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On a Restricted Admissibility Compulsory Vacation Queue with Essential and Optional Services

Abstract In this paper, we study a single server queueing system in which the customers arrive according to a Poisson process with two phases of services. The customers may require, with a certain probability, an optional secondary service upon completion of the two phases of essential service. The server takes a vacation compulsorily after completion of service to a customer. In addition the admission of the customer to the queue is based on a Bernoulli process. The model is studied using supplementary variable techniques. Some special models are derived and some illustrative cases are discussed. Also, the queue has been realized using simulation.

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1. Introduction Various queueing models have been studied extensively because of their applications in real life situations. In day to day life, we have numerous queueing situations in which all the arriving customers are given the essential service and merely some of them may ask additional optional service. Such a model was first studied by Madan [17]. The other works to be noted here are Madan [18], Medhi [21], Al-Jararah and Madan [1], Jinting Wang [7], Kalyanaraman et al. [11] and Jau-Chuan [6].

In some queueing situations, it can also be seen that, after serving a certain number of customers, the server becomes unavailable for a random period of time, such queueing systems are called vacation queueing systems and the random period is called vacation period. Various modifications have been made in the vacation period by researchers and therefore different systems have been modelled. One such system is a queueing system with compulsory server vacation. In the compulsory server vacation queue, the server takes vacation after completion of each and every service. The earlier work on vacation systems are by Miller [22], Cooper [2, 3], Levy and Yechiali [15],

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Keilson and Servi [13], Levy et al. [14] and Doshi [4]. Madan [16] analyzed the single server queue with compulsory server vacation. Takagi [27] made an elaborate study on vacation models in his book on Queueing Analysis and Tian and Zhang [28] extensively studied about the various vacation systems in their book titled Vacation Queueing Models: Theory and Applications. In a queueing system if the administrator feels that the customers arrive faster than they can be served, then he may adopt a policy of restricting the arriving customers. This will help him to prevent the system from becoming over-loaded. Rue and Rosenshine [25] gave an optimal control policy on the arrivals of an $M/M/1$ queue. Neuts [23] considered an $M/G/1$ queue with a restriction on the number of customers to be admitted during a service period or with restrictions on the time period at which the customers are admitted. Stidham [26] considered an optimal arrival policy queues and network of queues. Doshi [5] analyzed a queueing system in which each customer receives two services: (i) Batch service (ii) Individual service. Madan [19] considered a single server queue with two stages of heterogeneous service and deterministic server vacation. Madan and Abu-Dayyeh [20] investigated about a bulk queue with restricted admissibility of batches with Bernoulli schedule server vacation. In this article the server provides two phases of heterogeneous exponential services with single vacation policy.

Kalyanaraman and Suvitha [8] have considered a non-Markovian queue with two type of services and with compulsory server vacation. In addition the author's assumed that the admission of customers to the system is based on a Bernoulli process. The same author's [9, 10], consider a single server Bernoulli and multiple vacation queue with two type of services and with restricted admissibility. Zadeh [29] has considered a batch arrival queue with coxian-2 server vacation and with restricted admissibility. For this model the author obtains the probability generating function on number of customers in the queue and some performance measures. Kalyanaraman et al. [12] analyzed a bulk queue with modified Bernoulli vacation and restricted admissibility of customers in fuzzy environment.

Generally, the application of queueing theory for daily life problems has remained restricted. The perception seems to have grown that queueing analysis is riddled with too much of mathematical complications rather than with general insights to allow for a practical applications. At the same time the question regarding practical situations, still appears to be open from a queueing point of view. On the other hand simulation has proven to be the most powerful tool to model and evaluate, such realistic situations both in daily life and industrial environment. In order to narrow down a this discrepancy, a queueing model has been formulated using simulation study to analyze realistic situations both in daily life and Industrial environment. For the simulation on stochastic models, Ross [24] can be seen.

Queuing models with server vacations generally occur in the field of computer and communications, manufacturing and banking systems. Our study approximates these situations mathematically. In the banking sector, it can be considered that the service to the client is always qualified with some conditions. Also, it can be seen that the service is in two phases. Some additional services are provided by the bank to some special customers. In the same way the server also may take vacation to do some other work. This situation and some other such situations can be approximated using the model discussed in this article.

In this article, a single server infinite capacity Poisson arrival queue with two phases of essential services and an optional service, with compulsory server vacation and with restriction on arrivals has been studied. The corresponding mathematical model is defined in section 2. In section 3, the ergodic condition is derived. In section 4, the governing differential difference equations, the boundary conditions and the normalizing conditions are obtained. In section 5, the probability generating function of the number of customers in the queue when the server is on vacation, the probability generating function of the number of customers in the queue when the server provides i^{th} phase of service $(i = 1, 2, 3)$ and the probability generating function of the number of customers in queue irrespective of the server states are derived. Also, in section 6 some performance measures related to this queueing model are obtained from these probability generating functions and are given. In sections 7 and 8, a particular model is derived by assigning particular values to the parameters and a numerical study is carried out. In section 9, a simulation study has been carried out on the model discussed in this article. In section 10, a discussion regarding the numerical results has been given. In the final section a conclusion is given.

2. The Model In this article, a single server queueing system with two phases of essential services and an optional service and with compulsory server vacation has been considered. In addition to this, a restriction is imposed on the admissibility of the arriving customers. The arrival process follows a Poisson process with rate $\lambda(> 0)$ and a single server provides two phases of essential services called phase 1 service and phase 2 service respectively. As soon as the phase 2 service of a customer is completed, the customer may opt an optional service with probability q, in which case the optional service will immediately commence or the customer may opt to leave the system with probability $1-q$. The optional service may be called phase 3 service. The service time distributions of the phases are general probability distribution. The respective distribution functions are $B_i(x)$, $i = 1, 2, 3$, Laplace–Stieltjes transform (LST) $B_i^*(\theta)$, the density functions are $b_i(x)$ and finite moments $b_{im} = \int\limits_0^\infty t^m dB_i(t),\ m = 1, 2, 3$. After completion of phase 2 service or phase 3 service, the server will go for compulsory server vacation of random period and after completion of a vacation period the server returns to the system irrespective of the server state. This random period follows a general distribution with distribution function $V(x)$, Laplace–Stieltjes transform (LST) $V^*(\theta)$, density function $v(x)$ and finite moments $v_m = \int\limits_0^\infty t^m \, dV(t),\ m = 1, 2.$

Further, it is assumed that not all the arriving customers are allowed to join the system at all times. That is, the admission to the queue is based on the following two Bernoulli processes: Let r $(0 \leqslant r \leqslant 1)$ be the probability that an arriving customer will be allowed to join the system while the server is either busy or in an idle state and let p $(0 \leqslant p \leqslant 1)$ be the probability that an arriving customer will be allowed to join the system while the server is on vacation.

For analysis purposes the supplementary variable elapsed service time (elapsed vacation time) has been introduced. Let $\mu_i(x)dx$ be the conditional probability of completion of the i^{th} phase of service during the interval $(x, x + dx]$ given that the elapsed service time is x, so that $\mu_i(x) = \dfrac{b_i(x)}{1 - B_i(x)}, i = 1, 2, 3$ and let $\gamma(x)dx$ be the conditional probability of completion of the vacation during the interval $(x, x + dx]$ given that elapsed vacation time is x, so that $\gamma(x) = \dfrac{v(x)}{1 - V(x)}$. For further analysis the following notations are introduced:

$P_n^{(i)}(x, t)\, dx$ is the probability that at time t there are n customers in the queue excluding one in the i^{th} phase of service and the elapsed service time is in $(x, x + dx]$, $n \geqslant 0$, $i = 1, 2, 3$,

$V_n(x, t)dx$ is the probability that at time t, the server is on vacation with elapsed vacation time is in $(x, x + dx]$ and the number of customers in the queue is n, $n \geqslant 0$ and

$Q(t)$ is the probability that at time t, there are no customers in the system and the server is idle.

$P_n^{(i)}(x), V_n(x)$ and Q are the corresponding steady state probability densities.

