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Existence and stability of solutions for a system of quadratic integral equations in Banach algebras

Abstract. The aim of this paper is to prove the existence and stability of solutions of a system of quadratic integral equations in the Banach algebra of continuous and bounded functions on unbounded rectangle. The main tool used in our considerations is the multiple fixed point theorem which is a consequence of Darbo's fixed point theorem and the technique associated with measures of noncompactness. We also present an illustrative example.

1. Introduction

The theory of integral operators and integral equations is an important part of nonlinear analysis. It is caused by the fact that this theory is frequently applicable in other branches of mathematics and in mathematical physics, engineering, economics, biology as well in describing problems connected with real world (cf.[19, 25]). It is well-known that a lot of problems investigated in the theories of radiative transfer and neutron transport, and in the kinetic theory of gases and also several real world problems can be described with the help of various quadratic integral equations. Especially the quadratic integral equations of Chandrasekhar type can be very often encountered in several applications see [12, 18]. In the last years there appeared many papers devoted to the quadratic integral equations and several types of integral operators are investigated, we refer to the papers of Benchohra and *al.* [3], Banaś and *al.* [9, 14, 15, 16, 20, 21] and references therein.

On the other hand, the classical theory of integral operators and equations can be generalized with the help of the Stieltjes integral having kernels dependent

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on one or two variables. Such an approach was presented and developed in many research papers see the results of Abbas and Benchohra [1, 2, 7], Banaś and *al.* [8, 15, 17].

The goal of this paper is to investigate a finite system of nonlinear quadratic integral equations of Hadamard-Volterra-Stieltjes type applying the concept of multiple fixed point of condensing operators and we will use the technique associated with measures of noncompactness. There are some papers in this direction, Aghajani and *al.* [4, 5], Dhage [20] and references therein.

It is worthwhile mentioning that in applications, the most useful measures of noncompactness are those defined in an axiomatic way [6, 11]. Such a direction of investigations has been initiated in the papers [10, 13], where the authors introduced the so-called condition (m) related to the operation of multiplication in an algebra and playing a crucial role in the use of the technique of measures of noncompactness in Banach algebras setting. Moreover, if we apply an approach to the measure of noncompactness concept associated with a suitable axiomatic definition, then we create the possibility to characterize solutions of investigated operator equations see [9, 12, 13, 14, 16, 21].

Consider the following system of quadratic fractional integral equations

$$\begin{cases} u_1(x,y) = \varphi(x,y) + (Gu_1)(x,y)\displaystyle\int_1^x\int_1^y \left(\ln\frac{x}{t}\right)^{r_1-1}\left(\ln\frac{y}{s}\right)^{r_2-1} \\ \qquad\qquad \times \dfrac{g(t,s,u_1(t,s),\ldots,u_n(t,s))}{\Gamma(r_1)\Gamma(r_2)st} d_s h_1(y,s) d_t h_2(x,t), \\ u_2(x,y) = \varphi(x,y) + (Gu_2)(x,y)\displaystyle\int_1^x\int_1^y \left(\ln\frac{x}{t}\right)^{r_1-1}\left(\ln\frac{y}{s}\right)^{r_2-1} \\ \qquad\qquad \times \dfrac{g(t,s,u_2(t,s),\ldots,u_n(t,s),u_1(t,s))}{\Gamma(r_1)\Gamma(r_2)st} d_s h_1(y,s) d_t h_2(x,t), \\ \quad\vdots \qquad\qquad\qquad \vdots \\ u_n(x,y) = \varphi(x,y) + (Gu_n)(x,y)\displaystyle\int_1^x\int_1^y \left(\ln\frac{x}{t}\right)^{r_1-1}\left(\ln\frac{y}{s}\right)^{r_2-1} \\ \qquad\qquad \times \dfrac{g(t,s,u_n(t,s),\ldots,u_{n-2}(t,s),u_{n-1}(t,s))}{\Gamma(r_1)\Gamma(r_2)st} d_s h_1(y,s) d_t h_2(x,t), \end{cases} \tag{1}$$

where $x, y \in J = [1,+\infty)\times[1,b]$, $r_1, r_2 > 1$, $\varphi\colon J \to \mathbb{R}$ is a continuous and bounded function, $G\colon BC \to BC$ is a linear operator, $g\colon J \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function, $h_1\colon [1,b]\times[1,b] \to \mathbb{R}$, $h_2\colon [1,+\infty)\times[1,+\infty) \to \mathbb{R}$ are given functions, and $\Gamma(\cdot)$ is the Euler gamma function.

2. Preliminaries

Assume that $(X, \|\cdot\|)$ is an infinite dimensional Banach space with zero element θ. Denote by $B(x,r)$ the closed ball with center at x and radius r. We write B_r to denote the ball $B(\theta,r)$. If U is a subset of X then the symbols $\overline{U}$, $\operatorname{Conv} U$ stand for the closure and closed convex hull of U, respectively. Apart from this the symbol $\operatorname{diam} U$ will denote the diameter of a set U while $\|U\|$ denotes the norm of U; that is, $\|U\| = \sup\{\|u\| : \ u \in U\}$. Moreover, we denote by $\mathtt{M}_X$ the family of all nonempty and bounded subsets of X and by $\mathtt{N}_X$ its subfamily consisting of all relatively compact sets.

Here, we give some basic facts concerning measures of noncompactness, which is defined axiomatically in terms of some natural conditions.

Definition 2.1 ([6, 11])
A mapping $\psi\colon \mathtt{M}_X \to \mathbb{R}_+$ is said to be a measure of noncompactness in X if it satisfies the following conditions:

(A1) The family $\ker\psi = \{A \in \mathtt{M}_X : \ \psi(A) = 0\}$ is nonempty and $\ker\psi \subset \mathtt{N}_X$;

(A2) $A \subset B \to \psi(A) \le \psi(B)$;

(A3) $\psi(A) = \psi(\overline{A})$;

(A4) $\psi(A) = \psi(\operatorname{Conv} A)$;

(A5) $\psi(\lambda A + (1-\lambda)B) \le \lambda\psi(A) + (1-\lambda)\psi(B)$ for $\lambda \in [0,1]$;

(A6) If (A_n) is a sequence of closed sets from $\mathtt{M}_X$ such that $A_{n+1} \subset A_n$ ($n = 0, 1, 2, \ldots$) and if $\lim_{n\to\infty} \psi(A_n) = 0$, then the intersection set $A_\infty = \bigcap_{n=0}^{\infty} A_n$ is nonempty.

