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Law of the iterated logarithm type results for random vectors with infinite second moments

Abstract This survey paper is an extended version of the author's presentation at the conference in honor of Professor F. Thomas Bruss at the occasion of his retirement as Chair of Mathématiques Générales from the Unversité Libre de Bruxelles which was held September 9-11, 2015 in Brussels. I first present some results generalizing the classical Hartman-Wintner law of the iterated logarithm to 1-dimensional variables with infinite second moments and then I show how these results can be further extended to the d-dimensional setting. Finally, I look at general functional law of the iterated logarithm type results.

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1. Introduction. Let $X, X_1, X_2, \dots$ be i.i.d. random variables with finite variance σ^2 and mean zero. Set $S_n := \sum_{j=1}^n X_j, n \geqslant 1$, $Lt = \log(t \vee e), LLt = L(Lt), t \geqslant 0$. One of the fundamental results of probability is the law of the iterated logarithm(=LIL) which was first proved for the symmetric Bernoulli case by Khintchine in 1923. Later in 1929 Kolmogorov proved an LIL for independent, but not necessarily identically distributed random variables satisfying a certain bounding condition which directly implies that the LIL holds for i.i.d. **bounded** random variables (see, for instance, [1, Theorem 10.2.1]). Given this result, it was naturally to ask whether this result also holds under moment conditions and eventually the final result for this case was found by Hartman and Wintner in 1941.

Theorem 1.1 (Hartman-Wintner LIL)
If $\sigma^2 = \boldsymbol{E}[X^2] < \infty$ and $\boldsymbol{E}[X] = 0$, one has with probability one,

$$-\sigma = \liminf_{n\to\infty} \frac{S_n}{\sqrt{2nLLn}} \leqslant \limsup_{n\to\infty} \frac{S_n}{\sqrt{2nLLn}} = \sigma.$$

After the work of Hartman-Wintner it was still unclear for quite some time whether a further weakening of the above assumptions might be possible, but finally in 1966 Strassen [22] proved the following converse to the Hartman -Wintner LIL:

Given an arbitrary sequence of i.i.d. random variables $X_1, X_2, \ldots$. with partial sum sequence $S_n = \sum_{j=1}^n X_j, n \geqslant 1$, we have:

$$\limsup_{n\to\infty} |S_n|/\sqrt{nLLn} < \infty \text{ with prob. } 1 \;\Rightarrow \boldsymbol{E}X^2 < \infty \text{ and } \boldsymbol{E}X = 0.$$

For further historical background the reader is referred to the books [1, Section 10.2] and [16, Chapter 8] where also the LIL in Banach space is discussed.

Given the afore-mentioned result of Strassen one might think that there is no work left in connection with the LIL for sums of i.i.d. random variables in the infinite variance case. Looking more closely at this problem, however, shows that it is a question of choosing a more appropriate normalizing sequence. The classical sequence $\{\sqrt{2nLLn}\}$ is simply too "small" for sums of i.i.d random variables in the infinite variance case. Therefore, in order to obtain related results in this setting, one has to use different (bigger) normalizing sequences. A first result in this direction is due to Feller (1968) who obtained an LIL result for random variables in the domain of attraction to the normal distribution satisfying an extra integrability condition (see also [2] and [12] for some clarification). This result was further refined by Klass (1976, 1977). In particular, he obtained an LIL which is also valid for random variables outside the domain of attraction to the normal distribution.

A common feature of these results is that the authors find a method to define an increasing function $g : [0, \infty[\to]0, \infty[$ via the distribution of X such that one has under suitable assumptions on X with probability one,

$$0 < \limsup_{n\to\infty} |S_n|/g(n) < \infty.$$

In this case one speaks also of LIL behavior. These functions are often very difficult to determine. In some cases, they lead to LIL type results with nice normalizing sequences such as $\sqrt{n}(LLn)^\alpha$ for some $\alpha > 1/2$.

Motivated by this work, Einmahl and Li (2005) posed the problem:

Given a regular normalizing sequence c_n, when does one have with probability one,

$$0 < \limsup_{n\to\infty} |S_n|/c_n < \infty?$$

2. The basic result and its consequences. We now specify what we mean by a "regular" normalizing sequence. We assume that

$$c_n/\sqrt{n} \text{ is eventually non-decreasing} \tag{1}$$

and

$$\forall \epsilon > 0\ \exists m_\epsilon \geqslant 1 : c_n/c_m \leqslant (1+\epsilon)n/m, m_\epsilon \leqslant m \leqslant n. \tag{2}$$

Note that these assumptions are satisfied if one chooses normalizing sequences of the form $c_n = \sqrt{nh(n)}$ where $h : [0, \infty[\to [0, \infty[$ is increasing and slowly varying at infinity.

