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Composition of wavelet and Fourier transforms

Abstract The paper presents the basic properties of the serial composition of two transformations: wavelet and Fourier. Two types of transformations were obtained because wavelet and Fourier transformations do not commute. The consequences of a phenomenon known as a "wavelet crime" are presented. Using wavelets with compact support in the frequency domain (e.g. Meyer wavelets) leads to the representation of signals as sparse matrices. Speech signals were used to test the presented transforms.

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1. Introduction. The main goal of the work is to provide a new tool for the analysis of waveforms modeled by the functions of one variable. The new method of the analysis is the combination of two very frequently used transformations: Fourier and wavelet. Presentation of the basic properties of the proposed transformations is the main result of the conducted works.

The classic transformations presented in this paper have found applications in many fields of technology. The examples mentioned below deal with the processing of audio signals, in particular speech signals, due to the interest of the authors of this publication. For the same reason, the composition of classic transformations was used by the authors to analyze speech signals but there are many more possible applications.

Fourier Transform (FT) is the most frequently used tool in the signal analysis and modeling. The main reason is the utility of frequency representation. It is also important that mathematical models in the frequency domain are easier to use when comparing with models in the time domain.

Wavelet methods [2], [3], [7], [11] were studied extensively during the 1990. They are a useful tool for various signal processing applications. Wavelet Transforms (WT) are used to achieve the time-frequency representations of both digital and analog signals. Continuous Wavelet Transforms (CWT) are used to provide a local frequency analysis of analog signals and to track the frequency distribution over time. CWT is an integral operator in $L^2(\Re)$. It has the form of a scalar product and can be interpreted as a tool for finding similarities between signal and wavelet functions located in the time and frequency domains.

The paper presents the results of joining FT and WT. We considered the real value functions as the subject of transformations because such functions are usually the mathematical models used by engineers. If the initial transform is CWT and FT is calculated next, then the operation we call Fourier-Wavelet Transform (FWT). If the order of transforms is reversed, the operation we call as the Wavelet-Fourier Transform (WFT). Examples of FWT and WFT were tested for speech signals.

The definitions of FWT and WFT and their properties concern for example the theory of one-dimensional analog signals. Mathematical models of such signals are functions $s \in L^2(\Re)$. This way of presenting is clearer for readers than presentations referring to digital signals.

The computer calculations described at the end of this paper were performed using classic software for processing digital signals. From a mathematical point of view, digital signals are vectors $s_\Delta \in \Re^N$ which represent the discrete values of functions $s \in L^2(\Re)$.

2. Continuous wavelet transform

To examine how the distribution of frequencies changes in a signal, let us assume that a signal is represented by function $s(t)$. CWT has the form

$$\widetilde{s}_\psi(a,b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} s(t)\psi\left(\frac{t-b}{a}\right) dt, \tag{1}$$

where ψ is an arbitrarily chosen wavelet function which is a transform kernel. Signal s is a function of time t and, after transformation (1), we obtain a function which has two arguments. Variable b is the time in which surrounding the frequency properties of signal $s(t)$ are examined. The second variable is $a > 0$, representing a frequency band which has the central frequency

$$f_a = 2a^2 \int_0^{\infty} f|\widehat{\psi}(af)|^2 df, \tag{2}$$

where $\hat{\psi}(f)$ is FT of kernel $\psi(t)$. This means that (1) shows the strength of frequencies in the signal, near time b and simultaneously around frequency (2).

The speech signals analyzed in the paper cover the frequency band up to 8 kHz. Consequently, within the frequency band from 2 kHz to 8 kHz for

$a = 1$, the Meyer wavelet is defined by the formula

$$\widehat{\psi}(af) = \frac{\sqrt{a}}{\sqrt{2\pi}} \begin{cases} e^{\frac{\pi j a f}{6000}} \sin\left(\frac{\pi}{2}\left(\frac{a|f|}{2000} - 1\right)\right) \\ \quad \text{if } 2000 \leq a\,|f| \leq 4000 \\ e^{\frac{\pi j a f}{6000}} \cos\left(\frac{\pi}{2}\left(\frac{a|f|}{4000} - 1\right)\right) \\ \quad \text{if } 4000 \leq a\,|f| \leq 8000 \\ 0 \quad \text{if } a\,|f| \notin [2000, 8000] \end{cases} \tag{3}$$

where $j^2 = -1$.

Fig.1 presents the influence of scaling parameter a on the wavelet $a^{-1/2}\psi(t/a)$. We selected the Meyer wavelet as a kernel of CWT defined by the (1).