The family $\ker\psi$ described in (A1) is said to be *the kernel* of the measure of noncompactness ψ.

Remark 2.2
Observe that the intersection set A_∞ from (A6) is a member of the family $\ker\psi$. In fact, since $\psi(A_\infty) \le \psi(A_n)$ for any n, we infer that $\psi(A_\infty) = 0$.

In the sequel we will assume that the space X has the structure of a Banach algebra. In such a case we write uv in order to denote the product of elements $u, v \in X$. Similarly, we will write UV to denote the product of subsets U, V of X; that is, $UV = \{uv : \ u \in U, \, v \in V\}$.

Definition 2.3 ([10, 13])
One says that the measure of noncompactness μ defined on a Banach algebra X satisfies condition (m) if for arbitrary sets $U, V \in \mathtt{M}_X$ the following inequality is satisfied

$$\mu(UV) \le \|U\|\mu(V) + \|V\|\mu(U). \tag{m}$$

Remark 2.4
It turns out that the above defined condition (m) is very convenient in considerations connected with the use of the technique of measures of noncompactness in Banach algebras. Apart from this the majority of measures of noncompactness satisfy this condition.

We recall the well known fixed point theorem of Darbo.

Theorem 2.5 ([5, 6, 9])
Let Ω be a nonempty, bounded, closed and convex subset of a Banach space X and let $F\colon \Omega \to \Omega$ be a continuous mapping. Assume that there exists a constant $k \in [0,1)$ such that $\psi(FA) \le k\psi(A)$ for any nonempty subset A of Ω. Then F has a fixed point in the set Ω.

Remark 2.6
Let us denote by $\operatorname{Fix} F$ the set of all fixed points of the operator F which belongs to Ω. It can be shown that the set $\operatorname{Fix} F$ belongs to the family $\ker \psi$.

Theorem 2.7 ([11])
Let $X_1, X_2, \dots, X_n$ be Banach spaces and $\mu_1, \mu_2, \dots, \mu_n$ be measures of noncompactness in $X_1, X_2, \dots, X_n$, respectively. Assume that the function $\xi\colon \mathbb{R}_+^n \to \mathbb{R}_+$ is convex with $\xi(a_1, a_2, \dots, a_n) = 0$ if and only if $a_i = 0$ for $i = 1, 2, \dots, n$. Then

$$\mu(D) = \xi(\mu_1(D_1), \mu_2(D_2), \dots, \mu_n(D_n))$$

defines a measure of noncompactness in the product space $X_1 \times X_2 \times \cdots \times X_n$ where D_i denotes the natural projection of D into X_i for $i = 1, 2, \dots, n$.

Example 2.8 ([4])
Let μ be a measure of noncompactness on a Banach space X. Let $H_1(x,y) = \max(x,y)$ and $H_2(x,y) = x + y$ for any $(x,y) \in [0,\infty)^2$, then all the conditions of Theorem 2.7 are satisfied, i.e. H_1 and H_2 are convex with $H_1(x,y) = 0$ or $H_2(x,y) = 0$ if and only if $x = y = 0$. We conclude that $\widetilde{\mu}(D) = \max(\mu(D_1), \mu(D_2))$ and $\widetilde{\mu}(D) = \mu(D_1) + \mu(D_2)$ define measures of noncompactness in the space $X \times X$, where D_i, $i = 1, 2$ denote the natural projections of X.

Definition 2.9 ([20])
An element $(x,y) \in X \times X$ is called a coupled fixed point of a mapping $F\colon X \times X \to X$ if $F(x,y) = x$ and $F(y,x) = y$.

In what follows we will work in the Banach space BC consisting of all real functions defined, continuous and bounded on J. This space is furnished with the standard norm

$$\|u\| = \sup\{\|u(x,y)\| : (x,y) \in J\}.$$

The space BC has the structure of a Banach algebra, i.e. if $w, v \in BC$, we write wv to denote the product of elements u and v, and $(wv)(t,s) = w(t,s)v(t,s)$.

Remark 2.10
It is clear that the product space $\underbrace{BC \times \cdots \times BC}_{n}$ turns out to be a Banach space if equipped with the norm

$$\|(u_1, \dots, u_n)\| = \sum_{i=1}^{n} \|u_i\|_{BC}.$$

In order to define a measure of noncompactness in the space BC, fix a nonempty bounded subset Y of BC and $T > 1$. For $u \in Y$ and $\epsilon_1, \epsilon_2 > 0$ let us denote by $\omega^T(u, \epsilon_1, \epsilon_2)$ the modulus of continuity of the function u on the rectangle $[1, T] \times [1, b]$, i.e.

$$\begin{aligned}
\omega^T(u, \epsilon_1, \epsilon_2) &= \sup\{|u(x_2, y_2) - u(x_1, y_1)| : \ (x_1, y_1), (x_2, y_2) \in [1, T] \times [1, b], \\
&\qquad |x_2 - x_1| \leq \epsilon_1, \ |y_2 - y_1| \leq \epsilon_2\}; \\
\omega^T(Y, \epsilon_1, \epsilon_2) &= \sup\{\omega^T(u, \epsilon_1, \epsilon_2) : \ u \in Y\}; \\
\omega_0^T(Y) &= \lim_{\epsilon_1, \epsilon_2 \to 0} \omega^T(Y, \epsilon_1, \epsilon_2); \\
\omega_0(Y) &= \lim_{T \to \infty} \omega_0^T(Y).
\end{aligned}$$

If (t, s) is a fixed element from J, let us denote $Y(t, s) = \{u(t, s); \ u \in Y\}$ and

$$\operatorname{diam} Y(t, s) = \sup\{\|u(t, s) - v(t, s)\| : \ u, v \in Y\}.$$

Finally, consider the function ψ defined on the family $\mathtt{M}_X$ by the formula

$$\psi(Y) = \omega_0(Y) + \lim_{s \to \infty} \sup \operatorname{diam} Y(t, s).$$

It can be shown that the function ψ is a measure of noncompactness in the space BC. The kernel $\ker \psi$ consists of nonempty and bounded sets Y such that functions from Y are locally equicontinuous on J and the thickness of the bundle formed by functions from Y tends to zero at infinity. This property will permit us to characterize solutions of the system (1) considered in the next section.

