \overline{\mathbb{R}}_+$. To see this note that

$$\lim_{\zeta \to \mathring{\zeta}} \frac{\Lambda(\zeta, \cdot) - \Lambda(\mathring{\zeta}, \cdot)}{\zeta - \mathring{\zeta}} = \frac{\partial \Lambda}{\partial \zeta}(\mathring{\zeta}, \cdot) \quad \text{in } \underset{\sim}{L}_a(U),\ U \Supset \overline{\mathbb{R}}_+,\ \mathring{\zeta} \in W \setminus \overline{U}. \ \blacksquare$$

LEMMA 5.2. *Let* $G \in \mathfrak{L}_{(\omega)}(W \setminus \overline{\mathbb{R}}_+)$. *Then* $G \in \mathfrak{L}_{(\omega)}(W)$ *if and only if*

$$\int_\gamma G(z)\sigma(z)\,dz = 0 \quad \textit{for } \sigma \in \underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}_+), \tag{5.4}$$

where γ *encircles* $\overline{\mathbb{R}}_+$ *in the anticlockwise direction, lies in the set where both* G *and* σ *are holomorphic and* $\operatorname{dist}(\gamma, \overline{\mathbb{R}}_+) > 0$.

PROOF. If $G \in \mathfrak{L}_{(\omega)}(W)$ then $G\sigma$ is holomorphic on the domain U bounded by γ. Now the proof of (5.4) is standard. Take $0 < r < \infty$ and represent U as the union of $U^{1,r} = U \cap \{z \in \mathbb{C} : \operatorname{Re} z \le r\}$ and $U^{2,r} = U \setminus U^{1,r}$. Clearly $\int_{\partial U^{1,r}} G(z)\sigma(z)\,dz = 0$ for any $r > 0$ and by (5.2), $\lim_{r\to\infty} \int_{\partial U^{2,r}} G(z)\sigma(z)\,dz = 0$. Hence (5.4) follows.

For the converse take $w \in W \setminus \overline{\mathbb{R}}_+$ and choose a domain $P_{\gamma^-,\gamma^+} \subset W \setminus \overline{\mathbb{R}}_+$ bounded by an inner curve γ^- and an outer curve γ^+, both regular, rectifiable and encircling $\overline{\mathbb{R}}_+$, with $\operatorname{dist}(\gamma^-, \gamma^+) > 0$, $\operatorname{dist}(\gamma^-, \overline{\mathbb{R}}_+) > 0$ and such that $w \in P_{\gamma^-,\gamma^+}$.

By Lemma 5.1, $\Lambda(z, w) \in \underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}_+)$ and hence by assumption (5.4),

$$\int_{\gamma^-} G(z)\Lambda(z, w)\,dz = 0. \tag{5.5}$$

Take $r > \operatorname{Re} w$ and represent P_{γ^-,γ^+} as the union of $P^{1,r}_{\gamma^-,\gamma^+} = P_{\gamma^-,\gamma^+} \cap \{z \in \mathbb{C} : \operatorname{Re} z \le r\}$ and $P^{2,r}_{\gamma^-,\gamma^+} = P_{\gamma^-,\gamma^+} \setminus P^{1,r}_{\gamma^-,\gamma^+}$. By the Cauchy theorem

$$G(w) = \frac{1}{2\pi i} \int_{\partial P^{1,r}_{\gamma^-,\gamma^+}} G(z)\Lambda(z, w)\,dz \quad \text{for every } r > \operatorname{Re} w. \tag{5.6}$$

As in the first part of the proof we show that the integrals of $G(z)\Lambda(z, w)$ over the parts of γ^+ and γ^- with $\operatorname{Re} z > r$ and over the segment of the boundary of $P^{1,r}_{\gamma^-,\gamma^+}$ with $\operatorname{Re} z = r$ tend to zero as $r \to \infty$. Thus by (5.5) and (5.6) we get

$$G(w) = \frac{1}{2\pi i} \int_{\gamma^+} G(z)\Lambda(z, w)\,dz \tag{5.7}$$

and hence G is holomorphic in the domain V bounded by γ^+. Since by assumption $G \in \mathcal{O}(W \setminus \overline{\mathbb{R}}_+)$ it is holomorphic on W. To prove that $G \in \mathfrak{L}_{(\omega)}(W)$ we take first a tubular subset $\widetilde{W} \Subset V$ ($\operatorname{dist}(\widetilde{W}, \gamma^+) = \eta > 0$). Choose $a < \omega$ and let $a < b < \omega$. By the assumption on G we have $|G(z)| \le C|e^{-bz}|$ for $z \in \gamma^+$ and some $C < \infty$. Hence by (5.7) and (5.3) we get, with some constant $\widetilde{C}$,

$$\sup_{w\in\widetilde{W}} |e^{aw}G(w)| \le \frac{C}{2\pi} \sup_{w\in\widetilde{W}} \Big| \int_{\gamma^+} e^{aw-bz} \Lambda(z,w)\,dz\Big|$$
$$\le \frac{C}{2\pi} \sup_{w\in\widetilde{W}} \sup_{z\in\gamma^+} |e^{-a(z-w)}\Lambda(z,w)| \cdot \int_{\gamma^+} e^{-(b-a)\operatorname{Re} z}|dz| < \widetilde{C} < \infty.$$

To end the proof that $G \in \mathfrak{L}_{(\omega)}(W)$, i.e. to get the above estimate for an arbitrary tubular set $\widetilde{W} \Subset W$ we write $\widetilde{W} = \widetilde{W_1} \cup \widetilde{W_2}$ where $\widetilde{W_1} \Subset V$ (as before) and $\widetilde{W_2}$ is a tubular set $\widetilde{W_2} \Subset W \setminus \mathbb{R}_+$. The estimate in $\widetilde{W_2}$ holds by the assumption that $G \in \mathfrak{L}_{(\omega)}(W \setminus \overline{\mathbb{R}}_+)$. ∎

Proposition 5.2. *The space* $\mathfrak{L}_{(\omega)}(W)$ *is a closed subspace of* $\mathfrak{L}_{(\omega)}(W \setminus \overline{\mathbb{R}}_+)$.

Proof. Let $\mathfrak{L}_{(\omega)}(W) \ni G_\nu \to G$ in $\mathfrak{L}_{(\omega)}(W \setminus \overline{\mathbb{R}}_+)$ as $\nu \to \infty$. Hence by Lemma 5.2 for $\sigma \in \underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}_+)$ and $\gamma \subset W \setminus \mathbb{R}_+$ in the set of holomorphy of σ we have $\int_\gamma G_\nu(z)\sigma(z)\,dz = 0$ ($\nu = 1, 2, \ldots$), and to prove that $G \in \mathfrak{L}_{(\omega)}(W)$ it suffices to show that the same is true for G. Take any $\sigma \in \underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}_+)$ and $a < \omega$ such that $\sigma \in \underset{\sim}{L}_a(\overline{\mathbb{R}}_+)$. Then for $0 < \delta < \omega - a$ we get the estimate

$$\Big|\int_\gamma G(z)\sigma(z)\,dz\Big| = \Big|\int_\gamma (G(z) - G_\nu(z))\sigma(z)\,dz\Big|$$
$$\le C \sup_{z\in\gamma} \Big|e^{(\omega-\delta)z}(G(z) - G_\nu(z))\Big| \cdot \int_\gamma e^{-(\omega-a-\delta)\operatorname{Re} z}\,|dz|,$$

where the right-hand side converges to zero as $\nu \to \infty$. Hence $\int_\gamma G(z)\sigma(z)\,dz = 0$. ∎

Definition 5.2. The quotient space (2)

$$\mathfrak{Q}_{(\omega)}(\overline{\mathbb{R}}_+) = \mathfrak{L}_{(\omega)}(W \setminus \overline{\mathbb{R}}_+)/\mathfrak{L}_{(\omega)}(W)$$

is called the *space of Laplace hyperfunctions* on $\overline{\mathbb{R}}_+$ of *type* ω.

By Proposition 5.2 it is a Hausdorff topological space. A function $F \in \mathfrak{L}_{(\omega)}(W \setminus \overline{\mathbb{R}}_+)$ is called a *defining function* for the Laplace hyperfunction $f = F + \mathfrak{L}_{(\omega)}(W)$, denoted briefly by $f = [F]$.

Definition 5.3. We say that a sequence $f_\nu \in \mathfrak{Q}_{(\omega)}(\overline{\mathbb{R}}_+)$ ($\nu = 1, 2, \ldots$) is *convergent* if there exist defining functions F_ν such that F_ν converges in $\mathfrak{L}_{(\omega)}(W \setminus \overline{\mathbb{R}}_+)$ to some F. We set $\lim_{\nu\to\infty} f_\nu = f := [F]$.

Lemma 5.3. *Let* $\Psi \in \mathfrak{L}_{(\omega)}(W \setminus \overline{\mathbb{R}}_+)$, $W \Supset \overline{\mathbb{R}}_+$, *and for* $z \in W \setminus \overline{\mathbb{R}}_+$ *define* $\Psi^*(z) = \frac{1}{2\pi i}\int_\gamma \Psi(w)\Lambda(z,w)\,dw$, *where* γ *encircles* $\mathbb{R}_+$ *in the anticlockwise direction leaving* z *on*

(2) The correctness of this symbol (i.e. independence from W) will be clear from Theorem 5.1 below (cf. Corollary 5.2).

the right, and $\operatorname{dist}(\gamma, \partial(W \setminus \overline{\mathbb{R}}_+)) > 0$. *Then* $\Psi^* \in \mathfrak{L}_{(\omega)}(W \setminus \overline{\mathbb{R}}_+)$ *and*

$$\Psi - \Psi^* \in \mathfrak{L}_{(\omega)}(W). \tag{5.8}$$

PROOF. The proof that $\Psi^* \in \mathfrak{L}_{(\omega)}(W \setminus \overline{\mathbb{R}}_+)$ is similar to that of Proposition 5.1. We observe first that $\Psi^* \in \mathcal{O}(W \setminus \overline{\mathbb{R}}_+)$, take a tubular set $\widetilde{W} \Subset W \setminus \overline{\mathbb{R}}_+$, an $a < \omega$ and choose $0 < \varrho < \omega - a$ and a curve γ encircling $\overline{\mathbb{R}}_+$ in the anticlockwise direction and leaving $\widetilde{W}$ on the right. Then by the assumption on Ψ we get, with some constant $C < \infty$,

$$\sup_{z \in \widetilde{W}} |e^{az} \Psi^*(z)| \le C \sup_{z \in \widetilde{W}} \sup_{w \in \gamma} |e^{a(z-w)} \Lambda(z,w)| \cdot \int_\gamma e^{-\varrho \operatorname{Re} w} |dw| < \infty$$

and thus $\Psi^* \in \mathfrak{L}_{(\omega)}(W \setminus \overline{\mathbb{R}}_+)$.

In the proof of (5.8) we apply Lemma 5.2. Take $\sigma \in \underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}_+)$. Let $a < \omega$ and $U \Supset \overline{\mathbb{R}}_+$ be such that $\sigma \in \underset{\sim}{L}_a(U)$. Take curves γ and $\widetilde{\gamma}$ in U satisfying the usual conditions and moreover such that $\widetilde{\gamma}$ leaves γ on the left and $\operatorname{dist}(\widetilde{\gamma}, \gamma) > 0$.

Fix $w \in \gamma$, let $r > \operatorname{Re} w$, and split the oriented curve $\widetilde{\gamma}$ into two oriented curves: a bounded $\widetilde{\gamma}^{1,r}$ and an unbounded $\widetilde{\gamma}^{2,r}$ having a common bounded segment with the line $\operatorname{Re} z = r$. Clearly by the Cauchy formula applied to the function $\sigma_w(z) = \sigma(z) e^{-(z-w)^2}$ we get

$$\sigma(w) = \frac{1}{2\pi i} \int_{\widetilde{\gamma}^{1,r}} \sigma(z) \Lambda(z,w)\, dz \quad \text{for every } r > \operatorname{Re} w.$$

By the standard estimate $|\sigma(z)\Lambda(z,w)| \le C e^{-\varrho \operatorname{Re} z}$ with some $\varrho \in \overline{\mathbb{R}}_+$, $C = C(w) < \infty$ for $\operatorname{Re} z > a + 2 \operatorname{Re} w$, we have $\int_{\widetilde{\gamma}^{2,r}} \sigma(z)\Lambda(z,w)\, dz \to 0$ as $r \to \infty$. Thus

$$\sigma(w) = \frac{1}{2\pi i} \int_{\widetilde{\gamma}} \sigma(z) \Lambda(z,w)\, dz \tag{5.9}$$

and hence by applying the standard estimates and the Cauchy theorem,

$$\begin{aligned}\int_{\widetilde{\gamma}} \Psi^*(z)\sigma(z)\, dz &= \frac{1}{2\pi i} \int_\gamma \Psi(w) \Big(\int_{\widetilde{\gamma}} \sigma(z) \Lambda(z,w)\, dz \Big)\, dw \\ &= \int_\gamma \Psi(w)\sigma(w) dw = \int_{\widetilde{\gamma}} \Psi(w)\sigma(w)\, dw.\end{aligned}$$

In particular, to see that the change of order of integrations above was legitimate, observe (as in the proof of Lemma 5.1) that there exist $\varrho > 0$ and $\widetilde{C} < \infty$ such that

$$|\sigma(z)\Psi(w)\Lambda(z,w)| \le \widetilde{C} e^{\operatorname{Re}(z-w)(a - \operatorname{Re}(z-w))} e^{-\varrho \operatorname{Re} w} \quad \text{for } w \in \gamma,\ z \in \widetilde{\gamma},$$

which implies that the function $\sigma(z)\Psi(w)\Lambda(z,w)$ is integrable over $\gamma \times \widetilde{\gamma}$ since

$$\begin{aligned}\int_0^\infty \Big\{ \int_0^\infty e^{-(x-u)^2 + a(x-u)} e^{-\varrho u}\, dx \Big\}\, du &= \int_0^\infty \Big\{ \int_{-u}^\infty e^{-t^2 + at}\, dt \Big\} e^{-\varrho u}\, du \\ &\le \int_0^\infty e^{-\varrho u} \Big\{ \int_{-\infty}^\infty e^{-t^2 + at}\, dt \Big\}\, du < \infty. \ \blacksquare\end{aligned}$$

THEOREM 5.1. *There exists a natural topological isomorphism*

$$\mathfrak{Q}_{(\omega)}(\overline{\mathbb{R}}_+) \simeq \underset{\sim}{L}'_{(\omega)}(\overline{\mathbb{R}}_+), \quad \omega \in \mathbb{R} \cup \{\infty\},$$

given by

$$\mathfrak{Q}_{(\omega)}(\overline{\mathbb{R}}_+) \ni f = [F] \mapsto \mathcal{I}f \in \underset{\sim}{L}'_{(\omega)}(\overline{\mathbb{R}}_+),$$

where $F \in \mathfrak{L}_{(\omega)}(W \setminus \overline{\mathbb{R}}_+)$ *and the functional* $\mathcal{I}f$ *is given by* $\mathcal{I}f[\sigma] = -\int_\gamma F(z)\sigma(z)\,dz$ *for* $\sigma \in \underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}_+)$ *and* γ *encircling* $\overline{\mathbb{R}}_+$ *in the anticlockwise direction with* $\operatorname{dist}(\gamma, \overline{\mathbb{R}}_+) > 0$. *We assume that* $\sigma \in \underset{\sim}{L}_a(V)$ *with some* $a < \omega$, $V \Supset \overline{\mathbb{R}}_+$ *and that* $\gamma \subset V$. *The inverse mapping* $\mathcal{J}$ *is*

$$\underset{\sim}{L}'_{(\omega)}(\overline{\mathbb{R}}_+) \ni T \overset{\mathcal{J}}{\mapsto} [\mathcal{C}_\Lambda T] = \mathcal{C}_\Lambda T + \mathfrak{L}_{(\omega)}(W) \in \mathfrak{Q}_{(\omega)}(\overline{\mathbb{R}}_+),$$

where $\mathcal{C}_\Lambda T$, *defined in Definition* 5.1, *belongs to* $\mathfrak{L}_{(\omega)}(W \setminus \overline{\mathbb{R}}_+)$ *for every tubular neighbourhood* W *of* $\overline{\mathbb{R}}_+$ (*as asserted in Proposition* 5.1).

Proof. By assumptions on F, σ, γ there exists a tubular neighbourhood $V \Supset \overline{\mathbb{R}}_+$ and $a < \omega$ such that $|\int_\gamma F(z)\sigma(z)\,dz| \le C\varrho_{a,V}(\sigma)$ and $\mathcal{I}f[\sigma]$ is independent of the choice of γ encircling $\overline{\mathbb{R}}_+$ in V with $\operatorname{dist}(\gamma, \overline{\mathbb{R}}_+) > 0$. Thus the functional $\mathcal{I}f$ is in $\underset{\sim}{L}'_{(\omega)}(\overline{\mathbb{R}}_+)$ and by Lemma 5.2 it does not depend on the choice of the defining function F.

To prove that $\mathcal{J} = \mathcal{I}^{-1}$ we verify first that $\mathcal{I} \circ \mathcal{J} = \mathrm{Id}$ in $\underset{\sim}{L}'_{(\omega)}(\overline{\mathbb{R}}_+)$. Let $T \in \underset{\sim}{L}'_{(\omega)}(\overline{\mathbb{R}}_+)$ and $\sigma \in \underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}_+)$. Then by Definition 5.1 and (5.9) we get

$$(\mathcal{I} \circ \mathcal{J}T)[\sigma] = \frac{1}{2\pi i}\int_\gamma \sigma(z) T[\Lambda(z,w)]\,dz = T\left[\frac{1}{2\pi i}\int_\gamma \sigma(z)\Lambda(z,w)\,dz\right] = T[\sigma]$$

after passing to Riemann sums for the integral over γ (and proving their convergence in $\underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}_+)$). To prove that $\mathcal{J} \circ \mathcal{I}f = f$ for $f = [F] \in \mathfrak{Q}_{(\omega)}(\overline{\mathbb{R}}_+)$ observe that by Definition 5.1,

$$(\mathcal{C}_\Lambda(\mathcal{I}f))(\zeta) = -\frac{1}{2\pi i}(\mathcal{I}f)[\Lambda(\zeta,w)] = \frac{1}{2\pi i}\int_\gamma F(w)\Lambda(\zeta,w)\,dw = F^*(\zeta)$$

and hence by Lemma 5.3, $\mathcal{J} \circ \mathcal{I}f = [\mathcal{C}_\Lambda \mathcal{I}f] = [F] = f$.

To prove the continuity of $\mathcal{I}$ assume that $\lim_{\nu\to\infty} f_\nu = f$ in $\mathfrak{Q}_{(\omega)}(\overline{\mathbb{R}}_+)$ (cf. Definition 5.3), note that

$$|\mathcal{I}f_\nu[\sigma] - \mathcal{I}f[\sigma]| = \Big|\int_\gamma (F_\nu(z) - F(z))\sigma(z)\,dz\Big| \quad \text{for } \sigma \in \underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}_+)$$

and end the proof as in Proposition 5.2.

To prove the continuity of $\mathcal{J}$ assume that $T_\nu \to 0$ in $\underset{\sim}{L}'_{(\omega)}(\overline{\mathbb{R}}_+)$ as $\nu \to \infty$, i.e. $T_\nu[\sigma] \to 0$ for every $\sigma \in \underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}_+)$. By 2° and the Banach–Steinhaus theorem for every $a < \omega$ and tubular set $V \Supset \overline{\mathbb{R}}_+$ there exists a constant $C_{a,V} < \infty$ such that

$$|T_\nu[\sigma]| \le C_{a,V}\,\varrho_{a,V}(\sigma) \quad \text{for } \sigma \in \underset{\sim}{L}_a(W), \tag{5.10}$$

where W is an arbitrary open tubular set such that $V \Subset W$. Since $\mathcal{J}T_\nu = [\mathcal{C}_\Lambda T_\nu]$, it suffices to prove that $\mathcal{C}_\Lambda T_\nu \to 0$ in $\mathfrak{L}_{(\omega)}(W \setminus \overline{\mathbb{R}}_+)$ as $\nu \to \infty$. Clearly for every $\zeta \in W \setminus \overline{\mathbb{R}}_+$,

$$\mathcal{C}_\Lambda T_\nu(\zeta) = -\frac{1}{2\pi i} T_\nu[\Lambda(\zeta,w)] \underset{\nu\to\infty}{\longrightarrow} 0$$

since $\Lambda(\zeta,\cdot\,) \in \underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}_+)$. Take $V^1 \Subset W \setminus \overline{\mathbb{R}}_+$ and a tubular neighbourhood V^2 of $\overline{\mathbb{R}}_+$ such that $\operatorname{dist}(V^1, V^2) = \varrho > 0$. Then by (5.10) and Lemma 5.1 we get, proceeding as in the proof of Proposition 5.1,

$$\sup_{\zeta\in V^1}|e^{a\zeta}(\mathcal{C}_\Lambda T_\nu)(\zeta)| = \frac{1}{2\pi}\sup_{\zeta\in V^1}|e^{a\zeta}T_\nu[\Lambda(\zeta,w)]|$$
$$\le \frac{1}{2\pi}C_{a,V^2}\sup_{\zeta\in V^1}|e^{a\zeta}\sup_{w\in V^2}|e^{-aw}\Lambda(\zeta,w)|| \le \widetilde{C} < \infty.$$

Hence $e^{a\zeta}(\mathcal{C}_\Lambda T_\nu)$ is a sequence of holomorphic functions convergent to zero pointwise in $W\setminus\overline{\mathbb{R}}_+$ and uniformly bounded on V^1. By the Vitali theorem it is locally uniformly convergent on V^1.