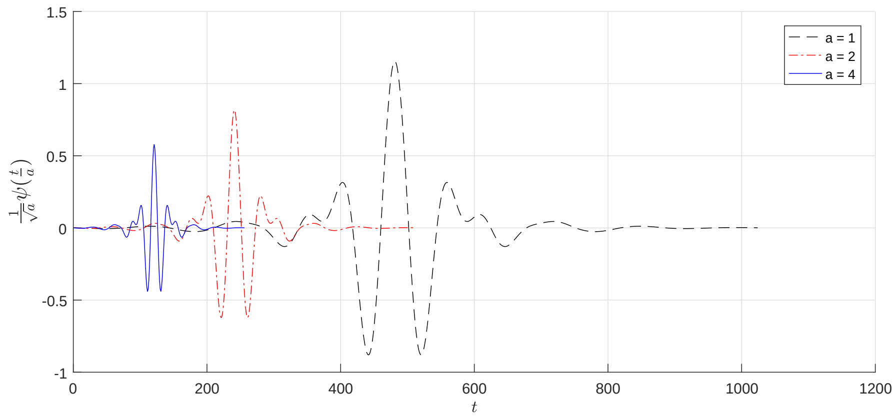

Figure 1: Meyer wavelet defined by (3) for scaling factor a equal to 1, 2 and 4

3. Fourier-wavelet transform

Let us define FWT as the combination of WT and FT

$$\widehat{\widetilde{s}}_{\psi}(a, f) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-2\pi j f b} \int_{-\infty}^{\infty} s(t)\psi\left(\frac{t-b}{a}\right) dt\, db. \tag{4}$$

FWT applications in speech technology are presented in [13] and [14] where the authors used FWT to analyze speech signals and studied the usefulness of FWT in speech recognition.

Using the property of FT which preserves the scalar products [8], from (1) we obtain

$$\widehat{\widetilde{s}}_{\psi}(a, f) = \sqrt{a}\widehat{\psi}^{*}(af)\,\widehat{s}(f). \tag{5}$$

This formula suggests an important interpretation of FWT. It appears that FWT is just a signal filtration. The characteristic of the filter is the function conjugated to the wavelet spectrum $\widehat{\psi}(af)$ multiplied by $\sqrt{a}$. The amplitude characteristic has either compact support (e.g. for Meyer wavelets) or is clearly greater than 0 for a certain frequency band only. Scaling parameter a shifts the amplitude characteristics and thus $\hat{\tilde{s}}_\psi(a,f)$ for fixed a depends on a part of the signal spectrum $\hat{s}(f)$. This property is strictly fulfilled if the CWT kernel has a compact support in the frequency domain. We then obtain a sparse representation [10] of the signal. Such forms can be used for lossless compression of signals. If the support of the wavelet spectrum is not compact, then FWT does not give a sparse representation. However, due to the assumed fundamental property of the wavelet functions, the representation of the signal in the form of FWT must include components with absolute values clearly smaller than other FWT values.

Fig. 2 shows the amplitude spectra of Meyer wavelet filters $\sqrt{a}|\widehat{\psi}^*(af)|$ for three values of parameter a.

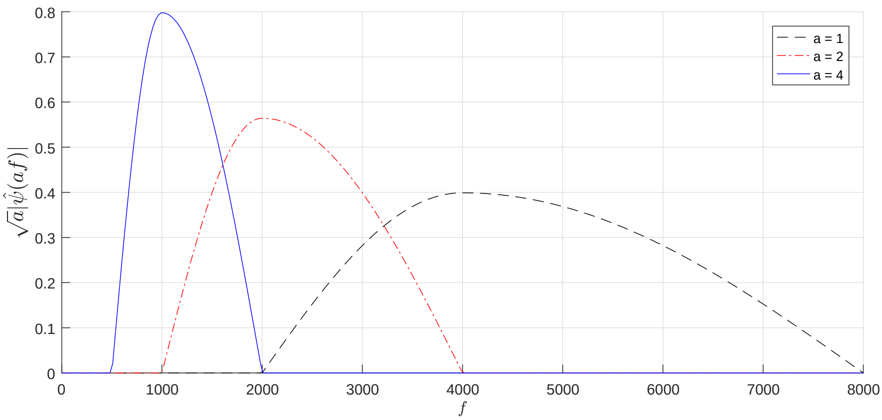

Figure 2: Amplitude characteristics of the filter defined by (5) for Meyer wavelet (3) and scaling factors a equal to 1, 2 and 4.