Theorem 2.11 ([10, 13])
The measure of noncompactness ψ satisfies condition (m).

Let $L^1(J, \mathbb{R})$ be the Banach space of functions $u \colon J \to \mathbb{R}$ that are Lebesgue integrable with the norm

$$\|u\|_{L^1} = \iint_J |u(x, y)| dy dx.$$

Definition 2.12
Let $r_1, r_2 \geq 0$, $\sigma = (1, 1)$ and $r = (r_1, r_2)$. For $u \in L^1(J, \mathbb{R})$, define the Hadamard partial fractional integral of order r by the expression

$$({}^{HS}I_\sigma^r u)(x, y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\ln \frac{x}{t}\right)^{r_1 - 1} \left(\ln \frac{y}{s}\right)^{r_2 - 1} \frac{u(t, s)}{st} ds dt,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

For more details about Hadamard fractional integral see [22].

Let a nondegenerate rectangle $I = [a, b] \times [c, d] \subset \mathbb{R}^2$ be given. We consider a real function $p \colon I \to \mathbb{R}$ defined on I. For a given sub-rectangle $\overline{I} = [a_1, b_1] \times [c_1, d_1] \subset I$, $a \leq a_1 \leq b_1 \leq b$, $c \leq c_1 \leq d_1 \leq d$ we set

$$m_p(\overline{I}) = p(b_1, d_1) - p(b_1, c_1) - p(a_1, d_1) + p(a_1, c_1).$$

Let us de define

$$\bigvee_I(p) = \sup \sum_i |m_p(\overline{I}_i)|,$$

where the supremum is taken over all finite systems of nonoverlapping rectangles $\overline{I}_i \subset I$ (i.e. for the interiors $\overline{I}_i^\circ$ of the rectangles $\overline{I}_i$ we assume that $\overline{I}_i^\circ \cap \overline{I}_j^\circ = \emptyset$, whenever $i \neq j$).

If $p\colon I = [a,b] \times [c,d] \to \mathbb{R}$ and $\gamma \in [c,d]$ (resp. $\alpha \in [a,b]$) is fixed, then we denote the usual variation of the function $p(s,\gamma)$ (resp. $p(\alpha,t)$) in the interval $[a,b]$ (resp. $[c,d]$) by $\bigvee_a^b p(\cdot,\gamma)$ (resp. $\bigvee_c^d p(\alpha,\cdot)$).

Definition 2.13 ([25])
The real function $p\colon I \to \mathbb{R}$ is of bounded variation on I if $\bigvee_I(p) < \infty$.

Lemma 2.14 ([25])
Let $p\colon I = [a,b] \times [c,d] \to \mathbb{R}$ be such that $\bigvee_I(p) < \infty$ and let $\bigvee_a^b p(\cdot,\gamma_0) < \infty$ for some $\gamma_0 \in [c,d]$. Then $\bigvee_a^b p(\cdot,\gamma) < \infty$ for all $\gamma \in [c,d]$ and

$$\bigvee_a^b p(\cdot,\gamma) \leq \bigvee_I(p) + \bigvee_a^b p(\cdot,\gamma_0).$$

Theorem 2.15 ([24])
If $f\colon [a,b] \to \mathbb{R}$ is monotone on $[a,b]$, then f is of bounded variation on $[a,b]$ and

$$\bigvee_a^b(f) = |f(b) - f(a)|.$$

Let f and g be functions defined on the interval $[a,b]$, the Stieltjes integral of f with respect to g is designated by

$$\int_a^b f(x)dg(x).$$

It is clear that the Riemann integral is a special case of the Stieltjes integral, obtained by setting $g(x) = x$. The Stieltjes integral exists under several conditions, we will restrict ourselves to only one theorem in this direction.

Theorem 2.16 ([24, 25])
The integral

$$\int_a^b f(x)dg(x),$$

exists if the function f is continuous on $[a,b]$ and g is of finite variation on $[a,b]$, and we have

$$\left|\int_a^b f(x)dg(x)\right| \leq \sup_{x\in[a,b]} |f(x)| \bigvee_a^b(g).$$

Lemma 2.17 ([24])
If the function f is continuous on $[a, b]$ and if the function g has a Riemann integrable derivative g' at every point of $[a, b]$, then

$$\int_a^b f(x)dg(x) = \int_a^b f(x)g'(x)dx.$$

For more properties of the Stieltjes integral see [23, 24, 25, 26].

We consider the Hadamard-Stieltjes fractional integral of a function $u\colon J \to \mathbb{R}$ of order $r = (r_1, r_2)$ of the form

$$\begin{aligned}&({}^{HS}I^r u)(x, y)\\ &\qquad = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \Big(\ln\frac{x}{t}\Big)^{r_1-1}\Big(\ln\frac{y}{s}\Big)^{r_2-1} \frac{u(t,s)}{st} d_s h_1(y,s) d_t h_2(x,t),\end{aligned}$$

were $h_1\colon [1, b] \times [1, b] \to \mathbb{R}$, $h_2\colon [1, +\infty) \times [1, +\infty) \to \mathbb{R}$, and the symbols d_t, d_s indicate the integration with respect to t, s respectively.