The probability generating functions for the probability densities $\left\{P_n^{(i)}(x)\right\}$, $\{V_n(x)\}$ are respectively defined as

$P^{(i)}(x, z) = \sum\limits_{n=0}^{\infty} z^n P_n^{(i)}(x), i = 1, 2, 3$ and $V(x, z) = \sum\limits_{n=0}^{\infty} z^n V_n(x)$ with $|z| \leqslant 1$.

3. The Ergodic Conditions. Let $L(t)$ be the number of customers in the queue at time t, let $\xi_1(t)$, $\xi_2(t)$, $\xi_3(t)$ be the elapsed service time of phase 1, phase 2, phase 3 respectively and $\xi_4(t)$ be the elapsed vacation time at t.

The server state of the system is

$$J(t) = \begin{cases} 0; & \text{the server is idle at time } t \\ 1; & \text{the server is in phase 1 service at time } t \\ 2; & \text{the server is in phase 2 service at time } t \\ 3; & \text{the server is in phase 3 service at time } t \\ 4; & \text{the server is in vacation at time } t \end{cases}$$

Then $\{X(t); t \geqslant 0\}$ is a Markov process, where

$$X(t) = (L(t), \xi_1(t), \xi_2(t), \xi_3(t), \xi_4(t))$$

with state space $\Omega = \{(i, x, y, z, u); i \geqslant 0, x, y, z, u \geqslant 0\}$.

First, we investigate the stability condition of our model.

Let $\{t_n; n \in Z_+\}$ be a sequence of time epochs at which a service completion occurs or vacation ends. Let $L_n = L(t_n+)$ be the number of customers in the queue just after the time t_n and $J_n = J(t_n+)$ be the server state after the time t_n forms a Markov chain, which is the embedded Markov chains of the Markov process $\{X(t); t \geqslant 0\}$. Its state space is $S = \{(i, n); i \geqslant 0, n = 0, 1, 2, 3, 4\}$.

Let $p_{i,j}^{n,m} = Pr\{L_{n+1} = j, J_{n+1} = m | L_n = i, J_n = n\}$ where $(j, m), (i, n) \in S$ are the transition probabilities of the Markov chain. Now

$$p_{0,j}^{0,2} = \int_0^\infty \frac{e^{-\lambda rt}(\lambda rt)^{j-1}}{(j-1)!} dB_1(t);\ j \geqslant 1$$

$$p_{i,j}^{1,2} = \int_0^\infty \frac{e^{-\lambda rt}(\lambda rt)^{j-i+1}}{(j-i+1)!} dB_1(t);\ i \geqslant 1, j \geqslant i-1$$

$$p_{i,j}^{2,3} = q\int_0^\infty \frac{e^{-\lambda rt}(\lambda rt)^{j-i+1}}{(j-i+1)!} dB_2(t);\ i \geqslant, j \geqslant i-1$$

$$p_{i,j}^{2,4} = (1-q)\int_0^\infty \frac{e^{-\lambda rt}(\lambda rt)^{j-i+1}}{(j-i+1)!} dB_2(t);\ i \geqslant 1, j \geqslant i-1$$

$$p_{i,j}^{3,4} = q\int_0^\infty \frac{e^{-\lambda rt}(\lambda rt)^{j-i+1}}{(j-i+1)!} dB_3(t);\ i \geqslant 1, j \geqslant i-1$$

$$p_{i,j}^{4,1} = \int_0^\infty \frac{e^{-\lambda pt}(\lambda pt)^{j-i+1}}{(j-i+1)!} dV(t);\ i \geqslant 1, j \geqslant i-1$$

$$p_{i,j}^{m,n} = 0;\ \text{otherwise}$$

Theorem 3.1 *The inequality* $\lambda r(b_{11} + b_{21} + qb_{31}) + \lambda pv_1 < 1$ *is a necessary and sufficient condition for the system to be stable.*

Proof It is easy to see that the Markov chain $\{(L_n, J_n); n \geqslant 1\}$ is irreducible and aperiodic. To prove the positive recurrence we use the Foster's criterion, which states that an irreducible, aperiodic Markov chain is positive recurrent if there exists a non-negative function $f((i,n)); (i,n) \in S$ and $\epsilon > 0$ such that the mean drift $\phi_{i,n} = E\big[f(L_n, J_n) - f(L_{n-1}, J_{n-1}) | (L_{n-1}, J_{n-1}) = (i,n)\big]$ is finite for all $(i,n) \in S$ and $\phi_{i,n} < -\epsilon$ for $(i,n) \in S$ except a finite number of points. In this case, we choose $f((i,n)) = i$ as a test function on the state space.

Now the mean drift

$$\phi_{i,n} = \begin{cases} \lambda r b_{11} + 1; & i = 0 \\ \lambda r(b_{11} + b_{21} + qb_{31}) + \lambda pv_1 - 1; & i \geqslant 1 \end{cases}$$

Obviously $\lambda r(b_{11} + b_{21} + qb_{31}) + \lambda pv_1 < 1$ is sufficient condition for ergodicity. The necessary condition follows from the Kaplan's condition: $\phi_{i,n} < \infty$ for all $(i,n) \in S$ there exists $(i_0, n) \in \Omega$ such that $\phi_{i,n} \geqslant 0$ for $i \geqslant i_0$. ∎

4. The Governing Equations The forward Kolmogorov equations related to the above defined model are

$$\frac{d}{dx}P_0^{(1)}(x) + (\lambda + \mu_1(x))P_0^{(1)}(x) = \lambda(1-r)P_0^{(1)}(x) \tag{1}$$

$$\frac{d}{dx}P_n^{(1)}(x) + (\lambda + \mu_1(x))P_n^{(1)}(x) = \lambda(1-r)P_n^{(1)}(x) + r\lambda P_{n-1}^{(1)}(x), n \geqslant 1 \tag{2}$$

$$\frac{d}{dx}P_0^{(2)}(x) + (\lambda + \mu_2(x))P_0^{(2)}(x) = \lambda(1-r)P_0^{(2)}(x) \tag{3}$$

$$\frac{d}{dx}P_n^{(2)}(x) + (\lambda + \mu_2(x))P_n^{(2)}(x) = \lambda(1-r)P_n^{(2)}(x) + r\lambda P_{n-1}^{(2)}(x), n \geqslant 1 \tag{4}$$

$$\frac{d}{dx}P_0^{(3)}(x) + (\lambda + \mu_3(x))P_0^{(3)}(x) = \lambda(1-r)P_0^{(3)}(x) \tag{5}$$

$$\frac{d}{dx}P_n^{(3)}(x) + (\lambda + \mu_3(x))P_n^{(3)}(x) = \lambda(1-r)P_n^{(3)}(x) + r\lambda P_{n-1}^{(3)}(x), n \geqslant 1 \tag{6}$$

$$\frac{d}{dx}V_0(x) + (\lambda + \gamma(x))V_0(x) = \lambda(1-p)V_0(x) \tag{7}$$

$$\frac{d}{dx}V_n(x) + (\lambda + \gamma(x))V_n(x) = \lambda(1-p)V_n(x) + p\lambda V_{n-1}(x), n \geqslant 1 \tag{8}$$

$$\lambda Q = \lambda(1-r)Q + \int_0^\infty \gamma(x)V_0(x)dx \tag{9}$$

The boundary conditions are

$$P_0^{(1)}(0) = r\lambda Q + \int_0^\infty \gamma(x)V_1(x)dx \tag{10}$$

$$P_n^{(1)}(0) = \int_0^\infty \gamma(x)V_{n+1}(x)dx, n \geqslant 1 \tag{11}$$

$$P_n^{(2)}(0) = \int_0^\infty \mu_1(x)P_n^{(1)}(x)dx, n \geqslant 0 \tag{12}$$

$$P_n^{(3)}(0) = q\int_0^\infty \mu_2(x)P_n^{(2)}(x)dx, n \geqslant 0 \tag{13}$$

$$V_n(0) = (1-q)\int_0^\infty \mu_2(x)P_n^{(2)}(x)dx + \int_0^\infty \mu_3(x)P_n^{(3)}(x)dx, n \geqslant 0 \tag{14}$$

The normalizing condition is

$$Q + \sum_{n=0}^{\infty}\int_0^\infty \left[P_n^{(1)}(x) + P_n^{(2)}(x) + P_n^{(3)}(x) + V_n(x)\right]dx = 1$$

5. The Analysis Multiplying equations (2), (4), (6) and (8) by z^n and summing from $n = 1$ to ∞ and then adding (1), (3), (5) and (7), we get

$$\frac{\frac{d}{dx}P^{(1)}(x,z)}{P^{(1)}(x,z)} = -T - \mu_1(x) \tag{15}$$

$$\frac{\frac{d}{dx}P^{(2)}(x,z)}{P^{(2)}(x,z)} = -T - \mu_2(x) \tag{16}$$

$$\frac{\frac{d}{dx}P^{(3)}(x,z)}{P^{(3)}(x,z)} = -T - \mu_3(x) \tag{17}$$

$$\frac{\frac{d}{dx}V(x,z)}{V(x,z)} = -R - \gamma(x) \tag{18}$$

where $T = \lambda r(1-z)$ and $R = \lambda p(1-z)$.