Using elementary properties of CWT and FT, we determined the properties of FWT:

1) conservation of energy

$$\int_{-\infty}^{\infty} s^2(t)dt = \frac{1}{\eta}\int_{-\infty}^{\infty} a^{-2}\int_{-\infty}^{\infty} \tilde{s}_\psi^2(a,b)dbda = \frac{1}{\eta}\int_{-\infty}^{\infty} a^{-2}\int_{-\infty}^{\infty} |\widehat{\tilde{s}}_\psi(a,f)|^2 df da$$

where $\eta = 2\pi\int_0^\infty \frac{|\widehat{\psi}(f)|^2}{f}df < \infty$,

2) for the time-shifted signal

$$s(t-\tau) \longleftrightarrow \tilde{s}_\psi(a,b-\tau) \longleftrightarrow \widehat{\tilde{s}}_\psi(a,f)e^{-2\pi j\tau f},$$

3) for the scaled signal

$$s(\gamma t) \longleftrightarrow \gamma^{-1/2}\tilde{s}_\psi(a\gamma,b\gamma) \longleftrightarrow \gamma^{-3/2}\widehat{\tilde{s}}_\psi(a\gamma,f/\gamma) \text{ for } \gamma > 0,$$

4) for the shift FWT in the frequency domain we obtain

$$\tilde{s}_\psi(a,b)\exp(2\pi j f_0 b) \longleftrightarrow \widehat{\tilde{s}}_\psi(a,f-f_0),$$

5) differentiation with respect to variable b gives

$$d^n\tilde{s}_\psi(a,b)/db^n \longleftrightarrow (2\pi j f)^n\widehat{\tilde{s}}_\psi(a,f),$$

6) differentiation with respect to variable f gives
$(-2\pi jb)^n \widetilde{s}_\psi(a,b) \longleftrightarrow d^n \widehat{\widetilde{s}}_\psi(a,f)/df^n$,
7) for integration of the wavelet transform
$\int_{-\infty}^{b} \widetilde{s}_\psi(a,\tau)\, d\tau \longleftrightarrow \widehat{\widetilde{s}}_\psi(a,f)/(2\pi jf)$,
where condition $\widehat{\widetilde{s}}_\psi(a,0)=0$ must be satisfied.

4. Wavelet-Fourier transform

FT and CWT are not commuting operations, so we obtain a separate method by determining the CWT for FT spectrum $\widehat{s}(f)$. WFT is therefore a two-step operation. First, spectrum $\widehat{s}(f)$ is determined and then the CWT gives

$$\widetilde{\widehat{s}}_\psi(a,\theta) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} \psi\left(\frac{f-\theta}{a}\right) \int_{-\infty}^{\infty} s(t) e^{-2\pi jft} dt\, df. \tag{6}$$

Using elementary properties of both transforms, we determine properties of WFT:
1) conservation of energy
$\frac{1}{\eta}\int_{-\infty}^{\infty} a^{-2} \int_{-\infty}^{\infty} |\widetilde{\widehat{s}}_\psi(a,\theta)|^2 d\theta\, da = 2\int_0^\infty |\widehat{s}(f)|^2 df = \int_{-\infty}^{\infty} s^2(t) dt$
2) shift in the frequency domain
$\widetilde{\widehat{s}}_\psi(a,\theta-f_0) \longleftrightarrow \widehat{s}(f-f_0) \longleftrightarrow s(t)\exp(2\pi jf_0 t)$,
3) scaled FT spectrum gives
$\sqrt{a_0}\,\widetilde{\widehat{s}}_\psi(a/a_0,\theta/a_0) \longleftrightarrow \widehat{s}(f/a_0) \longleftrightarrow |a_0| s(a_0 t)$.

5. Numerical procedures

To calculate the Discrete Fourier-Wavelet Transform (DFWT) for signal s(t) it is necessary to take its values for appropriately selected moments of time. This is a standard sampling procedure described by the Shannon theorem. The result is a representation of signal $s(t)$ by vector $s_\Delta \in \Re^N$.