Theorem 2.18
Let D be a nonempty, bounded and closed subset of a Banach space X and μ an arbitrary measure of noncompactness on X. If $F\colon D^n \to D$ is a continuous operator and there exists a constant $k \in [0, 1)$ such that

$$\mu(F(Q_1 \times Q_2 \times \cdots \times Q_n)) \le k \max(\mu(Q_1), \mu(Q_2), \ldots, \mu(Q_n))$$

for any $Q_1, Q_2, \ldots, Q_n$ of D, then the operator F has a multiple fixed point in D, i.e. there exist $x_1^*, \ldots, x_n^* \in D$ such that

$$\begin{cases} F(x_1^*, x_2^*, \ldots, x_n^*) = x_1^*,\\ F(x_2^*, x_3^*, \ldots, x_n^*, x_1^*) = x_2^*,\\ \qquad\vdots \qquad\qquad\qquad \vdots\\ F(x_n^*, x_1^*, \ldots, x_{n-2}^*, x_{n-1}^*) = x_n^*.\end{cases}$$

Proof. see [4].

Let us assume that Ω is a nonempty subset of the space BC and F is an operator on Ω^n with values in BC^n. Consider the following equation

$$(u_1, \ldots, u_n)(x, y) = F(u_1, \ldots, u_n)(x, y), \qquad (x, y) \in J. \tag{2}$$

Definition 2.19
A solution $u^* = (u_1(x, y), \ldots, u_n(x, y))$ of equation (2) is said to be globally attractive if for each solution $v^* = (v_1(x, y), \ldots, v_n(x, y))$ of (2) we have

$$\lim_{x\to\infty} \sup_{1\le i\le n} (u_i(x, y) - v_i(x, y)) = 0.$$

In the case when this limit is uniform, i.e. when for each $\epsilon > 0$ there exists $T > 0$ such that

$$\sup_{1\le i\le n} |u_i(x, y) - v_i(x, y)| < \epsilon, \qquad x \ge T,$$

we will say that solutions of (2) are uniformly globally attractive.

Remark 2.20
This concept of stability in the case of equation $(Fu)(x, y) = u(x, y)$ can be found in [14, 21].

3. Main results

We will give an existence and stability results for the system (1) applying Theorem 2.18 under the following assumptions:

(H_1) The function g is continuous and there exist continuous and bounded functions $p_i\colon J \to \mathbb{R}_+$, $i = 1, \dots, n$ such that

$$|g(x, y, u_1, \dots, u_n) - g(x, y, v_1, \dots, v_n)| \le \frac{1}{n} \sum_{i=1}^{n} \frac{p_i(x, y)|u_i - v_i|}{\max_{1\le i\le n} |u_i + v_i| + 1}$$

for all $u_i, v_i \in \mathbb{R}$, $i = 1, \dots, n$.

(H_2) The operator $G\colon BC \to BC$ is a bounded linear operator with spectral radius $r_\sigma(G) < 1$.

(H_3) The function $s \mapsto h_1(y, s)$ is continuous and of bounded variation on $[1, b]$ for each fixed $y \in [1, b]$, and the function $t \mapsto h_2(x, t)$ is continuous and of bounded variation on $[1, +\infty)$ for each $x \in [1, +\infty)$.

(H_4) There exists a constant $\eta > 0$ such that

$$\sup_{x\ge 1; 1\le y\le b} \left| \int_1^x \int_1^y \left(\ln \frac{x}{s}\right)^{r_1 - 1} [1+|g(t, s, 0, \dots, 0)|] d_s h_1(y, s) d_t h_2(x, t) \right| \le \eta,$$

with

$$\rho = \frac{\eta p^* (\ln b)^{r_2}}{\Gamma(r_1)\Gamma(r_2)} < 1 \quad \text{and} \quad \frac{1 + \rho}{1 - \rho} \|\varphi\| < 1,$$

where $p^* = \sup_{1\le i\le n} \|p_i\|$.

Remark 3.1

(i) In view of the assumption (H_1) we infer that

$$|g(x, y, u_1, \dots, u_n)| \le \frac{1}{n} \sum_{i=1}^{n} \frac{p_i(x, y)|u_i|}{\max_{1\le i\le n} |u_i| + 1} + |g(x, y, 0, \dots, 0)|$$

for all $(x, y) \in J$ and $u_i \in \mathbb{R}$.

(ii) The operator G is a bounded linear operator, i.e. the operator G is linear and there exists a constant $l > 0$ such that for each $u \in BC$,

$$\|Gu\| \le l\|u\|.$$

If spectral radius of G satisfies $r_\sigma(G) < 1$, we conclude that $\sup_{\|u\|\neq 0} \frac{\|Gu\|}{\|u\|} < 1$.

Theorem 3.2
Under the assumptions (H_1)–(H_4) *the system* (1) *has at least one solution* $u^* = (u_1^*, \dots, u_n^*)$. *Moreover, solutions of the system* (1) *are globally attractive.*

Proof. Consider the operator $F\colon \underbrace{BC \times \cdots \times BC}_{n} \to BC$ defined by

$$(F(u_1, \dots, u_n))(x,y) = \varphi(x,y) + (Gu_1)(x,y) \int_1^x \int_1^y \Big(\ln \frac{x}{t}\Big)^{r_1 - 1} \Big(\ln \frac{y}{s}\Big)^{r_2 - 1} \times \frac{g(t,s,u_1(t,s),\dots,u_n(t,s))}{\Gamma(r_1)\Gamma(r_2)st} d_s h_1(y,s) d_t h_2(x,t).$$

Observe that in view of our assumptions, for any function $u = (u_1, \dots, u_n)$ of the space $BC \times \cdots \times BC$ the function $F(u_1, \dots, u_n)$ is continuous on $J \times \cdots \times J$. Next, let us take an arbitrary function $u = (u_1, \dots, u_n) \in BC \times \cdots \times BC$. Using our assumptions, for a fixed $(x,y) \in J$ we have