Thus we have proved the following

Statement (A). *For every $a<\omega$ and every $V\Subset W\setminus\overline{\mathbb{R}}_+$ there exists $\widetilde{C}<\infty$ such that*

$$\sup_{\zeta\in V}|e^{a\zeta}(\mathcal{C}_\Lambda T_\nu)(\zeta)| \le \widetilde{C} < \infty \tag{5.11}$$

and the sequence $\{e^{a\zeta}(\mathcal{C}_\Lambda T_\nu)(\zeta)\}$ is locally uniformly convergent to zero on V.

To end the proof of Theorem 5.1 fix $\kappa<\omega$, $\mathring{V}\Subset W\setminus\overline{\mathbb{R}}_+$ and $\varepsilon>0$. Let $\mathring{V}\Subset V\Subset W\setminus\overline{\mathbb{R}}_+$, for any $R>0$ define $\mathring{V}_R=\mathring{V}\cap\{z:\operatorname{Re}z\le R\}$ and note that the closure of $\mathring{V}_R$ is a compact subset of V. Hence by Statement (A),

$$\sup_{\zeta\in\mathring{V}_R}|e^{\kappa\zeta}(\mathcal{C}_\Lambda T_\nu)(\zeta)| \xrightarrow[\nu\to\infty]{} 0. \tag{5.12}$$

Take a such that $\kappa<a<\omega$ and assume that

$$R\ge\frac{1}{a-\kappa}\left|\ln\frac{\varepsilon}{\widetilde{C}}\right|$$

with $\widetilde{C}$ from (5.11). Then by (A),

$$\sup_{\zeta\in\mathring{V}\setminus\mathring{V}_R}|e^{\kappa\zeta}(\mathcal{C}_\Lambda T_\nu)(\zeta)| \le \widetilde{C}\sup_{\zeta\in\mathring{V}\setminus\mathring{V}_R}|e^{(\kappa-a)\operatorname{Re}\zeta}| \le \widetilde{C}e^{-(a-\kappa)R}\le\varepsilon.$$

Hence by (5.12), $\sup_{\zeta\in\mathring{V}}|e^{\kappa\zeta}\mathcal{C}_\Lambda T_\nu(\zeta)|\le\varepsilon$ for ν large enough. This ends the proof. ∎

Remark 5.3. The continuity proof for $\mathcal{J}$ shows clearly that we made use of the fact that $\mathfrak{L}_{(\omega)}(W)$ is the projective limit both over $a<\omega$ and $\widetilde{W}\Subset W$ (cf. Remark 5.1).

Theorem 5.1 implies

Corollary 5.1. *A sequence of Laplace hyperfunctions $f_\nu\in\mathfrak{Q}_{(\omega)}$ converges to some $f\in\mathfrak{Q}_{(\omega)}$ if and only if one of the following two conditions holds*:

(i) *the sequence of defining functions $\mathcal{C}_\Lambda(\mathcal{I}f_\nu)$ converges to $\mathcal{C}_\Lambda(\mathcal{I}f)$ in $\mathfrak{L}_{(\omega)}(W\setminus\overline{\mathbb{R}}_+)$*,
(ii) *$\mathcal{I}f_\nu[\sigma]\to\mathcal{I}f[\sigma]$ as $\nu\to\infty$ for every $\sigma\in\underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}_+)$.*

In other words, each of the conditions (i) and (ii) is equivalent to that in Definition 5.3. Condition (ii) can be replaced by the assumption that for every $\sigma\in\underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}_+)$ the limit $\lim_{\nu\to\infty}\mathcal{I}f_\nu[\sigma]$ exists and is finite.

Corollary 5.2. *The quotient space $\mathfrak{L}_{(\omega)}(W\setminus\overline{\mathbb{R}}_+)/\mathfrak{L}_{(\omega)}(W)$ is independent of the choice of the complex tubular neighbourhood $W\Supset\overline{\mathbb{R}}_+$, i.e. $\mathfrak{L}_{(\omega)}(W\setminus\overline{\mathbb{R}}_+)/\mathfrak{L}_{(\omega)}(W)$ and $\mathfrak{L}_{(\omega)}(W_1\setminus\overline{\mathbb{R}}_+)/\mathfrak{L}_{(\omega)}(W_1)$ for any $W_1\Supset\overline{\mathbb{R}}_+$ are canonically isomorphic.*

6. Mellin hyperfunctions and Mellin distributions in one variable

6.1. Mellin hyperfunctions and Mellin analytic functionals. We introduce sectorial sets which correspond to tubular sets considered in Section 5 via the transformation

$$\mu : \mathbb{C} \ni \zeta \mapsto e^{-\zeta} \in \mathbb{C} \setminus \{0\}.$$

In particular, $I := (0,1] = \mu(\overline{\mathbb{R}}_+)$. Let $t > 0$ and $\theta_1 \le \theta_2$, $\theta_1, \theta_2 \in \mathbb{R}$. Define

$$I_t^{\theta_1,\theta_2} := \{\varrho e^{i\varphi} : 0 < \varrho \le t,\ \theta_1 \le \varphi \le \theta_2\}.$$

If $\theta_1 = -\theta$ and $\theta_2 = \theta \ge 0$, then $I_t^{-\theta,\theta} = \mu([\overline{\mathbb{R}}_+]_\theta)$ when $t = e^\theta$ and $[\overline{\mathbb{R}}_+]_\theta = [-\theta, \infty) + i[-\theta,\theta]$. If $\theta > 0$, then $[\overline{\mathbb{R}}_+]_\theta$ is a tubular neighbourhood of $\overline{\mathbb{R}}_+$ (see footnote ([1]), Section 5) and we say that $I_t^{-\theta,\theta}$ is a *sectorial neighbourhood* of I if $t > 1$. If $\theta > \pi$ then $I_t^{-\theta,\theta}$ is understood to be a sector in the universal covering space $(\widetilde{\mathbb{C} \setminus \{0\}}, a)$ of $\mathbb{C}\setminus\{0\}$ with base point $a \ne 0$ and projection $\Pi : (\widetilde{\mathbb{C} \setminus \{0\}}, a) \to \mathbb{C}\setminus\{0\}$. So

$$I_t^{-\theta,\theta} = \{w \in \widetilde{\mathbb{C} \setminus \{0\}} : 0 < |w| \le t,\ |\arg w| \le \theta\} \quad \text{for } t > 0,\ \theta \ge 0.$$

Recall that the covering space $(\widetilde{\mathbb{C} \setminus \{0\}}, a)$ is simply connected and the spaces $(\widetilde{\mathbb{C} \setminus \{0\}}, a)$ and $(\widetilde{\mathbb{C} \setminus \{0\}}, b)$ are isomorphic for $a, b \in \mathbb{C}\setminus\{0\}$. In the following we will take a fixed $a \in \mathbb{R}_+$ and omit it from the notation thus writing $\widetilde{\mathbb{C} \setminus \{0\}}$ for the space $(\widetilde{\mathbb{C} \setminus \{0\}}, a)$; we will also write $|w|$ for $|\Pi w|$.

Denoting by $\operatorname{Arg} w$ the principal argument of $w \in \mathbb{C}\setminus\{0\}$, i.e. $-\pi < \operatorname{Arg} w \le \pi$, we may consider the principal branch $\operatorname{Ln} w$ of $\ln w$ which is the single-valued function

$$\mathbb{C}\setminus\{0\} \ni w \mapsto \operatorname{Ln} w = \operatorname{Ln}|w| + i \operatorname{Arg} w.$$

The space $\widetilde{\mathbb{C} \setminus \{0\}}$ is the Riemann surface for the complex logarithmic function. The logarithm is a well defined single-valued function on that space:

$$\widetilde{\mathbb{C} \setminus \{0\}} \ni z \mapsto \ln z = \operatorname{Ln}|z| + i(\operatorname{Arg} \Pi z + 2k\pi)$$

with $k = 0, \pm1, \pm2, \ldots$ corresponding to the "level" of z in $\widetilde{\mathbb{C} \setminus \{0\}}$, and $|z| = |\Pi z|$.

Let $\theta_1 = \theta_2 = \theta \in \mathbb{R}$ and $t > 0$. Then $I_t^{\theta_1,\theta_2} = I_t^{\theta,\theta} = \{\varrho e^{i\theta} : 0 < \varrho \le t\}$; $I_t^{\theta,\theta}$ equals $I = (0,1]$ if $\theta = 0$ and $t = 1$. We see that

$$I_t^{\theta,\theta} = \mu[\mathbb{L}_+] \quad \text{where} \quad \mathbb{L}_+ = -\ln t - i\theta + \overline{\mathbb{R}}_+.$$

We say that V is a *sectorial neighbourhood* of $I_t^{\theta_1,\theta_2}$ where $\theta_1 \le \theta_2$, $\theta_1, \theta_2 \in \mathbb{R}$, $t > 0$, and write $I_t^{\theta_1,\theta_2} \Subset V$, if V is open in $\widetilde{\mathbb{C} \setminus \{0\}}$ and for some $0 < \varepsilon < M < \infty$,

$$I_{t+\varepsilon}^{\theta_1-\varepsilon,\theta_2+\varepsilon} \subset V \subset I_{t+M}^{\theta_1-M,\theta_2+M}. \tag{6.1}$$

More generally, $V \subset \widetilde{\mathbb{C} \setminus \{0\}}$ is called a *sectorial set* if it contains a sectorial neighbourhood of some $I_t^{\theta_1,\theta_2}$. We say that $S \subset \widetilde{\mathbb{C} \setminus \{0\}}$ is a *proper subset* of a sectorial set V, denoted by $S \Subset V$, if $\operatorname{pdist}(S, \partial V) > 0$, where the *polar distance* (pdist for short) between $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2} \in \widetilde{\mathbb{C} \setminus \{0\}}$ is defined as

$$\operatorname{pdist}(z_1, z_2) = \sqrt{(r_1 - r_2)^2 + (\theta_1 - \theta_2)^2}$$

and as usual

$$\operatorname{pdist}(A,B) = \inf_{\substack{z_1\in A\\ z_2\in B}} \operatorname{pdist}(z_1,z_2) \quad \text{for } A,B\subset \widetilde{\mathbb{C}\setminus\{0\}}.$$

Note that if $\operatorname{pdist}(S,\partial V) > 0$ then there exist $r>0$ and $\beta_1<\beta_2$ such that $S\cap\{z : |z|\le r\} \Subset I_r^{\beta_1,\beta_2} \Subset V$.

Next we define some spaces of holomorphic functions on an open sectorial set V having a power growth at zero. Let $\omega\in\mathbb{R}\cup\{\infty\}$ and $a<\omega$. Define the space

$$\mathfrak{M}_a(V) = \{F\in\mathcal{O}(V) : \sup_{z\in S}|z^{-a}F(z)| < \infty \text{ for every } S\Subset V\},$$

with the following convergence topology: $F_\nu\to 0$ in $\mathfrak{M}_a(V)$ as $\nu\to\infty$ if and only if $\sup_{z\in S}|z^{-a}F_\nu(z)|\to 0$ as $\nu\to\infty$ for any $S\Subset V$, and the space

$$\mathfrak{M}_{(\omega)}(V) = \varprojlim_{a<\omega}\mathfrak{M}_a(V)$$

with the projective limit topology.

Analogously we define the space $\mathfrak{M}_{(\omega)}(V\setminus I_t^{\theta_1,\theta_2})$ where V is an open sectorial neighbourhood of $I_t^{\theta_1,\theta_2}$ and denote by $\mathfrak{P}_{(\omega)}(I_t^{\theta_1,\theta_2})$ the quotient space

$$\mathfrak{P}_{(\omega)}(I_t^{\theta_1,\theta_2}) = \mathfrak{M}_{(\omega)}(V\setminus I_t^{\theta_1,\theta_2})/\mathfrak{M}_{(\omega)}(V).$$

The space $\mathfrak{P}_{(\omega)}(I)$ is called the space of *Mellin hyperfunctions* on $I=(0,1]=\mu(\overline{\mathbb{R}}_+)$. A function $G\in\mathfrak{M}_{(\omega)}(V\setminus I)$ is called a *defining function* for the Mellin hyperfunction $g=G+\mathfrak{M}_{(\omega)}(V)$, denoted briefly by $g=[G]$.

Definition 6.1 (cf. Definition 5.3). We say that a sequence $g_\nu\in\mathfrak{P}_{(\omega)}(I_t^{\theta_1,\theta_2})$ $(\nu = 1,2,\dots)$ is *convergent* if there exist defining functions G_ν such that G_ν converges in $\mathfrak{M}_{(\omega)}(V\setminus I_t^{\theta_1,\theta_2})$ to some G $(g_\nu=[G_\nu]=G_\nu+\mathfrak{M}_{(\omega)}(V))$. We then set $\lim_{\nu\to\infty} g_\nu = g := [G]$.

Now we define the spaces $\underset{\sim}{M}_a(V)$, where $a\in\mathbb{R}$ and V is an open sectorial neighbourhood of $I_t^{\theta_1,\theta_2}$, $\theta_1\le\theta_2$, $t>0$:

$$\underset{\sim}{M}_a(V) = \{\varphi\in\mathcal{O}(V) : \eta_{a,S}(\varphi)<\infty \text{ for every } S\Subset V\}$$

with the topology defined by the family of seminorms

$$\eta_{a,S}(\varphi) = \sup_{w\in S}|w^{a+1}\varphi(w)| \quad \text{for } S\Subset V.$$

Thus $\underset{\sim}{M}_a(V)$ is a complete Fréchet space with the topology defined by the family $\{\eta_{a,S}\}_{S\Subset V}$ of norms. Next for $\omega\in\mathbb{R}\cup\{\infty\}$ we consider the space

$$\underset{\sim}{M}_{(\omega)}(V) = \varinjlim_{a<\omega}\underset{\sim}{M}_a(V)$$

and thus $\varphi_\nu\to 0$ in $\underset{\sim}{M}_{(\omega)}(V)$ if and only if there exists $a<\omega$ such that $\varphi_\nu\to 0$ in $\underset{\sim}{M}_a(V)$.

Let $\theta_1\le\theta_2$, $t>0$ and $a\in\mathbb{R}$. Define

$$\underset{\sim}{M}_a(I_t^{\theta_1,\theta_2}) = \varinjlim_{V\Supset I_t^{\theta_1,\theta_2}}\underset{\sim}{M}_a(V),$$

where V ranges over sectorial neighbourhoods of $I_t^{\theta_1,\theta_2}$. For any $\omega\in\mathbb{R}\cup\{\infty\}$ we put

$$\underset{\sim}{M}_{(\omega)}(I_t^{\theta_1,\theta_2}) = \varinjlim_{a<\omega}\underset{\sim}{M}_a(I_t^{\theta_1,\theta_2})$$

and we also define

$$\underset{\sim}{M}_{-\infty}(I_t^{\theta_1,\theta_2}) = \varprojlim_{a\in\mathbb{R}} \underset{\sim}{M}_a(I_t^{\theta_1,\theta_2}) = \varprojlim_{a\in\mathbb{R}} \underset{\sim}{M}_{(a)}(I_t^{\theta_1,\theta_2}).$$

Note that

$$\underset{\sim}{M}_{(\omega)}(I_t^{\theta_1,\theta_2}) = \varinjlim_{V \geqslant I_t^{\theta_1,\theta_2}} \underset{\sim}{M}_{(\omega)}(V) = \varinjlim_{V \geqslant I_t^{\theta_1,\theta_2}} \varinjlim_{a<\omega} \underset{\sim}{M}_a(V).$$

1° (cf. 1° in Section 5). The space $\underset{\sim}{M}_a(I_t^{\theta_1,\theta_2})$ can be defined equivalently as follows:

$$\underset{\sim}{M}_a(I_t^{\theta_1,\theta_2}) = \varinjlim_{V \geqslant I_t^{\theta_1,\theta_2}} \underset{\sim}{M}_a(\overline{V}),$$

where $\overline{V}$ denotes the closure of V in $\widetilde{\mathbb{C}\setminus\{0\}}$ and

$$\underset{\sim}{M}_a(\overline{V}) = \{\varphi \in \mathcal{O}(V) \cap C^0(\overline{V}) : \eta_{a,V}(\varphi) < \infty\}$$

is a Banach space.

The space $\underset{\sim}{M}_{(\omega)}(I_t^{\theta_1,\theta_2})$ is complete; $\varphi \in \underset{\sim}{M}_{(\omega)}(I_t^{\theta_1,\theta_2})$ if and only if there exist $a < \omega$ and a sectorial neighbourhood V of $I_t^{\theta_1,\theta_2}$ such that $\varphi \in \mathcal{O}(V)$ and $\eta_{a,S}(\varphi) < \infty$ for every $S \leqslant V$.

Denote by $\underset{\sim}{M}'_a(V)$ the dual space of $\underset{\sim}{M}_a(V)$. Then $u \in \underset{\sim}{M}'_a(V)$ if and only if u is a linear functional on $\underset{\sim}{M}_a(V)$ and there exist a subset $S \leqslant V$ and a constant $C < \infty$ such that

$$|u[\varphi]| \le C\eta_{a,S}(\varphi) \quad \text{for } \varphi \in \underset{\sim}{M}_a(V).$$

The dual space $\underset{\sim}{M}'_{(\omega)}(V)$ is the space of linear functionals $u \in \underset{\sim}{M}'_a(V)$ for every $a < \omega$. Thus $u \in \underset{\sim}{M}'_{(\omega)}(V)$ if for every $a < \omega$, u is a linear functional on $\underset{\sim}{M}_a(V)$ and there exist a subset $S \leqslant V$ and a constant $C_a < \infty$ such that $|u[\varphi]| \le C_a\eta_{a,S}$ for $\varphi \in \underset{\sim}{M}_a(V)$.

The dual space $\underset{\sim}{M}'_{(\omega)}(I_t^{\theta_1,\theta_2})$ is called the space of *Mellin analytic functionals*. Sometimes it is denoted by $\underset{\sim}{M}'_{(\omega)}(\overline{I}_t^{\theta_1,\theta_2})$, where $\overline{I}_t^{\theta_1,\theta_2}$ is the radial compactification of $I_t^{\theta_1,\theta_2}$, to underline the fact that there exist Mellin analytic functionals carried by $\{0\}$ (see Example 6.1 below; cf. also the definition of the space $\underset{\sim}{L}'_{(\omega)}(\overline{\mathbb{R}}_+)$ in Section 5).

Note that by 1° we get (cf. 2° in Section 5)

2° $u \in \underset{\sim}{M}'_{(\omega)}(I_t^{\theta_1,\theta_2})$ if and only if for every $a < \omega$, u is a linear functional on $\underset{\sim}{M}_a(I_t^{\theta_1,\theta_2})$ and for every $S \geqslant I_t^{\theta_1,\theta_2}$ there exists a constant $C_{a,S}$ such that

$$|u[\varphi]| \le C_{a,S}\,\eta_{a,S}(\varphi) \quad \text{for } \varphi \in \underset{\sim}{M}_a(V),$$

where V is an arbitrary open sectorial set such that $S \leqslant V$.

Remark 6.1. The space $\underset{\sim}{M}_{(\omega)}(I)$ can be derived from the space $\underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}_+)$ by means of the transformation μ (see the beginning of this section):

$$\underset{\sim}{M}_{(\omega)}(I) = \left\{\frac{1}{z}\cdot\sigma\circ\mu^{-1}(z) : \sigma \in \underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}_+)\right\};$$

$u \in \underset{\sim}{M}'_{(\omega)}(I)$ if and only if there exists $T \in \underset{\sim}{L}'_{(\omega)}(\overline{\mathbb{R}}_+)$ such that $u = T\circ\mu^{-1}$.

Analogous relations hold between the spaces $\underset{\sim}{M}_a(\mu(W))$ and $\underset{\sim}{L}_a(W)$ (and their duals). The substitutions $T\circ\mu^{-1}$ for $T \in \underset{\sim}{L}'_{(\omega)}(\overline{\mathbb{R}}_+)$ and $u\circ\mu$ for $u \in \underset{\sim}{M}'_{(\omega)}(I)$ are defined (cf.

Subsection 4.10) by

$$T\circ\mu^{-1}[\varphi]=T[e^{-\zeta}\cdot\varphi\circ\mu(\zeta)]\qquad\text{for }\varphi\in\underset{\sim}{M}_{(\omega)}(I),$$
$$u\circ\mu[\psi]=u\left[\frac{1}{z}\cdot\psi\circ\mu^{-1}(z)\right]\qquad\text{for }\psi\in\underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}_{+}).$$

Another equivalent way of defining Mellin analytic functionals carried by $I_t^{\theta_1,\theta_2}$ is given in Theorem 6.1 and is similar to the method of defining Laplace analytic functionals in Section 5 (cf. also Theorem 4.4). We must keep track, this time, of the growth order of defining functions when approaching zero. To ensure those conditions at zero we replace the standard Cauchy kernel by the so-called *"logarithmic" Cauchy kernel*

$$\Gamma(z,w)=\frac{1}{w}\,\frac{e^{-(\ln z-\ln w)^2}}{\ln z-\ln w}. \tag{6.2}$$

Lemma 6.1. *Fix a sectorial neighbourhood $W\gg I_t^{\theta_1,\theta_2}$ and a sectorial set $V^1\ll W\setminus I_t^{\theta_1,\theta_2}$. Choose a sectorial set V^2 such that $I_t^{\theta_1,\theta_2}\ll V^2\ll W$ and* $\operatorname{pdist}(V^1,V^2)>0$. *Let $c\in\mathbb{R}$. Then there exists $C<\infty$ such that*

$$\sup_{w\in V^2}\ \sup_{\zeta\in V^1}|\zeta^{-c}w^{c+1}\Gamma(\zeta,w)|\le C<\infty. \tag{6.3}$$

In particular, for fixed $\zeta\in W\setminus I_t^{\theta_1,\theta_2}$ and $I_t^{\theta_1,\theta_2}\ll V\ll W$ with $\operatorname{dist}(\zeta,V)>0$ *we have* $\sup_{w\in V}|w^{c+1}\Gamma(\zeta,w)|<\infty$, *which means that $\Gamma(\zeta,\cdot\,)\in\underset{\sim}{M}_c(V)$ and hence (since c was arbitrary) $\Gamma(\zeta,\cdot\,)\in\underset{\sim}{M}_{(\omega)}(V)$ for every $\omega\in\mathbb{R}\cup\{\infty\}$, and $\Gamma(\zeta,\cdot\,)\in\underset{\sim}{M}_{-\infty}(V)$.*

Proof. Since $W\gg I_t^{\theta_1,\theta_2}$ by (6.1), $\arg\zeta-\arg w$ is bounded for any $\zeta,w\in W$, and

$$\sup_{w\in V^2}\ \sup_{\zeta\in V^1}\frac{1}{|\ln\zeta-\ln w|}$$

is bounded since $\operatorname{pdist}(V^1,V^2)>0$. Hence there exist $C_1\le C<\infty$ such that

$$\begin{aligned}&\sup_{w\in V^2}\ \sup_{\zeta\in V^1}|\zeta^{-c}w^{c+1}\Gamma(\zeta,w)|\\&\qquad\le\sup_{w\in V^2}\ \sup_{\zeta\in V^1}e^{-\operatorname{Re}(\ln\zeta-\ln w)(c+\operatorname{Re}(\ln\zeta-\ln w))}\cdot e^{(\arg\zeta-\arg w)^2}\cdot\frac{1}{|\ln\zeta-\ln w|}\\&\qquad\le C_1\sup_{\xi\in\mathbb{R}}e^{-\xi(c+\xi)}\le C.\ \blacksquare\end{aligned}$$

Remark 6.2. The function (6.2) considered as a function of the variable z with w fixed ($z\ne0,\ w\ne0$) has a simple pole at $z=w$ with residue 1. If we consider it as a function of w (with z fixed), the residue at $w=z$ equals -1.