The algorithms for DFWT calculations can use formula (4) or (5). In the first case, the calculation consists of two stages. In the first stage, the Discrete Wavelet Transform (DWT) [2], [6], [7], [11] for signal $s_\Delta \in \Re^N$ is computed. This is an iterative procedure used for computing DWT values for subsequent discrete variables $a = 2^{m-1}$, where $m = 1, 2, .., M$. The number of resolution levels M must be assumed, taking into account the required range of the lowest frequencies. Then, M vectors of

$$DWT = \{\{\widetilde{s}_\Delta(1,n)\}_{n=1}^{N/2}, \widetilde{s}_\Delta(2,n)\}_{n=1}^{N/4}, ..., \{\widetilde{s}_\Delta(M,n)\}_{n=1}^{N/2^M}\} \tag{7}$$

with different lengths are obtained. As a result, a two-dimensional, trapezoidal table is obtained. If the sample number N divided by the power of 2 does not give an integer value, then the result of the division should be rounded to the nearest integer. Next, the Fast Fourier Transform (FFT) should be applied separately for each vector. Finally, we obtain

$$DFWT = \{\{\widehat{\widetilde{s}}_\Delta(1,k)\}_{k=1}^{N/4}, \widehat{\widetilde{s}}_\Delta(2,k)\}_{k=1}^{N/8}, ..., \{\widehat{\widetilde{s}}_\Delta(M,k)\}_{k=1}^{N/2^{M+1}}\}. \tag{8}$$

The variable k determines the discrete frequency which has the maximum value equal to $F = 0.5N/T$, where time T is the length of analog signal s.

Very efficient procedures are used for generating DWT coefficients. However, to start this algorithm one needs to determine the initial sequence for certain (sufficiently large) resolution level M. The problem is that these coefficients are values of definite integrals, so their numerical determination requires time-consuming calculations. Common practice, in such a situation, is to use sampled values of the function as the initial coefficients in DWT. Such procedure is justified by the specific form of the integrated functions. They have values significantly different from zero only in the environments of the acquired function values. Such an approach is commonly used, despite the fact that it was charged by Strang and Nguyen to be a "wavelet crime" [11]. The error resulting from this approximation depends on the function being transformed. In engineering practice, it is widely believed that the benefit of computational efficiency is much more important than the losses caused by computational inaccuracies.

If the method of calculating DFWT is based on formula (5), a rectangular matrix $\widehat{\psi}_\Delta(a, k)$ is determined as a discrete representation of (3). Finally, the discrete transform $\widehat{\widetilde{s}}_\Delta(a, k) \in C^{M\times K}$ is determined as a matrix arising from the scalar products

$$\widehat{\widetilde{s}}_\Delta(a, k) = \widehat{\psi}_\Delta(a, k)\widehat{s}_\Delta(k). \tag{9}$$

This algorithm allows us to obtain DFWT as a sparse representation of signal s. In contrast, the first method gives faster calculations, but generally no representation in the form of a sparse matrix can be obtained.

The DWFT determination algorithm also involves two-steps. For a discrete signal s_Δ, the spectrum $\widetilde{s}_\Delta$ must be determined first using FFT. In the next stage, standard software should be used to determine DWT, assuming the required number of resolution levels M. Similarly to the procedure described by (8)

$$DWFT = \{\{\widetilde{\widehat{s}}_\Delta(1, k)\}_{k=1}^{N/4}, \widetilde{\widehat{s}}_\Delta(2, k)\}_{k=1}^{N/8}, \ldots\ldots, \{\widetilde{\widehat{s}}_\Delta(M, k)\}_{k=1}^{N/2^{M+1}}\} \tag{10}$$

is obtained as a two-dimensional, trapezoidal table.

6. Examples The speech signal (the sentence "She had your dark suit in greasy wash water all year") spoken by a male speaker was taken from the TIMIT corpus. Plots in Fig. 3 present the results of computations. Plots (b), (d), (e) and (f) present complex value modules.

Classic procedures for FFT and DWT were used for numerical calculations. A six-level DWT was computed using the Meyer decomposition filters. Fig.3 shows the results of DFWT calculations using two methods: based on formula (5) (plot (d)) and the combination of DWT and DFT (plot (e)). The second method uses standard DWT and FFT numerical procedures for transform (8). Theoretically, both methods should give the same results. The