Integration of the equations (15)–(18) leads to

$$P^{(1)}(x,z) = A_0(1 - B_1(x))e^{-Tx} \tag{19}$$

$$P^{(2)}(x,z) = A_1(1 - B_2(x))e^{-Tx} \tag{20}$$

$$P^{(3)}(x,z) = A_2(1 - B_3(x))e^{-Tx} \tag{21}$$

$$V(x,z) = A_3(1 - V(x))e^{-Rx} \tag{22}$$

Taking $x = 0$ in equations (19)–(22), the constants A_0, A_1, A_2 and A_3 are obtained as

$$A_i = P^{(i+1)}(0,z),\ \ i = 0,1,2 \tag{23}$$

$$A_3 = V(0,z) \tag{24}$$

Using equations (23) and (24) respectively in equations (19)–(22), we get

$$P^{(1)}(x,z) = P^{(1)}(0,z)(1-B_1(x))e^{-Tx} \tag{25}$$
$$P^{(2)}(x,z) = P^{(2)}(0,z)(1-B_2(x))e^{-Tx} \tag{26}$$
$$P^{(3)}(x,z) = P^{(3)}(0,z)(1-B_3(x))e^{-Tx} \tag{27}$$
$$V(x,z) = V(0,z)(1-V(x))e^{-Rx} \tag{28}$$

Applying a similar manipulation on equations (10) and (11) as in the case of (1)–(2) and using equations (9) and (28), we get

$$zP^{(1)}(0,z) = \lambda r(z-1)Q + V^*(R)V(0,z) \tag{29}$$

Multiplying equation (12) by z^n and summing from $n = 0$ to ∞ and using equation (25), we get

$$P^{(2)}(0,z) = B_1^*(T)P^{(1)}(0,z) \tag{30}$$

Applying a similar manipulations on equations (13) and (14) as in the case of (12) and using equations (26), (27) and (28), we get

$$P^{(3)}(0,z) = qB_2^*(T)P^{(2)}(0,z) \tag{31}$$
$$V(0,z) = (1-q)B_2^*(T)P^{(2)}(0,z) + B_3^*(T)P^{(3)}(0,z) \tag{32}$$

Using equation (30) in (31), we get

$$P^{(3)}(0,z) = qB_1^*(T)B_2^*(T)P^{(1)}(0,z) \tag{33}$$

Using equations (30) and (33) respectively in (32), we get

$$V(0,z) = (1-q+qB_3^*(T))B_1^*(T)B_2^*(T)P^{(1)}(0,z) \tag{34}$$

Using equation (34) in (29), we get

$$P^{(1)}(0,z) = \frac{\lambda r(z-1)Q}{z - V^*(R)B_1^*(T)B_2^*(T)(1-q+qB_3^*(T))} \tag{35}$$

Using equation (35) in (30), (33) and (34), we get

$$P^{(2)}(0,z) = \frac{\lambda r(z-1)B_1^*(T)Q}{z - V^*(R)B_1^*(T)B_2^*(T)(1-q+qB_3^*(T))} \tag{36}$$
$$P^{(3)}(0,z) = \frac{\lambda rq(z-1)B_1^*(T)B_2^*(T)Q}{z - V^*(R)B_1^*(T)B_2^*(T)(1-q+qB_3^*(T))} \tag{37}$$
$$V(0,z) = \frac{\lambda r(z-1)B_1^*(T)B_2^*(T)(1-q+qB_3^*(T))Q}{z - V^*(R)B_1^*(T)B_2^*(T)(1-q+qB_3^*(T))} \tag{38}$$

Integration of equations (25) to (28) by parts with respect to x and then using (35) to (38), we get

$$P^{(1)}(z) = \int_0^\infty P^{(1)}(x,z)dx = \frac{(B_1^*(T)-1)Q}{z - V^*(R)B_1^*(T)B_2^*(T)(1-q+qB_3^*(T))} \tag{39}$$

$$P^{(2)}(z) = \int_0^\infty P^{(2)}(x,z)dx = \frac{B_1^*(T)(B_2^*(T)-1)Q}{z - V^*(R)B_1^*(T)B_2^*(T)(1-q+qB_3^*(T))} \tag{40}$$

$$P^{(3)}(z) = \int_0^\infty P^{(3)}(x,z)dx = \frac{qB_1^*(T)B_2^*(T)(B_3^*(T)-1)Q}{z - V^*(R)B_1^*(T)B_2^*(T)(1-q+qB_3^*(T))} \tag{41}$$

$$V(z) = \int_0^\infty V(x,z)dx = \frac{rB_1^*(T)B_2^*(T)(1-q+qB_3^*(T))(V^*(R)-1)Q}{p[z - V^*(R)B_1^*(T)B_2^*(T)(1-q+qB_3^*(T))]} \tag{42}$$

The unknown idle probability Q is obtained using the normalizing condition $Q + P^{(1)}(1) + P^{(2)}(1) + P^{(3)}(1) + V(1) = 1$, as

$$Q = \frac{1 - \lambda r(b_{11} + b_{21} + qb_{31}) - \lambda p v_1}{1 - \lambda(p-r)v_1} \tag{43}$$

Equations (39)–(42) together with equation (43) are respectively, the probability generating functions of the number of customers in the queue when the server is, serving phase 1 service, serving phase 2 service and serving phase 3 service respectively and when the server is on vacation.

Here $Q > 0$ guarantees the existence of the probability generating functions in equations (39)–(42) and therefore the stability condition for the system takes the form $\lambda r(b_{11} + b_{21} + qb_{31}) + \lambda p v_1 < 1$, already theoretically proved this in section 3.

The probability generating function for the number of customers in the queue irrespective of server state is

$$\begin{aligned} U(z) &= Q + P^{(1)}(z) + P^{(2)}(z) + P^{(3)}(z) + V(z) \\ &= \frac{[1 - \lambda r(b_{11} + b_{21} + qb_{31}) - \lambda p v_1]}{p[1-\lambda(p-r)v_1]\,[z - V^*(R)B_1^*(T)B_2^*(T)(1-q+qB_3^*(T))]} \\ &\quad \times \{[(z-1)p - (p-r)(V^*(R)-1)B_1^*(T)B_2^*(T)(1-q+qB_3^*(T))]\} \end{aligned}$$

6. The Performance Measures Using straightforward calculations the following performance measures have been obtained:

(i) The mean number of customers in the queue

$$L_q = U^{'}(1)$$
$$= \frac{\lambda^2 r C_1}{2[1-\lambda(p-r)v_1][1-\lambda r(b_{11}+b_{21}+qb_{31})-\lambda p v_1]}$$

where

$$C_1 = r[1-\lambda(p-r)v_1][b_{12}+b_{22}+qb_{32}+2b_{11}b_{21}]$$
$$+ [1+\lambda(p-r)(b_{11}+b_{21}+qb_{31})][2qrb_{31}v_1+pv_2] + 2r[b_{11}+b_{21}]$$
$$\times [v_1(1+\lambda(p-r)(b_{11}+b_{21})+qb_{31}]$$

