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PSEUDODIFFERENTIAL OPERATORS WITH MULTIPLE CHARACTERISTICS AND GEVREY CLASSES

LUIGI RODINO

Dipartimento di Matematica, Università di Torino
Torino, Italy

We announce here some results obtained in collaboration with Liess in [7] and with Zanghirati in [10], concerning the propagation of Gevrey singularities and the Gevrey solvability of pseudodifferential operators with multiple characteristics.

Let us begin by recalling that a distribution u in Ω, Ω an open subset of $\mathbf{R}^n$, is said to be of class G^s, $1 < s < \infty$, at $x_0 \in \Omega$ if there exists a neighborhood $V \subset \Omega$ of x_0 where u is C^∞ and satisfies

$$\sup_{x \in V} |D^\alpha u(x)| \leqslant C^{|\alpha|+1} |\alpha|^{s|\alpha|} \tag{1}$$

for a constant C independent of α. We denote by $G^s(\Omega)$ the set of all $u \in C^\infty(\Omega)$ which are of class G^s at every $x_0 \in \Omega$, and write $G_0^s(\Omega)$ for $G^s(\Omega) \cap C_0^\infty(\Omega)$. The space of s-ultradistributions $G^{(s)'}(\Omega)$ and the space of s-ultradistributions with compact support $G_0^{(s)'}(\Omega)$ are defined as the duals of $G_0^s(\Omega)$, $G^s(\Omega)$, respectively. Fix then $(x_0, \xi_0) \in \Omega \times (\mathbf{R}^n \setminus 0)$; for $u \in G^{(s)'}(\Omega)$ we write $(x_0, \xi_0) \notin \mathrm{WF}_s u$ if there exists $\varphi \in G_0^s(\Omega)$ with $\varphi(x) = 1$ in a neighborhood of x_0, and positive constants C and ε such that

$$|(\varphi u)\hat{\ }(\xi)| \leqslant C \exp[-\varepsilon |\xi|^{1/s}] \tag{2}$$

for all ξ in a conic neighborhood of ξ_0. We recall that the projection on Ω of the Gevrey wave front set $\mathrm{WF}_s u$ is the Gevrey singular support, s-sing supp u, defined as the complement of the largest open subset of Ω where u is of class G^s.

The Gevrey classes G^s play an important role in the theory of linear partial differential equations as intermediate spaces between the spaces of C^∞ and analytic functions; in particular, whenever the properties of a certain operator differ in the C^∞ and in the analytic category, it is natural to test the behavior of the operator on the classes G^s and the related wave front sets.

In this order of ideas we shall consider in Ω an analytic pseudodifferential operator $P = p(x, D)$, whose principal symbol will be assumed to vanish exactly of order $k \geqslant 1$ on a regular submanifold of codimension one in the cotangent space. To be precise, we shall suppose $p(x, \xi) \sim \sum_{j=0}^{\infty} p_{m-j}(x, \xi)$, where $p_{m-j}(x, \xi)$ is positively homogeneous of degree $m-j$ in ξ (i.e. P is classical) and we shall assume that the principal part $p_m(x, \xi)$ satisfies the following condition in a conic neighborhood Γ of a fixed point $(x_0, \xi_0) \in \Omega \times(\boldsymbol{R}^n \setminus 0)$:

(3) *We may write* $p_m(x, \xi) = q_{m-k}(x, \xi) a_1(x, \xi)^k$, *where* $q_{m-k}(x, \xi)$ *is an elliptic symbol homogeneous of order* $m-k$, *and the first order symbol* $a_1(x, \xi)$ *is real-valued and of principal type, i.e.* $d_{x,\xi} a_1(x, \xi)$ *never vanishes and is not parallel to* $\sum_{h=1}^{n} \xi_h dx_h$ *on* $\Sigma = \{(x, \xi) \in \Gamma\colon a_1(x, \xi) = 0\} \neq \emptyset$.

This hypothesis is sufficient to deduce the non-analytic-hypoellipticity of P and the propagation of the analytic wave front set along the bicharacteristic strips associated to P (see Bony–Schapira [1]), whereas to obtain a similar result in the C^∞ category it is necessary to add the so-called Levi condition on the lower order terms (see Chazarain [3], Sjöstrand [8]). We shall begin by giving a result of propagation for G^s singularities under the following ϱ-Levi condition, $0 < \varrho < 1$:

(4) *Let* A *be a classical analytic pseudodifferential operator whose principal symbol is given by the function* $a_1(x, \xi)$ *in* (3); *then* P *can be written in* Γ *in the form* $P = \sum_{j=0}^{k} Q_j A^{k-j}$, *where* Q_j, $j = 0, \ldots, k$, *are classical analytic pseudodifferential operators of order* $\leqslant m-k+\varrho j$.