$$\begin{aligned}
&|(F(u_1, \dots, u_n))(x,y)| \\
&\quad \le |\varphi(x,y)| + |(Gu_1)(x,y)| \int_1^x \int_1^y \Big(\ln \frac{x}{t}\Big)^{r_1 - 1} \Big(\ln \frac{y}{s}\Big)^{r_2 - 1} \\
&\qquad \times \frac{|g(t,s,u_1(t,s),\dots,u_n(t,s))|}{\Gamma(r_1)\Gamma(r_2)st} d_s h_1(y,s) d_t h_2(x,t) \\
&\quad \le |\varphi(x,y)| + \frac{|u_1(x,y)|}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \Big(\ln \frac{x}{t}\Big)^{r_1 - 1} \Big(\ln \frac{y}{s}\Big)^{r_2 - 1} \\
&\qquad \times \left[\frac{1}{n} \sum_{i=1}^n \frac{p_i |u_i(t,s)|}{\max_{1 \le i \le n} |u_i(t,s)| + 1} + |g(t,s,0,\cdots,0)| \right] d_s h_1(y,s) d_t h_2(x,t) \\
&\quad \le \|\varphi\| + \frac{p^* (\ln b)^{r_2}}{\Gamma(r_1)\Gamma(r_2)} \|u\| \int_1^x \int_1^y \Big(\ln \frac{x}{t}\Big)^{r_1 - 1} \\
&\qquad \times [1 + |g(t,s,0,\dots,0)|] d_s h_1(y,s) d_t h_2(x,t),
\end{aligned}$$

hence we obtain

$$\|F(u_1, \dots, u_n)\| \le \|\varphi\| + \frac{\eta p^* (\ln b)^{r_2}}{\Gamma(r_1)\Gamma(r_2)} \|u\|,$$

so the function $F(u_1, \dots, u_n)$ is bounded on $J \times \cdots \times J$. Then $Fu \in BC$.

We take

$$r = \frac{\|\varphi\|}{1 - \rho}.$$

We deduce that the operator F transforms the set $B_r \times \cdots \times B_r$ into the ball B_r.

Further, let $(u_m)=(u_1,\cdots,u_n)_m\subset B_r\times\cdots\times B_r$ such that $\lim\limits_{m\to\infty}u_{i_m}=u_i$ for $i=1,\dots,n$ we get

$$\begin{aligned}
&|(F(u_{1_m},\dots,u_{n_m}))(x,y)-(F(u_1,\dots,u_n))(x,y)|\\
&\qquad=\Bigg|\varphi(x,y)+(Gu_{1_m})(x,y)\int_1^x\int_1^y\Big(\ln\frac{x}{t}\Big)^{r_1-1}\Big(\ln\frac{y}{s}\Big)^{r_2-1}\\
&\qquad\quad\times\frac{|g(t,s,u_{1_m}(t,s),\dots,u_{n_m}(t,s))|}{\Gamma(r_1)\Gamma(r_2)st}d_sh_1(y,s)d_th_2(x,t)\\
&\qquad\quad-\varphi(x,y)+(Gu_1)(x,y)\int_1^x\int_1^y\Big(\ln\frac{x}{t}\Big)^{r_1-1}\Big(\ln\frac{y}{s}\Big)^{r_2-1}\\
&\qquad\quad\times\frac{|g(t,s,u_1(t,s),\dots,u_n(t,s))|}{\Gamma(r_1)\Gamma(r_2)st}d_sh_1(y,s)d_th_2(x,t)\Bigg|\\
&\qquad\le\Bigg|(Gu_{1_m})(x,y)\int_1^x\int_1^y\Big(\ln\frac{x}{t}\Big)^{r_1-1}\Big(\ln\frac{y}{s}\Big)^{r_2-1}\\
&\qquad\quad\times\frac{|g(t,s,u_{1_m}(t,s),\dots,u_{n_m}(t,s))|}{\Gamma(r_1)\Gamma(r_2)st}d_sh_1(y,s)d_th_2(x,t)\\
&\qquad\quad-(Gu_1)(x,y)\int_1^x\int_1^y\Big(\ln\frac{x}{t}\Big)^{r_1-1}\Big(\ln\frac{y}{s}\Big)^{r_2-1}\\
&\qquad\quad\times\frac{|g(t,s,u_1(t,s),\dots,u_n(t,s))|}{\Gamma(r_1)\Gamma(r_2)st}d_sh_1(y,s)d_th_2(x,t)\\
&\qquad\quad+(Gu_1)(x,y)\int_1^x\int_1^y\Big(\ln\frac{x}{t}\Big)^{r_1-1}\Big(\ln\frac{y}{s}\Big)^{r_2-1}\\
&\qquad\quad\times\frac{|g(t,s,u_{1_m}(t,s),\dots,u_{n_m}(t,s))|}{\Gamma(r_1)\Gamma(r_2)st}d_sh_1(y,s)d_th_2(x,t)\\
&\qquad\quad-(Gu_1)(x,y)\int_1^x\int_1^y\Big(\ln\frac{x}{t}\Big)^{r_1-1}\Big(\ln\frac{y}{s}\Big)^{r_2-1}\\
&\qquad\quad\times\frac{|g(t,s,u_{1_m}(t,s),\dots,u_{n_m}(t,s))|}{\Gamma(r_1)\Gamma(r_2)st}d_sh_1(y,s)d_th_2(x,t)\Bigg|\\
&\qquad\le\frac{\eta p^*(\ln b)^{r_2}}{\Gamma(r_1)\Gamma(r_2)}\|Gu_{1_m}-Gu_1\|+\frac{p^*\|Gu_1\|}{\Gamma(r_1)\Gamma(r_2)}\sup_{1\le i\le n}\|u_{i_m}-u_i\|\\
&\qquad\quad\times\int_1^x\int_1^y\Big(\ln\frac{x}{t}\Big)^{r_1-1}\Big(\ln\frac{y}{s}\Big)^{r_2-1}d_sh_1(y,s)d_th_2(x,t)\\
&\qquad\le\frac{\eta p^*(\ln b)^{r_2}}{\Gamma(r_1)\Gamma(r_2)}\sup_{1\le i\le n}\|u_{i_m}-u_i\|+\frac{(\ln b)^{r_2}p^*\|Gu_1\|}{\Gamma(r_1)\Gamma(r_2)}\sup_{1\le i\le n}\|u_{i_m}-u_i\|\\
&\qquad\quad\times\int_1^x\int_1^y\Big(\ln\frac{x}{t}\Big)^{r_1-1}d_sh_1(y,s)d_th_2(x,t).
\end{aligned}$$

Since $\lim_{m\to\infty} \sup_{1\le i\le n} \|u_{i_m} - u_i\| = 0$, we obtain

$$\lim_{m\to\infty} \|F(u_{1_m}, \dots, u_{n_m}) - F(u_1, \dots, u_n)\| = 0.$$

Then the operator F is continuous on $B_r \times \cdots \times B_r$.