THEOREM 2.1 *Let $X, X_1, X_2, \ldots$ be i.i.d. mean zero random variables. Assume that*

$$\sum_{n=1}^{\infty} \boldsymbol{P}\{|X| \geqslant c_n\} < \infty, \tag{3}$$

where c_n satisfies conditions (1) and (2) Set

$$\alpha_0 = \sup \left\{ \alpha \geqslant 0 : \sum_{n=1}^{\infty} n^{-1} \exp\left(-\frac{\alpha^2 c_n^2}{2n\sigma_n^2} \right) = \infty \right\},$$

where $\sigma_n^2 := \boldsymbol{E}[X^2 I\{|X| \leqslant c_n\}]$. Then we have with probability one,

$$-\alpha_0 = \liminf_{n\to\infty} S_n/c_n \leqslant \limsup_{n\to\infty} S_n/c_n = \alpha_0. \tag{4}$$

The proofs of Theorem 2.1 and all other results in this section can be found in [8]. See also [19] for a related result.

Furthermore, one can show that under the assumptions of Theorem 2.1 with probability one, the set of all limit points of the sequence $\{S_n/c_n : n \geqslant 1\}$ is equal to the interval $[-\alpha_0, \alpha_0]$. We call this set the **cluster set** and we denote it by $C(\{S_n/c_n : n \geqslant 1\})$. Note that this set is in principle a random set, but an easy application of the $0-1$ law of Hewitt-Savage, shows that whenever $c_n \to \infty$ as $n \to \infty$, this random set is with probability one equal to a deterministic set A which is just the interval $[-\alpha_0, \alpha_0]$ in the above case.

If condition (3) is not satisfied, we trivially have that with probability one, $\limsup_{n\to\infty} |S_n|/c_n = \infty$ and the sequence c_n is too small for an LIL type result. This can also happen if (3) is satisfied and the parameter α_0 is infinite. (An example for this situation is $c_n = \sqrt{n}(LLn)^{1/4}$ in the finite variance case.)

On the other hand, we can also have $\alpha_0 = 0$ and we then obtain a stability result. (Example: If we choose $c_n = n$ one can show that $\alpha_0 = 0$ and we obtain the strong law of large numbers for mean zero random variables as a corollary of our theorem.)

The conclusion is that we have LIL behavior w.r.t. the normalizing sequence $\{c_n\}$ if and only if condition (3) is satisfied and we have for the above parameter

$$\boxed{0 < \alpha_0 < \infty}$$

The exact determination of this parameter can be difficult, but it turned out that in a special case one can infer a precise result from Theorem 2.1 which leads to a somewhat surprising generalization of the Hartman-Wintner LIL.

We call this result "the law of a very slowly varying function". In order to formulate our result, we first need to define what we mean by a very slowly varying function.

Recall that a non-decreasing function $h : [0, \infty[\to]0, \infty[$ is called slowly varying at infinity if we have $h(2t)/h(t) \to 1$ as $t \to \infty$. We require now more, namely that one can replace the factor "2" in the definition by certain functions f satisfying $f(t) \to \infty$ as $t \to \infty$.

To be more specific, set for $\tau > 0$, $f_\tau(t) := \exp((Lt)^\tau), t \geqslant 0$ and let $\mathcal{H}_0$ be the class of all non-decreasing functions $h : [0, \infty[\to]0, \infty[$ so that

$$\lim_{t \to \infty} h(t f_\tau(t))/h(t) = 1, 0 < \tau < 1. \tag{5}$$

An easy calculation shows that $(LLt)^p$, $p > 0$ and $(Lt)^r$, $r > 0$ are functions in $\mathcal{H}_0$ which we call the class of the very slowly varying functions.