By Lemma 6.1 the following definition makes sense.

Definition 6.2. Let $\omega\in\mathbb{R}\cup\{\infty\}$ and $u\in\underset{\sim}{M}'_{(\omega)}(I_t^{\theta_1,\theta_2})$. The function

$$\Phi(z):=(\mathcal{C}_\Gamma u)(z):=-\frac{1}{2\pi i}u[\Gamma(z,\cdot\,)]\quad\text{for }z\in\widetilde{\mathbb{C}\setminus\{0\}}\setminus I_t^{\theta_1,\theta_2} \tag{6.4}$$

will be called the *logarithmic Cauchy transform* of u.

If $u \in \underset{\sim}{M}'_{(\omega)}(I)$ (i.e. $\theta_1 = \theta_2 = 0$, $t = 1$) then $u = T \circ \mu^{-1}$ with some $T \in L'_{(\omega)}(\overline{\mathbb{R}}_+)$ and by Definition 5.1 we get

$$\mathcal{C}_\Lambda T = \mathcal{C}_\Gamma u \circ \mu.$$

THEOREM 6.1. *There exists a natural topological isomorphism*

$$\mathfrak{P}_{(\omega)}(I_t^{\theta_1,\theta_2}) \simeq \underset{\sim}{M}'_{(\omega)}(I_t^{\theta_1,\theta_2})$$

given by ([1])

$$\mathfrak{P}_{(\omega)}(I_t^{\theta_1,\theta_2}) \ni g = [G] \mapsto \mathcal{I}_M g \in \underset{\sim}{M}'_{(\omega)}(I_t^{\theta_1,\theta_2}),$$

where $G \in \mathfrak{M}_{(\omega)}(V \setminus I_t^{\theta_1,\theta_2})$, *the functional* $\mathcal{I}_M g$ *is given by*

$$\mathcal{I}_M g[\varphi] = -\int_\gamma G(z)\varphi(z)\,dz \quad \text{for } \varphi \in \underset{\sim}{M}_{(\omega)}(\widetilde{V}),\ I_t^{\theta_1,\theta_2} \Subset \widetilde{V}, \tag{6.5}$$

and γ *is an arbitrary rectifiable curve with both end-points at zero, encircling* $I_t^{\theta_1,\theta_2}$ *once in the anticlockwise direction and contained in* $(V\setminus I_t^{\theta_1,\theta_2})\cap\widetilde{V}$ *with* $\operatorname{pdist}(\gamma, \partial(I_t^{\theta_1,\theta_2})) > 0$. *The inverse mapping* $\mathcal{J}_M$ *is*

$$\underset{\sim}{M}'_{(\omega)}(I_t^{\theta_1,\theta_2}) \ni u \overset{\mathcal{J}_M}{\mapsto} [\mathcal{C}_\Gamma u] = \Phi + \mathfrak{M}_{(\omega)}(V) \in \mathfrak{P}(I_t^{\theta_1,\theta_2}).$$

Theorem 6.1 is a slight generalization of the corresponding result for the Laplace spaces stated in Theorem 5.1 and its proof can be given in a completely analogous way. So we only point out the main steps of the proof giving references to Section 5 instead of proofs. For simplicity we take $\theta_1 = -\theta$, $\theta_2 = \theta \geq 0$ and denote $I_t^{-\theta,\theta}$ by I^θ.

LEMMA 6.2 (cf. Lemma 5.2). *Let* $G \in \mathfrak{M}_{(\omega)}(V \setminus I^\theta)$. *Then* $G \in \mathfrak{M}_{(\omega)}(V)$ *if and only if*

$$\int_\gamma G(z)\varphi(z)\,dz = 0 \quad \text{for } \varphi \in \underset{\sim}{M}_{(\omega)}(I^\theta),$$

where γ *encircles* I^θ *in the anticlockwise direction*, *lies in the set where both* G *and* φ *are holomorphic and* $\operatorname{pdist}(\gamma, I^\theta) > 0$.

By Lemma 6.2 it follows at once that the linear transformation $\mathcal{I}_M$ described in Theorem 6.1 is well defined. Moreover, Lemma 6.2 yields (see the proof of Proposition 5.2)

PROPOSITION 6.1. *The space* $\mathfrak{M}_{(\omega)}(V)$ *is a closed subspace in* $\mathfrak{M}_{(\omega)}(V\setminus I^\theta)$ *and hence* $\mathfrak{P}_{(\omega)}(I^\theta)$ *is a Hausdorff topological space.*

DEFINITION 6.3. We say that a sequence $g_\nu \in \mathfrak{P}_{(\omega)}(I^\theta)$, $\nu \in \mathbb{N}$, is *convergent* if there exist functions G_ν, $\nu \in \mathbb{N}$, such that G_ν converges in $\mathfrak{M}_{(\omega)}(V \setminus I^\theta)$ to some G. We set $\lim_{\nu\to\infty} g_\nu = g := [G]$.

PROPOSITION 6.2 (cf. Proposition 5.1). *Let* $u \in \underset{\sim}{M}'_{(\omega)}(I^\theta)$ *and* $\omega \in \mathbb{R}\cup\{\infty\}$. *Then* $\mathcal{C}_\Gamma u \in \mathfrak{M}_{(\omega)}(W \setminus I^\theta)$ *for every sectorial neighbourhood* W *of* I^θ. *Thus the linear transformation* $\mathcal{J}_M$ *described in Theorem* 6.1 *is well defined.*

([1]) We use the subscript M to distinguish the assignments $\mathcal{I}_M, \mathcal{J}_M$ defined below from $\mathcal{I}, \mathcal{J}$ from Theorem 5.1.

In the proof that $\mathcal{J}_M$ is the inverse of $\mathcal{I}_M$ we use the following (counterpart of Lemma 5.3 for the "logarithmic" Cauchy kernel):

LEMMA 6.3. *Let* $V \gg I^\theta$, $\omega \in \mathbb{R} \cup \{\infty\}$, $G \in \mathfrak{M}_{(\omega)}(V \setminus I^\theta)$ *and* $g = G + \mathfrak{M}_{(\omega)}(V)$. *Let*

$$G^*(z) = \frac{1}{2\pi i} \int_\gamma G(w)\Gamma(z,w)\,dw \quad \textit{for } z \in V \setminus I^\theta,$$

where γ *is a curve in* V *encircling* I^θ *anticlockwise, leaving* z *on the right and* $\operatorname{pdist}(\gamma, \partial(V \setminus I^\theta)) > 0$. *Then* $G^* \in \mathfrak{M}_{(\omega)}(V \setminus I^\theta)$, $G^* - G \in \mathfrak{M}_{(\omega)}(V)$ *and thus* $g = G^* + \mathfrak{M}_{(\omega)}(V)$.

PROOF OF THEOREM 6.1. Now we prove that $\mathcal{I}_M \circ \mathcal{J}_M = \mathrm{Id}$ on $\underset{\sim}{M}'_{(\omega)}(I^\theta)$ and $\mathcal{J}_M \circ \mathcal{I}_M = \mathrm{Id}$ on $\mathfrak{M}_{(\omega)}(V \setminus I^\theta)/\mathfrak{M}_{(\omega)}(V)$. Let $u \in \underset{\sim}{M}'_{(\omega)}(I^\theta)$ and $\varphi \in \underset{\sim}{M}_{(\omega)}(I^\theta)$. We prove that $(\mathcal{I}_M \circ \mathcal{J}_M u)[\varphi] = u[\varphi]$. Observe that by (6.5), the definition of $\mathcal{J}_M$ and by (6.4) we get

$$(6.6) \qquad (\mathcal{I}_M \circ \mathcal{J}_M u)[\varphi] = -\int_\gamma \mathcal{C}_\Gamma u(z)\varphi(z)\,dz = \frac{1}{2\pi i}\int_\gamma u[\Gamma(z,w)]\varphi(z)\,dz = u\bigg[\frac{1}{2\pi i}\int_\gamma \Gamma(z,w)\varphi(z)\,dz\bigg]$$

after passing to Riemann sums for the integral over γ (and proving their convergence in $\underset{\sim}{M}_{(\omega)}(I^\theta)$). Now, it is easy to note that by (6.2) and Remark 6.2,

$$\frac{1}{2\pi i}\int_\gamma \varphi(z)\Gamma(z,w)\,dz = \varphi(w),$$

which in view of (6.6) leads to the desired result.

Next take $g = G + \mathfrak{M}_{(\omega)}(V)$, $G \in \mathfrak{M}_{(\omega)}(V \setminus I^\theta)$ and let $u = \mathcal{I}_M g$. By the definition of $\mathcal{J}_M$, $\mathcal{J}_M \circ \mathcal{I}_M g = \mathcal{C}_\Gamma(\mathcal{I}_M g) + \mathfrak{M}_{(\omega)}(V)$. Hence by (6.4) and the definition of $\mathcal{I}_M$,

$$\mathcal{C}_\Gamma(\mathcal{I}_M g)(z) = -\frac{1}{2\pi i}(\mathcal{I}_M g)[\Gamma(z,\cdot\,)] = \frac{1}{2\pi i}\int_\gamma G(w)\Gamma(z,w)\,dw \quad \text{for } z \in \widetilde{\mathbb{C} \setminus \{0\}} \setminus I_t^{\theta_1,\theta_2}.$$

Thus Lemma 6.3 shows that $\mathcal{J}_M \circ \mathcal{I}_M g = \int_\gamma G(w)\Gamma(z,w)\,dw + \mathfrak{M}_{(\omega)}(V) = g$.

Therefore we have proved that $\mathcal{I}_M$ is an algebraic isomorphism between the spaces $\mathfrak{P}_{(\omega)}(I^\theta)$ and $\underset{\sim}{M}'_{(\omega)}(I^\theta)$.

To prove the continuity of $\mathcal{I}_M$ assume that $\lim_{\nu\to\infty} g_\nu = g$ in $\mathfrak{P}_{(\omega)}(I^\theta)$ (cf. Definition 6.1), i.e. there exist G_ν, G such that $g_\nu = [G_\nu]$, $g = [G]$ and $\lim_{\nu\to\infty} G_\nu = G$ in $\mathfrak{M}_{(\omega)}(V \setminus I^\theta)$. Take $\varphi \in \underset{\sim}{M}_c(V^*)$, where $\gamma \subset V^*$, $V^* \gg I^\theta$ and $c < \omega$. Thus $\sup_{z\in\gamma} |z^{c+1}\varphi(z)| < \infty$. Note that $|\mathcal{I}_M g_\nu[\varphi] - \mathcal{I}_M g[\varphi]| = \left|\int_\gamma (G_\nu(z) - G(z))\varphi(z)\,dz\right|$ and end the proof as that of Proposition 6.1 (cf. the proof of continuity of $\mathcal{I}$ in Theorem 5.1).

The continuity of $\mathcal{J}_M$ can be proved analogously to that of $\mathcal{J}$ in Theorem 5.1 (cf. also Remark 5.3). ■

Theorem 6.1 has the following corollaries.

COROLLARY 6.1. *The convergence* $g_\nu \to g$ *in* $\mathfrak{P}_{(\omega)}(I_t^{\theta_1,\theta_2})$ *defined in Definition* 6.1 *is equivalent to* $\mathcal{I}_M g_\nu \to \mathcal{I}_M g$ *in* $\underset{\sim}{M}'_{(\omega)}(I_t^{\theta_1,\theta_2})$, *i.e.* $(\mathcal{I}_M g_\nu)[\varphi] \to (\mathcal{I}_M g)[\varphi]$ *for every*

$\varphi \in \underset{\sim}{M}_{(\omega)}(I_t^{\theta_1,\theta_2})$. *Moreover, Definition* 6.1 *can assume the following equivalent form*: $g_\nu \to g$ *in* $\mathfrak{P}_{(\omega)}(I_t^{\theta_1,\theta_2})$ *if* $\mathcal{C}_\Gamma(\mathcal{I}_M g_\nu) \to \mathcal{C}_\Gamma(\mathcal{I}_M g)$ *in* $\underset{\sim}{M}_{(\omega)}(V \setminus I_t^{\theta_1,\theta_2})$.

Corollary 6.2. *The quotient space* $\mathfrak{M}_{(\omega)}(V \setminus I_t^{\theta_1,\theta_2})/\mathfrak{M}_{(\omega)}(V)$ *is independent of the choice of the complex neighbourhood* $V \succcurlyeq I_t^{\theta_1,\theta_2}$, *i.e. the spaces* $\mathfrak{M}_{(\omega)}(V \setminus I_t^{\theta_1,\theta_2})/\mathfrak{M}_{(\omega)}(V)$ *and* $\mathfrak{M}_{(\omega)}(W \setminus I_t^{\theta_1,\theta_2})/\mathfrak{M}_{(\omega)}(W)$ *for any* $W \succcurlyeq I_t^{\theta_1,\theta_2}$ *are canonically isomorphic.*

For $\theta = 0$ and $t = 1$ (then $I_t^\theta = I = (0,1]$) we have

$$\mathfrak{M}_{(\omega)}(V \setminus I)/\mathfrak{M}_{(\omega)}(V) \simeq \underset{\sim}{M}'_{(\omega)}(I);$$

thus the space $\mathfrak{P}_{(\omega)}(I)$ of Mellin hyperfunctions on I does not depend on the choice of the sectorial neighbourhood V of I.

In the case of $\theta_1 = \theta_2 = 0$, $t = 1$, Theorem 6.1 is related to Theorem 5.1 by the following commuting diagram in which all the operations are continuous:

$$\begin{array}{ccc}
\mathfrak{M}_{(\omega)}(V \setminus I)/\mathfrak{M}_{(\omega)}(V) \ni g & \xrightarrow{\ \mathcal{I}_M\ } & \mathcal{I}_M g \in \underset{\sim}{M}'_{(\omega)}(I) \\
\mu^{-1}\uparrow\ \downarrow\mu & & \mu^{-1}\uparrow\ \downarrow\mu \\
\mathfrak{L}_{(\omega)}(W \setminus \overline{\mathbb{R}}_+)/\mathfrak{L}_{(\omega)}(W) \ni g \circ \mu & \xrightarrow{\ \mathcal{I}\ } & \mathcal{I}(g \circ \mu) = (\mathcal{I}_M g) \circ \mu \in \underset{\sim}{L}'_{(\omega)}(\overline{\mathbb{R}}_+).
\end{array}$$

One could deduce Theorem 6.1 from Theorem 5.1 simply by applying the transformation μ (and μ^{-1}). Instead we outlined a direct proof. Similarly, the existence of a Laplace hyperfunction carried by the point ∞ (cf. [M-Y]) implies the existence of a Mellin hyperfunction with carrier at zero, i.e. a non-zero Mellin hyperfunction in $\bigcap_{t>0} \underset{\sim}{M}'_{(\omega)}((0,t])$. Below we give a direct proof of the latter statement.

Example 6.1 (Mellin hyperfunction with carrier at $\{0\}$). The construction of a Mellin hyperfunction carried by $\{0\}$ is done in two steps. Let $I_t = (0,t]$, $t > 0$.

I. We construct a Mellin hyperfunction T such that

$$T \in \underset{\sim}{M}'_{(0)}(I_t) \quad \text{for every} \quad 0 < t < e^{-1}.$$

By the arbitrariness of t we note that the carrier of T is contained in $\{0\}$.

II. We prove that $T \neq 0$.

Step I. Let

$$H_\tau = \{w \in \mathbb{C} : |w| \le \tau,\ |\ln|w| \cdot \operatorname{Arg} w| \le \pi/2\}, \quad \tau \in (0, e^{-1})$$

(see Fig. 2). For every sectorial set $S \succcurlyeq I_t$ $(0 < t < e^{-1})$ we have $H_\tau \subset S$ for sufficiently small τ. Let

$$\Psi(w) = \exp(e^{(\operatorname{Ln} w)^2}).$$

Then

$$|\Psi(w)| \le \exp(\cos(2\ln|w|\operatorname{Arg} w) \cdot e^{(\ln|w|)^2 - (\operatorname{Arg} w)^2}).$$

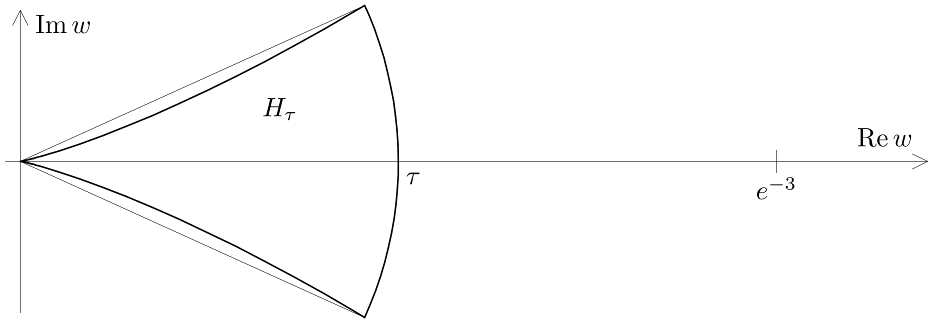

Fig. 2

It is easy to see that on ∂H_τ we have

$$|\Psi(w)| \le \exp(e^{(\ln\tau)^2}) = C_1(\tau) < \infty \quad \text{when } |w| = \tau,$$
$$|\Psi(w)| = \exp(-e^{(\ln|w|)^2-(\operatorname{Arg} w)^2}) \le \exp(-\lambda e^{(\ln|w|)^2}) \le 1$$

when

$$|2\ln|w|\operatorname{Arg} w| = \pi \quad \text{and} \quad \lambda = \exp\biggl(-\frac{\pi^2}{4}\frac{1}{|\ln\tau|^2}\biggr).$$

Fix $0 < t < e^{-1}$ and $\varphi \in \underset{\sim}{M}_{(0)}(I_t)$. There exist $c < 0$ and $V \geqslant I_t$ such that $\sup_{z\in S}|z^{c+1}\varphi(z)| < \infty$ for any bounded proper sectorial set $S \leqslant V$. Choose $S \leqslant V$ and τ small enough that $H_\tau \subset S \leqslant V$. Then

$$T[\varphi] = -\frac{1}{2\pi i}\int_{\partial H_\tau}\Psi(w)\varphi(w)\,dw, \tag{6.7}$$

where ∂H_τ is the boundary of H_τ anticlockwise oriented, is well defined (does not depend on the choice of τ). Moreover,

$$|T[\varphi]| \le \frac{1}{2\pi}(C_1(\tau)+1)\sup_{w\in S}|w^{c+1}\varphi(w)|\int_{\partial H_\tau} w^{-c-1}|dw| \le C_{c,S}\sup_{w\in S}|w^{c+1}\varphi(w)|,$$

hence $T \in \underset{\sim}{M}'_{(0)}(I_t)$.

Step II. Let $0 < \delta < 1$, $c < 0$, $t < e^{-(1+|c|)/\delta}$ and define $\psi(w) = \operatorname{Ln} w \cdot e^{\operatorname{Ln} w(-\delta \operatorname{Ln} w - 1)}$ for $w \in V = \{w \in \mathbb{C} : |w| < t,\ |\operatorname{Arg} w| \le \pi/2\}$. We show that $\psi \in \underset{\sim}{M}_c(V)$ and hence $\psi \in \underset{\sim}{M}_{(0)}(I_t)$. Indeed, choose a subsector $S \leqslant V$ and observe that

$$\begin{aligned}\sup_{w\in S}|w^{c+1}\psi(w)| &= \sup_{w\in S}|\operatorname{Ln} w|e^{c\ln|w| - \delta(\ln|w|)^2 + \delta(\operatorname{Arg} w)^2}\\ &\le \sup_{w\in S}((\ln|w|)^2 + \pi^2/4)^{1/2}e^{\delta\pi^2/4}e^{-|\ln|w||} < \infty,\end{aligned}$$

i.e. $\psi \in \underset{\sim}{M}_c(V)$.

