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NILPOTENT MINIMUM LOGIC NM AND PRETABULARITY

Abstract

This paper deals with pretabularity of fuzzy logics. For this, we first introduce two systems NM^{nfp} and $\mathrm{NM}^{\overline{\frac{1}{2}}}$, which are expansions of the fuzzy system NM (Nilpotent minimum logic), and examine the relationships between NM^{nfp} and the another known extended system NM^{-}. Next, we show that NM^{nfp} and $\mathrm{NM}^{\overline{\frac{1}{2}}}$ are pretabular, whereas NM is not. We also discuss their algebraic completeness.

Keywords: Pretabularity, nilpotent minimum logic, algebraic semantics, fuzzy logic, finite model property.

1. Fuzzy logic and pretabularity

This paper is a contribution to the study of pretabularity of fuzzy logics. In general, a logic L is said to be pretabular if it does not itself have a finite characteristic matrix (algebra, or frame), but every normal extension of it does (see [4, 7, 8, 11, 13]). Note that Dunn (and Meyer) [3, 5] investigated the pretabularity of the semi-relevance logic RM (R with mingle) and the Dummett-Gödel logic G. One interesting fact is that these systems can be also regarded as *fuzzy* logics.[1] Then, a natural question is now raised as follows.

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[1] According to Cintula (and Běhounek) [1, 2], a (weakly implicative) logic L is said to be *fuzzy* if it is complete with respect to (w.r.t.) linearly ordered matrices (or algebras) and *core fuzzy* if it is complete w.r.t. *standard* algebras (i.e., algebras on the real unit interval $[0, 1]$).

Which fuzzy logics are pretabular?

This question, on the one hand, is not interesting in the sense that most basic fuzzy logics such as UL (Uninorm logic), MTL (Monoidal t-norm logic), and BL (Basic fuzzy logic) are not pretabular because such logics have some axiomatic extensions (henceforth, extensions for short) without finite characteristic matrices. On the other hand, it is interesting in that while, since then, no further pretabular fuzzy logics have been introduced, we can still introduce other concrete fuzzy logic systems.

We introduce two new pretabular systems as fuzzy logics, which we shall call the fixed-pointed nilpotent minimum logic $\mathrm{NM}^{\overline{\frac{1}{2}}}$ and the non-fixed-pointed nilpotent minimum logic NM^{nfp}. These two are the systems expanding and extending, respectively, the well-known fuzzy system NM (Nilpotent minimum logic) [6].[2] In particular, the system NM^{nfp} can be regarded as a Hilbert-style presentation of NM^{-} (the NM with (BP) below), which is one of the extensions of NM introduced in [9, 10]. For this purpose, we first introduce these two systems and examine the relationship between NM^{nfp} and NM^{-}. We then show that NM^{nfp} and $\mathrm{NM}^{\overline{\frac{1}{2}}}$ are pretabular while NM is not. We also discuss their algebraic completeness.

2. Nilpotent minimum logics

The nilpotent minimum logic NM can be based on a countable propositional language with formulas Fm built inductively as usual from a set of propositional variables VAR, binary connectives $\rightarrow, \&, \wedge$, and constant $\mathbf{F}$, with defined connectives: (df1) $\neg A := A \rightarrow \mathbf{F}$; (df2) $A \vee B := ((A \rightarrow B) \rightarrow B) \wedge ((B \rightarrow A) \rightarrow A)$; (df3) $A \leftrightarrow B := (A \rightarrow B) \wedge (B \rightarrow A)$.

The constant $\mathbf{T}$ is defined as $\mathbf{F} \rightarrow \mathbf{F}$. For the rest of this paper, we use the customary notations and terminology, and the axiom systems to provide a consequence relation.

We start with the following axiomatizations of NM and its two expansions.

Definition 1.

(i) ([6]) NM consists of the following axiom schemes and rules:
A1. $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$;

[2]For the definitions of expansion and extension, see Definition 9 in [2].

A2. $(A\&B) \rightarrow A$;
A3. $(A\&B) \rightarrow (B\&A)$;
A4. $(A \wedge B) \rightarrow A$;
A5. $(A \wedge B) \rightarrow (B \wedge A)$;
A6. $(A\&(A \rightarrow B)) \rightarrow (A \wedge B)$;
A7. $(A \rightarrow (B \rightarrow C)) \leftrightarrow ((A\&B) \rightarrow C)$;
A8. $((A \rightarrow B) \rightarrow C) \rightarrow (((B \rightarrow A) \rightarrow C) \rightarrow C)$;
A9. $\mathbf{F} \rightarrow A$;
A10. $\neg\neg A \rightarrow A$;
A11. $((A\&B) \rightarrow \mathbf{F}) \vee ((A \wedge B) \rightarrow (A\&B))$;
$A \rightarrow B, A \vdash B$ (modus ponens, mp);
$A, B \vdash A \wedge B$ (adjunction, adj).