Remark 3.3

By the hypothesis (H_4) we notice that the quantity

$$\sup_{(x,y)\in J} \left| \int_1^x \int_1^y \left(\ln \frac{x}{t} \right)^{r_1-1} d_s h_1(y,s) d_t h_2(x,t) \right|$$

is finite.

Now, we consider the operator T defined on $BC \times \cdots \times BC$ by

$$\begin{aligned} &T(u_1, \dots, u_n)(x,y) \\ &\quad = \int_1^x \int_1^y \left(\ln \frac{x}{t} \right)^{r_1-1} \left(\ln \frac{y}{s} \right)^{r_2-1} \frac{g(t,s,u_1(t,s),\dots,u_n(t,s))}{\Gamma(r_1)\Gamma(r_2)st} d_s h_1(y,s) d_t h_2(x,t). \end{aligned}$$

For an arbitrarily function $u = (u_1, \dots, u_n)$ of the space $BC \times \cdots \times BC$ and a fixed (x,y) in J, using our assumptions, we get

$$\begin{aligned} &|T(u_1, \dots, u_n)(x,y)| \\ &\quad = \Bigg| \int_1^x \int_1^y \left(\ln \frac{x}{t} \right)^{r_1-1} \left(\ln \frac{y}{s} \right)^{r_2-1} \frac{g(t,s,u_1(t,s),\dots,u_n(t,s))}{\Gamma(r_1)\Gamma(r_2)st} \\ &\qquad \times d_s h_1(y,s) d_t h_2(x,t) \Bigg| \\ &\quad \le \int_1^x \int_1^y \left(\ln \frac{x}{t} \right)^{r_1-1} \left(\ln \frac{y}{s} \right)^{r_2-1} \Bigg[\frac{1}{n} \sum_{i=1}^n \frac{p_i |u_i|(t,s)}{\max_{1\le i\le n} |u_i(t,s)| + 1} \\ &\qquad + |g(t,s,0,\cdots,0)| \Bigg] \frac{d_s h_1(y,s) d_t h_2(x,t)}{\Gamma(r_1)\Gamma(r_2)} \\ &\quad \le \frac{\eta (\ln b)^{r_2} p^*}{\Gamma(r_1)\Gamma(r_2)} \sup_{1\le i\le n} |u_i(t,s)|, \end{aligned}$$

thus

$$\|T(u_1, \dots, u_n)\| \le \rho \sup_{1\le i\le n} \|u_i\|.$$

Clearly the operator T transforms the set $B_r \times \cdots \times B_r$ into the ball B_r and

$$F(B_r \times \cdots \times B_r) \subseteq \varphi + G(B_r) \cdot T(B_r \times \cdots \times B_r).$$

Next, we take a nonempty $Y \subset B_r$, using Definition 2.3 and applying Theorem 2.11, we obtain

$$\begin{aligned}\psi(F(Y \times \cdots \times Y)) &\le \psi(G(Y) \cdot T(Y \times \cdots \times Y)) \\ &\le \|G(Y)\|\psi(T(Y \times \cdots \times Y)) + \|T(Y \times \cdots \times Y)\|\psi(G(Y)) \\ &\le \frac{\|\varphi\|}{1-\rho}\psi(Y) + \rho\frac{\|\varphi\|}{1-\rho}\psi(Y),\end{aligned}$$

hence

$$\psi(F(Y \times \cdots \times Y)) \le \frac{1+\rho}{1-\rho}\|\varphi\|\psi(Y).$$

Finally, in view of Theorem 2.18 we deduce that F has at least one multiple fixed point in B_r which is a solution of the system (1). Moreover, taking into account the fact that the set $\operatorname{Fix} F \in \ker\psi$ and the characterization of sets belonging to $\ker\psi$ (Remark 2.6) we conclude that all solutions of the system (1) are globally attractive in the sense of Definition 2.19.

4. Example

We consider the following system of integral equations

$$\begin{cases} u_1(x,y) = \dfrac{1}{5s+t} + e^{-2y-x}u_1(x,y)\displaystyle\int_1^x\int_1^y \left(\ln\frac{x}{t}\right)^{r_1-1}\left(\ln\frac{y}{s}\right)^{r_2-1} \\ \qquad\times \dfrac{n^{-1}e^{-3st}}{(\max_{1\le i\le n}|u_i(t,s)|+1)\Gamma(r_1)\Gamma(r_2)st}d_s(\arctan(ys))d_t\Big(\dfrac{1}{tx}\Big), \\ u_2(x,y) = \dfrac{1}{5s+t} + e^{-2y-x}u_2(x,y)\displaystyle\int_1^x\int_1^y \left(\ln\frac{x}{t}\right)^{r_1-1}\left(\ln\frac{y}{s}\right)^{r_2-1} \\ \qquad\times \dfrac{n^{-1}e^{-3st}}{(\max_{1\le i\le n}|u_i(t,s)|+1)\Gamma(r_1)\Gamma(r_2)st}d_s(\arctan(ys))d_t\Big(\dfrac{1}{tx}\Big), \\ \quad\vdots \qquad\qquad\qquad \vdots \\ u_n(x,y) = \dfrac{1}{5s+t} + e^{-2y-x}u_n(x,y)\displaystyle\int_1^x\int_1^y \left(\ln\frac{x}{t}\right)^{r_1-1}\left(\ln\frac{y}{s}\right)^{r_2-1} \\ \qquad\times \dfrac{n^{-1}e^{-3st}}{(\max_{1\le i\le n}|u_i(t,s)|+1)\Gamma(r_1)\Gamma(r_2)st}d_s(\arctan(ys))d_t\Big(\dfrac{1}{tx}\Big), \end{cases} \tag{3}$$

where $(x,y) \in J = [1,+\infty) \times [1,\pi]$, and $(r_1,r_2) = \left(\frac{3}{2},\frac{3}{2}\right)$. Set

$$\varphi(x,y) = \frac{1}{5s+t}, \quad \|\varphi\| = \frac{1}{6}, \quad h_1(y,t) = \arctan(ys), \quad h_2(x,s) = \frac{1}{tx},$$
$$(Gu_i)(x,y) = e^{-2y-x}u_i(x,y)$$

and

$$g(t,s,u_1(t,s),\dots,u_n(t,s)) = \frac{n^{-1}e^{-3st}}{\max_{1\le i\le n}|u_i(t,s)|+1}, \qquad (t,s) \in J.$$