Thus by (6.7),

$$T[\psi] = -\frac{1}{2\pi i}\int_{\partial H_\tau}\Psi(w)\psi(w)\,dw = -\frac{1}{2\pi i}\int_{\partial H_\tau}\frac{\operatorname{Ln} w}{w}e^{-\delta(\operatorname{Ln} w)^2}\exp(e^{(\operatorname{Ln} w)^2})\,dw.$$

Substituting $z = e^{(\mathrm{Ln}\, w)^2}$ in the last integral and suitably deforming the contour (which does not affect $T[\psi]$), we get after integration by parts

$$T[\psi] = -\frac{1}{4\pi i\delta}\int_\gamma z^{-\delta}e^z\,dz$$

where γ consists of the interval $(-\infty,-\varepsilon]$ run over twice in opposite directions and of the positively oriented circle $|z|=\varepsilon$. Thus (cf. [W, Chapter 6])

$$T[\psi] = -\frac{1}{4\pi i\delta}2i\sin\pi\delta\int_0^\infty x^{-\delta}e^{-x}\,dx = -\frac{1}{2\pi}\sin\pi\delta\; \Gamma(-\delta)\neq 0,$$

where Γ is the Euler function.

Analytic functionals $A'(I_r^\theta)$. A special case of Mellin analytic functionals are the analytic functionals carried by sets denoted briefly by I_r^θ,

$$I_r^\theta := I_{r,t}^{-\theta,\theta} = \{z\in\widetilde{\mathbb{C}\setminus\{0\}} : r\le|z|\le t,\ |\arg z|\le\theta\}\quad\text{for some}\quad t>r>0.$$

They are defined as the dual $A'(I_r^\theta)$ of the space

$$A(I_r^\theta) = \varinjlim_{U\supset I_r^\theta} H(\overline{U})$$

where $H(\overline{U})$ denotes the space of continuous functions on $\overline{U}$ which are holomorphic on U and U ranges over open neighbourhoods of I_r^θ in $\widetilde{\mathbb{C}\setminus\{0\}}$ (cf. also Subsection 3.2). A proof similar to that of the Köthe Theorem 4.4 shows that there exists a natural topological isomorphism

$$\mathcal{O}(U\setminus I_r^\theta)/\mathcal{O}(U)\simeq A'(I_r^\theta), \tag{6.8}$$

where $U\subset\widetilde{\mathbb{C}\setminus\{0\}}$ is an open neighbourhood of I_r^θ. Lemma 4.2 is now replaced by

LEMMA 6.4. *Let* $G\in\mathcal{O}(U\setminus I_r^\theta)$ (U *open in* $\widetilde{\mathbb{C}\setminus\{0\}}$, $I_r^\theta\subset U$). *Then* $G\in\mathcal{O}(U)$ *if and only if* $\int_\gamma G(z)\varphi(z)\,dz=0$ (γ *encircles* I_r^θ) *for every* $\varphi\in A(I_r^\theta)$.

By Proposition 4.5 and Remark 6.2, when defining the isomorphism (6.8) we may take both the standard Cauchy kernel and the logarithmic Cauchy kernel; this will be used in the proof of Proposition 6.3 below.

$A'(I_r^\theta)$ can be regarded as a subspace of $\underset{\sim}{M}'_{(\omega)}(I^\theta)$ for any $\omega\in\mathbb{R}\cup\{\infty\}$ and $I^\theta = I_t^{-\theta,\theta}$, $\theta\ge 0$. We write

$$A'(I_r^\theta)\hookrightarrow \underset{\sim}{M}'_{(\omega)}(I^\theta)\quad\text{for any }\omega\in\mathbb{R}\cup\{\infty\}.$$

Indeed, if $u\in A'(I_r^\theta)$ then (taking $U=\widetilde{\mathbb{C}\setminus\{0\}}$ in (6.8)), $u=G+\mathcal{O}(\widetilde{\mathbb{C}\setminus\{0\}})$ with $G\in\mathcal{O}(\widetilde{\mathbb{C}\setminus\{0\}}\setminus I_r^\theta)$ and the following definition makes sense:

$$\widetilde{u}[\varphi] = -\int_\gamma G(z)\varphi(z)\,dz\quad\text{for }\varphi\in\underset{\sim}{M}_{(\omega)}(I^\theta), \tag{6.9}$$

where γ is a rectifiable curve encircling I_r^θ once in the anticlockwise direction which is contained in the set where G and φ are holomorphic and $\mathrm{dist}(\gamma,0)>0$. In fact, take $\varphi\in\underset{\sim}{M}_{(\omega)}(I^\theta)$ and let $a<\omega$ and $V\succcurlyeq I^\theta$ be such that $\varphi\in\underset{\sim}{M}_a(V)$. Take $I^\theta\preccurlyeq S\preccurlyeq V$ and $\gamma\subset S$. Then $|\widetilde{u}[\varphi]|\le C_{a,S}\sup_{z\in S}|\varphi(z)z^{a+1}|$ since $G(z)z^{-a-1}$ is holomorphic on γ. Thus $\widetilde{u}\in\underset{\sim}{M}'_{(\omega)}(I^\theta)$ by 2°.

By the Cauchy theorem the functional $\widetilde{u}$ does not depend on the choice of the defining function G and the curve γ, and hence (6.9) gives the map

$$A'(I_r^\theta) \ni u \mapsto \widetilde{u} \in \underset{\sim}{M}'_{(\omega)}(I^\theta). \tag{6.10}$$

To show that (6.10) is injective assume that $\widetilde{u} = 0$, i.e. $\int_\gamma G(z)\varphi(z)\,dz = 0$ for $\varphi \in \underset{\sim}{M}_{(\omega)}(I^\theta)$ and hence also for $\varphi \in A(I_r^\theta)$. Thus by Lemma 6.4, $G \in \mathcal{O}(U)$ and by isomorphism (6.8), $u = 0$.

The following proposition characterizes $A'(I_r^\theta)$ as a subspace of $\underset{\sim}{M}'_{(\omega)}(I^\theta)$ in the spirit of Theorem 6.1.

PROPOSITION 6.3. *$A'(I_r^\theta)$ coincides with the subspace of $\underset{\sim}{M}'_{(\omega)}(I^\theta)$ (for every $\omega \in \mathbb{R}\cup\{\infty\}$) formed by those $u \in \underset{\sim}{M}'_{(\omega)}(I^\theta)$ for which there exists a defining function G such that*:

(i) *$G \in \mathcal{O}(V \setminus I_r^\theta)$ where V is a sectorial neighbourhood of I^θ in $\widetilde{\mathbb{C}\setminus\{0\}}$,*
(ii) *for any $a < \omega$ and any proper sectorial subset $\widetilde{S} \Subset V \setminus I_r^\theta$* ([2])

$$|G(z)| \le C_{a,\widetilde{S}}|z|^a \quad \textit{for } z \in \widetilde{S}.$$

The defining function G of $u \in \underset{\sim}{M}'_{(\omega)}(I^\theta)$ satisfying conditions (i), (ii) *is also a defining function of the extension of u to a linear continuous functional on $A(I_r^\theta)$.*

PROOF. Suppose first that $u \in A'(I_r^\theta)$. Then (as in 2°) u is linear on $A(I_r^\theta)$ and for every V with $\widetilde{\mathbb{C}\setminus\{0\}} \Supset V \Supset I_r^\theta$ there exists a constant C_V such that

$$|u[\varphi]| \le C_V \sup_{w\in V}|\varphi(w)| \quad \text{for } \varphi \in H(\overline{U}),$$

where U is an arbitrary open set in $\widetilde{\mathbb{C}\setminus\{0\}}$ such that $V \Subset U$. The logarithmic defining function of u in the isomorphism (6.8),

$$\mathcal{C}_\Gamma u(z) = -\frac{1}{2\pi i}u[\Gamma(z,w)],$$

is holomorphic on $\widetilde{\mathbb{C}\setminus\{0\}} \setminus I_r^\theta$. Fix a sectorial set $\widetilde{S} \Subset \widetilde{\mathbb{C}\setminus\{0\}} \setminus I_r^\theta$ and $a \in \mathbb{R}$. Choose $S \Supset I_r^\theta$ with $\operatorname{pdist}(\widetilde{S}, S) > 0$ and $\inf_{w\in S}|w| > 0$. Then (see the proof of Lemma 6.1) there exists $C < \infty$ such that

$$\sup_{w\in S}|\Gamma(z,w)| \le C|z|^a \quad \text{for } z \in \widetilde{S}.$$

Thus we get

$$|\mathcal{C}_\Gamma u(z)| \le \frac{1}{2\pi}C\sup_{w\in S}|\Gamma(z,w)| \le \widetilde{C}|z|^a \quad \text{for } z \in \widetilde{S}$$

with a suitable constant $\widetilde{C} < \infty$.

Let now $u \in \underset{\sim}{M}'_{(\omega)}(I^\theta)$ have a defining function satisfying (i) and (ii). By the isomorphism of Theorem 6.1,

$$u[\varphi] = -\int_\gamma G(z)\varphi(z)\,dz \quad \text{for } \varphi \in \underset{\sim}{M}_{(\omega)}(V),\ I^\theta \Subset V$$

([2]) Clearly it is enough to take $\widetilde{S} \subset V \cap \{z \in \widetilde{\mathbb{C}\setminus\{0\}} : |z| < r\}$.

with γ as in Theorem 6.1. Let

$$\widetilde{S} = \{z \in \widetilde{\mathbb{C}\setminus\{0\}} : 0 < |z| \le \widetilde{r},\ |\arg z| \le \widetilde{\theta}\}$$

where $0 < \widetilde{r} < r$, and $\widetilde{\theta} > \theta$ is such that $\widetilde{S} \Subset V$. Then there exists $\kappa < \omega$ such that $\sup_{z\in\widetilde{S}} |z^{\kappa+1}\varphi(z)| < \infty$. Hence by assumption (ii) we get for $\kappa < a < \omega$ and $z \in \widetilde{S}$,

$$|G(z)\varphi(z)| \le C^*_{\widetilde{S}} |z|^{a-\kappa-1} \quad \text{where } C^*_{\widetilde{S}} < \infty. \tag{6.11}$$

Let $\gamma_s = \{se^{i\tau} : |\tau| \le \widetilde{\theta}\}$ for $s \le \widetilde{r}$. Then $\gamma_s \subset \widetilde{S}$ and by (6.11),

$$\Big|\int_{\gamma_s} G(z)\varphi(z)\,dz\Big| \le 2\widetilde{\theta} C^*_{\widetilde{S}} s^{a-\kappa} \underset{s\to 0}{\longrightarrow} 0.$$

In a similar way the integral over the radii $\{\tau e^{\pm i\widetilde{\theta}} : 0 < \tau \le s\}$ tends to zero as $s \to 0$. By the Cauchy theorem and assumption (i) this shows that

$$u[\varphi] = -\int_{\gamma} G(z)\varphi(z)\,dz = -\int_{\lambda} G(z)\varphi(z)\,dz$$

where λ is any curve encircling I^θ_r once in the anticlockwise direction and contained in $V \setminus I^\theta_r$. Thus u extends to a functional in $A'(I^\theta_r)$ with defining function G. ∎

Remark 6.3. Assumption (ii) is essential. An example of a function $G \in \mathfrak{M}_{(\omega)}(V\setminus I^\theta)$ with $G \in \mathcal{O}(V \setminus I^\theta_r)$ and $\widetilde{\mathbb{C}\setminus\{0\}} \supset V \Supset I^\theta$ which does not satisfy (ii) is provided in the case $\omega = 0$, $\theta = 0$, by the function Ψ constructed in Example 6.1. In fact, $\Psi \in \mathcal{O}(V) \subset \mathcal{O}(V \setminus [r,t])$ and hence $[\Psi]_{\operatorname{mod}\mathcal{O}(V)} = 0$. Moreover, $\Psi \in M_{(0)}(V \setminus (0,t])$ and there exists $\psi \in \underset{\sim}{M}_{(0)}((0,t])$ such that $T[\psi] \ne 0$ where $T \in \underset{\sim}{M}'_{(0)}((0,t])$ corresponds to $[\Psi]_{\operatorname{mod}\mathfrak{M}_{(0)}(V)}$ under the isomorphism of Theorem 6.1. If Ψ satisfied condition (ii) then by Proposition 6.3, T would extend to a functional $\widetilde{T} \in A'([r,t])$ corresponding to the same defining function Ψ: $\widetilde{T} \simeq [\Psi]_{\operatorname{mod}\mathcal{O}(V)} = 0$. Moreover, since $\widetilde{T}$ is an extension of T we must have $\widetilde{T}[\psi] = T[\psi] \ne 0$, which is a contradiction.

6.2. Mellin distributions (3)**.** Let $t\in\mathbb{R}_+$ and $I = (0,t]$. In Subsection 2.3 we denoted by $C^\infty_{(0)}(I)$ ($C^\infty(I)$, resp.) the space of "extrinsically" smooth functions on I defined as restrictions to I of functions in $C^\infty_0(\Omega)$ ($C^\infty(\Omega)$, resp.), where I is relatively closed in an open set $\Omega \subset \mathbb{R}$. On the other hand, we defined the class $\widetilde{C}^\infty(I)$ of "intrinsically" smooth functions on I by formula (2.3) and observed that $\widetilde{C}^\infty(I) = C^\infty(I)$ by the Seeley extension theorem. Below, following §5 of [Sz-Z1] we give a review of notions and theorems useful to introduce the space of Mellin distributions on I and we present its most important properties.

Definition 6.4. For $a \in \mathbb{R}$ we introduce the space

$$M_a = M_a(I) = \Big\{\varphi \in \widetilde{C}^\infty(I) : \sup_{x\in I}\Big|x^{a+j+1}\Big(\frac{d}{dx}\Big)^j\varphi(x)\Big| < \infty,\ j \in \mathbb{N}_0\Big\}$$

(3) The presentation below follows [Sz-Z1, §5] where the theory of Mellin distributions in several variables is presented in detail.

equipped with the topology defined by the sequence of seminorms

$$\varrho^*_{a,j}(\varphi) = \sup_{x\in I}\left|x^{a+j+1}\left(\frac{d}{dx}\right)^j\varphi(x)\right|, \quad j\in\mathbb{N}_0.$$

Note that the topological space $M_a(I)$ is defined "intrinsically", and that $\{\varrho^*_{a,j}\}_{j\in\mathbb{N}_0}$ can be replaced by the increasing sequence of (semi)norms

$$q_{a,k}(\varphi) = \sum_{j\le k}\varrho^*_{a,j}(\varphi), \quad k\in\mathbb{N}_0,$$

without changing the topology in M_a. Also the same topology is given by the sequence of seminorms

$$\varrho_{a,j}(\varphi) = \sup_{x\in I}\left|x^{a+1}\left(x\frac{d}{dx}\right)^j\varphi(x)\right|, \quad j\in\mathbb{N}_0\ (^4).$$

THEOREM 6.2 (see [Sz-Z1, Theorem 5.1]). *Let $a\in\mathbb{R}$ and $0<t<\widetilde{t}\in\mathbb{R}_+$. Then for every $0<\varepsilon<\widetilde{t}-t$, $\varepsilon<t$ there exists a continuous linear extension mapping*

$$\mathcal{E}_\varepsilon : M_a((0,t])\to M_a((0,\widetilde{t}])$$

such that for every $\varphi\in M_a((0,t])$, $(\mathcal{E}_\varepsilon\varphi)(x)=0$ if $t+\varepsilon<x<\widetilde{t}$.

From this theorem it is easy to show

PROPOSITION 6.4. *The space $M_a(I)$ is complete.*

Note the topological inclusions

$$D(I)\subset M_a(I)\subset M_b(I) \quad \text{for } a\le b$$

with $D(I)$ defined in Subsection 2.3.

Denote by $M'_a(I)$ the dual space of $M_a(I)$; $u\in M'_a(I)$ if and only if u is a linear functional on $M_a(I)$ and for some $m\in\mathbb{N}_0$ and $C=C_a<\infty$,

$$|u[\varphi]|\le C\sum_{j=0}^{m}\varrho_{a,j}(\varphi) \quad \text{for } \varphi\in M_a(I).$$

We then say that u is of *Mellin order not greater than m*.

PROPOSITION 6.5. *The set $C^\infty_{(0)}(I)$ is not dense in $M_a(I)$ and consequently $M'_a(I)$ is not a subspace of the space $D'(I)$ of distributions for any $a\in\mathbb{R}$.*

Let $\omega\in\mathbb{R}\cup\{\infty\}$. We define the function space $M_{(\omega)}(I)$ as the inductive limit $M_{(\omega)}(I)=\varinjlim_{a<\omega}M_a(I)$.

The following topological inclusion is clear:

$$D(I)\subset M_{(\omega)}(I).$$

Moreover, we have

PROPOSITION 6.6. *The set $C^\infty_{(0)}(I)$ is dense in $M_{(\omega)}(I)$.*

(^{4}) Note that $\sup_{x\in I}|(xd/dx)^j(x^{a+1}\varphi(x))|$, $j\in\mathbb{N}_0$, also constitutes a system of seminorms on $M_a(I)$ equivalent to $\{\varrho_{a,j}\}_{j\in\mathbb{N}_0}$.

It follows from the above properties of $M_{(\omega)}(I)$ that $M'_{(\omega)}(I)$—the dual space of $M_{(\omega)}(I)$—is a subspace of $D'(I)$. Therefore the elements of $M'_{(\omega)}(I)$ are called *Mellin distributions*. The totality of Mellin distributions is denoted by $M'(I)$:

$$M'(I) = \bigcup_{\omega\in\mathbb{R}\cup\{\infty\}} M'_{(\omega)}(I) = \bigcup_{\omega\in\mathbb{R}} M'_{(\omega)}(I).$$

We also set $M'_{(\infty)}(I) = \varprojlim_{\omega\in\mathbb{R}} M'_{(\omega)}(I)$.

REMARK 6.4. The principal feature of Mellin distributions is that they are concentrated on I, i.e. there are no Mellin distributions supported by $\{0\}$ (contrary to the Mellin hyperfunctions carried by $\{0\}$, cf. Example 6.1), which is due to the fact that $C^\infty_{(0)}(I)$ is dense in $M_{(\omega)}(I)$.

For $k \in \mathbb{N}_0$ and $a \in \mathbb{R}$ define

$$M^k_a(I) = \{\varphi \in C^k(I) : \varrho_{a,j}(\varphi) < \infty \text{ for } 0 \le j \le k\}.$$

The dual space of $M^k_a(I)$ is exactly the subspace of M'_a formed by those elements of M'_a whose Mellin order is not greater than k. Let $M^k_{(\omega)}(I) = \varinjlim_{a<\omega} M^k_a(I)$.

It follows easily from the proof of Proposition 6.6 given in [Sz-Z1, Proposition 3, p. 41] that $C^\infty_{(0)}(I)$ is dense in $M^k_{(\omega)}(I)$. We shall use this fact in the proof of

PROPOSITION 6.7. *$\underset{\sim}{M}_{(\omega)}(I)$ is dense in $M^k_{(\omega)}(I)$ for any $k \in \mathbb{N}_0$* (5).

PROOF. By the density of $C^\infty_{(0)}(I)$ in $M^k_{(\omega)}(I)$ it suffices to prove that the functions from $C^\infty_{(0)}(I)$ can be approximated by functions in $\underset{\sim}{M}_a(I)$ in the topology of $M^k_a(I)$ for any $a < \omega$ (since $M^k_{(\omega)}(I) = \varinjlim_{a<\omega} M^k_a(I)$). To prove the latter, note that (6)

$$M^k_a(I) \simeq M^k_{-1}(I), \quad \underset{\sim}{M}_a(I) \simeq \underset{\sim}{M}_{-1}(I) \quad \text{for any } a \in \mathbb{R}$$

and hence it is enough to find for any $\psi \in C^\infty_{(0)}(I)$ a sequence of functions $\sigma_\nu \in \underset{\sim}{M}_{-1}(I)$ such that

$$\sup_{x\in I}\left|\left(x\frac{d}{dx}\right)^j(\sigma_\nu(x) - \psi(x))\right| \underset{\nu\to\infty}{\longrightarrow} 0 \quad \text{for } j = 0, 1, \ldots, k.$$

Take $\psi \in C^\infty_{(0)}(I)$. There exists $\widetilde{\psi} \in C^\infty_0(\mathbb{R}_+)$ such that $\psi = \widetilde{\psi}|_I$. Since $\widetilde{\psi}$ is flat at zero (of any order), $\frac{1}{x}\left(x\frac{d}{dx}\right)^j\widetilde{\psi}$ is continuous in $[0, t]$ for any $j \in \mathbb{N}_0$ and by the Weierstrass theorem there exists a sequence $\{W_\nu\}$ of polynomials such that

$$\sup_{x\in \overline{I}}\left|W_\nu(x) - \frac{1}{x}\left(x\frac{d}{dx}\right)^k\widetilde{\psi}(x)\right| \underset{\nu\to\infty}{\longrightarrow} 0.$$

(5) To be more precise: $\underset{\sim}{M}_{(\omega)}(I)|_I$ is dense in $M^k_{(\omega)}(I)$ for any $k \in \mathbb{N}_0$.