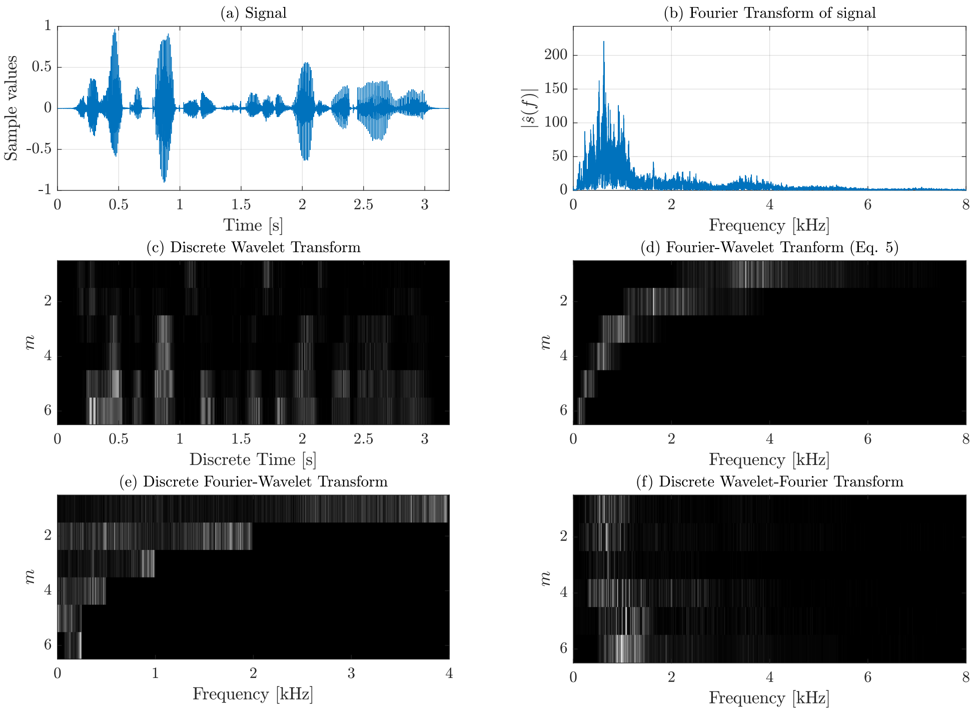

Figure 3: Speech signal (plot (a)) and its: amplitude spectrum (plot (b)), DWT described by (7) (plot (c)), DFWT transform defined by (9) (plot (d)), DWFT obtained from classical DWT and FFT (plot (e)) and DWFT (plot (f)) obtained by FFT and then DWT. The Meyer wavelet defined by (3) was used to compute WT for scaling factors m from 1 to 6.

differences are clearly visible and result from a specific, approximate determination of DWT. The main reason for the deviations is the wavelet multiresolution analysis procedure which is applied instead of the time-consuming numerical approximation of the integral presented in (4). This approximation is known as wavelet crime [9]. The algorithms which speed up the calculation cause unavoidable errors. Their sizes usually do not exceed the limits of tolerance and for sufficiently regular signals they may even be negligibly small. However, this approach has been analyzed and tested by authors of [1], [4], [5], [12] and they suggest some improvements.

Plot (f) of Fig.3 presents DWFT computed by applying FFT and DWT. It is clear that DFWT (plots (d) and (e)) and DWFT (plot (f)) give entirely different results.

7. Conclusions. This paper is addressed to the readers who use transformations that have character of frequency-based analysis. The combination of such two classic transformations has created new possibilities for a related nature analysis. The examples presented above illustrate how to use the proposed methods of the analysis. Their usefulness is not discussed in this publication. This requires a separate work for which the authors of this paper are preparing.

Frequency analysis is the most commonly used application in a wide variety of speech signal procedures. These methods include not only FT and WT but also cosine transform and mel-frequency cepstral coefficients. It can be expected that the proposed FWT and WFT algorithms will be useful in increasing the diversity of frequency analysis methods. This paper presents some of the fundamental properties of the hybrid Fourier-wavelet and wavelet-Fourier transforms.

The presented properties of composite transforms result from the elementary properties of both WT and FT components. The kernel $\exp(-2\pi jft)$ of FT is uniquely defined. It makes it possible to find many properties of FT. This leads to the detection of several relationships between FWT and WT. On the other hand, the kernel ψ of WT is not strictly defined and can be chosen arbitrary. This lack of explicitness limits the number of relationships between WFT and FT or WT.

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Złożenie transformacji falkowej i Fouriera

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Streszczenie W pracy przedstawione są podstawowe własności szeregowego złożenia dwóch transformacji: falkowej i Fouriera. Uzyskano dwa rodzaje transformacji ponieważ transformacje falkowe i Fouriera nie są przemienne. Przedstawione są konsekwencje zjawiska zwanego "przestępstwem falkowym". Zastosowanie falek ze zwartymi nośnikami w dziedzinie częstotliwości (np. falki Meyera) prowadzi do reprezentacji sygnałów w postaci macierzy rzadkich. Sygnały mowy zostały użyte do przetestowania przedstawionych transformacji.

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Słowa kluczowe: Transformacja falkowa, transformacja Fouriera, metody numeryczne, systemy rzadkie.

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