It is clear that the system (3) can be written as the system (1). Let us show that conditions (H_1)–(H_4) hold. The function $s \mapsto \arctan(ys)$ is continuous and nondecreasing for each fixed $y_0 \in [1,\pi]$, so it is of bounded variation on $[1,\pi]\times[1,\pi]$, and

$$\bigvee_{s=1}^{s=\pi} h_2(y,s) \le \arctan(\pi^2) - \arctan(1).$$

The function $t \mapsto \frac{1}{tx}$ is continuous and decreasing for each fixed $x_0 \in [1,+\infty)$, and

$$\lim_{t\to+\infty} \frac{1}{tx} = 0,$$

so it is of bounded variation on $[1,+\infty)\times[1,+\infty)$, and for each fixed $x_0 \in [1,+\infty)$

$$d_t h_1(x_0,t) = -\frac{1}{x_0 t^2} dt.$$

We have also

$$\begin{aligned}
&|g(t,s,u_1,\dots,u_{n)} - g(t,s,v_1,\dots,v_{n)}| \\
&\quad = \Big|\frac{n^{-1}e^{-3st}}{\max_{1\le i\le n}|u_i|+1} - \frac{n^{-1}e^{-3st}}{\max_{1\le i\le n}|v_i|+1}\Big| \\
&\quad = \Big|\frac{n^{-1}e^{-3st}(\max_{1\le i\le n}|u_i| - \max_{1\le i\le n}|v_i|)}{(\max_{1\le i\le n}|u_i|+1)(\max_{1\le i\le n}|v_i|+1)}\Big| \\
&\quad \le \frac{n^{-1}e^{-3st}\max_{1\le i\le n}|u_i-v_i|}{\max_{1\le i\le n}|u_i+v_i|+1} \\
&\quad \le \frac{n^{-1}e^{-3st}\sum_{i=1}^n|u_i-v_i|}{\max_{1\le i\le n}|u_i+v_i|+1} \\
&\quad \le \frac{1}{n}\sum_{i=1}^n \frac{e^{-3st}|u_i-v_i|}{\max_{1\le i\le n}|u_i+v_i|+1},
\end{aligned}$$

so for all $i = 1,\dots,n$ we have $p_i(t,s) = e^{-3st}$ and $\sup\limits_{1\le i\le n}\|p_i\| = e^{-3}$.

Obviously the operator G is linear and for each u in BC, we get

$$\sup_{\|u\|\neq 0} \frac{\|Gu\|}{\|u\|} \le e^{-3}.$$

Next, for a fixed (x,y) in J, we obtain

$$\begin{aligned}
&\Big|\int_1^x\int_1^y \Big(\ln\frac{x}{t}\Big)^{r_1-1}[1+|g(t,s,0,\dots,0)|]d_s h_1(y,s)d_t h_2(x,t)\Big| \\
&\quad = \Big|\int_1^x\int_1^y \Big(\ln\frac{x}{t}\Big)^{\frac{3}{2}-1} n^{-1}e^{-3st} d_s(\arctan(ys))d_t\Big(\frac{1}{tx}\Big)\Big| \\
&\quad + \Big|\int_1^x\int_1^y \Big(\ln\frac{x}{t}\Big)^{\frac{3}{2}-1} d_s(\arctan(ys))d_t\Big(\frac{1}{tx}\Big)\Big|
\end{aligned}$$

$$\leq \left| \ln x \int_1^x n^{-1} e^{-3t} \Big(\int_1^y d_s(\arctan(ys)) \Big) d_t \Big(\frac{1}{tx} \Big) \right|$$

$$\leq \ln x[\arctan(\pi^2) - \arctan(1)] \left| \int_1^x -\frac{n^{-1}e^{-3t}}{xt^2} dt \right|$$

$$\leq \ln x[\arctan(\pi^2) - \arctan(1)] \left| \int_1^x \frac{n^{-1}e^{-3t}}{x} dt \right|$$

$$\leq [\arctan(\pi^2) - \arctan(1)] \left| \frac{-\ln x}{3xe^{3x}} + \frac{\ln x}{3e^3 x} \right|,$$

since

$$\lim_{x\to\infty} \left| \frac{-\ln x}{3xe^{3x}} + \frac{\ln x}{3e^3 x} \right| = 0,$$

then the function $q(x) = \left| \frac{-\ln x}{3xe^{3x}} + \frac{\ln x}{3e^3 x} \right|$ is bounded, i.e. there exists $\alpha > 0$ such that $\sup_{x\geq 1} q(x) \leq \alpha$. We take

$$\eta = \alpha[\arctan(\pi^2) - \arctan(1)].$$

It follows that

$$\rho = \frac{\eta p^* (\ln b)^{r_2}}{\Gamma(r_1)\Gamma(r_2)} = \frac{\eta(\ln \pi)^{3/2}}{e^3 \Gamma(3/2)\Gamma(3/2)} < 1,$$

and

$$\frac{1+\rho}{1-\rho}\|\varphi\| = \frac{1+\rho}{6(1-\rho)} < 1.$$

Consequently from Theorem 3.2 the system (3) has at least solution in BC and solutions of the system (3) are globally attractive.

References

[1] Abbas, Saïd et al. "Existence and attractivity results for coupled systems of nonlinear Volterra-Stieltjes multidelay fractional partial integral equations." *Abstr. Appl. Anal.* (2018): Art. ID 8735614. Cited on 204.