(6) The following operations of multiplication are topological isomorphisms (here $x, c, b \in \mathbb{R}$, $z \in \mathbb{C}$):

$$z^x : \underset{\sim}{M}_c(I^\theta) \to \underset{\sim}{M}_{c-x}(I^\theta), \quad M'_c(I^\theta) \to M'_{c+x}(I^\theta),$$
$$x^b : M_{(c)}(I) \to M_{(c-b)}(I), \quad M'_{(c)}(I) \to M'_{(c+b)}(I),$$
$$x^b : M^k_c(I) \to M^k_{c-b}(I), \quad (M^k_c(I))' \to (M^k_{c+b}(I))'.$$

Hence for the polynomials $\widetilde{W}_\nu(x) = xW_\nu(x)$ we have

$$\sup_{x\in\bar{I}}\left|\frac{1}{x}\left(\widetilde{W}_\nu(x) - \left(x\frac{d}{dx}\right)^k\widetilde{\psi}(x)\right)\right| \underset{\nu\to\infty}{\longrightarrow} 0. \tag{6.12}$$

Now by an argument similar to that in the proof of Proposition 3.1 applied to the operator

$$Jf(x) = \int_0^x \frac{1}{s} f(s)\,ds \quad \text{for } x \in \bar{I} \text{ and } \frac{1}{s}f(s) \in C^0(\bar{I})$$

we shall establish that

$$\sup_{x\in\bar{I}}\left|\left(x\frac{d}{dx}\right)^j (J^k\widetilde{W}_\nu(x) - \widetilde{\psi}(x))\right| \underset{\nu\to\infty}{\longrightarrow} 0, \quad j = 0,1,\ldots,k.$$

Observe that $(d/dx)^j\widetilde{\psi}(x)|_{x=0} = 0$ for $j \in \mathbb{N}_0$ and hence ([7])

$$J^p\left(\left(x\frac{d}{dx}\right)^k\widetilde{\psi}\right)(x) = \left(x\frac{d}{dx}\right)^{k-p}\widetilde{\psi}(x) \quad \text{for } x \in \bar{I}, \quad p = 0,1,\ldots,k.$$

Thus

$$\left|J^p\left(\widetilde{W}_\nu(x) - \left(x\frac{d}{dx}\right)^k\widetilde{\psi}(x)\right)\right| = \left|J^p\widetilde{W}_\nu(x) - \left(x\frac{d}{dx}\right)^{k-p}\widetilde{\psi}(x)\right|, \quad p = 0,1,\ldots,k. \tag{6.13}$$

On the other hand, by the definition of J we get the estimate

$$\sup_{x\in\bar{I}}\left|J^p\left(\widetilde{W}_\nu(x) - \left(x\frac{d}{dx}\right)^k\widetilde{\psi}(x)\right)\right| \le t\sup_{x\in\bar{I}}\left|\frac{1}{x}\left(\widetilde{W}_\nu(x) - \left(x\frac{d}{dx}\right)^k\widetilde{\psi}(x)\right)\right|, \quad p = 1,\ldots,k.$$

Thus by (6.13) and (6.12) we get

$$\sup_{x\in\bar{I}}\left|J^p\widetilde{W}_\nu(x) - \left(x\frac{d}{dx}\right)^{k-p}\widetilde{\psi}(x)\right| \underset{\nu\to\infty}{\longrightarrow} 0 \quad \text{for } p = 0,1,\ldots,k,$$

i.e. (cf. footnote ([7]))

$$\sup_{x\in I}\left|\left(x\frac{d}{dx}\right)^{k-p}(J^k\widetilde{W}_\nu(x) - \widetilde{\psi}(x))\right| \underset{\nu\to\infty}{\longrightarrow} 0 \quad \text{for } p = 0,1,\ldots,k,$$

which ends the proof since $\psi(x) = \widetilde{\psi}(x)$ for $x \in I$, and the polynomials $\sigma_\nu(x) = J^k\widetilde{W}_\nu(x)$ are clearly in $\underset{\sim}{M}_{-1}(I)$. ∎

Below we consider Mellin distributions as Mellin hyperfunctions and describe their defining functions.

Proposition 6.8. *There exist natural topological imbeddings*

$$\underset{\sim}{M}_c(I) \hookrightarrow M_c(I), \quad \underset{\sim}{M}_{(\omega)}(I) \hookrightarrow M_{(\omega)}(I)$$

with $I = (0,t]$, given be the restriction mapping $\varphi(z) \mapsto \varphi(x)$ with $x = \operatorname{Re} z$. By duality they induce a natural imbedding

$$M'_{(\omega)}(I) \hookrightarrow \underset{\sim}{M}'_{(\omega)}(I).$$

([7]) Note that the relation $(xd/dx)^jJ^kf(x) = J^{k-j}f(x)$ holds for $\frac{1}{x}f(x) \in C^0(\bar{I})$, $j = 0,1,\ldots,k$ and will be applied later to the polynomials $\widetilde{W}_\nu$.

Proof. Let $\varphi \in \underset{\sim}{M}_c(I)$. Hence there exist $T > t$, $\theta > 0$ and $V \ggcurly I_T^\theta$, where $I_T^\theta = \{z : 0 < |z| \le T,\ |\arg z| \le \theta\}$, such that $\varphi \in \mathcal{O}(V)$ and

$$\sup_{z\in I_T^\theta} |z^{c+1}\varphi(z)| = C_0 < \infty.$$

Fix $k \in \mathbb{N}$. We shall prove in k steps that

$$\sup_{x\in I} \left| \left(x\frac{d}{dx}\right)^k (x^{c+1}\varphi(x))\right| < \infty. \tag{6.14}$$

Indeed, let

$$\varrho = \tan\frac{\theta}{k+1},\quad \theta_j = \theta - j\frac{\theta}{k+1},\quad T_j = T - j\frac{T-t}{k+1},\quad j = 1,\dots,k,$$
$$z \in I_{T_1}^{\theta_1},\quad r \le \min\left(\frac{T-t}{k+1}, \varrho|z|\right),\quad \gamma = \partial B(z,r)$$

and let $\psi(z) = z^{c+1}\varphi(z)$. Note that for $\zeta = z + re^{i\varphi}$, $0 < \varphi \le 2\pi$ we have $|\zeta| \le T$, $|\arg\zeta| \le \theta$ and thus by the Cauchy theorem

$$\psi'(z) = \frac{1}{2\pi i}\int_\gamma \frac{\psi(\zeta)}{(\zeta-z)^2}\,d\zeta$$

and hence

$$\left|\left(z\frac{d}{dz}\right)\psi\right| \le \frac{T_1}{r}\sup|\psi| = \frac{T_1}{r}C_0 < \infty \quad \text{for } z \in I_{T_1}^{\theta_1}.$$

Repeating this k times we get

$$\left|\left(z\frac{d}{dz}\right)^k\psi\right| \le \frac{T_1\dots T_k}{r^k}C_0 < \infty \quad \text{for } z \in I_{T_k}^{\theta_k},$$

which gives (6.14), i.e. $\varphi \in M_c(I)$ (cf. footnote ([4])).

To prove the continuity of the imbedding $\underset{\sim}{M}_c(I) \hookrightarrow M_c(I)$ take $\varphi_\nu \to 0$ in $\underset{\sim}{M}_c(I)$ as $\nu \to \infty$, and let $\sup_{z\in I_T^\theta} |z^{c+1}\varphi_\nu(z)| =: C_{0,\nu} \to 0$. Fix $k \in \mathbb{N}_0$; then

$$\sup_{x\in I}\left|\left(x\frac{d}{dx}\right)^k (x^{c+1}\varphi_\nu(x))\right| \le \frac{T_1\dots T_k}{r^k}C_{0,\nu}.$$

One proves the remaining imbeddings in the standard way. ∎

Theorem 6.1 and Proposition 6.8 imply

Corollary 6.3 (Imbedding of Mellin distributions in Mellin hyperfunctions). *There exists a natural topological imbedding*

$$M'_{(\omega)}(I) \hookrightarrow \mathfrak{P}_{(\omega)}(I).$$

The image of $M'_{(\omega)}(I)$ under this imbedding will be described in Subsection 7.3.

7. Laplace distributions $L'_{(\omega)}(\overline{\mathbb{R}}_+)$

7.1. Definitions and basic properties of Laplace distributions. The present subsection is mostly a reformulation of the results of Subsection 6.2 and therefore contains only statements without proofs which the reader can easily supply by passing to exponential variables.

Now we proceed to the definition of the space $L'_{(\omega)}(\overline{\mathbb{R}}_+)$ of Laplace distributions of type $\omega \in \mathbb{R} \cup \{\infty\}$ supported by $\overline{\mathbb{R}}_+$. Following Subsections 2.3 and 2.4 we denote by $C^k(\overline{\mathbb{R}}_+)$, $k \in \mathbb{N}_0 \cup \{\infty\}$ the space of restrictions to $\overline{\mathbb{R}}_+$ of functions in $C^k(\mathbb{R})$ (1). This space can also be defined intrinsically (cf. Proposition 2.3):

$$C^k(\overline{\mathbb{R}}_+) = \{\varphi \in C^k(\mathbb{R}_+) : (d/dw)^j \varphi(w) \text{ extends continuously to } \overline{\mathbb{R}}_+ \text{ for every } j \in \mathbb{N}_0 \text{ if } k = \infty, \text{ and for every } j \le k \text{ if } k \in \mathbb{N}_0\} \ (^2).$$

Let $a \in \mathbb{R}$ and $k \in \mathbb{N}_0$, and define the space

$$L^k_a(\overline{\mathbb{R}}_+) = \{\sigma \in C^k(\overline{\mathbb{R}}_+) : q_{a,k}(\sigma) < \infty\}$$

with the norm $q_{a,k} = \sum_{j=0}^k \gamma_{a,j}$ where (3)

$$\gamma_{a,j}(\sigma) = \sup_{w \in \overline{\mathbb{R}}_+} |e^{-aw}(d/dw)^j \sigma(w)|.$$

Clearly we have

$$L^{k+1}_a(\overline{\mathbb{R}}_+) \subset L^k_a(\overline{\mathbb{R}}_+) \quad \text{and} \quad L^k_a(\overline{\mathbb{R}}_+) \subset L^k_{\widetilde{a}}(\overline{\mathbb{R}}_+) \quad \text{for } a < \widetilde{a}.$$

Thus we can define the projective limit of $L^k_a(\overline{\mathbb{R}}_+)$ (4):

$$L_a(\overline{\mathbb{R}}_+) = \varprojlim_{k \in \mathbb{N}_0} L^k_a(\overline{\mathbb{R}}_+)$$

and for any $\omega \in \mathbb{R} \cup \{\infty\}$ we define the inductive limit

$$L^k_{(\omega)}(\overline{\mathbb{R}}_+) = \varinjlim_{a<\omega} L^k_a(\overline{\mathbb{R}}_+).$$

Clearly $L_a(\overline{\mathbb{R}}_+) \subset L_{\widetilde{a}}(\overline{\mathbb{R}}_+)$ for $a < \widetilde{a}$ and we define

$$L_{(\omega)}(\overline{\mathbb{R}}_+) = \varinjlim_{a<\omega} L_a(\overline{\mathbb{R}}_+).$$

REMARK 7.1. For any $\varepsilon > 0$ the function $\sigma \in L_a(\overline{\mathbb{R}}_+)$ admits an extension $\widetilde{\sigma} \in C^\infty(\mathbb{R})$ with $\widetilde{\sigma}(w) = 0$ for $w \le -\varepsilon$ (cf. footnote (1)).

The spaces defined above (Laplace spaces) are isomorphic to the Mellin spaces (introduced in Subsection 6.2) under the exponential change of variables:

$$\mu : \mathbb{R} \to \mathbb{R}_+, \quad \mu(y) = e^{-y} \quad \text{for } y \in \mathbb{R}.$$

For example, $\sigma \in L_a(\overline{\mathbb{R}}_+)$ if and only if there exists $\varphi \in M_a(I)$ $(I = (0,1])$ such that $\sigma(y) = e^{-y}\varphi(e^{-y})$.

Mellin spaces can be regarded as defined on a "compactification" of the underlying set for the Laplace spaces. In particular, they are useful for deriving Propositions 7.1–7.5 below (cf. Propositions 6.4–6.8).

(1) For a given $\varphi \in C^k(\overline{\mathbb{R}}_+)$ and $\varepsilon > 0$ there exists a $C^k(\mathbb{R})$-function $\widetilde{\varphi}$ with $\widetilde{\varphi}(w) = \varphi(w)$ for $w \ge 0$, and $\widetilde{\varphi}(w) = 0$ for $w \le -\varepsilon$.

(2) Note that the definition of $C^\infty(\overline{\mathbb{R}}_+)$ can be given also in the following form:

$$C^\infty(\overline{\mathbb{R}}_+) = \{\varphi \in C^\infty(\mathbb{R}_+) : \sup_{x \in \mathbb{R}_+} |(d/dx)^j \varphi(x)| < \infty \text{ for any } j \in \mathbb{N}_0\}.$$

(3) By $(d/dw)^j \sigma(w)$ for $w = 0$ we understand $\lim_{w \to 0_+} (d/dw)^j \sigma(w)$.

(4) Clearly $L_a(\overline{\mathbb{R}}_+) = \{\sigma \in C^\infty(\overline{\mathbb{R}}_+) : \gamma_{a,j}(\sigma) < \infty,\ j \in \mathbb{N}_0\}$ in analogy to Definition 6.4.

PROPOSITION 7.1. *The spaces* $L^k_a(\overline{\mathbb{R}}_+)$ $(k \in \mathbb{N}_0,\ a \in \mathbb{R})$ *and* $L_a(\overline{\mathbb{R}}_+)$ *are complete.*

PROPOSITION 7.2. *The set* $C^\infty_{(0)}(\overline{\mathbb{R}}_+)$ *of restrictions to* $\overline{\mathbb{R}}_+$ *of functions in* $C^\infty_0(\mathbb{R})$ *is dense in* $L_{(\omega)}(\overline{\mathbb{R}}_+)$ *and in* $L^k_{(\omega)}(\overline{\mathbb{R}}_+)$ *for any* $k \in \mathbb{N}_0$.

Hence by Proposition 1.1 the dual space $L'_{(\omega)}(\overline{\mathbb{R}}_+)$ is a subspace of $D'(\overline{\mathbb{R}}_+)$ (= the dual space of $C^\infty_0(\overline{\mathbb{R}}_+)$). We call it the *space of Laplace distributions on* $\overline{\mathbb{R}}_+$.

PROPOSITION 7.3. $C^\infty_{(0)}(\overline{\mathbb{R}}_+)$ *is not dense in* $L_a(\overline{\mathbb{R}}_+)$.

We also set

$$L'_{(-\infty)}(\overline{\mathbb{R}}_+) = \varinjlim_{b\in\mathbb{R}} L'_{(b)}(\overline{\mathbb{R}}_+), \quad L'_{(\infty)}(\overline{\mathbb{R}}_+) = \varprojlim_{b\in\mathbb{R}} L'_{(b)}(\overline{\mathbb{R}}_+).$$

PROPOSITION 7.4. $\underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}_+)$ *is dense in* $L^k_{(\omega)}(\overline{\mathbb{R}}_+)$ *for any* $k \in \mathbb{N}_0$ (5).

PROPOSITION 7.5. *There exist natural topological imbeddings*

$$\underset{\sim}{L}_c(\overline{\mathbb{R}}_+) \hookrightarrow L_c(\overline{\mathbb{R}}_+), \quad \underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}_+) \hookrightarrow L_{(\omega)}(\overline{\mathbb{R}}_+)$$

given by restriction to the real axis. By duality they induce a natural imbedding

$$L'_{(\omega)}(\overline{\mathbb{R}}_+) \hookrightarrow \underset{\sim}{L}'_{(\omega)}(\overline{\mathbb{R}}_+).$$

For the proof analogous to that of Proposition 6.8 we take $\sigma \in \underset{\sim}{L}_a(\overline{\mathbb{R}}_+)$, $k \in \mathbb{N}$, and show that there exists an infinite rectangle $P \Supset \overline{\mathbb{R}}_+$ such that

$$\sup_{z\in P} |(d/dz)^j(e^{-az}\sigma(z))| = C_j < \infty, \quad j = 0, 1, \ldots, k,$$

and hence also $\sup_{z\in P} \left|e^{-az}(d/dz)^j\sigma(z)\right| < \infty$, $j = 0, 1, \ldots, k$.

The Laplace distributions are characterized by the following structure theorem (cf. [Sz-Z1, p. 164]).

THEOREM 7.1. *Let* $\omega \in \mathbb{R} \cup \{\infty\}$. *A distribution* $T \in D'(\overline{\mathbb{R}}_+)$ *is in* $L'_{(\omega)}(\overline{\mathbb{R}}_+)$ *if and only if for every* $\kappa < \omega$ *there exist* $m_\kappa \in \mathbb{N}_0$ *and measurable functions* $T_{j,\kappa}$ *on* $\mathbb{R}$ *with support in* $\overline{\mathbb{R}}_+$ *for* $0 \le j \le m_\kappa$ *such that*

$$|T_{j,\kappa}(\alpha)| \le C_\kappa e^{-\kappa\alpha} \quad \text{for } 0 \le \alpha < \infty \tag{7.1}$$

almost everywhere with some constant $C_\kappa > 0$ *and*

$$T = \sum_{j=0}^{m_\kappa} \left(\frac{d}{d\alpha}\right)^j T_{j,\kappa} \quad \text{in } L'_{(\kappa)}(\overline{\mathbb{R}}_+). \tag{7.2}$$

REMARK 7.2. Observe that the sum in (7.2) cannot in general be reduced to a single summand (as can be done e.g. for tempered distributions in $S'(\mathbb{R})$). An example is provided by $T = \delta_{(0)} \in L'_{(\infty)}(\overline{\mathbb{R}}_+)$. Indeed, if $\delta_{(0)} = (d/d\alpha)^m T_m$ then we must have $T_m(\alpha) = \alpha^{m-1}/(m-1)!$ for $\alpha > 0$ and T_m does not satisfy (7.1) for $\kappa > 0$. On the other hand, for every $\omega \in \mathbb{R}$ we have

$$\delta_{(0)} = \omega e^{-\omega\alpha} Y + \frac{d}{d\alpha} e^{-\omega\alpha} Y$$

(5) To be more precise: $\underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}_+)_{|\mathbb{R}}$ is dense in $L^k_{(\omega)}(\mathbb{R})$ for any $k \in \mathbb{N}_0$.

where Y is the Heaviside function. This means that $\delta_{(0)} = P(D)T$, where $P(D) = \omega + d/d\alpha$ and $T = e^{-\omega\alpha}Y$.

Definition 7.1. We say that a Laplace distribution $T \in L'_{(\omega)}(\overline{\mathbb{R}}_+)$ is of *Laplace order not greater than* $m \in \mathbb{N}_0$ iff for every $\kappa < \omega$ there exists a constant $C_\kappa < \infty$ such that

$$|T[\sigma]| \le C_\kappa \sum_{j=0}^{m} \gamma_{a,j}(\sigma) \quad \text{for } \sigma \in L_a(\overline{\mathbb{R}}_+).$$

If such an m does not exist, T is said to be of *infinite order*.

It follows from Theorem 7.1 that for every $\kappa < \omega$ the Laplace distribution $T \in L'_{(\omega)}(\overline{\mathbb{R}}_+)$ is of finite order as a Laplace distribution in the space $L'_{(\kappa)}(\overline{\mathbb{R}}_+)$.

Remark 7.3. The principal feature of Laplace distributions in $L'_{(\omega)}(\overline{\mathbb{R}}_+)$ is that (in contrast to Laplace analytic functionals) there are no Laplace distributions carried by $\{\infty\}$. This is due to the fact that $C^\infty_{(0)}(\overline{\mathbb{R}}_+)$ is dense in $L_{(\omega)}(\overline{\mathbb{R}}_+)$ for any $\omega \in \mathbb{R}$ (cf. Remark 6.4).

Theorem 5.1 and Proposition 7.5 imply

Corollary 7.1 (Imbedding of Laplace distributions in Laplace hyperfunctions). *There exists a natural topological imbedding*

$$L'_{(\omega)}(\overline{\mathbb{R}}_+) \hookrightarrow \mathfrak{Q}_{(\omega)}(\overline{\mathbb{R}}_+).$$

7.2. Imbedding of Laplace distributions in Laplace hyperfunctions. The aim of this subsection is to describe the image of $L'_{(\omega)}(\overline{\mathbb{R}}_+)$ under the imbedding of Corollary 7.1.

We shall need the following slight generalization of the Arzelà theorem to the case of functions defined on an unbounded interval.

Lemma 7.1. *Let $\{F_\nu\}_{\nu\in\mathbb{N}}$ be a sequence of functions equicontinuous on compact subsets of $\overline{\mathbb{R}}_+$ and suppose that F_ν tends uniformly to zero at infinity, i.e. for every $\varepsilon > 0$ there exists $r > 0$ such that*

$$|F_\nu(x)| \le \varepsilon \quad \textit{for } x \ge r \textit{ and all } \nu \in \mathbb{N}. \tag{7.3}$$

Further suppose that the F_ν are uniformly bounded in the sup norm. Then $\{F_\nu\}$ is precompact in $C^0(\overline{\mathbb{R}}_+)$ in the sup norm.

Proof. Set $G_\nu(t) = F_\nu(-\ln t)$ for $0 < t \le 1$, and $G_\nu(0) = 0$. Then $G_\nu \in C^0([0,1])$ and by (7.3) for every $\varepsilon > 0$, $|G_\nu(t_1) - G_\nu(t_2)| \le 2\varepsilon$ for $t_1, t_2 \in [0, e^{-r}]$. Since for any $R > 0$, $\{F_\nu\}$ is equicontinuous on $[0,R]$ and the function $[e^{-R}, 1] \ni t \mapsto -\ln t$ is uniformly continuous, the sequence $\{G_\nu\}$ is equicontinuous on $[e^{-R}, 1]$. Thus $\{G_\nu\}$ is equicontinuous on $[0,1]$ and since it is uniformly bounded in the sup norm on $[0,1]$, the classical Arzelà theorem applies. ∎

Lemma 7.2. *Let $p \in \mathbb{N}_0$, $\kappa, \widetilde{\kappa} \in \mathbb{R}$, $\kappa < \widetilde{\kappa}$, $\mathfrak{P} = L^p_{\widetilde{\kappa}}$, $\mathfrak{Q} = L^{p+1}_\kappa$ and $\mathcal{K}_{\mathfrak{Q}} = \{\sigma \in \mathfrak{Q} : q_{\kappa,p+1}(\sigma) \le 1\}$. Then $\mathcal{K}_{\mathfrak{Q}}$ is precompact in $\mathfrak{P}$.*

Proof. Suppose first $p = 0$. Take $\sigma_\nu \in \mathcal{K}_{\mathfrak{Q}}$, $\nu = 1, 2, \ldots$ Then $|e^{-\kappa x}\sigma_\nu(x)| \le 1$, $|e^{-\kappa x}\sigma'_\nu(x)| \le 1$ for $x \in \overline{\mathbb{R}}_+$, $\nu \in \mathbb{N}$ and hence also

$$|e^{-\widetilde{\kappa}x}\sigma_\nu(x)| \le e^{-(\widetilde{\kappa}-\kappa)x} \le 1 \quad \text{for } x \in \overline{\mathbb{R}}_+,\ \nu \in \mathbb{N} \tag{7.4}$$

and similarly $|e^{-\widetilde{\kappa}x}\sigma'_\nu(x)| \le 1$ for $x \in \overline{\mathbb{R}}_+$, $\nu \in \mathbb{N}$ since $\kappa < \widetilde{\kappa}$. Thus $|(e^{-\widetilde{\kappa}x}\sigma_\nu(x))'| \le |\kappa| + 1$, which gives the equicontinuity of $\{e^{-\widetilde{\kappa}x}\sigma_\nu\}$ on $\overline{\mathbb{R}}_+$. Moreover, by (7.4) for every $\varepsilon > 0$ there exists $r > 0$ such that $|e^{-\widetilde{\kappa}x}\sigma_\nu(x)| \le \varepsilon$ for $x \ge r$. Now the assertion follows from Lemma 7.1.

