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A SOUND INTERPRETATION OF LEŚNIEWSKI'S EPSILON IN MODAL LOGIC KTB

Abstract

In this paper, we shall show that the following translation I^M from the propositional fragment $\mathbf{L_1}$ of Leśniewski's ontology to modal logic **KTB** is sound: for any formula ϕ and ψ of $\mathbf{L_1}$, it is defined as

(M1) $I^M(\phi \vee \psi) = I^M(\phi) \vee I^M(\psi)$,
(M2) $I^M(\neg\phi) = \neg I^M(\phi)$,
(M3) $I^M(\epsilon ab) = \Diamond p_a \supset p_a . \wedge . \Box p_a \supset \Box p_b . \wedge . \Diamond p_b \supset p_a$,

where p_a and p_b are propositional variables corresponding to the name variables a and b, respectively. In the last section, we shall give some comments including some open problems and my conjectures.

Keywords: Leśniewski's ontology, propositional ontology, translation, interpretation, modal logic, KTB, soundness, Grzegorczyk's modal logic.

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1. Introduction and I^M

Inoué [9] initiated a study of interpretations of Leśniewski's epsion ϵ in the modal logic **K** and its certain extensions. That is, Ishimoto's propositional fragment $\mathbf{L_1}$ (Ishimoto [12]) of Leśniewski's ontology **L** (refer to Urbaniak [19]) is partially embedded in **K** and in the extensions, respectively, by the following translation I from $\mathbf{L_1}$ to them: for any formula ϕ and ψ of $\mathbf{L_1}$, it is defined as

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(I1) $I(\phi \vee \psi) = I(\phi) \vee I(\psi)$,

(I2) $I(\neg\phi) = \neg I(\phi)$,

(I3) $I(\epsilon ab) = p_a \wedge \Box(p_a \equiv p_b)$,

where p_a and p_b are propositional variables corresponding to the name variables a and b, respectively. Here, "$\mathbf{L_1}$ is partially embedded in $\mathbf{K}$ by I" means that for any formula ϕ of a certain decidable nonempty set of formulas of $\mathbf{L_1}$ (i.e. decent formulas (see § 3 of Inoué [10])), ϕ is a theorem of $\mathbf{L_1}$ if and only if $I(\phi)$ is a theorem of $\mathbf{K}$. Note that I is sound. The paper [10] also proposed similar partial interpretations of Leśniewski's epsilon in certain von Wright-type deontic logics, that is, ten Smiley-Hanson systems of monadic deontic logic and in provability logic **GL**, respectively. (See Åqvist [1] and Boolos [3] for those logics.)

The interpretation I is however not faithful. A counterexample for the faithfulness is, for example, $\epsilon ac \wedge \epsilon bc. \supset .\epsilon ab \vee \epsilon cc$ (for the details, see [10]). Blass [2] gave a modification of the interpretation and showed that his interpretation T is faithful, using Kripke models. Inoué [11] called the translation *Blass translation* (for short, *B-translation*) or *Blass interpretation* (for short, *B-interpretation*). The translation B from $\mathbf{L_1}$ to $\mathbf{K}$ is defined as follows: for any formula ϕ and ψ of $\mathbf{L_1}$,

(B1) $B(\phi \vee \psi) = B(\phi) \vee B(\psi)$,

(B2) $B(\neg\phi) = \neg B(\phi)$,

(B3) $B(\epsilon ab) = p_a \wedge \Box(p_a \supset p_b) \wedge .p_b \supset \Box(p_b \supset p_a)$,

where p_a and p_b are propositional variables corresponding to the name variables a and b, respectively. Inoué [11] extended Blass's faithfulness result for many normal modal logics, provability logic and von Wright-type deontic logics including **K4**, **KD**, **KB**, **KD4**, etc, **GL** and ten Smiley-Hanson systems of monadic deontic logic, using model constructions based on Hintikka formula (cf. Kobayashi and Ishimoto [13]).

In this paper, we first propose a translation I^M from $\mathbf{L_1}$ in modal logic **KTB**, which will be specified in § 2.

DEFINITION 1.1. A translation I^M of Leśniewski's propositional ontology $\mathbf{L_1}$ in modal logic **KTB** is defined as follows: for any formula ϕ and ψ of $\mathbf{L_1}$,

(M1) $I^M(\phi \vee \psi) = I^M(\phi) \vee I^M(\psi)$,
(M2) $I^M(\neg\phi) = \neg I^M(\phi)$,
(M3) $I^M(\epsilon ab) = \Diamond p_a \supset p_a. \wedge .\Box p_a \supset \Box p_b. \wedge .\Diamond p_b \supset p_a$,

where p_a and p_b are propositional variables corresponding to the name variables a and b, respectively.

We call I^M to be *M-translation* or *M-interpretation*.

In the following § 2, we shall collect the basic preliminaries for this paper. In § 3, using proof theory, we shall show that I^M is sound, as the main theorem of this paper. In § 4, we shall give some comments including some open problems and my conjectures.