If in (4) we set $\varrho = 0$, we obtain the standard C^∞ Levi condition, which implies the validity of (4) for any ϱ, $0 < \varrho < 1$; also observe that (3) and (4) are invariant under changes of variables and conjugation by elliptic Fourier integral operators.

Consider then the Hamiltonian vector field associated to $a_1(x, \xi)$ in (3),

$$H_{a_1} = \sum_{h=1}^{n} \left(\frac{\partial a_1}{\partial \xi_h} \frac{\partial}{\partial x_h} - \frac{\partial a_1}{\partial x_h} \frac{\partial}{\partial \xi_h} \right).$$

The related integral curves on the characteristic manifold Σ are called the *bicharacteristic strips* of P; their definition is independent of the choice of a_1 in (3). Let us assume $(x_0, \xi_0) \in \Sigma$ and write γ_0 for the restriction to Γ of the bicharacteristic strip through (x_0, ξ_0). In the following we shall denote by $G^{(s)\prime}(\Gamma)$ the factor space $G^{(s)\prime}(\Omega)/\sim$, where $f \sim g$ means that $\Gamma \cap \mathrm{WF}_s(f-g) = \emptyset$. For $u \in G^{(s)\prime}(\Gamma)$, $\mathrm{WF}_s u$ is a well-defined conic closed subset of Γ; the operator P can be regarded as a linear map on $G^{(s)\prime}(\Gamma)$.

THEOREM 1. *Under the preceding hypotheses on* P, *and in particular under* *the assumption* (3) *and the ϱ-Levi condition* (4), *let s be any real number with* $1 < s < 1/\varrho$; *then, taking Γ sufficiently small*:

(i) *There exists* $u \in G^{(s)'}(\Gamma)$ *with* $Pu = 0$ *and* $\mathrm{WF}_s u = \gamma_0$.

(ii) *If u is in* $G^{(s)'}(\Gamma)$ *with* $Pu = 0$, *then* $(x_0, \xi_0) \in \mathrm{WF}_s u$ *implies* $\gamma_0 \subset \mathrm{WF}_s u$.

(iii) *For every* $v \in G^{(s)'}(\Gamma)$ *there exists* $u \in G^{(s)'}(\Gamma)$ *such that* $Pu = v$.

Theorem 1 is proved in Rodino–Zanghirati [10] by using a suitable calculus of Gevrey pseudodifferential operators of infinite order (cf. Zanghirati [11], [12], Cattabriga [2]). Let us add here some remarks concerning the validity of (3), (4). Basing on the characteristic submanifold $\Sigma = \{(x, \xi) \in \Gamma\colon\ p_m(x, \xi) = 0\}$, which is smooth of codimension one in $T^*\Omega \backslash 0$ and transverse to the fibres $x = $ constant, we may equivalently express (3) by writing that

$$C^{-1} d_\Sigma(x, \xi)^k \leqslant |p_m(x, \xi)|/|\xi|^m \leqslant C d_\Sigma(x, \xi)^k \tag{5}$$

for a suitable constant C and for all $(x, \xi) \in \Gamma$, $|\xi| \geqslant 1$ ($d_\Sigma(x, \xi)$ is the distance from $(x, \xi/|\xi|)$ to Σ). Concerning the ϱ-Levi condition (4), it is easy to see that it is satisfied for a given ϱ, $1/2 \leqslant \varrho < 1$, if and only if

$$|p_{m-j}(x, \xi)|/|\xi|^{m-j} \leqslant C d_\Sigma(x, \xi)^{k-j/(1-\varrho)}, \qquad 0 \leqslant j < k(1-\varrho), \tag{6}$$

for some constant C and for all $(x, \xi) \in \Gamma$, $|\xi| \geqslant 1$. For $0 < \varrho < 1/2$ the estimates (6) do not have an invariant meaning, and the equivalence with (4) fails.