The general case of $p \in \mathbb{N}$ is obtained by applying the above reasoning to the consecutive derivatives of σ up to order p. ∎

Lemma 7.2 and Proposition 1.1 yield

Corollary 7.2. *Let $p \in \mathbb{N}_0$, $\kappa, \widetilde{\kappa} \in \mathbb{R}$, $\kappa < \widetilde{\kappa}$, and $T_\nu \in (L^p_{\widetilde{\kappa}})'$ for $\nu \in \mathbb{N}$. Assume that $T_\nu \to 0$ in $(L^p_{\widetilde{\kappa}})'$ as $\nu \to \infty$. Then there exists a sequence $\varepsilon_\nu \to 0_+$ such that*

$$|T_\nu[\sigma]| \le \varepsilon_\nu q_{\kappa,p+1}(\sigma) \quad \textit{for } \sigma \in L^{p+1}_\kappa.$$

Remark 7.4. Lemma 7.1 remains true in $\overline{\mathbb{R}}^n_+$ for $n > 1$ if we assume that (7.3) holds for $\|x\| \ge r$. Also Lemma 7.2 and Corollary 7.2 can be formulated in arbitrary dimension replacing $L^p_{\widetilde{\kappa}}$ and L^{p+1}_κ by $L^p_{\widetilde{\kappa}}(\overline{\mathbb{R}}^n_+)$ and $L^{p+1}_\kappa(\overline{\mathbb{R}}^n_+)$ with $\kappa, \widetilde{\kappa} \in \mathbb{R}^n$, $\kappa < \widetilde{\kappa}$.

In the sequel we use the notation of Subsection 7.1 and Section 5, e.g. a tubular set $W \Supset \overline{\mathbb{R}}_+$ and the modified Cauchy kernel Λ.

Lemma 7.3. *Fix a tubular set W with $(\overline{\mathbb{R}}_+)_\varepsilon \subset W \subset (\overline{\mathbb{R}}_+)_M$, $0 < \varepsilon < M < \infty$. Let $\zeta = \alpha + i\beta \in W \setminus \overline{\mathbb{R}}_+$. Then for any $\kappa \in \mathbb{R}$ and $j \in \mathbb{N}_0$ there exists $\widetilde{C}_j = \widetilde{C}_j(M) < \infty$ such that*

$$(\operatorname{dist}(\zeta, \overline{\mathbb{R}}_+))^{j+1} e^{\alpha\kappa} \gamma_{\kappa,j}(\Lambda(\alpha + i\beta, \cdot\,)) \le \widetilde{C}_j \tag{7.5}$$

and hence for any $k \in \mathbb{N}_0$ we get, with some constant $C_k < \infty$,

$$(\operatorname{dist}(\zeta, \overline{\mathbb{R}}_+))^{k+1} e^{\alpha\kappa} q_{\kappa,k}(\Lambda(\alpha + i\beta, \cdot\,)) \le C_k \quad \textit{for any } \zeta \in W \setminus \overline{\mathbb{R}}_+. \tag{7.6}$$

Thus $\Lambda(\zeta, \cdot\,) \in L_\kappa(\overline{\mathbb{R}}_+)$ for any $\kappa \in \mathbb{R}$ and $\zeta \in W \setminus \overline{\mathbb{R}}_+$.

Proof. By definition of $\gamma_{\kappa,j}$ and Λ we get, for any $\zeta \in W \setminus \overline{\mathbb{R}}_+$,

$$\operatorname{dist}(\zeta, \overline{\mathbb{R}}_+) e^{\alpha\kappa} \gamma_{\kappa,0}\big(\Lambda(\alpha + i\beta, \cdot\,)\big) \le e^{M^2} \sup_{w \in \overline{\mathbb{R}}_+} e^{(\alpha-w)(\kappa-(\alpha-w))}.$$

Similarly,

$$(\operatorname{dist}(\zeta, \overline{\mathbb{R}}_+))^2 e^{\alpha\kappa} \gamma_{\kappa,1}(\Lambda(\alpha + i\beta, \cdot\,)) \le e^{M^2} \sup_{w \in \overline{\mathbb{R}}_+} e^{(\alpha-w)(\kappa-(\alpha-w))}(2|\alpha - w|^2 + 4M|\alpha - w| + 2M^2 + 1).$$

In general, for any $j \in \mathbb{N}_0$ there exist constants $c_{j,l} = c_{j,l}(M)$ such that for any $\zeta = \alpha + i\beta \in W \setminus \overline{\mathbb{R}}_+$,

$$(\operatorname{dist}(\alpha + i\beta, \overline{\mathbb{R}}_+))^{j+1} e^{\alpha\kappa} \gamma_{\kappa,j}(\Lambda(\alpha + i\beta, \cdot\,)) \le e^{M^2} \sup_{w \in \overline{\mathbb{R}}_+} e^{(\alpha-w)(\kappa-(\alpha-w))} \sum_{l=0}^{2j} c_{j,l}|\alpha - w|^l.$$

Put $\xi = \alpha - w$ and observe that $\lim_{\xi\to\pm\infty} e^{\xi(\kappa-\xi)}|\xi|^l = 0$ for any $l \in \mathbb{N}_0$. Hence we deduce (7.5) and therefore (7.6). ■

PROPOSITION 7.6. *Let $T \in L'_{(\omega)}(\overline{\mathbb{R}}_+)$, $\omega \in \mathbb{R} \cup \{\infty\}$. Then the function*

$$\Psi(z) = -\frac{1}{2\pi i} T[\Lambda(z,\,\cdot\,)] \quad \textit{for } z \in \mathbb{C} \setminus \overline{\mathbb{R}}_+ \tag{7.7}$$

satisfies the conditions:

(i) $\Psi \in \mathcal{O}(\mathbb{C} \setminus \overline{\mathbb{R}}_+)$,

(ii) *for any $\kappa < \omega$ there exist $k = k(\kappa)$ and $C^* = C^*(\kappa) < \infty$ such that*

$$|\Psi(\alpha + i\beta)| \le C^* \frac{e^{-\kappa\alpha}}{\big(\operatorname{dist}(\alpha + i\beta, \overline{\mathbb{R}}_+)\big)^k} \le C^* \frac{e^{-\kappa\alpha}}{|\beta|^k} \quad \textit{for } \alpha + i\beta \in W \setminus \overline{\mathbb{R}}_+,$$

(iii) *for any $\kappa < \omega$ and $0 < \varepsilon < M$ there exists $C < \infty$ such that*

$$|\Psi(\alpha + i\beta)| \le C e^{-\alpha\kappa} \quad \textit{for } \alpha + i\beta \in \mathbb{R}_M \setminus (\overline{\mathbb{R}}_+)_\varepsilon.$$

PROOF. By Proposition 7.5 we consider $T \in L'_{(\omega)}(\overline{\mathbb{R}}_+)$ as a functional from $\underset{\sim}{L}'_{(\omega)}(\overline{\mathbb{R}}_+)$, hence by Theorem 5.1, $T = \Psi + \mathfrak{L}_{(\omega)}(W)$ and (i) holds. To show (ii) note that since $T \in L'_{(\omega)}(\overline{\mathbb{R}}_+) = \varprojlim_{\kappa<\omega} L'_\kappa(\overline{\mathbb{R}}_+)$, for every $\kappa < \omega$ there are $l = l(\kappa) \in \mathbb{N}_0$ and $C < \infty$ such that

$$|T[\sigma]| \le C q_{\kappa,l}(\sigma) \quad \text{for } \sigma \in L^l_\kappa(\overline{\mathbb{R}}_+). \tag{7.8}$$

By Lemma 7.3, $\Lambda(z,\,\cdot\,) \in L_\kappa(\overline{\mathbb{R}}_+)$ for $z \in W \setminus \overline{\mathbb{R}}_+$ and since $L_\kappa(\overline{\mathbb{R}}_+) = \varprojlim_{k\in\mathbb{N}_0} L^k_\kappa(\overline{\mathbb{R}}_+)$ we get by (7.7) and (7.8),

$$|\Psi(z)| \le \frac{C}{2\pi} q_{\kappa,l}(\Lambda(z,\,\cdot\,)),$$

and hence the estimates in (ii) and (iii) follow from Lemma 7.3. ■

For technical reasons it is often convenient to introduce the space $L'_{(\omega)}(\Omega)$, where $\omega \in \mathbb{R} \cup \{\infty\}$ and Ω is an open set with $\Omega \subset \xi + \mathbb{R}_+$ for some $\xi \in \mathbb{R}$.

Let $a \in \mathbb{R}$. We denote by $L_a(\Omega)$ the space

$$L_a(\Omega) = \{\sigma \in C^\infty(\Omega) : \operatorname{dist}(\partial\Omega, \operatorname{supp}\sigma) > 0 \ (^6),\ \gamma_{a,j}(\sigma) < \infty \text{ for } j \in \mathbb{N}_0\},$$

where

$$\gamma_{a,j}(\sigma) = \sup_{\alpha\in\Omega} \big|e^{-\alpha a}(d/d\alpha)^j \sigma(\alpha)\big| < \infty,$$

with the topology defined by $\{\gamma_{a,j}\}_{j\in\mathbb{N}_0}$.

Define

$$L_{(\omega)}(\Omega) = \varinjlim_{a<\omega} L_a(\Omega)$$

and let $L'_{(\omega)}(\Omega)$ be the dual space of $L_{(\omega)}(\Omega)$.

REMARK 7.5. Note that for any Ω described above and any $b < 0$ the function $\Omega \ni \alpha \mapsto e^{\alpha b}$ is integrable.

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(6) Hence $\operatorname{dist}(\xi, \operatorname{supp}\sigma) > 0$.

Lemma 7.4. *Let* $C > 0$, $\kappa \in \mathbb{R}$, $r \in \mathbb{R}_+$, $\kappa \in \mathbb{N}_0$, Ω *open*, $\Omega \subset \xi + \overline{\mathbb{R}}_+$, $\xi \in \mathbb{R}$. *Assume that* $F \in \mathcal{O}(\Omega + i(0,r))$ *is such that*

$$|F(\alpha+i\beta)| \le C\frac{e^{-\alpha\kappa}}{\beta^k} \quad \text{for } \alpha+i\beta \in \Omega + i(0,r). \tag{7.9}$$

Take $a < \kappa$ *and let* $\widetilde{M} = \int_\Omega e^{(a-\kappa)\alpha}\,d\alpha$. *Then there exists* $T^k \in L'_a(\Omega)$ *such that*

$$\lim_{\beta\to 0_+} \int_\Omega F(\alpha+i\beta)\sigma(\alpha)\,d\alpha = T^k[\sigma] \quad \text{for } \sigma \in L_a(\Omega) \tag{7.10}$$

and

$$|T^k[\sigma]| \le \begin{cases} C\widetilde{M}(\gamma_{a,0}(\sigma) + r\gamma_{a,1}(\sigma)) & \text{if } k = 0,\\ C\widetilde{M}\sum_{j=0}^{k+2} b_{k,j}\gamma_{a,j}(\sigma) & \text{if } k \ge 1, \end{cases} \tag{7.11}$$

where $b_{k,j}$ *are constants independent of* F.

Before passing to the proof of Lemma 7.4 note the following corollary.

Corollary 7.3. *Let* $C_\nu \to 0_+$ *as* $\nu \to \infty$ *and let* $F_\nu \in \mathcal{O}(\Omega + i(0,r))$ *satisfy* (7.9) *with* F_ν, C_ν *in place of* F, C, *respectively. Then there exist* $T^k_\nu \in L'_a(\Omega)$ *which satisfy* (7.11) *and* (7.10) *with* F_ν *in place of* F *and* C_ν *in place of* C. *Hence* $T^k_\nu \to 0$ *in* $L'_a(\Omega)$ *as* $\nu \to \infty$.

Proof of Lemma 7.4. Let $0 < \delta < r$, $z = \alpha + i\beta \in \Omega + i(0, r-\delta)$ and define

$$\mathcal{J}F(z) = \int_z^{z+i\delta} F(\zeta)\,d\zeta = i\int_\beta^{\beta+\delta} F(\alpha+is)\,ds. \tag{7.12}$$

Then

$$\frac{d}{dz}\mathcal{J}F(z) = F(z+i\delta) - F(z). \tag{7.13}$$

We begin the proof with the case $k = 0$. Then

$$\begin{aligned} &|F(\alpha+i\beta)| \le Ce^{-\alpha\kappa} && \text{for } \alpha+i\beta \in \Omega + i(0,r),\\ &|\mathcal{J}F(\alpha+i\beta)| \le C\delta e^{-\alpha\kappa} && \text{for } \alpha+i\beta \in \Omega + i(0,r-\delta),\\ &|\mathcal{J}F(\alpha+i\beta') - \mathcal{J}F(\alpha+i\beta'')| \le 2|\beta'-\beta''|Ce^{-\alpha\kappa} && \\ & && \text{for } \beta',\beta'' \in (0,r-\delta), \alpha \in \Omega. \end{aligned} \tag{7.14}$$

Hence there exists a function F^1 continuous on Ω such that

$$\mathcal{J}F(\,\cdot\,+i\beta) \xrightarrow[\beta\to 0_+]{} F^1 \text{ locally uniformly on } \Omega, \quad |F^1(\alpha)| \le C\delta e^{-\alpha\kappa} \text{ for } \alpha \in \Omega. \tag{7.15}$$

Let $\sigma \in L_a(\Omega)$. By (7.14) we get

$$\left|\mathcal{J}F(\alpha+i\beta)\frac{d\sigma}{d\alpha}\right| \le C\delta\gamma_{a,1}(\sigma)e^{\alpha(a-\kappa)}$$

and since $a < \kappa$, (7.13) and (7.15) yield

$$\lim_{\beta\to 0_+}\int_\Omega F(\alpha+i\beta)\sigma(\alpha)\,d\alpha = \int_\Omega F(\alpha+i\delta)\sigma(\alpha)\,d\alpha + \int_\Omega F^1(\alpha)\frac{d\sigma}{d\alpha}\,d\alpha = T^0[\sigma]$$

where $T^0 = F(\cdot + i\delta) - dF^1/d\alpha$. By (7.14) and (7.15) we have $|T^0[\sigma]| \le C\widetilde{M}(\gamma_{a,0}(\sigma) + \delta\gamma_{a,1}(\sigma))$, which proves (7.11) for $k = 0$ since $\delta < r$.

Now let $k = 1$. Thus we assume that

$$|F(\alpha + i\beta)| \le C\frac{e^{-\alpha\kappa}}{\beta} \quad \text{for } \alpha + i\beta \in \Omega + i(0, r). \tag{7.16}$$

Then for $0 < 2\delta < r$ we get

$$\mathcal{J}^2 F(z) = \int\limits_z^{z+i\delta} (\mathcal{J}F)(\zeta)\, d\zeta, \quad |\mathcal{J}^2 F(z)| \le \delta|\ln\delta| C e^{-\alpha\kappa} \quad \text{for } 0 < \beta < r - 2\delta \tag{7.17}$$

and

$$F(z) = \frac{d^2}{dz^2}\mathcal{J}^2 F(z) - F(z + 2i\delta) + 2F(z + i\delta). \tag{7.18}$$

By the case $k = 0$ just examined we deduce from (7.17) that there exists $T_1 \in L'_a(\Omega)$ such that

$$\lim_{\beta\to 0_+} \int\limits_\Omega \mathcal{J}^2 F(\alpha + i\beta)\sigma(\alpha)\, d\alpha = T_1[\sigma],$$

$$|T_1[\sigma]| \le C\widetilde{M}\delta|\ln\delta|(\gamma_{a,0}(\sigma) + \delta\gamma_{a,1}(\sigma)) \quad \text{for } \sigma \in L_a(\Omega)$$

and similarly it follows from (7.16) that there exists $T_2 \in L'_a(\Omega)$ such that

$$\lim_{\beta\to 0_+} \int\limits_\Omega (-F(z + 2i\delta) + 2F(z + i\delta))\sigma(\alpha)\, d\alpha = T_2[\sigma] \quad \text{for } \sigma \in L_a(\Omega),$$

$$|T_2[\sigma]| \le C\widetilde{M}\left(\frac{5}{2\delta}\gamma_{a,0}(\sigma) + \frac{5}{2}\gamma_{a,1}(\sigma)\right).$$

Thus by (7.18) we get

$$\lim_{\beta\to 0_+} \int\limits_\Omega F(z)\sigma(\alpha)\, d\alpha = \lim_{\beta\to 0_+} \int\limits_\Omega \mathcal{J}^2 F(\alpha + i\beta)\frac{d^2\sigma}{d\alpha^2}\, d\alpha + T_2[\sigma]$$

$$= T_1\left[\frac{d^2\sigma}{d\alpha^2}\right] + T_2[\sigma] = T^1[\sigma],$$

where $T^1 = d^2T_1/d\alpha^2 + T_2$, and

$$|T^1[\sigma]| \le C\widetilde{M}\left(\delta^2\gamma_{a,0}\left(\frac{d^2\sigma}{d\alpha^2}\right) + \delta^3\gamma_{a,1}\left(\frac{d^2\sigma}{d\alpha^2}\right) + \frac{5}{2\delta}\gamma_{a,0}(\sigma) + \frac{5}{2}\gamma_{a,1}(\sigma)\right)$$

$$\le C\widetilde{M}\sum_{j=0}^{3} b_{1,j}\gamma_{a,j}(\sigma)$$

with suitable constants $b_{1,j}$ $(j = 1, 2, 3)$. This ends the proof in the case $k = 1$.

To prove the lemma for any $k \in \mathbb{N}$, take $p \in \mathbb{N}$, $p \ge 1$, and assume that the lemma is true for any $l \le p$. Let $F \in \mathcal{O}(\Omega + i(0, r))$ with

$$|F(\alpha + i\beta)| \le C\frac{e^{-\alpha\kappa}}{\beta^{p+1}} \quad \text{for } \alpha + i\beta \in \Omega + i(0, r).$$

Then by (7.12),

$$\mathcal{J}F(\alpha+i\beta)\in\mathcal{O}(\Omega+i(0,r-\delta)),\quad |\mathcal{J}F(\alpha+i\beta)|<C\frac{1}{p}\frac{e^{-\alpha\kappa}}{\beta^p}\quad\text{for } \alpha\in\Omega,\ \beta<r-\delta$$

and hence by the inductive assumption

$$\lim_{\beta\to 0_+}\int_\Omega \mathcal{J}F(\alpha+i\beta)\sigma(\alpha)\,d\alpha=T^p[\sigma],\quad |T^p[\sigma]|\le C\widetilde{M}\sum_{j=0}^{p+2}b_{p,j}\gamma_{a,j}(\sigma).$$

Now formula (7.13) yields

$$\lim_{\beta\to 0_+}\int_\Omega F(\alpha+i\beta)\sigma(\alpha)\,d\alpha=T^p\left[\frac{d\sigma}{d\alpha}\right]+\int_\Omega F(\alpha+i\delta)\sigma(\alpha)\,d\alpha=T^{p+1}[\sigma]$$

where $T^{p+1}=F(\,\cdot\,+i\delta)-\frac{d}{d\alpha}T^p\in L'_a(\Omega)$ and

$$\begin{aligned}|T^{p+1}[\sigma]|&\le C\widetilde{M}\biggl(\frac{1}{\delta^{p+1}}\gamma_{a,0}(\sigma)+\frac{1}{\delta^p}\gamma_{a,1}(\sigma)+\sum_{j=0}^{p+2}b_{p,j}\gamma_{a,j}\Bigl(\frac{d\sigma}{d\alpha}\Bigr)\biggr)\\&\le CM\biggl(\sum_{j=0}^{p+3}b_{p+1,j}\gamma_{a,j}(\sigma)\biggr)\end{aligned}$$

with suitable constants $b_{p+1,j}$. ∎

Definition 7.2. Let $r\in\mathbb{R}_+$, $\xi\in\mathbb{R}$, $\Omega\subset\xi+\overline{\mathbb{R}}_+$ and let $F\in\mathcal{O}(\Omega+i(0,r))$. Assume that for some $\omega\in\mathbb{R}\cup\{\infty\}$ and for any $\beta>0$ close to zero the functional

$$L_{(\omega)}(\Omega)\ni\sigma\mapsto u_\beta[\sigma]:=\int_\Omega F(\alpha+i\beta)\sigma(\alpha)\,d\alpha$$

belongs to $L'_{(\omega)}(\Omega)$ and $\lim_{\beta\to 0_+}u_\beta$ exists. Then we say that $b_+^L F=\lim_{\beta\to 0_+}u_\beta$ (belonging to $L'_{(\omega)}(\Omega)$) is a *Laplace distributional boundary value of F from above* on Ω. Similarly for $F\in\mathcal{O}(\Omega-i(0,r))$ we define the *Laplace distributional boundary value of F from below* on Ω, $b_-^L F=\lim_{\beta\to 0_-}u_\beta$ if the limit exists.