2. Propositional ontology $\mathbf{L_1}$ and modal logic KTB

Let us recall a formulation of $\mathbf{L_1}$, which was introduced in [12]. The Hilbert-style system of it, denoted again by $\mathbf{L_1}$, consists of the following axiom-schemata with a formulation of classical propositional logic **CP** as its axiomatic basis:

(Ax1) $\epsilon ab \supset \epsilon aa$,

(Ax2) $\epsilon ab \wedge \epsilon bc. \supset \epsilon ac$,

(Ax3) $\epsilon ab \wedge \epsilon bc. \supset \epsilon ba$,

where we note that every atomic formula of $\mathbf{L_1}$ is of the form ϵab for some name variables a and b and a possible intuitive interpretation of ϵab is 'the a is b'. We note that (Ax1), (Ax2) and (Ax3) are theorems of Leśniewski's ontology (see Słupecki [17]).

The modal logic **K** is the smallest logic which contains all instances of classical tautology and all formulas of the forms $\Box(\phi \supset \psi) \supset .\Box\phi \supset \Box\psi$ being closed under modus ponens and the rule of necessitation (for **K** and basics for modal logic, see Bull and Segerberg [4], Chagrov and Zakharyaschev [5], Fitting [6], Hughes and Cresswell [8] and so on).

We recall the naming of modal logics as follows (refer to e.g. Poggiolesi [15] and Ono [14], also see Bull and Segerberg [4]):

KT: **K** + $\Box\phi \supset \phi$ (**T**, reflexive relation)
KB: **K** + $\phi \supset \Box\Diamond\phi$ (**B**, symmetric relation)
KTB: **KT** + **B** (reflexive and symmetric relation).

3. The soundness of I^M

THEOREM 3.1. (Soundness) *For any formula ϕ of $\mathbf{L_1}$, we have*

$$\vdash_{\mathbf{L_1}} \phi \ \Rightarrow \vdash_{\mathbf{KTB}} I^M(\phi).$$

PROOF: Let ϕ be a formula of $\mathbf{L_1}$. We shall prove the meta-implication by induction on derivation.
BASIS.

(Case 1) We shall first treat the case for (Ax1). Let a and b be name variables. Then we have the following inferences in **KTB**:

(∗) $I^M(\epsilon ab)$ (Assumption)

(1.1) $\Diamond p_a \supset p_a$ from (∗) and Definition 1.1) †

(1.2) $\Box p_a \supset \Box p_a$ (true in **K**) †

(1.3) $\Diamond p_a \supset p_a. \wedge .\Box p_a \supset \Box p_a. \wedge .\Diamond p_a \supset p_a$ (from (1.1) and (1.2))

(1.4) $I^M(\epsilon aa)$ (from (1.3) and Definition 1.1)

(1.5) $I^M(\epsilon ab \supset \epsilon aa)$ (from (∗), (1.4) and Definition 1.1).

(Case 2) Next we shall deal with the case of (Ax2). Let a, b and c be name variables. Then we have the following inferences in **KTB**:

(∗∗) $I^M(\epsilon ab \wedge \epsilon bc)$ (Assumption)

(2.1) $I^M(\epsilon ab)$ (from (∗∗) and Definition 1.1)

(2.2) $I^M(\epsilon bc)$ (from (∗∗) and Definition 1.1)

(2.3) $\Diamond p_a \supset p_a. \wedge .\Box p_a \supset \Box p_b. \wedge .\Diamond p_b \supset p_a$ (from (2.1) and Def 1.1)

(2.4) $\Diamond p_b \supset p_b. \wedge .\Box p_b \supset \Box p_c. \wedge .\Diamond p_c \supset p_b$ (from (2.2) and Def 1.1)

(2.5) $\Diamond p_a \supset p_a$ (from (2.3)) †

(2.6) $\Box p_a \supset \Box p_b$ (from (2.3))

(2.7) $\Box p_b \supset \Box p_c$ (from (2.4))

(2.8) $\Box p_a \supset \Box p_c$ (from (2.6) and (2.7)) †

(2.9) $\Diamond p_b \supset p_a$ (from (2.3))

(2.10) $\Box(\Diamond p_b \supset p_a)$ (from (2.9) and the rule of necessitation)

(2.11) $\Box\Diamond p_b \supset \Box p_a$ (from (2.10) with a true inference in **K**)

(2.12) $\Box p_a \supset p_a$ (true in **KT**)

(2.13) $\Box\Diamond p_b \supset p_a$ (from (2.11) and (2.12))

(2.14) $p_b \supset \Box\Diamond p_b$ (true in **KB**)
(2.15) $\Diamond p_c \supset p_b$ (from (2.4))
(2.16) $\Diamond p_c \supset p_a$ (from (2.13) and (2.14) and (2.15)) †
(2.17) $\Diamond p_a \supset p_a.\wedge.\Box p_a \supset \Box p_c.\wedge.\Diamond p_c \supset p_a$ (from (2.5), (2.8) and (2.16))
(2.18) $I^M(\epsilon ac)$ (from (2.17) and Definition 1.1)
(2.19) $I^M(\epsilon ab \wedge \epsilon bc. \supset \epsilon ac)$ (from (**), (2.18) and Definition 1.1).