(ii)
- Non-fixed-pointed nilpotent minimum logic NM^{nfp} is NM plus $(A \vee \neg A) \rightarrow ((A\&A) \vee (\neg A\&\neg A))$ (Non-fixed-point, Nfp).
- Fixed-pointed nilpotent minimum logic $\mathrm{NM}^{\overline{\frac{1}{2}}}$ is NM plus $\overline{\frac{1}{2}}$ and $\overline{\frac{1}{2}} \leftrightarrow \neg\overline{\frac{1}{2}}$ (Fixed-point, Fp).[3]

For convenience, '¬,' '∧,' '∨,' and '→' are used ambiguously as propositional connectives and as algebraic operators, but context should clarify their meaning.

The algebraic counterpart of L $\in \{NM, NM^{nfp}, NM^{\overline{\frac{1}{2}}}\}$ is defined as follows.

Definition 2.

(i) An *NM-algebra* is a structure $\mathcal{A} = (A, \top, \bot, \wedge, \vee, *, \rightarrow, \neg)$, where $\neg x := x \rightarrow \bot$ for all $x \in A$ and $x \vee y := ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)$ for all $x, y \in A$, such that:
  - $(A, \top, \bot, \wedge, \vee)$ is a bounded lattice with top element $\top$ and bottom element $\bot$;
  - $(A, *, \top)$ is an integral commutative monoid;
  - $y \leq x \rightarrow z$ iff $x * y \leq z$ (residuation);
  - $\top = (x \rightarrow y) \vee (y \rightarrow x)$ (prelinearity);

[3]The constant $\overline{\frac{1}{2}}$ does not necessarily correspond to the actual fraction $\frac{1}{2}$. Since the standard negation $\neg x$ is defined as $1 - x$ in $[0, 1]$ and $\frac{1}{2}$ has the role of fixed-point in that $\frac{1}{2} = \neg\frac{1}{2}$ in $[0, 1]$, $\frac{1}{2}$ is used as a representative of fixed-point. Therefore, here we use $\overline{\frac{1}{2}}$ as the constant for denoting a fixed-point element of any algebra.

- $\neg\neg x = x$ (involution);
- $\top = ((x*y) \to \bot) \vee ((x \wedge y) \to (x*y))$ (weak nilpotent minimum).

(ii)
- An *NM^{nfp}-algebra* is an NM-algebra satisfying $x \vee \neg x \leq (x * x) \vee (\neg x * \neg x)$ (non-fixed-point).
- An *$NM^{\frac{1}{2}}$-algebra* is an NM-algebra with $\frac{1}{2}$ satisfying $\frac{1}{2} = \neg\frac{1}{2}$ (fixed-point).

Consider the system NM^{-}, which is NM plus (BP) $\neg(\neg(A \ \& \ A) \ \& \ \neg(A \ \& \ A)) \leftrightarrow (\neg(\neg A \ \& \ \neg A) \ \& \ \neg(\neg A \ \& \ \neg A))$. This system was introduced as the logic with semantics on $[0, 1]$ minus the fixed-point in [9]. Let linearly ordered algebras be chains. We finally consider the relationships between NM^{nfp} and NM^{-}.

Theorem 1.

(1) ([9]) *A nontrivial NM-chain satisfies* $(BP^{\mathcal{A}})$ $\neg(\neg(x*x) * \neg(x*x)) = \neg(\neg x * \neg x) * \neg(\neg x * \neg x)$ *iff it does not contain a fixed-point.*

(2) A nontrivial NM-chain satisfies (non-fixed-point) iff it does not contain a fixed-point.