(ii) The variance of the number of customers in the queue

$$V_{L_q} = U^{''}(1) + U^{'}(1) - [U^{'}(1)]^2$$
$$= \frac{\lambda^2 r}{12[1-\lambda(p-r)v_1]^2[1-\lambda r(b_{11}+b_{21}+qb_{31})-\lambda p v_1]^2}$$
$$\times \Big\{ 2[1-\lambda(p-r)v_1][C_2 + 3C_1(1-\lambda r(b_{11}+b_{21}+qb_{31})-\lambda p v_1)]$$
$$- 3\lambda^2 r C_1^2 \Big\}$$

where

$$U^{''}(1) = \frac{\lambda^2 r C_2}{6[1-\lambda(p-r)v_1][1-\lambda r(b_{11}+b_{21}+qb_{31})-\lambda p v_1]^2}$$
$$C_2 = 2\lambda[1-\lambda r(b_{11}+b_{21}+qb_{31})-\lambda p v_1][[1+\lambda(p-r)(b_{11}+b_{21}+qb_{31})]$$
$$\times [3r(b_{11}+b_{21})(2qrb_{31}v_1+pv_2)+p^2v_3+3pqrb_{31}v_2] + 6r^2 b_{11}b_{21}v_1$$
$$\times [1+\lambda(p-r)(b_{11}+b_{21})] + 6r^2 q b_{11}b_{21}b_{31} + 3qr^2 b_{32}(b_{11}+b_{21})$$
$$+ 3\lambda r^2 v_1(p-r)[b_{11}b_{12}+b_{21}b_{22}+q^2 b_{31}b_{32}] + r^2[b_{13}+b_{23}+qb_{33}]$$
$$\times [1-\lambda(p-r)v_1] + 3r^2 v_1[b_{12}+b_{22}+qb_{32}] + 3r^2[b_{12}(b_{21}+qb_{31})$$
$$+ b_{22}(b_{11}+qb_{31})]] + 3\lambda^2 C_1[2pqrb_{31}v_1 + 2r(b_{11}+b_{21})(rqb_{31}+pv_1)$$
$$+ p^2 v_2 + r^2(b_{12}+b_{22}+qb_{32}) + 2r^2 b_{11}b_{21}]$$

(iii) The expected waiting time in the queue

$$W_q = \frac{L_q}{\lambda^{'}}$$
$$= \frac{\lambda C_1}{2[1-\lambda r(b_{11}+b_{21}+qb_{31})-\lambda p v_1]}$$

where

$$\lambda^{'} = \text{actual arrival rate}$$
$$= \lambda r[P^{(1)}(1)+P^{(2)}(1)+P^{(3)}(1)+Q] + \lambda p V(1)$$
$$= \frac{\lambda r}{1-\lambda(p-r)v_1}$$

(iv) The variance of the waiting time in the queue

$$V_{W_q} = \frac{V_{L_q}}{\lambda'^2}$$
$$= \frac{1}{12r[1-\lambda r(b_{11}+b_{21}+qb_{31})-\lambda pv_1]^2}\Big\{2[1-\lambda(p-r)v_1][C_2+3C_1$$
$$\times(1-\lambda pv_1-\lambda r(b_{11}+b_{21}+qb_{31}))]-3\lambda^2 rC_1^2\Big\}$$

(v) Utilization factor which is the fraction of time that the server is busy

$$\rho = \lambda r(b_{11}+b_{21}+qb_{31})+\lambda pv_1$$

(vi) The mean number of customers in the system

$$L = L_q+\rho$$
$$= \frac{\lambda^2 rC_1}{2[1-\lambda(p-r)v_1][1-\lambda r(b_{11}+b_{21}+qb_{31})-\lambda pv_1]}+\lambda r(b_{11}+b_{21}+qb_{31})$$
$$+\lambda pv_1$$

(vii) The mean response time defined as Mean time a customer spends in the system which is equal to Mean number of customers in the system $\times$ Mean inter-arrival time

$$M = \frac{L}{\lambda'}$$
$$= \frac{\lambda C_1}{2[1-\lambda r(b_{11}+b_{21}+qb_{31})-\lambda pv_1]}+\frac{1}{r}[r(b_{11}+b_{21}+qb_{31})+pv_1]$$
$$\times[1-\lambda(p-r)v_1]$$

(viii) The mean number of customers in the queue when the server is busy

$$L_{q_b} = P^{(1)'}(1)+P^{(2)'}(1)+P^{(3)'}(1)$$
$$= \frac{\lambda^2 r}{2[1-\lambda(p-r)v_1][1-\lambda r(b_{11}+b_{21}+qb_{31})-\lambda pv_1]}\Big\{r(1-\lambda pv_1)$$
$$\times[b_{12}+b_{22}+qb_{32}+2b_{11}b_{21}+2qb_{31}(b_{11}+b_{21})]+\lambda p[b_{11}+b_{21}+qb_{31}]$$
$$\times[2qrb_{31}v_1+2rv_1(b_{11}+b_{21})+pv_2]\Big\}$$

(ix) The mean number of customers in the queue when the server is on vacation

$$L_{qv} = V'(1)$$
$$= \frac{\lambda^2 r}{2[1-\lambda(p-r)v_1][1-\lambda r(b_{11}+b_{21}+qb_{31})-\lambda pv_1]}\Big\{2rv_1(b_{11}+b_{21})$$
$$\times(1+\lambda qrb_{31})-\lambda r(b_{11}+b_{21}+qb_{31})[2rv_1(b_{11}+b_{21}+qb_{31})+pv_2]+pv_2$$
$$+2rv_1[qb_{31}+\lambda rb_{11}b_{21}]+\lambda r^2 v_1[b_{12}+b_{22}+qb_{32}]\Big\}$$

7. Particular Model In this section, a particular model has been derived by taking standard distributions to service times and vacation time. The arrival process is Poisson with parameter λ. The service times (all phase 1, phase 2 and phase 3) are negative exponential with parameters μ_1 for phase 1, μ_2 for phase 2 and μ_3 for phase 3 and the vacation time is hyper exponential with parameters $q_1, q_2(q_1+q_2=1), \theta_1$ and θ_2. The formulae the idle probability, the probability generating function for the number of customers in the queue irrespective of server state, the mean number of customers in the queue, the variance of the number of customers in the queue, the expected waiting time in the queue, the variance of the waiting time in the queue, the utilization factor, the mean number of customers in the system, the mean response time, the mean number of customers in the queue when the server is busy and the mean number of customers in the queue when the server is on vacation are derived in section 5 and 6 are used. The results of the formulae's as follows:

$$
\begin{aligned}
Q &= \frac{E_0}{\mu_1\mu_2\mu_3[\theta_1\theta_2-\lambda(p-r)(q_1\theta_2+q_2\theta_1)]} \\
U(z) &= \frac{(z-1)E_0}{\mu_1\mu_2\mu_3E_1[\theta_1\theta_2-\lambda(p-r)(q_1\theta_2+q_2\theta_1)]}\Big\{[T^2+T(\mu_1+\mu_2)+\mu_1\mu_2] \\
&\quad\times(T+\mu_3)[R^2+R(\theta_1+\theta_2)+\theta_1\theta_2]-\lambda\mu_1\mu_2(p-r)(T(1-q)+\mu_3) \\
&\quad\times[R+\theta_1(1-q_1)+\theta_2(1-q_2)]\Big\} \\
L_q &= \frac{\lambda^2rE_3}{\mu_1\mu_2\mu_3E_2[\theta_1\theta_2-\lambda(p-r)(q_1\theta_2+q_2\theta_1)]} \\
V_{L_q} &= \frac{\lambda^2rE_4}{\theta_1\theta_2\mu_1^2\mu_2^2\mu_3^2E_2^2[\theta_1\theta_2-\lambda(p-r)(q_1\theta_2+q_2\theta_1)]^2} \\
W_q &= \frac{\lambda E_3}{\mu_1\mu_2\mu_3\theta_1\theta_2E_2} \\
V_{W_q} &= \frac{E_4}{r\mu_1^2\mu_2^2\mu_3^2\theta_1^3\theta_2^3E_2^2} \\
\rho &= \frac{\lambda r\theta_1\theta_2[\mu_3(\mu_1+\mu_2)+q\mu_1\mu_2]+\lambda p(q_1\theta_2+q_2\theta_1)\mu_1\mu_2\mu_3}{\mu_1\mu_2\mu_3\theta_1\theta_2} \\
L &= \frac{\lambda^2rE_3}{\mu_1\mu_2\mu_3E_2[\theta_1\theta_2-\lambda(p-r)(q_1\theta_2+q_2\theta_1)]} \\
&\quad+\frac{\lambda[p\mu_1\mu_2\mu_3(q_1\theta_2+q_2\theta_1)+r\theta_1\theta_2(\mu_3(\mu_1+\mu_2)+q\mu_1\mu_2)]}{\theta_1\theta_2\mu_1\mu_2\mu_3} \\
M &= \frac{[p\mu_1\mu_2\mu_3(q_1\theta_2+q_2\theta_1)+r\theta_1\theta_2(\mu_3(\mu_1+\mu_2)+q\mu_1\mu_2)]}{r\theta_1^2\theta_2^2\mu_1\mu_2\mu_3} \\
&\quad\times[\theta_1\theta_2-\lambda(p-r)(q_1\theta_2+q_2\theta_1)]+\frac{\lambda E_3}{\mu_1\mu_2\mu_3E_2\theta_1\theta_2}
\end{aligned}
$$

$$L_{q_b} = \frac{\lambda^2 r}{\mu_1\mu_2\mu_3 E_2[\theta_1\theta_2 - \lambda(p-r)(q_1\theta_2 + q_2\theta_1)]}\Big\{r\theta_1\theta_2[\theta_1\theta_2 - \lambda p(q_1\theta_2$$
$$+ q_2\theta_1)][\mu_3^2(\mu_1^2 + \mu_2^2 + \mu_1\mu_2) + q\mu_1\mu_2(\mu_3(\mu_1+\mu_2) + \mu_1\mu_2)] + \lambda p$$
$$\times [\mu_3(\mu_1+\mu_2) + q\mu_1\mu_2][r\theta_1\theta_2(q_1\theta_2 + q_2\theta_1)(\mu_3(\mu_1+\mu_2) + q\mu_1\mu_2)$$
$$+ p\mu_1\mu_2\mu_3(q_1\theta_2^2 + q_2\theta_1^2)]\Big\}$$

$$L_{qv} = \frac{\lambda^2 r}{\mu_1\mu_2\mu_3 E_2[\theta_1\theta_2 - \lambda(p-r)(q_1\theta_2 + q_2\theta_1)]}\Big\{[\mu_1\mu_2\mu_3 - \lambda r(\mu_3(\mu_1+\mu_2)$$
$$+ q\mu_1\mu_2)][p\mu_1\mu_2\mu_3(q_1\theta_2^2 + q_2\theta_1^2) + r\theta_1\theta_2(q_1\theta_2 + q_2\theta_1)(\mu_3(\mu_1+\mu_2)$$
$$+ q\mu_1\mu_2)] + \lambda r^2\theta_1\theta_2(q_1\theta_2 + q_2\theta_1)[\mu_3^2(\mu_1^2 + \mu_2^2 + \mu_1\mu_2) + q\mu_1\mu_2(\mu_3(\mu_1$$
$$+ \mu_2) + \mu_1\mu_2)]\Big\}$$

where

$$E_0 = \mu_1\mu_2\mu_3(\theta_1\theta_2 - \lambda p(q_1\theta_2 + q_2\theta_1)) - \lambda r\theta_1\theta_2(\mu_3(\mu_1+\mu_2) + q\mu_1\mu_2)$$
$$E_1 = z(T+\mu_3)[T^2 + T(\mu_1+\mu_2) + \mu_1\mu_2][R^2 + R(\theta_1+\theta_2) + \theta_1\theta_2]$$
$$- \mu_1\mu_2(T(1-q) + \mu_3)[R(q_1\theta_1 + q_2\theta_2) + \theta_1\theta_2]$$
$$E_2 = \mu_1\mu_2\mu_3[\theta_1\theta_2 - \lambda p(q_1\theta_2 + q_2\theta_1)] - \lambda r\theta_1\theta_2[\mu_3(\mu_1+\mu_2) + q\mu_1\mu_2]$$
$$E_3 = r\theta_1\theta_2[\theta_1\theta_2 - \lambda(p-r)(q_1\theta_2 + q_2\theta_1)][\mu_3^2(\mu_1^2 + \mu_2^2 + \mu_1\mu_2) + q\mu_1^2\mu_2^2]$$
$$+ \mu_1\mu_2[qr\theta_1\theta_2(q_1\theta_2 + q_2\theta_1) + p\mu_3(q_1\theta_2^2 + q_2\theta_1^2)][\mu_1\mu_2\mu_3 + \lambda(p-r)$$
$$\times (\mu_3(\mu_1+\mu_2) + q\mu_1\mu_2)] + r\theta_1\theta_2\mu_3(\mu_1+\mu_2)[\mu_3(q_1\theta_2 + q_2\theta_1)$$
$$\times (\lambda(p-r)(\mu_1+\mu_2) + \mu_1\mu_2) + q\mu_1\mu_2\theta_1\theta_2]$$
$$E_4 = [\theta_1\theta_2 - \lambda(p-r)(q_1\theta_2 + q_2\theta_1)][E_2(2\lambda E_5 + \theta_1\theta_2\mu_1\mu_2\mu_3 E_3) + 2\lambda^2 E_3$$
$$\times [\mu_1^2\mu_2^2 p\mu_3(qr\theta_1\theta_2(q_1\theta_2 + q_2\theta_1) + p\mu_3(q_1\theta_2^2 + q_2\theta_1^2)) + r^2\theta_1^2\theta_2^2(q\mu_1^2\mu_2^2$$
$$+ \mu_3^2(\mu_1^2 + \mu_2^2)) + r\mu_1\mu_2\mu_3\theta_1\theta_2((\mu_1+\mu_2)(qr\theta_1\theta_2 + p\mu_3(q_1\theta_2 + q_2\theta_1))$$
$$+ r\mu_3\theta_1\theta_2)]] - \lambda^2 r\theta_1\theta_2 E_3^2$$
$$E_5 = \mu_1\mu_2\mu_3[\mu_1\mu_2\mu_3 + \lambda(p-r)(\mu_3(\mu_1+\mu_2) + q\mu_1\mu_2)][qr^2\theta_1^2\theta_2^2(\mu_1+\mu_2)$$
$$\times (q_1\theta_2 + q_2\theta_1) + p^2\mu_1\mu_2\mu_3(q_1\theta_2^3 + q_2\theta_1^3) + pr\theta_1\theta_2(q_1\theta_2^2 + q_2\theta_1^2)(q\mu_1\mu_2$$
$$+ \mu_3(\mu_1+\mu_2))] + r^2\theta_1^3\theta_2^3[\mu_3(\mu_1+\mu_2)(\mu_3^2(\mu_1^2+\mu_2^2) + q\mu_1^2\mu_2^2) + q\mu_1\mu_2$$
$$\times (\mu_3^2(\mu_1^2 + \mu_2^2 + \mu_1\mu_2) + \mu_1^2\mu_2^2)] + r^2\theta_1^2\theta_2^2\mu_1\mu_2(q_1\theta_2 + q_2\theta_1)[\lambda(p-r)$$
$$\times (\mu_3^3(\mu_1+\mu_2) + q(q-1)\mu_1^2\mu_2^2) + \mu_3^3(\mu_1^2+\mu_2^2) + q\mu_1^2\mu_2^2\mu_3]$$