Since (6) is a consequence of (5) in the case $k(1-\varrho) \leqslant 1$, that is, $k \leqslant 2$ and $\varrho \geqslant (k-1)/k$, or else $k = 1$, from Theorem 1 we have the following

COROLLARY 2. *Suppose* $k \geqslant 2$. *Under the only condition* (5) *for the principal symbol* $p_m(x, \xi)$ *of the operator P, the conclusions of Theorem* 1 *hold for* $1 < s < k/(k-1)$.

When $k = 1$, i.e. when P is of real principal type, the conclusions of Theorem 1 hold for all s, $1 < s < \infty$ (the result on propagation in this case is well known; see Hörmander [4]). For $k \geqslant 2$ the conclusions of Theorem 1 fail in general for $s \geqslant 1/\varrho$, i.e. $s \geqslant k/(k-1)$ in Corollary 2, and the study of the corresponding G^s regularity requires then a further analysis of the lower order terms. In this connection we shall first recall some results on Gevrey hypoellipticity from Liess–Rodino [5], [6], Rodino [9]. Let us argue, for the sake of simplicity, on the model operator

$$P = D_{x_n}^k + \sum_{j=1}^{k} Q_j D_{x_n}^{k-j}, \tag{7}$$

where the Q_j, $j = 1, \ldots, k$, are classical analytic pseudodifferential operators of order $\leqslant \varrho j$, defined in a conic neighborhood Γ of $x_0 = 0$, ξ_0

$= (\xi_{1,0}, \ldots, \xi_{n-1,0}, 0)$. The operator P in (7) satisfies (3) on the characteristic manifold $\Sigma = \{\xi_n = 0\}$ and the ϱ-Levi condition (4) is also satisfied. Theorem 1 then applies for $1 < s < 1/\varrho$, the bicharacteristic strips being now the parallels on Σ to the x_n-axis. Let us remark that every operator satisfying (3) and (4) can actually be reduced to the form (7) by conjugation with analytic Fourier integral operators and multiplication by elliptic factors.

Denoting by $q_j(x, \xi)$ the principal symbol of Q_j, we may now state:

THEOREM 3. *Suppose that for every* $(x, \xi) \in \Sigma$, *with* $\xi' = (\xi_1, \ldots, \xi_{n-1}) \neq 0$, *and for every* $\tau \in \mathbf{R}$

$$I(x, \xi', \tau) = \tau^k + \sum_{j=1}^{k} q_j(x; \xi', 0)\tau^{k-j} \neq 0; \tag{8}$$

then for $1/\varrho \leqslant s < \infty$ *the operator* P *in* (7) *is* G^s*-hypoelliptic in* Γ, *i.e.* $\mathrm{WF}_s Pu = \mathrm{WF}_s u$ *for all* $u \in G^{(s)\prime}(\Gamma)$.

When $1/2 \leqslant \varrho < 1$ it is easy to give a geometric invariant formulation of Theorem 3, referring to symbols which satisfy (5), (6); we shall limit ourselves here to the case $\varrho = (k-1)/k$, $k \geqslant 2$, already considered in Corollary 2.

Under the assumption (5) on $p_m(x, \xi)$ let us define for any $\gamma \in \Sigma$ and for any C^∞ vector field Y in a neighbourhood of γ,

$$J(\gamma, Y) = (k!)^{-1}(Y^k p_m)(\gamma) + p'_{m-1}(\gamma), \tag{9}$$

where p'_{m-1} denotes the subprincipal symbol $p'_{m-1} = p_{m-1} + (i/2)\sum_{h=1}^{n} \partial^2 p_m/\partial x_h \partial \xi_h$.

COROLLARY 4. *Let* P *be any classical analytic pseudodifferential operator satisfying* (5) (*i.e.* (3)). *Suppose* $J(\gamma, Y) \neq 0$ *for every* γ *and* Y. *Then* P *is* G^s*-hypoelliptic in* Γ *for* $k/(k-1) \leqslant s < \infty$.

In fact, after conjugation by Fourier integral operators and multiplication by elliptic factors, the operators P in Corollary 2 and Corollary 4 can be written in the form $P = D^k_{x_n} + Q$, with Q of order $k-1$; we may then apply Theorem 3 for $\varrho = (k-1)/k$, observing that the ranges of $I(x, \xi', \tau)$ in (8) and $J(\gamma, Y)$ in (9) coincide in this case. Theorem 3 was proved in Rodino [9] by using a Gevrey version of the classes $S^m_{\varrho,\delta}$ of Hörmander (see also Liess-Rodino [5], [6]).