Proof: For the left-to-right direction of (2), we assume that there is an element $x > \bot$ such that $x = \neg x$ and show that $x \vee \neg x > (x*x) \vee (\neg x * \neg x)$. Let $x = \neg x$. Then, since $x * x = \neg x * \neg x = \bot$, we have that $x \vee \neg x > (x * x) \vee (\neg x * \neg x) = \bot$. For the right-to-left direction of (2), assume that $x \neq \neg x$ for all $x \in A$. First, consider the case $x < \neg x$. Using (weak nilpotent minimum), we can obtain that $x * x = \bot$ and $\neg x * \neg x = \neg x$ and thus $x \vee \neg x = \neg x = (x*x) \vee (\neg x * \neg x)$; therefore, $x \vee \neg x \leq (x*x) \vee (\neg x * \neg x)$. Consider the case $\neg x < x$. Its proof is analogous to that of the case $x < \neg x$. $\square$

Corollary 1. *A nontrivial NM-chain satisfies* $(BP^{\mathcal{A}})$ *iff it satisfies (non-fixed-point).*

Now consider the systems NM^{nfp} and NM^{-} synthetically. We can show the following.

Theorem 2. *The system NM^{nfp} proves:*
(BP) $\neg(\neg(A\&A)\&\neg(A\&A)) \leftrightarrow (\neg(\neg A\&\neg A)\&\neg(\neg A\&\neg A))$.

Proof: First, note that the following are theorems of NM: (a) $(A \to B) \vee (B \to A)$; (b) $A \to (\neg A \to B)$; (c) $A \to (B \to A)$; (d) $\neg\neg A \leftrightarrow A$; (e) $((A\&A) \vee (\neg A\&\neg A)) \to (A \vee \neg A)$.

($\Rightarrow$) 1. $(A \to \neg A) \vee (\neg A \to A)$ (a);
2. $((A \to \neg A)\&(A \to \neg A)) \vee ((\neg A \to A)\&(\neg A \to A))$ (1, $\mathbf{T}\&\mathbf{T} \leftrightarrow \mathbf{T}$);
3. $((A \to \neg A)\&(A \to \neg A)) \to (\neg((A \to \neg A)\&(A \to \neg A)) \to ((\neg A \to A)\&(\neg A \to A)))$ (b);
4. $((\neg A \to A)\&(\neg A \to A)) \to (\neg((A \to \neg A)\&(A \to \neg A)) \to ((\neg A \to A)\&(\neg A \to A)))$ (c);
5. $\neg((A \to \neg A)\&(A \to \neg A)) \to ((\neg A \to A)\&(\neg A \to A))$ (2, 3, 4, adj, mp);
6. $\neg(\neg(A\&A)\&\neg(A\&A)) \to (\neg(\neg A\&\neg A)\&\neg(\neg A\&\neg A))$ (5, d, (df4) $A\&B := \neg(A \to \neg B)$).

($\Leftarrow$) 1. $(A \vee \neg A) \leftrightarrow ((A\&A) \vee (\neg A\&\neg A))$ (e, Nfp, adj, df3);
2. $\neg(A \vee \neg A) \leftrightarrow \neg((A\&A) \vee (\neg A\&\neg A)) \leftrightarrow \mathbf{F}$ (1, df1, A9, adj);
3. $(A \wedge \neg A) \leftrightarrow (\neg(A\&A) \wedge \neg(\neg A\&\neg A)) \leftrightarrow \mathbf{F}$ (2, d, De Morgan);
4. $(\neg(A\&A)\&\neg(A\&A)) \wedge (\neg(\neg A\&\neg A)\&\neg(\neg A\&\neg A)) \leftrightarrow \mathbf{F}$ (3, $\mathbf{F}\&\mathbf{F} \leftrightarrow \mathbf{F}$);
5. $\neg(\neg(A\&A)\&\neg(A\&A)) \vee \neg(\neg(\neg A\&\neg A)\&\neg(\neg A\&\neg A)) \leftrightarrow \mathbf{T}$ (4, $\neg\mathbf{F} \leftrightarrow \mathbf{T}$, De Morgan);
6. $(\neg(\neg A\&\neg A)\&\neg(\neg A\&\neg A)) \to \neg(\neg(A\&A)\&\neg(A\&A))$ (4, 5, Boolean property). □

Then, from Theorem 2, the following question arises when we just think of the systems synthetically.

- *Open Problem*: Does the system NM^- prove (Nfp) synthetically?

According to Corollary 1, it seems possible to show this since the conditions $(BP^{\mathcal{A}})$ and (non-fixed-point) both correspond to the condition 'no fixed-point.' However, we have not yet proved this. To the author, it seems that the correct axiomatization of the extension of NM with the semantics on $[0,1]^-$, i.e., $[0,1] \setminus \{\frac{1}{2}\}$, is not the axiomatization of NM^-, but that of NM^{nfp}.