Remark: The corresponding results of Erlang-k with parameter θ are given by

$$Q = \frac{\mu_1\mu_2\mu_3(\theta - \lambda p) - \lambda r\theta(\mu_3(\mu_1+\mu_2) + q\mu_1\mu_2)}{\mu_1\mu_2\mu_3(\theta - \lambda(p-r))}$$
$$L_q = \frac{\lambda^2 r E_7}{2k\mu_1\mu_2\mu_3(\theta - \lambda(p-r))[\mu_1\mu_2\mu_3(\theta - \lambda p) - \lambda r\theta(\mu_3(\mu_1+\mu_2) + q\mu_1\mu_2)]}$$

$$U(z) = \frac{[\mu_1\mu_2\mu_3(\theta-\lambda p) - \lambda r\theta(\mu_3(\mu_1+\mu_2)+q\mu_1\mu_2)]}{p\mu_1\mu_2\mu_3E_6(\theta-\lambda(p-r))}\Big\{p(z-1)(T+\mu_3)$$
$$\times (k\theta+R)^k[T^2+T(\mu_1+\mu_2)+\mu_1\mu_2] - \mu_1\mu_2(p-r)(T(1-q)+\mu_3)$$
$$\times [(k\theta)^k - (k\theta+R)^k]\Big\}$$
$$V_{L_q} = \frac{(\lambda^2 rE_8/(\theta-\lambda(p-r))^2)}{12k^2\mu_1^2\mu_2^2\mu_3^2\theta[\mu_1\mu_2\mu_3(\theta-\lambda p)-\lambda r\theta(\mu_3(\mu_1+\mu_2)+q\mu_1\mu_2)]^2}$$
$$W_q = \frac{\lambda E_7}{2k\theta\mu_1\mu_2\mu_3[\mu_1\mu_2\mu_3(\theta-\lambda p)-\lambda r\theta(\mu_3(\mu_1+\mu_2)+q\mu_1\mu_2)]}$$
$$V_{W_q} = \frac{E_8}{12rk^2\mu_1^2\mu_2^2\mu_3^2\theta^3[\mu_1\mu_2\mu_3(\theta-\lambda p)-\lambda r\theta(\mu_3(\mu_1+\mu_2)+q\mu_1\mu_2)]^2}$$
$$\rho = \frac{\lambda r\theta[\mu_3(\mu_1+\mu_2)+q\mu_1\mu_2]+\lambda p\mu_1\mu_2\mu_3}{\mu_1\mu_2\mu_3\theta}$$
$$L = \frac{(\lambda r/2k\theta\mu_1\mu_2\mu_3)[\lambda r\theta E_7+2kE_5(\theta-\lambda(p-r))]}{(\theta-\lambda(p-r))[\mu_1\mu_2\mu_3(\theta-\lambda p)-\lambda r\theta(\mu_3(\mu_1+\mu_2)+q\mu_1\mu_2)]}$$
$$M = \frac{\lambda r\theta E_7+2kE_5(\theta-\lambda(p-r))}{2kr\theta^2\mu_1\mu_2\mu_3[\mu_1\mu_2\mu_3(\theta-\lambda p)-\lambda r\theta(\mu_3(\mu_1+\mu_2)+q\mu_1\mu_2)]}$$
$$L_{q_b} = \frac{(\lambda^2 r/2k\mu_1\mu_2\mu_3)}{(\theta-\lambda(p-r))[\mu_1\mu_2\mu_3(\theta-\lambda p)-\lambda r\theta(\mu_3(\mu_1+\mu_2)+q\mu_1\mu_2)]}$$
$$\times \Big\{2kr\theta(\theta-\lambda p)[\mu_3^2(\mu_1^2+\mu_2^2+\mu_1\mu_2)+q\mu_1\mu_2(\mu_3(\mu_1+\mu_2)+\mu_1\mu_2)]$$
$$+\lambda p[\mu_3(\mu_1+\mu_2)+q\mu_1\mu_2][\mu_1\mu_2(2kqr\theta+p\mu_3(k+1))+2kr\theta\mu_3$$
$$\times (\mu_1+\mu_2)]\Big\}$$
$$L_{qv} = \frac{(\lambda^2 r/2k\mu_1\mu_2\mu_3)}{(\theta-\lambda(p-r))[\mu_1\mu_2\mu_3(\theta-\lambda p)-\lambda r\theta(\mu_3(\mu_1+\mu_2)+q\mu_1\mu_2)]}$$
$$\times \Big\{p\mu_1^2\mu_2^2\mu_3^2(k+1)+2\lambda k\theta r^2[\mu_3^2(\mu_1^2+\mu_2^2+\mu_1\mu_2)+q\mu_1\mu_2(\mu_3(\mu_1$$
$$+\mu_2)+\mu_1\mu_2)]+[\mu_3(\mu_1+\mu_2)+q\mu_1\mu_2][r\mu_1\mu_2\mu_3(2k\theta-\lambda p(k+1))$$
$$-2\lambda kr^2\theta(\mu_3(\mu_1+\mu_2)+q\mu_1\mu_2)]\Big\}$$

where

$$E_6 = z(T+\mu_1)(T+\mu_2)(T+\mu_3)(k\theta+R)^k - \mu_1\mu_2(k\theta)^k(T(1-q)+\mu_3)$$
$$E_7 = 2kr\theta[(\theta-\lambda(p-r))(\mu_3^2(\mu_1^2+\mu_2^2+\mu_1\mu_2)+q\mu_1^2\mu_2^2)+\mu_3(\mu_1+\mu_2)$$
$$\times (\lambda\mu_3(p-r)(\mu_1+\mu_2)+\mu_1\mu_2(q\theta+\mu_3))]+\mu_1\mu_2[\mu_1\mu_2\mu_3+\lambda(p-r)$$
$$\times (\mu_3(\mu_1+\mu_2)+q\mu_1\mu_2)](2qkr\theta+p\mu_3(k+1))$$
$$E_8 = 2(\theta-\lambda(p-r))[(\mu_1\mu_2\mu_3(\theta-\lambda p)-\lambda r\theta(\mu_3(\mu_1+\mu_2)+q\mu_1\mu_2))(3k\theta$$
$$\times \mu_1\mu_2\mu_3E_7+2\lambda E_9)+3\lambda^2E_7[p^2\mu_1^2\mu_2^2\mu_3^2(k+1)+2kr\theta(\mu_1\mu_2\mu_3$$
$$\times (pq\mu_1\mu_2+(\mu_1+\mu_2)(qr\theta+p\mu_3))+r\theta(\mu_3^2(\mu_1^2+\mu_2^2+\mu_1\mu_2)$$
$$+q\mu_1^2\mu_2^2))]]-3\lambda^2r\theta E_7^2$$

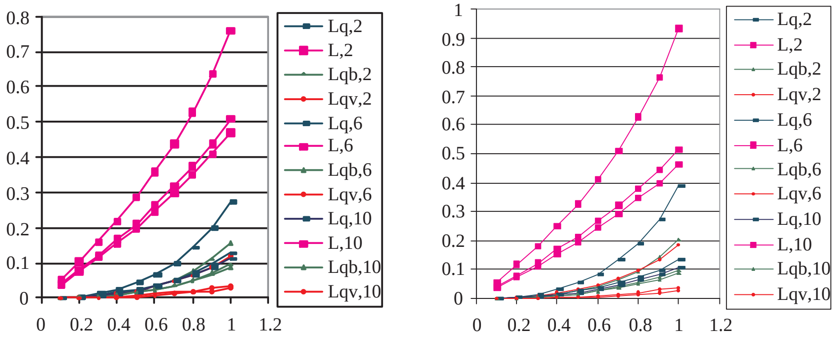
$$
\begin{aligned}
E_9 = {} & \mu_1\mu_2\mu_3[\mu_1\mu_2\mu_3 + \lambda(p-r)(\mu_3(\mu_1+\mu_2)+q\mu_1\mu_2)][3kr\theta(\mu_1+\mu_2)(2kqr\theta \\
& + p\mu_3(k+1)) + p\mu_1\mu_2(k+1)(3kqr\theta + p\mu_3(k+2))] + 6r^2k^2\theta^2[\mu_1\mu_2\mu_3(\mu_1 \\
& + \mu_2)(\mu_3^2(\lambda(p-r)+\theta) + q\theta\mu_1\mu_2) + \mu_3^2[\mu_1\mu_2(\mu_3+q\theta)(\mu_1^2+\mu_2^2+\mu_1\mu_2) \\
& + \mu_3\theta(\mu_1^3+\mu_2^3)] + q\mu_1^3\mu_2^3[\theta+\mu_3+\lambda(p-r)(q-1)]]
\end{aligned}
$$

Special case: $M/G/1$ queue with compulsory vacation

$$
\begin{aligned}
Q &= 1 + \lambda(B^{*'}(0) + V^{*'}(0)) \\
L_q &= \frac{\lambda^2[V''(0) + 2B^{*'}(0)V^{*'}(0) + B^{*''}(0)]}{2[1+\lambda(B^{*'}(0)+V^{*'}(0))]} \\
W_q &= \frac{\lambda[V''(0) + 2B^{*'}(0)V^{*'}(0) + B^{*''}(0)]}{2[1+\lambda(B^{*'}(0)+V^{*'}(0))]}
\end{aligned}
$$

The results coincide with the results of Madan [16], with exponential service time and vacation time.