If $F\in\mathcal{O}(\Omega+i((-r,0)\cup(0,r)))$ has Laplace distributional boundary values $b_+^L F$ (from above) and $b_-^L F$ (from below) we say that $b^L F=b_+^L F-b_-^L F$ is the *Laplace distributional boundary value of F* on Ω (LDBV in short). Clearly it belongs to $L'_{(\omega)}(\Omega)$ and

$$(b^L F)[\sigma]=\lim_{\beta\to 0_+}\int_\Omega(F(\alpha+i\beta)-F(\alpha-i\beta))\sigma(\alpha)\,d\alpha\quad\text{for } \sigma\in L_{(\omega)}(\Omega).$$

By Lemma 7.4 we get

Corollary 7.4. *Let $W\Supset\overline{\mathbb{R}}_+$, Ω open, $\overline{\mathbb{R}}_+\subset\Omega\Subset W\cap\mathbb{R}$. Let $\Psi\in\mathcal{O}(W\setminus\overline{\mathbb{R}}_+)$, $\omega\in\mathbb{R}\cup\{\infty\}$ and assume that for any $\kappa<\omega$ and $k=k(\kappa)\in\mathbb{N}_0$,*

$$|\Psi(\alpha+i\beta)|\le C_\kappa\frac{e^{-\alpha\kappa}}{|\beta|^k}\quad\textit{for } \alpha+i\beta\in W\setminus\overline{\mathbb{R}}_+.$$

Then Ψ has LDBV on Ω,

$$(b_+^L\Psi)[\widetilde{\sigma}]=\lim_{\beta\to 0_+}\int_\Omega\Psi(\alpha+i\beta)\widetilde{\sigma}(\alpha)\,d\alpha,\quad (b_-^L\Psi)[\widetilde{\sigma}]=\lim_{\beta\to 0_+}\int_\Omega\Psi(\alpha-i\beta)\widetilde{\sigma}(\alpha)\,d\alpha$$

for $\widetilde{\sigma} \in L_{(\omega)}(\Omega)$, $b_+^L\Psi, b_-^L\Psi \in L'_{(\omega)}(\Omega)$, $b^L\Psi = b_+^L\Psi - b_-^L\Psi \in L'_{(\omega)}(\Omega)$ *and* $\operatorname{supp} b^L\Psi \subset \overline{\mathbb{R}}_+$ (*since* Ψ *is holomorphic when* $\alpha < 0$). *For any* $\sigma \in L_{(\omega)}(\overline{\mathbb{R}}_+)$ *denote by* $\widetilde{\sigma}$ *its extension to* $L_{(\omega)}(\Omega)$. *Then the functional*

$$T[\sigma] = (b^L\Psi)[\widetilde{\sigma}] \quad \text{for } \sigma \in L_{(\omega)}(\overline{\mathbb{R}}_+)$$

does not depend on the choice of Ω, *and* $T \in L'_{(\omega)}(\overline{\mathbb{R}}_+)$. *More explicitly*,

$$T[\sigma] = \lim_{\beta\to 0_+} \int_{\Omega} (\Psi(\alpha + i\beta) - \Psi(\alpha - i\beta))\widetilde{\sigma}(\alpha)\, d\alpha \quad \text{for } \sigma \in L_{(\omega)}(\overline{\mathbb{R}}_+)$$

and we write $T = b^L\Psi$.

To describe the image of $L'_{(\omega)}(\overline{\mathbb{R}}_+)$ under the imbedding of Corollary 7.1 we introduce a family of subspaces of $\mathfrak{L}_\kappa(W\setminus\overline{\mathbb{R}}_+)$ indexed by $k \in \mathbb{N}_0$. Denote by $\mathfrak{L}^k_\kappa(W\setminus\overline{\mathbb{R}}_+)$ the space of all $\Psi \in \mathcal{O}(W \setminus \overline{\mathbb{R}}_+)$ satisfying for every proper subset $V \Subset W$ (7) the estimate

$$\sup_{\alpha+i\beta\in V} |e^{\alpha\kappa}\Psi(\alpha + i\beta)(\operatorname{dist}(\alpha + i\beta, \overline{\mathbb{R}}_+))^k| \le C_V < \infty$$

with the topology given by the family of norms

$$\theta_V(\Psi) = \theta^k_{\kappa,V}(\Psi) = \sup_{\alpha+i\beta\in V} |e^{\alpha\kappa}\Psi(\alpha + i\beta)(\operatorname{dist}(\alpha + i\beta, \overline{\mathbb{R}}_+))^k|,$$

where V runs through all proper subsets of W.

For $k \in \mathbb{N}_0$ and $\omega \in \mathbb{R} \cup \{\infty\}$ define the space

$$\mathfrak{L}^k_{(\omega)}(W \setminus \overline{\mathbb{R}}_+) = \varprojlim_{\kappa<\omega} \mathfrak{L}^k_\kappa(W \setminus \overline{\mathbb{R}}_+)$$

with the projective limit topology.

Clearly for every $k \in \mathbb{N}_0$ and $\omega \in \mathbb{R} \cup \{\infty\}$ the following topological inclusions hold:

$$\begin{aligned} &\mathfrak{L}^0_\kappa(W \setminus \overline{\mathbb{R}}_+) \subset \mathfrak{L}^k_\kappa(W \setminus \overline{\mathbb{R}}_+) \subset \mathfrak{L}_\kappa(W \setminus \overline{\mathbb{R}}_+),\\ &\mathfrak{L}^0_{(\omega)}(W \setminus \overline{\mathbb{R}}_+) \subset \mathfrak{L}^k_{(\omega)}(W \setminus \overline{\mathbb{R}}_+) \subset \mathfrak{L}_{(\omega)}(W \setminus \overline{\mathbb{R}}_+). \end{aligned} \tag{7.19}$$

Let

$$\mathfrak{L}^\infty_{(\omega)}(W \setminus \overline{\mathbb{R}}_+) = \varprojlim_{\kappa<\omega} \varinjlim_{k\in\mathbb{N}_0} \mathfrak{L}^k_\kappa(W \setminus \overline{\mathbb{R}}_+).$$

From Proposition 5.2 we get

Corollary 7.5. $\mathfrak{L}_{(\omega)}(W)$ *is a closed subspace of* $\mathfrak{L}^\infty_{(\omega)}(W \setminus \overline{\mathbb{R}}_+)$ *and hence the quotient space* $\mathfrak{L}^\infty_{(\omega)}(W \setminus \overline{\mathbb{R}}_+)/\mathfrak{L}_{(\omega)}(W)$ *is a Hausdorff topological space. In analogy with Definition* 5.3 *the sequence topology can be given explicitly as follows*: $h_\nu \to 0$ *in* $\mathfrak{L}^\infty_{(\omega)}(W \setminus \overline{\mathbb{R}}_+)/\mathfrak{L}_{(\omega)}(W)$ *if there exist defining functions* $\Psi_\nu \in \mathfrak{L}^\infty_{(\omega)}(W \setminus \overline{\mathbb{R}}_+)$ *with* $h_\nu = [\Psi_\nu] \operatorname{mod} \mathfrak{L}_{(\omega)}(W)$ *and for every* $\kappa < \omega$ *there exists* $k = k(\kappa)$ *such that* $\Psi_\nu \to 0$ *in* $\mathfrak{L}^k_\kappa(W \setminus \overline{\mathbb{R}}_+)$, *i.e.* $\Psi_\nu \in \mathcal{O}(W \setminus \overline{\mathbb{R}}_+)$ *and*

$$\sup_{\alpha+i\beta\in V} |e^{\alpha\kappa}\Psi_\nu(\alpha + i\beta)(\operatorname{dist}(\alpha + i\beta, \overline{\mathbb{R}}_+))^k| \xrightarrow[\nu\to\infty]{} 0$$

for every tubular set $V \Subset W$.

(7) And not of $W \setminus \overline{\mathbb{R}}_+$ as for the spaces $\mathfrak{L}_\kappa(W \setminus \overline{\mathbb{R}}_+)$.

Proof. Let $\mathfrak{L}_{(\omega)}(W) \ni G_\nu \to G$ in $\mathfrak{L}^\infty_{(\omega)}(W \setminus \overline{\mathbb{R}}_+)$ as $\nu \to \infty$. Hence for any $\kappa < \omega$ there exists $k(\kappa)$ such that $G_\nu - G \to 0$ in $\mathfrak{L}^{k(\kappa)}_\kappa(W \setminus \overline{\mathbb{R}}_+)$ and by (7.19), $G_\nu - G \to 0$ in $\mathfrak{L}_{(\omega)}(W \setminus \overline{\mathbb{R}}_+)$. By Proposition 5.2, $G \in \mathfrak{L}_{(\omega)}(W)$. ■

Let $W \Supset \overline{\mathbb{R}}_+$, Ω open, $\overline{\mathbb{R}}_+ \subset \Omega \Subset W \cap \mathbb{R}$. To derive Theorem 7.2 below (which can be regarded as a variant of Theorem 5.1) observe that the map $\mathcal{I}$ defined in Theorem 5.1 is also defined on $\mathfrak{L}^\infty_{(\omega)}(W \setminus \overline{\mathbb{R}}_+)/\mathfrak{L}_{(\omega)}(W)$. Hence

$$\mathfrak{L}^\infty_{(\omega)}(W \setminus \overline{\mathbb{R}}_+)/\mathfrak{L}_{(\omega)}(W) \ni h = \Psi + \mathfrak{L}_{(\omega)}(W) \overset{\mathcal{I}}{\mapsto} \mathcal{I}h \in \underset{\sim}{L}'_{(\omega)}(\overline{\mathbb{R}}_+),$$

where

$$\mathcal{I}h[\sigma] = -\int_\gamma \Psi(z)\sigma(z)\,dz \quad \text{for } \sigma \in \underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}_+) \tag{7.20}$$

and γ is an arbitrary regular curve encircling $\overline{\mathbb{R}}_+$ in the anticlockwise direction and contained in the set where both σ and Ψ are holomorphic.

Lemma 7.5. *Let W be a tubular neighbourhood of $\overline{\mathbb{R}}_+$ and let $\Psi \in \mathfrak{L}^\infty_{(\omega)}(W \setminus \overline{\mathbb{R}}_+)$. Then*

$$(b^L\Psi)[\sigma] = -\int_\gamma \Psi(z)\sigma(z)\,dz \quad \textit{for } \sigma \in \underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}_+),$$

where γ is an arbitrary regular curve encircling $\overline{\mathbb{R}}_+$ in the anticlockwise direction and contained in the set where both σ and Ψ are holomorphic.

Proof. Let $\sigma \in \underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}_+)$ and let $\eta > 0$ and $\varrho > 0$ be small enough that σ is holomorphic on $P_\eta = \{z \in \mathbb{C} : -\varrho \le \operatorname{Re} z,\ |\operatorname{Im} z| \le \eta\} \subset W$. Denote by $\gamma = \partial P_\eta$ and $\gamma_\beta = \partial P_\beta$ for $0 < \beta < \eta$ the curves oriented in the anticlockwise direction. By assumption on σ and Ψ, $\int_{\gamma_\beta} \Psi(z)\sigma(z)\,dz = \int_\gamma \Psi(z)\sigma(z)\,dz$ for $0 < \beta < \eta$ and $\lim_{\beta \to 0_+} \int_{-\varrho-i\beta}^{-\varrho+i\beta} \Psi(z)\sigma(z)\,dz = 0$. Hence

$$\begin{aligned} -\int_\gamma \Psi(z)\sigma(z)\,dz \\ = \lim_{\beta \to 0_+} \Big(\int_{-\varrho}^{\infty} \Psi(\alpha + i\beta)\sigma(\alpha + i\beta)\,d\alpha - \int_{-\varrho}^{\infty} \Psi(\alpha - i\beta)\sigma(\alpha - i\beta)\,d\alpha \Big). \end{aligned} \tag{7.21}$$

By Corollary 7.4, $b^L\Psi \in L'_{(\omega)}(\overline{\mathbb{R}}_+)$ exists, $\operatorname{supp} b^L\Psi \subset \overline{\mathbb{R}}_+$ and

$$(b^L\Psi)[\sigma] = \lim_{\beta \to 0_+} \int_{-\varrho}^{\infty} \Psi(\alpha + i\beta)\sigma(\alpha)\,d\alpha - \lim_{\beta \to 0_+} \int_{-\varrho}^{\infty} \Psi(\alpha - i\beta)\sigma(\alpha)\,d\alpha \tag{7.22}$$

for $\sigma \in \underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}_+)$.

Now we consider some special cases.

Case 1: $\Psi \in \varinjlim_{\kappa<\omega} \mathfrak{L}^0_\kappa(W \setminus \overline{\mathbb{R}}_+)$ and hence for every $\kappa < \omega$ there exists $C_\kappa < \infty$ such that $\sup_{\alpha+i\beta \in P_\eta} |\Psi(\alpha + i\beta)| \le C_\kappa e^{-\alpha\kappa}$. Let $\sigma \in \underset{\sim}{L}_a(\overline{\mathbb{R}}_+)$ with $a < \omega$ and $\sup_{z \in P_\eta} |e^{-az}\sigma(z)| < C$. To prove that

$$\lim_{\beta \to 0_+} \int_{-\varrho}^{\infty} \Psi(\alpha \pm i\beta)(\sigma(\alpha \pm i\beta) - \sigma(\alpha))\,d\alpha = 0$$

choose κ such that $a < \kappa < \omega$ and observe that for any $\delta > 0$ there exists $M < \infty$ such that

$$\Big|\int_M^\infty \Psi(\alpha \pm i\beta)\sigma(\alpha \pm i\beta)\,d\alpha\Big| \le CC_\kappa \int_M^\infty e^{-\alpha(\kappa-a)}\,d\alpha < \delta,$$

$$\Big|\int_M^\infty \Psi(\alpha \pm i\beta)\sigma(\alpha)\,d\alpha\Big| < \delta$$

and that $\lim_{\beta\to 0_+}\int_{-\varrho}^M \Psi(\alpha\pm i\beta)(\sigma(\alpha\pm i\beta)-\sigma(\alpha))\,d\alpha = 0$. Then the existence of the limits in (7.22) yields

$$\lim_{\beta\to 0_+}\int_{-\varrho}^\infty \Psi(\alpha \pm i\beta)\sigma(\alpha \pm i\beta)\,d\alpha = \lim_{\beta\to 0_+}\int_{-\varrho}^\infty \Psi(\alpha \pm i\beta)\sigma(\alpha)\,d\alpha$$

and by (7.21) and (7.22) it follows that

$$\begin{aligned}&-\int_\gamma \Psi(z)\sigma(z)\,dz\\ &\qquad = \lim_{\beta\to 0_+}\int_{-\varrho}^\infty \Psi(\alpha + i\beta)\sigma(\alpha)\,d\alpha - \lim_{\beta\to 0_+}\int_{-\varrho}^\infty \Psi(\alpha - i\beta)\sigma(\alpha)\,d\alpha = b^L\Psi[\sigma].\end{aligned}$$

Case 2: $\Psi \in \varinjlim_{\kappa<\omega} \varinjlim_{k\in\mathbb{N}} \mathfrak{L}^k_\kappa(W\setminus\overline{\mathbb{R}}_+)$, $\omega \le 0$. Fix $a < \omega$ and take κ with $a < \kappa < \omega$. Hence $\kappa < 0$ and

$$|\Psi(\alpha + i\beta)| < C_\kappa \frac{e^{-\alpha\kappa}}{(\operatorname{dist}(\alpha + i\beta, \overline{\mathbb{R}}_+))^k} \quad \text{for } \alpha + i\beta \in P_\eta.$$

Now we proceed as in the proof of Theorem 3.2, but this time we choose one base point $\mathring{z} = -\varrho$ and define only one operation $\mathcal{J}$ (instead of $\mathcal{J}_\pm$). For any $z = \alpha + i\beta \in P_\eta \setminus \mathbb{R}$ with $0 < \beta < \eta$ (the case $-\eta < \beta < 0$ is left to the reader) define

$$(\mathcal{J}\Psi)(z) = \int_{\mathring{z}}^z \Psi(\zeta)\,d\zeta = \int_{\mathring{z}}^{-\varrho+i\eta} \Psi(z)\,dz + \int_{-\varrho+i\eta}^{\alpha+i\eta} \Psi(z)\,dz + \int_{\alpha+i\eta}^{\alpha+i\beta} \Psi(z)\,dz.$$

Hence $\frac{d}{dz}(\mathcal{J}\Psi)(z) = \Psi(z)$ and

$$|A| := \Big|\int_{\mathring{z}}^{-\varrho+i\eta} \Psi(z)\,dz\Big| = \Big|\int_0^\eta \Psi(-\varrho + it)\,dt\Big| < \infty.$$

If $k > 1$ we get for $\alpha \ge -\varrho$ the estimates

$$\begin{aligned}|(\mathcal{J}\Psi)(\alpha + i\beta)| &\le |A| + \Big|\int_{-\varrho}^\alpha \Psi(s + i\eta)\,ds\Big| + \Big|\int_\eta^\beta \Psi(\alpha + it)\,dt\Big|\\ &\le |A| + \frac{C_k}{\eta^k}\int_{-\varrho}^\alpha e^{-s\kappa}ds + C_\kappa e^{-\alpha\kappa}\int_\eta^\beta \frac{dt}{t^k}\\ &\le |A| + \frac{C_\kappa}{\eta^k}\frac{1}{|\kappa|}e^{-\alpha\kappa} + C_\kappa e^{-\alpha\kappa}\frac{1}{k-1}\frac{1}{\beta^{k-1}}\\ &\le e^{-\alpha\kappa}\left(|A|e^{-\varrho\kappa} + \frac{C_\kappa}{\eta^k}\frac{1}{|\kappa|} + \frac{C_\kappa}{k-1}\frac{1}{\beta^{k-1}}\right) \le \widetilde{C}e^{-\alpha\kappa}\frac{1}{\beta^{k-1}}\end{aligned}$$

since by assumption $\kappa < 0$ we have $\alpha\kappa < -\varrho\kappa$.

If $k = 1$ we get the estimate $|\mathcal{J}\Psi(\alpha + i\beta)| \le Ce^{-a\kappa}|\ln\beta|$.

Thus after $k + 1$ iterations we arrive at the function $\mathcal{J}^{k+1}\Psi \in \mathcal{O}(P_\eta \setminus \overline{\mathbb{R}}_+)$ satisfying the estimate

$$|\mathcal{J}^{k+1}\Psi(\alpha + i\beta)| \le Ce^{-\alpha\kappa} \quad \text{on } P_\eta$$

and such that

$$\frac{d^{k+1}}{dz^{k+1}}\mathcal{J}^{k+1}\Psi = \Psi.$$

Thanks to the above formulae we can reduce the case under consideration to Case 1. Indeed,

$$\begin{aligned}
&\lim_{\beta\to 0_+}\int_{-\varrho}^{\infty}\Psi(\alpha+i\beta)(\sigma(\alpha+i\beta)-\sigma(\alpha))\,d\alpha\\
&\quad= (-1)^{k+1}\lim_{\beta\to 0_+}\int_{-\varrho}^{\infty}\mathcal{J}^{k+1}\Psi(\alpha+i\beta)\frac{d^{k+1}}{d\alpha^{k+1}}(\sigma(\alpha+i\beta)-\sigma(\alpha))\,d\alpha\\
&\quad= (-1)^{k+1}\lim_{\beta\to 0_+}\Big(\int_{-\varrho}^{M}\mathcal{J}^{k+1}\Psi(\alpha+i\beta)\frac{d^{k+1}}{d\alpha^{k+1}}(\sigma(\alpha+i\beta)-\sigma(\alpha))\,d\alpha\\
&\qquad-\int_{M}^{\infty}\mathcal{J}^{k+1}\Psi(\alpha+i\beta)\frac{d^{k+1}}{d\alpha^{k+1}}\sigma(\alpha)\,d\alpha+\int_{M}^{\infty}\mathcal{J}^{k+1}\Psi(\alpha+i\beta)\frac{d^{k+1}}{d\alpha^{k+1}}\sigma(\alpha+i\beta)\,d\alpha\Big).
\end{aligned}$$

Take $\delta > 0$. As in Case 1 we can find M large enough that

$$\Big|\int_{M}^{\infty}\mathcal{J}^{k+1}\Psi(\alpha+i\beta)\frac{d^{k+1}}{d\sigma^{k+1}}\sigma(\alpha)\,d\alpha\Big| < \delta.$$

To estimate the second integral over (M,∞) by δ we proceed as in the proof of Proposition 7.5. Hence

$$\lim_{\beta\to 0_+}\int_{-\varrho}^{\infty}\Psi(\alpha+i\beta)(\sigma(\alpha+i\beta)-\sigma(\alpha))\,d\alpha = 0$$

and similarly

$$\lim_{\beta\to 0_+}\int_{-\varrho}^{\infty}\Psi(\alpha-i\beta)(\sigma(\alpha-i\beta)-\sigma(\alpha))\,d\alpha = 0.$$

Now by (7.21) and (7.22) we get (as in Case 1)

$$\begin{aligned}
-\int_{\gamma}\Psi(z)\sigma(z)\,dz &= \lim_{\beta\to 0_+}\int_{-\varrho}^{\infty}\Psi(\alpha+i\beta)\sigma(\alpha)\,d\alpha - \lim_{\beta\to 0_+}\int_{-\varrho}^{\infty}\Psi(\alpha-i\beta)\sigma(\alpha)\,d\alpha\\
&= b^L\Psi[\sigma] \quad \text{for } \sigma \in \underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}_+).
\end{aligned}$$

Now, let ω be an arbitrary real number. Suppose that for any $\kappa < \omega$ there exists C_κ such that

$$|\Psi(\alpha+i\beta)| < C_\kappa\frac{e^{-\alpha\kappa}}{(\mathrm{dist}(\alpha+i\beta,\overline{\mathbb{R}}_+))^{k(\kappa)}} \quad \text{for } \alpha+i\beta \in P_\eta.$$