3. Pretabularity

For L $\in \{NM, NM^{nfp}, NM^{\overline{\frac{1}{2}}}\}$, by an *L-algebra*, we henceforth denote any of NM-, NM^{nfp}-, and NM^{nfp}-algebras. By *1* and *0*, we express $\top$ and $\bot$, respectively, on the real unit interval $[0,1]$ or on a subset of it with top and bottom elements $1, 0$. We refer to L-algebras on such a carrier set as S^L*-algebras*. S^L-algebras are defined as follows:

DEFINITION 3. The operations for an S^L-algebra are defined as follows.

(1) ([6]) Let the carrier set S be $[0, 1]$. An S^{NM}*-algebra* is an algebra satisfying: T1. $x \wedge y = min(x, y)$; T2. $x \vee y = max(x, y)$; T3. $x \rightarrow y = 1$ if $x \leq y$, and otherwise $x \rightarrow y = max(1 - x, y)$; T4. $\neg x = 1 - x$.[4]

(2) Let the carrier set S be a subset of $[0, 1]$ with top and bottom elements $1, 0$.
- An $S^{NM^{nfp}}$*-algebra* is an S^{NM}-algebra whose carrier set S has no fixed-point.
- *An* $S^{NM^{\frac{1}{2}}}$*-algebra* is an S^{NM}-algebra whose carrier set S has $\frac{1}{2}$, a fixed-point.

By $S^L_{[0,1]}$*-algebra*, we henceforth denote the S^L-algebra on $[0, 1]$; by $S^L_{[0,1]^-}$*-algebra*, the S^L-algebra on $[0, 1] \setminus \{\frac{1}{2}\}$; by S^L_n*-algebra*, the S^L-algebra whose elements are in $\{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$. Generalizing, S*-algebra* refers to any algebra whose elements form a chain with the greatest and least elements, and whose operations are defined in an analogous way.

Note that S-algebras having $\frac{1}{2}$ as an element x such that $x = \neg x$ are said to be *fixed-pointed*, and otherwise *non-fixed-pointed.* A logic L is said to be *fixed-pointed* if L is characterized by an S-algebra having a fixed-point, and otherwise is *non-fixed-pointed.* An extension of L is said to be *proper* if it does not have exactly the same theorems as L.

DEFINITION 4.

(i) (Tabularity) A logic L is *tabular* if L has some finite characteristic algebra.

(ii) (Pretabularity) A logic L is *pretabular* if (a) L is not tabular and (b) every proper extension of L has some finite characteristic algebra.

Now, we show that $\mathrm{L} \in \{NM^{nfp}, NM^{\overline{\frac{1}{2}}}\}$ is pretabular, but the systems NM is not. We first introduce some known pretabular logics.

FACT 1. ([3, 5]) Each of RM and G is pretabular.

[4] In general, the *involutive* negation is defined as the negation n satisfying $n(n(x)) = x$ for all $x \in [0, 1]$. Since any involutive negation $[0, 1]$ can be isomorphic to $1 - x$, for convenience, we take this definition.

We then divide the work into a number of propositions following the line in [3, 5].

PROPOSITION 1. *Let $\mathcal{X}$ be an extension of $L \in \{NM^{nfp}, NM^{\overline{\frac{1}{2}}}\}$, $\mathcal{A}$ be an $\mathcal{X}$-algebra, and $a \in \mathcal{A}$ be such that $a < \top$. Then, there is a homomorphism h of $\mathcal{A}$ onto an S-algebra which is an $\mathcal{X}$-algebra, such that $h(a) < 1$.*

PROOF: The proof is analogous to Theorem 3 in [3] and Theorem 11.10.4 in [4]. □

PROPOSITION 2.

(i) Let L be the system NM^{nfp}. Let $S_1^L, S_2^L, S_4^L, S_6^L, \ldots$, i.e., S_1^L and $S_{2n}^L, 1 \leq n \in N$, be the sequence of S^L-algebras relabeled in order as $M_1^L, M_2^L, M_3^L, \ldots$. If a sentence A is valid in M_i^L, then A is valid in M_j^L, for all j, $j \leq i$.