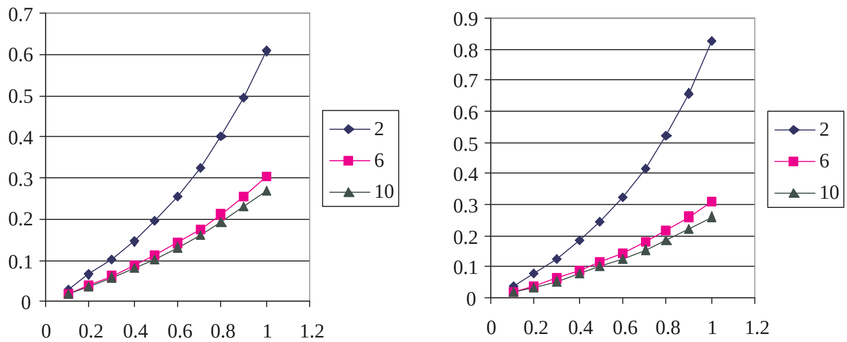
8. The Numerical Study In this section, some numerical examples related to the particular model analyzed in section 7 are given. The phase 1 service rate $\mu_1 = 3$, phase 2 service rate $\mu_2 = 4$, phase 3 service rate $\mu_3 = 5$, and the probabilities $r = 0.4, p = 0.8, q = 0.3, q_1 = 0.4, q_2 = 0.6, k = 3$ are fixed. The graphs of the mean number of customers in the queue (L_q), the mean number of customers in the system (L), the mean number of customers in the queue when the server is busy (L_{qb}), the mean number of customers in the queue when the server is on vacation (L_{qv}), and the expected waiting time (W_q) are drawn by varying arrival rate. In figure 1a, the graphs represent the mean value by fixing $\theta_2 = 7.0$ and varying θ_1 as $2, 6,$ and 10. The graphs in figure 1b are obtained by fixing $\theta_1 = 7.0$ and varying θ_2 as $2, 6,$ and 10. The graphs in figures 1a and 1b represent the effect of arrival rate (0.1 to 1) on L_q, L, L_{qb}, and L_{qv} for different values of θ_1 (figure 1a) and θ_2 (figure 1b). It is noticed that L_q, L, L_{qb}, L_{qv} increases sharply as the arrival rate increases for the lower values of $\theta_1(\theta_2)$ but increases gradually for higher values of $\theta_1(\theta_2)$. Figures 2a and 2b represent the expected waiting time in the queue against arrival rate for varying vacation rate. The graphs of the expected waiting time in the queue increase for the growing values of the arrival rate and increases sharply for lower values of $\theta_1(\theta_2)$. Tables 1- 4 for the variance of the number of customers in the queue (V_{L_q}), the variance of the waiting time in the queue (V_{W_q}), the utilization factor (ρ), and the mean response time (M) against the varying arrival rate are given. The results are obtained by varying the vacation rates. Tables 1 and 2 represents the variance of the number of customers in the queue and the variance of the waiting time in the queue. The utilization factors are tabulated in table 3. Table 4 gives the mean response time. All the tables show that the corresponding performance values increases for increasing values of arrival rate λ.



(a) Fixed $\theta_2 = 7.0$ and varying θ_1 as $2, 6$, and 10.

(b) Fixed $\theta_1 = 7.0$ and varying θ_2 as $2, 6$, and 10.

Figure 1: Arrival rate vs. the mean number of customer in the queue.



(a) Fixed $\theta_2 = 7.0$ and varying θ_1 as $2, 6$, and 10.

(b) Fixed $\theta_1 = 7.0$ and varying θ_2 as $2, 6$, and 10.

Figure 2: Arrival rate vs. the expected waiting time in the queue.

9. A Simulation Study In this section, a simulation model has been generated corresponding to the model discussed in this article. From the simulation model the quantity, mean number of customers in the queue has been calculated. The simulation is based on discrete event simulation. To do the simulation, the following variable has been used:

Time variable$= t$.

Countable variables: $N_A =$ Number of arrivals during time t

$N_D =$ Number of departure during time t

Table 1: The variance of the number of customers in the queue

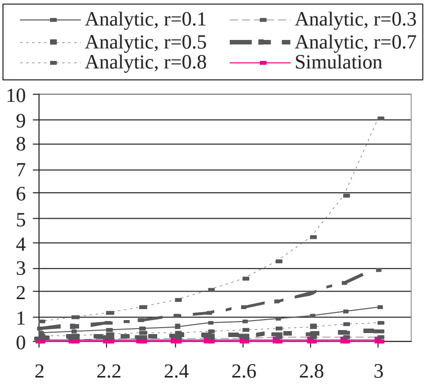
Arrival rate (λ)	$\theta_1 = 2$, $\theta_2 = 7$	$\theta_1 = 7$, $\theta_2 = 2$	$\theta_1 = 6$, $\theta_2 = 7$	$\theta_1 = 7$, $\theta_2 = 6$	$\theta_1 = 10$, $\theta_2 = 7$	$\theta_1 = 7$, $\theta_2 = 10$
0.1	0.0013	0.0016	0.0008	0.0008	0.0007	0.0007
0.2	0.0060	0.0074	0.0034	0.0035	0.0032	0.0031
0.3	0.0156	0.0197	0.0085	0.0086	0.0077	0.0075
0.4	0.0322	0.0416	0.0166	0.0169	0.0151	0.0146
0.5	0.0588	0.0777	0.0287	0.0293	0.0258	0.0250
0.6	0.0996	0.1350	0.0459	0.0469	0.0409	0.0395
0.7	0.1606	0.2241	0.0696	0.0712	0.0615	0.0592
0.8	0.2507	0.3615	0.1017	0.1042	0.0891	0.0854
0.9	0.3829	0.5734	0.1448	0.1487	0.1255	0.1199
1.0	0.5775	0.9042	0.2021	0.2080	0.1733	0.1650

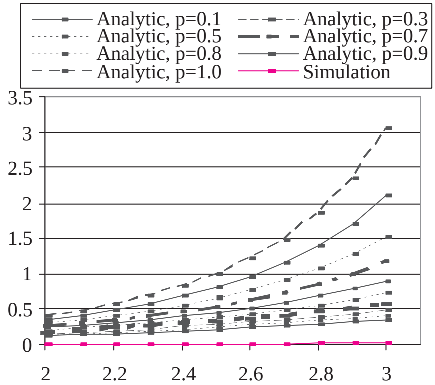
Table 2: The variance of the waiting time in the queue