[2] Abbas, Saïd, and Mouffak Benchohra, and Juan J. Nieto Roig. "Global attractivity of solutions for nonlinear fractional order Riemann-Liouville Volterra-Stieltjes partial integral equations." *Electron. J. Qual. Theory Differ. Equ.* (2012): Article no. 81. Cited on 204.

[3] Abbas, Saïd et al. "Existence and stability results for nonlinear fractional order Riemann-Liouville Volterra-Stieltjes quadratic integral equations." *Appl. Math. Comput.* 247 (2014): 319-328. Cited on 203.

[4] Aghajani, Asadollah, and Ali Shole Haghighi. "Existence of solutions for a system of integral equations via measure of noncompactness." *Novi Sad J. Math.* 44, no. 1 (2014): 59-73. Cited on 204, 206 and 209.

[5] Aghajani, Asadollah, Reza Allahyari, and Mohammad Mursaleen. "A generalization of Darbo's theorem with application to the solvability of systems of integral equations." *J. Comput. Appl. Math.* 260 (2014): 68-77. Cited on 204 and 206.

[6] Akhmerov, R.R. et al. *Measures of Noncompactness and Condensing Operators* Vol. 55 of *Operator Theory: Advances and Applications.* Basel: Birkhäuser Verlag, 1992. Cited on 204, 205 and 206.

[7] Baghdad, Said, and Mouffak Benchohra. "Global existence and stability results for Hadamard-Volterra-Stieltjes integral equations." *Commun. Fac. Sci. Univ. Ank. Ser. A1. Math. Stat.* 68, no. 2 (2019): 1387-1400. Cited on 204.

[8] Banaś, Józef. "Existence results for Volterra-Stieltjes quadratic integral equations on an unbounded interval." *Math. Scand.* 98, no. 1 (2006): 143-160. Cited on 204.

[9] Banaś, Józefet et al. "Monotonic solutions of a class of quadratic integral equations of Volterra type." *Comput. Math. Appl.* 49, no. 5-6 (2005): 943-952. Cited on 203, 204 and 206.

[10] Banaś, Józef, and Szymon Dudek. "The technique of measures of noncompactness in Banach algebras and its applications to integral equations." *Abstr. Appl. Anal.* (2013): Art. ID 537897. Cited on 204, 205 and 207.

[11] Banaś, Józef, and Kazimierz Goebel. *Measures of noncompactness in Banach spaces* vol. 60 of *Lecture Notes in Pure and Applied Mathematics.* New York: Marcel Dekker, Inc., 1980. Cited on 204, 205 and 206.

[12] Banaś, Józef, Millenia Lecko, and Wagdy Gomaa El-Sayed. "Existence theorems for some quadratic integral equations." *J. Math. Anal. Appl.* 222, no. 1 (1998): 276-285. Cited on 203 and 204.

[13] Banaś, Józef, and Leszek Olszowy. "On a class of measures of noncompactness in Banach algebras and their application to nonlinear integral equations." *Z. Anal. Anwend.* 28, no. 4 (2009): 475-498. Cited on 204, 205 and 207.

[14] Banaś, Józef, and Donal O'Regan. "On existence and local attractivity of solutions of a quadratic Volterra integral equation of fractional order." *J. Math. Anal. Appl.* 345, no. 1 (2008): 573-582. Cited on 203, 204 and 210.

[15] Banaś, Józef, Méndez, and Kishin B. Sadarangani. "On a class of Urysohn-Stieltjes quadratic integral equations and their applications." *J. Comput. Appl. Math.* 113, no. 1-2 (2000): 35-50. Cited on 203 and 204.

[16] Banaś, Józef, and Beata Rzepka. "Monotonic solutions of a quadratic integral equation of fractional order." *J. Math. Anal. Appl.* 332, no. 2 (2007): 1371-1379. Cited on 203 and 204.

[17] Banaś, Józef, and Kishin B. Sadarangani. "Solvability of Volterra-Stieltjes operator-integral equations and their applications." *Comput. Math. Appl.* 41, no. 12 (2001): 1535-1544. Cited on 204.

[18] Chandrasekhar, Subrahmanyan. *Radiative transfer.* New York: Dover Publications, 1960. Cited on 203.

[19] Corduneanu, Constantin. *Integral equations and stability of feedback systems.* Vol. 104 of *Mathematics in Science and Engineering.* New York, London: Academic Press, 1973. Cited on 203.

[20] Dhage, Bapurao Chandrabhan. "A coupled hybrid fixed point theorem involving the sum of two coupled operators in a partially ordered Banach space with applications." *Differ. Equ. Appl.* 9, no. 4 (2017): 453-477. Cited on 203, 204 and 206.

[21] Dhage, Bapurao Chandrabhan and Sidheshwar Sangram Bellale. "Local asymptotic stability for nonlinear quadratic functional integral equations." *Electron. J. Qual. Theory Differ. Equ.* (2008): Art. no. 10. Cited on 203, 204 and 210.

[22] Kilbas, Anatoly Aleksandrovich, Hari Mohan Srivastava, and Juan J. Trujillo. *Theory and applications of fractional differential equations* Vol. 204 of *North-Holland Mathematics Studies*. Amsterdam: Elsevier Science B.V., 2006. Cited on 207.

[23] Lebesgue, Henri Leon. *Leçons sur l'intégration et la recherche des fonctions primitives professées au Collège de France. Cambridge Library Collection.* Cambridge: Cambridge University Press, 2009. Reprint of the 1904 original. Cited on 209.

[24] Isidor Pavlovič Natanson. *Theory of functions of a real variable* Vol. 85 of *North-Holland Mathematics Studies*. Berlin: Akademie-Verlag, 1981. Cited on 208 and 209.

[25] Schwabik, Štefan, Milan Tvrdý and Otto Vejvoda. *Differential and integral equations.* Dordrecht, Boston, London: D. Reidel Publishing Co., 1979. Cited on 203, 208 and 209.

[26] Sikorski, Roman. *Funkcje rzeczywiste*. Warszawa: Państwowe Wydawnictwo Naukowe, 1958. Cited on 209.

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