(ii) Let L be the system NM$^{\overline{\frac{1}{2}}}$. Let $S_1^L, S_3^L, S_5^L, S_7^L, \ldots$, i.e., $S_{2n-1}^L, 1 \leq n \in N$, be the sequence of S^L-algebras relabeled in order as $M_1^L, M_2^L, M_3^L, \ldots$. If a sentence A is valid in M_i^L, then A is valid in M_j^L, for all j, $j \leq i$.

PROOF: Since each S_j^L is (isomorphic to) a subalgebra or a homomorphic image of S_i^L, (i) and (ii) are immediate. □

PROPOSITION 3. *In S^{NM}-algebras, when i is even (≥ 4), S_i^{NM} validates a sentence A that is not valid in any odd-valued $S_j^{NM}, 3 \leq j \leq i$.*

PROOF: The claim can be verified by considering the sentence (Nfp), which is valid in every even-valued S_i^{NM}, but not in S_3^{NM} (and thus not in any odd-valued $S_j^{NM}, j \geq 3$). □

Remark 1. Proposition 3 implies that every valid sentence in $S_{[0,1]}^{NM}$ must be valid in $S_{[0,1]^-}^{NM}$, but there is a valid sentence in $S_{[0,1]^-}^{NM}$ that is not in $S_{[0,1]}^{NM}$.

Now, we recall the concept of a Lindenbaum-Tarski algebra. Let L $\in \{NM^{nfp}, NM^{\overline{\frac{1}{2}}}\}$ and T be a theory in L. We define $[A] = \{B : T \vdash_L A \leftrightarrow B\}$ and L $= \{[A] : A \in Fm\}$. The *Lindenbaum-Tarski algebra* $\mathbf{Lind}_T$ w.r.t. L and T is L-algebra having the domain L, operations $\#^{\mathbf{Lind}}([A_1], \ldots, [A_n]) = [\#(A_1, \ldots, A_n)]$, where $\# \in \{\wedge, \&, \rightarrow\}$, and the top and bottom elements are [**T**] and [**F**], respectively. We call this algebra the Lindenbaum-Tarski algebra $\mathcal{A}(L)$.

Where $\mathcal{X}$ is a propositional system and $\mathbf{V}$ is a set of atomic sentences, let $\mathcal{X}/\mathbf{V}$ be that propositional system like $\mathcal{X}$ except that its sentences contain no atomic sentences other than those in $\mathbf{V}$ and thus $\mathcal{A}(\mathcal{X}/\mathbf{V})$ be its corresponding Lindenbaum-Tarski algebra. The following is obvious.

PROPOSITION 4. *Let $\mathcal{X}$ be an extension of $L \in \{NM^{nfp}, NM^{\overline{\frac{1}{2}}}\}$. Then, $\mathcal{A}(\mathcal{X}/\boldsymbol{V})$ is an $\mathcal{X}$-algebra and is characteristic for $\mathcal{X}/\boldsymbol{V}$, since any non-theorem may be falsified under the canonical evaluation v_c, which sends every sentence A to $[A]$, where $[A]$ is the set of all sentences B such that $B \leftrightarrow A$.*

Also, it follows from Propositions 1 and 4 that:

PROPOSITION 5. *Let $\mathcal{X}$ be an extension of $L \in \{NM^{nfp}, NM^{\overline{\frac{1}{2}}}\}$. Then, if a sentence A is not a theorem of $\mathcal{X}$, there is some S^L-algebra S^L_n such that S^L_n is an $\mathcal{X}$-algebra and A is not valid in S^L_n.*

PROOF: If A is not a theorem of $\mathcal{X}$, then, by Proposition 4, A is falsifiable in the $\mathcal{X}$-algebra $\mathcal{A}(\mathcal{X}/\mathbf{V})$, where $\mathbf{V}$ is the set of sentential variables occurring in A, by the canonical evaluation v_c. However, since $[A]$ is undesignated in $\mathcal{A}(\mathcal{X}/\mathbf{V})$, then, by Proposition 1, there is a homomorphism h of $\mathcal{A}(\mathcal{X}/\mathbf{V})$ onto an S^L-algebra S^L such that S^L is an $\mathcal{X}$-algebra and $h([A]) < 1$ in S^L. However, the composition of h and v_c, $h \circ v_c(B) = h([B])$, is an evaluation that falsifies A in S^L. Note that an S^L-subalgebra, the image $h(\mathcal{A}(\mathcal{X}/\mathbf{V})$, is finitely generated since it is the homomorphic image of $\mathcal{A}(\mathcal{X}/\mathbf{V})$, which is finitely generated by the elements $[p]$ such that $p \in \mathbf{V}$. Thus, this algebra is finitely generated by the elements $[p]$ such that $p \in \mathbf{V}$. It is obvious that every finitely generated S^L-subalgebra is finite and isomorphic to some S^L_n. Thus, this algebra is isomorphic to some S^L_n, which completes the proposition. □

If $\mathcal{X}$ is L itself, we have the following completeness theorem as a corollary.