When the hypothesis of Theorem 3 is not satisfied, i.e. $I(x, \xi', \tau)$ in (8) vanishes somewhere, results on nonhypoellipticity, propagation and solvability can be proved for $1/\varrho \leqslant s < \infty$ under a suitable condition on $I(x, \xi', \tau)$. Namely:

THEOREM 5. *Let* P *be defined by* (7) *and suppose further* $0 < \varrho \leqslant 1/2$. *Assume that*

(10) $I(x, \xi', \tau)$ *in* (8) *is real-valued, and* $\partial_\tau I(x, \xi', \tau) \neq 0$ *for all* x, ξ', τ *on the manifold* $\{I(x, \xi', \tau) = 0,\ \xi' \neq 0\} \neq \emptyset$.

Then the conclusions of Theorem 1 are also valid for $1/\varrho \leqslant s < \infty$.

The hypothesis $0 < \varrho \leqslant 1/2$ is essential; in fact, for $1/2 < \varrho < 1$ it is possible to give examples of operators which satisfy (10) and are G^s-hypoelliptic for sufficiently large values of s. The proof of Theorem 5, as well as the discussion of the case $1/2 < \varrho < 1$, will be found in the forthcoming paper [7].

When $\varrho = 1/2$ it is easy to express Theorem 5 in a geometric invariant way, referring to symbols which satisfy (5), (6). Let us end here with the analysis of the case $\varrho = 1/2$, $k = 2$. For any operator P satisfying (3) with $k = 2$, we have from Theorem 1 and Corollary 2 the results on nonhypoellipticity, propagation and solvability in G^s for $1 < s < 2$. Multiplying by an elliptic factor, we may further assume that the principal symbol of P takes real positive values in a conic neighborhood of the characteristic manifold Σ; from Corollary 4 we then deduce that

$$p'_{m-1}(\gamma) \notin \mathbf{R}_- \cup \{0\} \quad \textit{for all } \gamma \in \Sigma \tag{11}$$

implies the G^s-hypoellipticity for $2 \leqslant s < \infty$. Finally, when (11) is not satisfied, we may apply Theorem 5 observing that the condition (10) means in this case $p'_{m-1}(\gamma) < 0$ on Σ:

COROLLARY 6. *Assume P satisfies* (3) *in Γ with $k = 2$; suppose $p_m \geqslant 0$ in Γ and $p'_{m-1} < 0$ on the characteristic manifold Σ. Then the conclusions of Theorem* 1 *hold for $2 \leqslant s < \infty$.*

References

[1] J. M. Bony et P. Schapira, *Propagation des singularités analytiques pour les solutions des équations aux dérivées partielles*, Ann. Inst. Fourier (Grenoble) 26 (1) (1976), 81–140.

[2] L. Cattabriga, Seminario Analisi Matematica, Bologna 1984.

[3] J. Chazarain, *Propagation des singularités pour une classe d'opérateurs à caractéristiques multiples et résolubilité locale*, Ann. Inst. Fourier (Grenoble) 24 (1) (1974), 209–223.

[4] L. Hörmander, *Uniqueness theorems and wave front sets for solutions of linear differential equations with analytic coefficients*, Comm. Pure Appl. Math. 24 (1971), 671–704.

[5] O. Liess and L. Rodino, *A general class of Gevrey type pseudodifferential operators*, in: Journées Équations aux Dérivées Partielles, Saint-Jean-de-Monts 1983, conf. no. 6.

[6] –, –, *Inhomogeneous Gevrey classes and related pseudodifferential operators*, Boll. Un. Mat. Ital. (6) 3-C (1984), 233–323.

[7] –, –, in preparation.

[8] L. Rodino, *Analytic and Gevrey hypoellipticity*, Publ. Sém. Math., Rennes 1983.

[9] L. Rodino and L. Zanghirati, *Propagation of Gevrey singularities for pseudodifferential operators with multiple characteristics*, Comm. Partial Differential Equations 11 (1986), 673–711.

[10] J. Sjöstrand, *Propagation of singularities for operators with multiple involutive characteristics*, Ann. Inst. Fourier (Grenoble) 26 (1) (1976), 141–155.

[11] L. Zanghirati, *Operatori pseudodifferenziali di tipo Gevrey di ordine infinito*, Seminario Analisi Matematica, Bologna 1984.

[12] –, *Pseudodifferential operators of infinite order and Gevrey classes*, Ann. Univ. Ferrara Sez. VII 31 (1985), 197–219.