Define $\Psi^*(z)=\Psi(z)e^{z\omega}$. Thus

$$|\Psi^*(\alpha+i\beta)|<C_\kappa\frac{e^{-\alpha\lambda}}{(\operatorname{dist}(\alpha+i\beta,\overline{\mathbb{R}}_+))^{k(\kappa)}}\quad\text{for } \alpha+i\beta\in P_\eta,$$

where $\lambda=\kappa-\omega<0$. By Case 2,

$$-\int_\gamma \Psi^*(z)\sigma(z)\,dz=b^L\Psi^*[\sigma]\quad\text{for } \sigma\in \underset{\sim}{L}_{(0)}(\overline{\mathbb{R}}_+)$$

and hence by the isomorphism

$$e^{z\omega}:\underset{\sim}{L}_{(0)}(\overline{\mathbb{R}}_+)\ni\sigma\mapsto e^{z\omega}\sigma(z)=:\chi(z)\in\underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}_+)$$

we get

$$-\int_\gamma \Psi(z)\chi(z)\,dz=(b^L\Psi)[\chi]\quad\text{for } \chi\in\underset{\sim}{L}_{(\omega)}(\overline{\mathbb{R}}_+).$$

If $\Psi\in\mathfrak{L}^\infty_{(\infty)}(W\setminus\overline{\mathbb{R}}_+)$ then $\Psi\in\mathfrak{L}^\infty_{(\omega)}(W\setminus\overline{\mathbb{R}}_+)$ for any $\omega\in\mathbb{R}$ and the last formula holds for $\chi\in\underset{\sim}{L}_{(\infty)}(\overline{\mathbb{R}}_+)$. ∎

THEOREM 7.2. *Let W be any tubular neighbourhood $W\Supset\overline{\mathbb{R}}_+$ and let $\omega\in\mathbb{R}\cup\{\infty\}$. There exists a natural topological isomorphism*

$$\mathfrak{L}^\infty_{(\omega)}(W\setminus\overline{\mathbb{R}}_+)/\mathfrak{L}_{(\omega)}(W)\simeq L'_{(\omega)}(\overline{\mathbb{R}}_+)$$

given by

$$\mathfrak{L}^\infty_{(\omega)}(W\setminus\overline{\mathbb{R}}_+)/\mathfrak{L}_{(\omega)}(W)\ni h=\Psi+\mathfrak{L}_{(\omega)}(W)\overset{\mathcal{I}}{\mapsto}\mathcal{I}h\in L'_{(\omega)}(\overline{\mathbb{R}}_+)$$

(*here $\mathcal{I}h$ denotes the extension of* (7.20) *equal to $b^L\Psi$ by Lemma* 7.5). *The inverse mapping $\mathcal{J}$ is*

$$L'_{(\omega)}(\overline{\mathbb{R}}_+)\ni T\overset{\mathcal{J}}{\mapsto}\mathcal{C}_\Lambda T+\mathfrak{L}_{(\omega)}(W)\in\mathfrak{L}^\infty_{(\omega)}(W\setminus\overline{\mathbb{R}}_+)/\mathfrak{L}_{(\omega)}(W)$$

where $\Psi=\mathcal{C}_\Lambda T$ belongs to $\mathfrak{L}^\infty_{(\omega)}(W\setminus\overline{\mathbb{R}}_+)$ (see Proposition 7.6).

REMARK 7.6. The isomorphism of Theorem 7.2 is compatible with that of Theorem 5.1 (therefore we use the same symbol T for objects isomorphic via Theorems 7.2 and 5.1).

PROOF. We only have to show the continuity of $\mathcal{I}$ and $\mathcal{J}$. To prove the continuity of $\mathcal{J}$ take $L'_{(\omega)}(\overline{\mathbb{R}}_+)\ni T_\nu\to0$ as $\nu\to\infty$. This means that $T_\nu[\sigma]\to0$ for every $\sigma\in L_\kappa(\overline{\mathbb{R}}_+)$, $\kappa<\omega$. By the definition of $\mathcal{J}$,

$$\mathcal{J}T_\nu=\mathcal{C}_\Lambda T_\nu+\mathfrak{L}_{(\omega)}(W),\quad \mathcal{C}_\Lambda T_\nu(\zeta)=-\frac{1}{2\pi i}T_\nu[\Lambda(\zeta,\cdot)]\quad\text{for } \zeta\in\mathbb{C}\setminus\overline{\mathbb{R}}_+.$$

We have to show that for every $\kappa<\omega$,

$$\mathcal{C}_\Lambda T_\nu\xrightarrow[\nu\to\infty]{}0\quad\text{in }\varinjlim_{k\in\mathbb{N}_0}\mathfrak{L}^k_\kappa(W\setminus\overline{\mathbb{R}}_+).$$

Fix $\kappa<\omega$. It suffices to find $k\in\mathbb{N}_0$ such that

$$\mathcal{C}_\Lambda T_\nu\xrightarrow[\nu\to\infty]{}0\quad\text{in }\mathfrak{L}^k_\kappa(W\setminus\overline{\mathbb{R}}_+).\tag{7.23}$$

Take $\widetilde{\kappa}$ with $\kappa<\widetilde{\kappa}<\omega$. Hence, by assumption, $T_\nu[\sigma]\to0$ for every $\sigma\in L_{\widetilde{\kappa}}(\overline{\mathbb{R}}_+)=\varprojlim_{k\in\mathbb{N}_0}L^k_{\widetilde{\kappa}}(\overline{\mathbb{R}}_+)$ (see Subsection 7.1). Since $L^k_{\widetilde{\kappa}}(\overline{\mathbb{R}}_+)$ are Banach spaces, by the Banach–Steinhaus theorem there exist $p\in\mathbb{N}_0$ and $C_{\widetilde{\kappa}}<\infty$ such that

$$|T_\nu[\sigma]|\le C_{\widetilde{\kappa}}q_{\widetilde{\kappa},p}(\sigma)\quad\text{for } \sigma\in L_{\widetilde{\kappa}}(\overline{\mathbb{R}}_+),\ \nu\in\mathbb{N}.$$

Hence by the Hahn–Banach theorem $T_\nu \in (L^p_{\widetilde{\kappa}}(\overline{\mathbb{R}}_+))'$:

$$|T_\nu[\sigma]| \le C_{\widetilde{\kappa}} q_{\widetilde{\kappa},p}(\sigma) \quad \text{for } \sigma \in L^p_{\widetilde{\kappa}}(\overline{\mathbb{R}}_+)$$

and $T_\nu \to 0$ in $(L^p_{\widetilde{\kappa}})'$. Then by Corollary 7.2 there exist $\varepsilon_\nu \to 0$ such that

$$|T_\nu[\sigma]| \le \varepsilon_\nu q_{\kappa,p+1}(\sigma) \quad \text{for } \sigma \in L^{p+1}_\kappa(\overline{\mathbb{R}}_+). \tag{7.24}$$

By Lemma 7.3, $\sigma_\zeta(w) := \Lambda(\zeta, w) \in L_\kappa(\overline{\mathbb{R}}_+)$ for $\zeta \in W \setminus \overline{\mathbb{R}}_+$, hence $\sigma_\zeta \in L^{p+1}_\kappa(\overline{\mathbb{R}}_+)$ can be inserted in (7.24). To prove (7.23) it is enough to show that for $k = p+1$,

$$\sup_{\alpha+i\beta\in V\Subset W} |e^{\alpha\kappa}(\mathcal{C}_\Lambda T_\nu)(\alpha+i\beta)(\operatorname{dist}(\alpha+i\beta, \overline{\mathbb{R}}_+))^{p+2}| \xrightarrow[\nu\to\infty]{} 0.$$

This follows by Definition 5.1, inequalities (7.24) and (7.6):

$$\begin{aligned}\sup_{\alpha+i\beta\in V} &\bigl|e^{\alpha\kappa} T_\nu[\Lambda(\alpha+i\beta, \cdot)](\operatorname{dist}(\alpha+i\beta, \overline{\mathbb{R}}_+))^{p+2}\bigr| \\ &\le \varepsilon_\nu \sup_{\alpha+i\beta\in V} |e^{\alpha\kappa}(\operatorname{dist}(\alpha+i\beta, \overline{\mathbb{R}}_+))^{p+2} q_{\kappa,p+1}(\Lambda(\alpha+i\beta, \cdot))| \le C_{p+1}\varepsilon_\nu,\end{aligned}$$

where $\varepsilon_\nu \to 0$.

To prove the continuity of $\mathcal{I}$ take $h_\nu \to 0$ in $\mathfrak{L}^\infty_{(\omega)}(W\setminus\overline{\mathbb{R}}_+)/\mathfrak{L}_{(\omega)}(W)$, where $h_\nu = [\psi_\nu] \bmod \mathfrak{L}_{(\omega)}(W)$ with $\psi_\nu \to 0$ in $\mathfrak{L}^\infty_{(\omega)}(W\setminus\overline{\mathbb{R}}_+) = \varprojlim_{\kappa<\omega} \varinjlim_{k\in\mathbb{N}_0} \mathfrak{L}^k_\kappa(W\setminus\overline{\mathbb{R}}_+)$. We have to show that $\mathcal{I}h_\nu \to 0$ in $L'_{(\omega)}(\overline{\mathbb{R}}_+)$. Fix $a < \omega$ and select $a < \kappa < \omega$. By assumption there exists $k = k(\kappa)$ such that $\psi_\nu \to 0$ in $\mathfrak{L}^k_\kappa(W\setminus\overline{\mathbb{R}}_+)$. Let Ω be open with $\overline{\mathbb{R}}_+ \subset \Omega \subset \mathbb{R}$ and $r > 0$ be such that $V = \Omega + i(-r, r) \Subset W$. Then $\psi_\nu \in \mathcal{O}(V\setminus\mathbb{R})$ and

$$C_\nu = \sup_{\alpha+i\beta\in V} e^{\alpha\kappa}|\psi_\nu(\alpha+i\beta)|(\operatorname{dist}(\alpha+i\beta, \overline{\mathbb{R}}_+))^k \xrightarrow[\nu\to\infty]{} 0.$$

Hence $\{\psi_\nu\}$ satisfies the assumptions of Lemma 7.4 (for $V_+ = \Omega + i(0, r)$ and $V_- = \Omega - i(0, r)$) with C_ν instead of C and thus by Corollary 7.3 (see Definition 7.2) there exist $T_\nu \in L'_a(\Omega)$ with $T_\nu \to 0$ in $L'_a(\Omega)$ and $T_\nu = b^L\psi_\nu$. Since $\mathcal{I}h_\nu = b^L\psi_\nu$ and $a < \omega$ was arbitrary we get $\mathcal{I}h_\nu \to 0$ in $L'_{(\omega)}(\overline{\mathbb{R}}_+)$. ■

Theorems 7.2 and 5.1 (see also Remark 7.6) have the following corollary which completes Proposition 7.6.

Corollary 7.6. *Let $T \in \underset{\sim}{L}'_{(\omega)}(\overline{\mathbb{R}}_+)$ and $\Psi = \mathcal{C}_\Lambda T$. Then T is a Laplace distribution in $L'_{(\omega)}(\overline{\mathbb{R}}_+)$ if and only if Ψ satisfies* (i) *and* (ii) *of Proposition* 7.6.

Proof. Indeed, $\Psi \in \underset{\sim}{L}^\infty_{(\omega)}(W\setminus\overline{\mathbb{R}}_+)$ and

$$L'_{(\omega)}(\overline{\mathbb{R}}_+) \ni \mathcal{I}(\mathcal{C}_\Lambda T + \mathfrak{L}_{(\omega)}(W)) = \mathcal{I}(\mathcal{J}T) = T. \ ■$$

Next we write Theorem 7.2 in the following equivalent form:

Theorem 7.2′. *Let W be any tubular neighbourhood of $\overline{\mathbb{R}}_+$ and let $\omega \in \mathbb{R}$. There exists a natural topological isomorphism*

$$\varprojlim_{\kappa<\omega} \varinjlim_{k\in\mathbb{N}_0} (\mathfrak{L}^k_\kappa(W\setminus\overline{\mathbb{R}}_+)/\mathfrak{L}_\kappa(W)) \simeq L'_{(\omega)}(\overline{\mathbb{R}}_+). \tag{7.25}$$

PROOF. This follows from Theorem 7.2 by a purely set-theoretic argument. Let $F\in \bigcup_{k\in\mathbb{N}_0}\mathfrak{L}^k_{(\kappa)}(W\setminus\overline{\mathbb{R}}_+)$ for every $\kappa<\omega$. Then the set equality

$$F+\bigcap_{\kappa<\omega}\mathfrak{L}_\kappa(W)=\bigcap_{\kappa<\omega}(F+\mathfrak{L}_{(\kappa)}(W))$$

is interpreted as a canonical isomorphism of the quotient spaces ([8])

$$(\varprojlim_{\kappa<\omega}\varinjlim_{k\in\mathbb{N}_0}\mathfrak{L}^k_\kappa(W\setminus\overline{\mathbb{R}}_+))/\mathfrak{L}_{(\omega)}(W)\simeq\varprojlim_{\kappa<\omega}\varinjlim_{k\in\mathbb{N}_0}(\mathfrak{L}^k_\kappa(W\setminus\overline{\mathbb{R}}_+)/\mathfrak{L}_\kappa(W)).\ \blacksquare$$

7.3. Imbedding of Mellin distributions in Mellin hyperfunctions. Recall (see Section 6) that $I=(0,1]=\mu(\overline{\mathbb{R}}_+)$, where $\mu(\zeta)=e^{-\zeta}$ for $\zeta\in\mathbb{C}$. Then $\zeta=-\ln z=\mu^{-1}(z)$ for $z\in\mathbb{C}\setminus\{0\}$,

$$\operatorname{dist}(\zeta,\overline{\mathbb{R}}_+)=\begin{cases}|\zeta| & \text{for } \zeta\in\mathbb{C},\ \operatorname{Re}\zeta<0,\\ |\operatorname{Im}\zeta| & \text{for } \zeta\in\mathbb{C},\ \operatorname{Re}\zeta\ge 0,\end{cases}$$

and for $z\in\widetilde{\mathbb{C}\setminus\{0\}}$,

$$\operatorname{dist}(-\ln z,\overline{\mathbb{R}}_+)=\begin{cases}((\ln|z|)^2+(\arg z)^2)^{1/2} & \text{for } |z|>1,\\ |\arg z| & \text{for } 0<|z|\le 1.\end{cases}$$

In order to formulate for the Mellin distributions $M'_{(\omega)}(I)$ a statement analogous to Theorem 7.2 we introduce the spaces $\mathfrak{M}^k_\kappa(V\setminus I)$ for $k\in\mathbb{N}_0$, $\kappa\in\mathbb{R}$, $I\Subset V$, where $V\subset\widetilde{\mathbb{C}\setminus\{0\}}$ is a sectorial set ([9]):

$$\mathfrak{M}^k_\kappa(V\setminus I)=\{F\in\mathcal{O}(V\setminus I):\sup_{z\in S}|F(z)(\operatorname{dist}(-\ln z,\overline{\mathbb{R}}_+))^k z^{-\kappa}|<\infty,\ \text{where } S\Subset V\setminus I\}$$

and the space $\mathfrak{M}^k_{(\omega)}(V\setminus I)=\varprojlim_{\kappa<\omega}\mathfrak{M}^k_\kappa(V\setminus I)$ equipped with the projective limit topology given by the family of norms

$$\varrho^k_{\kappa,S}(F)=\sup_{z\in S}|F(z)(\operatorname{dist}(-\ln z,\overline{\mathbb{R}}_+))^k z^{-\kappa}|,$$

where S runs through proper subsets of $V\setminus I$ and $\kappa<\omega$.

Define

$$\mathfrak{M}^\infty_{(\omega)}(V\setminus I)=\varprojlim_{\kappa<\omega}\varinjlim_{k\in\mathbb{N}_0}\mathfrak{M}^k_\kappa(V\setminus I).$$

As in Corollary 7.5, $\mathfrak{M}_{(\omega)}(V)$ is a closed subspace of $\mathfrak{M}^\infty_{(\omega)}(V\setminus I)$ and hence the quotient space $\mathfrak{M}^\infty_{(\omega)}(V\setminus I)/\mathfrak{M}_{(\omega)}(V)$ is a Hausdorff topological space.

We get the following variant of Theorem 6.1 for the subspace $M'_{(\omega)}(I)$ of $\underset{\sim}{M}'_{(\omega)}(I)$ directly from Theorem 7.2 and the fact that composition with μ is a topological isomorphism of the corresponding spaces.

THEOREM 7.3. *Let V be any sectorial neighbourhood $\Supset I$ and let $\omega\in\mathbb{R}\cup\{\infty\}$. There exists a natural topological isomorphism*

$$\mathfrak{M}^\infty_{(\omega)}(V\setminus I)/\mathfrak{M}_{(\omega)}(V)\simeq M'_{(\omega)}(I).$$

([8]) Cf. [Ko2, §1].

([9]) See the definitions at the beginning of Section 6.

Lemma 7.6 (a counterpart of Lemma 7.3). *Fix a sectorial set V. Let $\Gamma(z,x)$ be defined by (6.2) for $x \in I=(0,1]$, $z\in \widetilde{\mathbb{C}\setminus\{0\}}\setminus I$. Then for every $\kappa\in\mathbb{R}$, $j\in\mathbb{N}_0$ and $S=\{z\in \widetilde{\mathbb{C}\setminus\{0\}} : |\arg z|\le\theta, 0<|z|\le r$ with $0<\theta<\infty$, $0<r<\infty\}$ there exists $C_{\kappa,j,S}<\infty$ such that for $z\in S\setminus I$,*

$$(\operatorname{dist}(-\ln z,\overline{\mathbb{R}}_+))^{j+1}\sup_{0<x\le 1}\left|x^{\kappa+1}\left(x\frac{d}{dx}\right)^j\Gamma(z,x)\right|<C_{\kappa,j,S}|z|^\kappa.$$

Hence the function $\varphi_z(x)=\Gamma(z,x)$ for $z\in\widetilde{\mathbb{C}\setminus\{0\}}\setminus I$, $x\in I$, belongs to $M_\kappa(I)$ for any $\kappa\in\mathbb{R}$.

Proof. Since the proof goes along the same lines as that of Lemma 7.3, we give it only for $j=0$. If $z\in S\setminus I$ and $x\in(0,1]$ we put $\xi=\operatorname{Re}(\ln z-\ln x)$ and get, as in the proof of Lemma 6.1,

$$|x^{\kappa+1}\Gamma(z,x)|\le e^{-\xi(\kappa+\xi)}e^{(\arg z)^2}\frac{|z|^\kappa}{|\ln z-\ln x|}\le A_{\kappa,S}\frac{|z|^\kappa}{|\ln z-\ln x|}$$

where $A_{\kappa,S}=e^{\theta^2}\sup_{\xi\in\mathbb{R}}e^{-\xi(\kappa+\xi)}<\infty$.

Note that

$$\sup_{0<x\le1}\frac{1}{|\ln z-\ln x|}=\frac{1}{\operatorname{dist}(-\ln z,\overline{\mathbb{R}}_+)}.$$

Thus

$$\sup_{0<x\le1}|x^{\kappa+1}\Gamma(z,x)|\le A_{\kappa,S}\frac{|z|^\kappa}{\operatorname{dist}(-\ln z,\overline{\mathbb{R}}_+)}$$

and hence our assertion follows with $C_{\kappa,0,S}=A_{\kappa,S}$. ■

Proposition 7.7. *Let $u\in M'_{(\omega)}(I)$ where $I=(0,1]$. Then the function $\Phi(z)=-\frac{1}{2\pi i}u[\Gamma(z,x)]$ for $z\in\widetilde{\mathbb{C}\setminus\{0\}}\setminus I$ with Γ defined by (6.2) satisfies the conditions:*

(i) $\Phi\in\mathcal{O}(\widetilde{\mathbb{C}\setminus\{0\}}\setminus I)$,

(ii) *for any $\kappa<\omega$ there exists $k=k(\kappa)\in\mathbb{N}$ such that for any $0<\theta<\infty$ and $1<r<\infty$ there exist $C_{\kappa,\theta}<\infty$ satisfying*

$$|\Phi(z)|\le C_{\kappa,\theta}\frac{|z|^\kappa}{(\operatorname{dist}(-\ln z,\overline{\mathbb{R}}_+))^k}\quad \textit{for } z\in S\setminus I$$

where $S=\{z\in\widetilde{\mathbb{C}\setminus\{0\}}:|\arg z|\le\theta,\ 0<|z|\le r\}$.

Proof. By Proposition 6.8 we consider u as a functional from $\underset{\sim}{M}'_{(\omega)}(I)$, hence by Theorem 6.1, $u=\Phi+\mathfrak{M}_{(\omega)}(V)$. Clearly Φ satisfies (i). To show (ii) note that the assumption $u\in M'_{(\omega)}(I)=\varinjlim_{\kappa<\omega}M'_\kappa(I)$ yields that for every $\kappa<\omega$ there exist $l\in\mathbb{N}_0$ and $C<\infty$ such that

$$|u[\varphi]|\le C\sum_{j=0}^{l}\sup_{0<x\le1}\left|x^{\kappa+1}\left(x\frac{d}{dx}\right)^j\varphi(x)\right|\quad\text{for }\varphi\in M_\kappa(I)$$

and hence the estimate in (ii) follows from Lemma 7.6. ■

Theorem 7.3 has the following corollary which completes Proposition 7.7.

COROLLARY 7.7. *Let* $u \in \underset{\sim}{M}'_{(\omega)}(I)$, $I = (0,1]$, $\Phi = \mathcal{C}_\Gamma u$. *Then* u *is a Mellin distribution in* $M'_{(\omega)}(I)$ *if and only if* Φ *satisfies* (i) *and* (ii) *of Proposition* 7.7.

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