Arrival rate (λ)	$\theta_1 = 2$, $\theta_2 = 7$	$\theta_1 = 7$, $\theta_2 = 2$	$\theta_1 = 6$, $\theta_2 = 7$	$\theta_1 = 7$, $\theta_2 = 6$	$\theta_1 = 10$, $\theta_2 = 7$	$\theta_1 = 7$, $\theta_2 = 10$
0.1	0.7970	0.9621	0.4841	0.4909	0.4486	0.4377
0.2	0.8956	1.0949	0.5237	0.5315	0.4833	0.4710
0.3	1.0109	1.2531	0.5681	0.5770	0.5220	0.5080
0.4	1.1466	1.4434	0.6181	0.6285	0.5652	0.5492
0.5	1.3077	1.6748	0.6749	0.6868	0.6138	0.5953
0.6	1.5007	1.9596	0.7394	0.7534	0.6685	0.6472
0.7	1.7343	2.3157	0.8134	0.8298	0.7306	0.7059
0.8	2.0210	2.7692	0.8987	0.9181	0.8014	0.7726
0.9	2.3781	3.3601	0.9979	1.0209	0.8827	0.8489
1.0	2.8313	4.1520	1.1141	1.1418	0.9767	0.9367

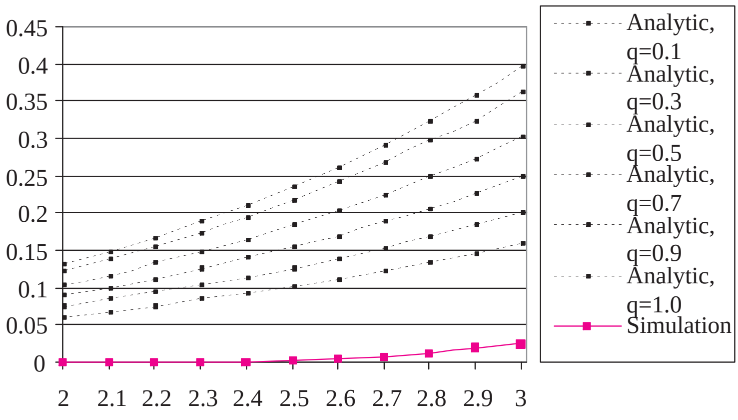
System state variable:
n = Number of customers in the system at time t.
Event type:
A = arrival, D = Departure.
$S1$ = Service completion(essential), $S2$ = Service completion (optional).
V = Vacation completion, I = Server Idle state.

The simulation process has been initialized by assuming the initial time $t = 0$ and the counting variables $N_A = N_D = 0$. To update the system, we move along the time axis until we encounter the next event. Using random numbers, time points has be generated corresponding to the events A,D,S1,S2 and V. For inter arrival time(A), second optional service time(S2) and va-





(a) (b)



(c)

Figure 3: Varying the probability of admission and optional service probability.

Table 3: Utilization factor

Arrival rate (λ)	$\theta_1 = 2$, $\theta_2 = 7$	$\theta_1 = 7$, $\theta_2 = 2$	$\theta_1 = 6$, $\theta_2 = 7$	$\theta_1 = 7$, $\theta_2 = 6$	$\theta_1 = 10$, $\theta_2 = 7$	$\theta_1 = 7$, $\theta_2 = 10$
0.1	0.0486	0.0543	0.0379	0.0383	0.0358	0.0351
0.2	0.0972	0.1086	0.0758	0.0766	0.0716	0.0702
0.3	0.1458	0.1629	0.1138	0.1149	0.1074	0.1053
0.4	0.1944	0.2172	0.1517	0.1532	0.1432	0.1404
0.5	0.2430	0.2715	0.1896	0.1915	0.1790	0.1755
0.6	0.2915	0.3258	0.2275	0.2298	0.2147	0.2106
0.7	0.3401	0.3801	0.2655	0.2681	0.2505	0.2457
0.8	0.3887	0.4344	0.3034	0.3064	0.2863	0.2808
0.9	0.4373	0.4887	0.3413	0.3447	0.3221	0.3159
1.0	0.4859	0.5430	0.3792	0.3830	0.3579	0.3510

Table 4: The mean response time

Arrival rate (λ)	$\theta_1 = 2$, $\theta_2 = 7$	$\theta_1 = 7$, $\theta_2 = 2$	$\theta_1 = 6$, $\theta_2 = 7$	$\theta_1 = 7$, $\theta_2 = 6$	$\theta_1 = 10$, $\theta_2 = 7$	$\theta_1 = 7$, $\theta_2 = 10$
0.1	1.2311	1.3746	0.9610	0.9706	0.9076	0.8905
0.2	1.2514	1.3969	0.9757	0.9853	0.9220	0.9047
0.3	1.2761	1.4255	0.9923	1.0020	0.9379	0.9205
0.4	1.3063	1.4617	1.0110	1.0209	0.9557	0.9380
0.5	1.3428	1.5073	1.0322	1.0424	0.9756	0.9574
0.6	1.3870	1.5646	1.0563	1.0669	0.9979	0.9792
0.7	1.4407	1.6365	1.0837	1.0947	1.0228	1.0034
0.8	1.5060	1.7274	1.1149	1.1265	1.0509	1.0305
0.9	1.5860	1.8432	1.1505	1.1630	1.0825	1.0610
1.0	1.6848	1.9928	1.1915	1.2050	1.1184	1.0954

cation time(V), We generate different exponential random variables, using random number U and for first essential service time random variable, We generate Erlang variable with two phase using random numbers. The simulation logic has been explained below:

$T =$ Time at which the simulation is to stop, $t =$ time, $t_A =$ time of next arrival.

$t_S =$ time of next service (second optional) completion.

$t_V =$ time of next vacation completion.

If $t_A > T$ or $t_S > T$, The simulation stops and the values of n, N_A and N_D are the most resent values.

If $t_A < t_S < T$, set $N_A = N_A + 1, n = n + 1$.

If $t_S < t_A < T$, set $N_D = N_D + 1, n = n - 1$.

If $t_V < t_A$, set, $N_A = N_A + 1, N_D = N_D, n = n + 1$.

The parameters $p, q, \mu_1, \mu_2, \mu_3$ and θ are to be fixed,value of λ has been varied. The aim is to give a meaningful analyzes of the model, by using non-stationary simulation. The result of mean number of customers have been simulated by fixing the study period as 8 units. The results are presented using graph. Also using the analytical formula in section 6, by taking exponential distribution for optional service, inter arrival time and vacation time and by taking Erlang distribution for phase, the mean value has been calculated. given. The analytical results are generated by varying the probability of admission (r, p) and optional service probability (q). The corresponding graphs are presented in the figures 3a, 3b and 3c. From figures 3a and 3b, it is clear that the simulation results almost nearer to the result corresponding to the probability $r = 0.1$ (figure 3a), $p = 0.1$ (figure 3b) on the other hand if the probability is 0.8 the difference is considerably large. Also from figure 3c we have the same experiences as for as the probability q is concern.

10. Results and Discussion In sections 8 and 9, we have presented the numerical results corresponding to the model discussed in this paper. We have used analytical results obtained and the corresponding simulation results. The results shows that the mean number of customers is almost a linear function with respect to arrival rate for simulation model. Whereas the mean function is an increasing curve in the case of analytic results. This study may be very helpful in the fields of computer system, wireless communication system and production system.

11. Conclusion In this paper we consider a single server queue with compulsory server vacation. The customers are admitted to queue using Bernoulli processes. The single server provides two phases of essential services and an optional service. Using the supplementary variable technique the probability generating functions of the number of customers in the queue for different server states are obtained. Some performance measures are calculated from the probability generating functions and we perform numerical study by assuming particular values to the parameters. Further, we present a simulation study.

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System kolejkowy z obowiązkową usługa dwufazową i obowiązkowymi przestojami, usługami na życzenie i ograniczona dostepnością.

Rathinasabapathy Kalyanaraman and V. Suvitha

Streszczenie Tematem tej pracy jest model serwera systemu kolejkowego z poissonowskim strumieniem zgłoszeń i obsługą podzieloną na dwie obowiązkowe fazy. Klienci mogą wymagać z pewnym prawdopodobieństwem opcjonalnej usługi dodatkowej po zakończeniu dwóch etapów podstawowych usługi. Po zakończeniu obsługi klienta następuje obowiązkowy przestój. Przy zajętym serwerze klienci dołączaja do kolejki zgodnie z procesem Bernoulliego. Przy badaniu systemu stosowane są metody zmiennych dodatkowych. Pewne szczególne przypadki są analizowane szczegółowo. Ilustracja są wyniki eksperymenów symulacyjnych.

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