(Case 3) Lastly we shall proceed to the case of (Ax3). Let a, b and c be name variables. Then we also have the following inferences in **KTB**:

(***) $I^M(\epsilon ab \wedge \epsilon bc)$ (Assumption)
(3.1) $I^M(\epsilon ab)$ (from (***) and Definition 1.1)
(3.2) $I^M(\epsilon bc)$ (from (***) and Definition 1.1)
(3.3) $\Diamond p_a \supset p_a. \wedge .\Box p_a \supset \Box p_b. \wedge .\Diamond p_b \supset p_a$ (from (3.1) and Def 1.1)
(3.4) $\Diamond p_b \supset p_b. \wedge .\Box p_b \supset \Box p_c. \wedge .\Diamond p_c \supset p_b$ (from (3.2) and Def 1.1)
(3.5) $\Diamond p_b \supset p_b$ (from (3.4)) †
(3.6) $\Diamond p_b \supset p_a$ (from (3.3))
(3.7) $\Box(\Diamond p_b \supset p_a)$ (from (3.6) and the rule of necessitation)
(3.8) $\Box\Diamond p_b \supset \Box p_a$ (from (3.7) with a true inference in **K**)
(3.9) $p_b \supset \Box\Diamond p_b$ (true in **KB**)
(3.10) $\Box p_b \supset p_b$ (true in **KT**)
(3.11) $\Box p_b \supset \Box p_a$ (from (3.8) and (3.9) and (3.10)) †
(3.12) $\Diamond p_a \supset p_a$ (from (3.3))
(3.13) $p_a \supset \Box\Diamond p_a$ (true in **KB**)
(3.14) $\Diamond p_a \supset \Box\Diamond p_a$ (from (3.12) and (3.13))
(3.15) $\Box(\Diamond p_a \supset p_a)$ (from (3.12) and the rule of necessitation)
(3.16) $\Box\Diamond p_a \supset \Box p_a$ (from (3.15) with a true inference in **K**)
(3.17) $\Diamond p_a \supset \Box p_a$ (from (3.14) and (3.16))
(3.18) $\Box p_a \supset \Box p_b$ (from (3.3))
(3.19) $\Diamond p_a \supset \Box p_b$ (from (3.17) and (3.18))
(3.20) $\Box p_b \supset p_b$ (true in **KT**)
(3.21) $\Diamond p_a \supset p_b$ (from (3.19) and (3.20)) †

(3.22) $\Diamond p_b \supset p_b. \wedge .\Box p_b \supset \Box p_a. \wedge .\Diamond p_a \supset p_b$

(from (3.5), (3.11) and (3.21))

(3.23) $I^M(\epsilon ba)$ (from (3.22) and Definition 1.1)

(3.24) $I^M(\epsilon ab \wedge \epsilon bc. \supset \epsilon ba)$ (from $(***)$, (3.23) and Definition 1.1).

Induction Steps. The induction step is easily dealt with. Suppose that ϕ and $\phi \supset \psi$ are theorems of $\mathbf{L_1}$. By induction hypthesis, $I^M(\phi)$ and $I^M(\phi \supset \psi)$ ($\leftrightarrow I^M(\phi) \supset I^M(\psi)$) are theorems of **KTB**. By modus ponens, we obtain $\vdash_{\mathbf{KTB}} I^M(\psi)$. Thus this completes the proof the theorem. □

4. Comments

One motive from which I wrote [9] and [10] is that I wished to understand Leśniewski's epsilon ϵ on the basis of my recognition that Leśniewski's epsilon would be a variant of truth-functional equivalence $\equiv$. Namely, my original approach to the interpretation of ϵ was to express the deflection of ϵ from $\equiv$ in terms of Kripke models. Another (hidden) motive of mine for I^M is to interpret $\mathbf{L_1}$ in intuitionistic logic and bi-modal logic. It is well-known that Leśniewski's epsilon can be interpreted by the Russellian-type definite description in classical first-order predicate logic with equality (see [12]). Takano [18] proposed a natural set-theoretic interpretation for the epsilon. To repeat, I do not deny the interpretation using the Russellian-type definite description and a set-theoretic one. I wish to obtain another interpretation of Leśniewski's epsilon having a more propositional character. We have the following direct open problems.

Open problem 1: Is I^M faithful?

Open problem 2: Find the set of other translations and modal logics in which $\mathbf{L_1}$ is embedded. I think that there seems to be many possibilities.

Open problem 3: Can $\mathbf{L_1}$ be embedded in **S4.2**? (See e.g. Hamkins and Löwe [7].)

Open problem 4: Can $\mathbf{L_1}$ be embedded in Grzegorczyk's modal Logic? (See e.g. Savateev and Shamkanov [16])

My conjectures are the following.

Conjecture 4.1. I^M is faithful.

CONJECTURE 4.2. t seems that $\mathbf{L_1}$ cannot be embedded in intuitionistic propositional logic.

CONJECTURE 4.3. It seems that $\mathbf{L_1}$ can well be embedded in intuitionistic modal propositional logic.

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