COROLLARY 2. (*Completeness*) *For $L \in \{NM^{nfp}, NM^{\overline{\frac{1}{2}}}\}$ and the set of S^L-algebras $\mathcal{S}^{\mathcal{L}}$, if a sentence A is valid in $\mathcal{S}^{\mathcal{L}}$, then A is a theorem of L.*

PROOF: By proposition 5, we have that if a sentence A is not a theorem of L, there is some S^L-algebra S^L_n such that A is not valid in S^L_n. Thus, by contraposition, we obtain the claim. □

Finally, we turn to a proof of our principal results.

Theorem 3.

(i) $L \in \{NM^{nfp}, NM^{\overline{\frac{1}{2}}}\}$ *is pretabular.*

(ii) *NM is not pretabular.*

Proof: For (i), we show that every proper extension of L has a finite characteristic algebra. Let $M_1^L, M_2^L, M_3^L, \ldots$ be the sequence of S^L-algebras defined in Proposition 2. Let I be the set of indices of those S^L-algebras that are $\mathcal{X}$-algebras, where $\mathcal{X}$ is the given proper extension of L.

First, if I contains an infinite number of indices, then I contains every index because of Proposition 2. However, since every S^L-algebra M_i^L is an L-algebra, it follows from Proposition 5 and Corollary 2 that $\mathcal{X}$ is identical with L, which contradicts the hypothesis that $\mathcal{X}$ is a proper extension of L.

Second, if I contains only a finite number of indices, then, by Proposition 2, there must be some index i such that I contains exactly those indices less then or equal to i. By construction, S_i^L is an $\mathcal{X}$-algebra. Let a sentence A not be a theorem of $\mathcal{X}$. Then, by Proposition 5, A is not valid in some $\mathcal{X}$-algebra M_h^L, and, by our choice of i, $h \leq i$. However, by Proposition 2, A is not valid in M_i^L. Therefore, M_i^L is the desired finite characteristic algebra.

L itself has no finite characteristic algebra, which can easily be shown by a proof similar to that of Sugihara in [12]. Therefore, it can be ensured that L is pretabular.

(ii) directly follows from (i), Proposition 3, and Remark 1. (Note that the system NM^{nfp} is a pretabular extension of NM.) □

We finally remark some relationships between the results in Theorem 3 and algebraic results introduced in [9, 10].

Remark 2.

(1) The fact that NM^{nfp} is pretabular but NM is not can be *algebraically* obtained as a consequence of the full description of the lattice of subvarieties of the variety $\mathcal{NM}$ (see Theorems 2 and 3 and Figure 2 in [9] and Figure 1 in [10]).

(2) Pretabularity is a property related to logics whose associated varieties of algebras are locally finite. A variety of algebras is said to be *locally finite* if each of its finitely generated members is a finite algebras. We first note that the variety $\mathcal{NM}$ is locally finite (see [9, 10]).

Thus, since the varieties $\mathcal{NM}^{nfp}$ (the variety of non-fixed-pointed NM-algebras) and $\mathcal{NM}^{\overline{\frac{1}{2}}}$ (the variety of fixed-pointed NM-algebras) are subvarieties of $\mathcal{NM}$, $\mathcal{NM}^{nfp}$ and $\mathcal{NM}^{\overline{\frac{1}{2}}}$ are locally finite. These results show that every pretabular variety is locally finite, but not conversely.

4. Concluding remarks

We showed that the two fuzzy systems NM^{nfp}, $\mathrm{NM}^{\overline{\frac{1}{2}}}$ are pretabular while NM is not. We also showed that NM^{nfp} and NM^{-} are semantically equivalent. However, we have not yet shown this syntactically. This problem should be addressed in future research. We also have another interesting question as follows: Let L_1 and L_2 be two pretabular logics complete w.r.t. characteristic algebras S^{L_1} and S^{L_2}, and consider the logic L induced by the ordinal sum $S^{L_1} \oplus S^{L_2}$. Then, we can ask: Under which condition L is pretabular?

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