Theorem 2.2 (Law of a very slowly varying function)
Let $X, X_1, X_2, \ldots$ be i.i.d. random variables. Set $\Psi(t) = \sqrt{th(t)}, t \geqslant 0$, where $h \in \mathcal{H}_0$ and let $\lambda \geqslant 0$. Then we have with probability one,

$$-\lambda = \liminf_{n \to \infty} S_n/\sqrt{nh(n)} \leqslant \limsup_{n \to \infty} S_n/\sqrt{nh(n)} = \lambda$$

if and only if

(i) $\sum_{n=1}^{\infty} \boldsymbol{P}\{|S_n| \geqslant \sqrt{nh(n)}\} < \infty$ *and* $\boldsymbol{E}[X] = 0$

(ii) $\limsup_{x \to \infty} \frac{\Psi^{-1}(xLLx)}{x^2 LLx} \boldsymbol{E}[X^2 \mathbb{1}_{\{|X| \leqslant x\}}] = \frac{\lambda^2}{2}$

Moreover, one can show (see [4, Remark 2]) that if one additionally assumes that $h(t) = O(\exp((LLt)^\gamma))$ for some $0 < \gamma < 1$, then condition (ii) implies that the series in (i) is finite. In this case the law of a very slowly varying function holds if and only if (ii) is satisfied and $\boldsymbol{E}[X] = 0$.

Obviously, this is the case if $h(t) = (LLt)^p$ with $p \geqslant 1$ and we obtain the following result.

Theorem 2.3 (General law of the iterated logarithm)
Let $X, X_1, X_2, \ldots$ be i.i.d. random variables. Then we have for any $p \geqslant 1$ and $\lambda \geqslant 0$ with probability one,

$$-\lambda = \liminf_{n \to \infty} S_n/\sqrt{2n(LLn)^p} \leqslant \limsup_{n \to \infty} S_n/\sqrt{2n(LLn)^p} = \lambda$$

if and only if $\limsup_{x \to \infty} \boldsymbol{E}[X^2 \mathbb{1}_{\{|X| \leqslant x\}}]/(LLx)^{p-1} = \lambda^2$ *and* $\boldsymbol{E}[X] = 0$.

If $p = 1$, the Hartman-Wintner LIL and its converse follow from Theorem 2.3 via monotone convergence.

Choosing $h(t) = (Lt)^r$ in Theorem 2.2, one obtains a law of the logarithm. In this case, the series condition in (i) is crucial and it cannot be dropped. It is easy to see that this series in (i) is finite if and only if $\boldsymbol{E}[X^2/(L|X|)^r] < \infty$ and we thus have,

Theorem 2.4 (Law of the logarithm)
Let $X, X_1, X_2, \ldots$ be i.i.d. random variables. Then we have for any fixed $r > 0$ and $\lambda \geqslant 0$ with probability one,

$$-\lambda = \liminf_{n\to\infty} S_n/\sqrt{2n(Ln)^r} \leqslant \limsup_{n\to\infty} S_n/\sqrt{2n(Ln)^r} = \lambda$$

if and only if

$$\limsup_{x\to\infty} \frac{LLx}{(Lx)^r}\boldsymbol{E}[X^2 \mathbb{1}_{\{|X|\leqslant x\}}] = 2^r\lambda^2, \boldsymbol{E}[X^2/(L|X|)^r] < \infty \text{ and } \boldsymbol{E}[X] = 0.$$

These results represent only a small subsample of the possible corollaries of our basic Theorem 2.1. We mention that one can also infer Feller's LIL and the two-sided version of the Klass LIL from it. One can also look at arbitrary slowly varying functions h, but then one has, in general, less tight bounds for α_0 than in the law of a very slowly varying function. The interested reader is referred to [8].

3. Results for multidimensional random vectors. We now look at the d-dimensional case ($d \geqslant 2$), where we denote the Euclidean norm by $|\cdot|$. We state first the d-dimensional version of the Hartman-Wintner LIL. As in the previous section we denote the (d-dimensional) partial sums $\sum_{j=1}^n X_j$ by $S_n, n \geqslant 1$.

Theorem 3.1 *Let $X, X_1, X_2, \ldots$ be i.i.d. random vectors and assume that $\boldsymbol{E}|X|^2 < \infty$ and $\boldsymbol{E}X = 0$. Then we have with probability one,*

$$\limsup_{n\to\infty} \frac{|S_n|}{\sqrt{2nLLn}} = \sigma,$$

where $\sigma^2 =$ is the largest eigenvalue of the covariance matrix Σ of X.

Moreover, we have with probability one, $C(\{S_n/c_n : n \geqslant 1\}) = K_\Sigma = \{\Sigma^{1/2}x : |x| \leqslant 1\}$.

Here we denote for any symmetric non-negative definite matrix A the unique symmetric non-negative definite matrix B satisfying $A = B^2$ by $A^{1/2}$ and we see that the cluster set is an ellipsoid (possibly degenerate) in this case.

The lim sup-part of this result can be proven directly via exponential inequalities, but it also follows from the d-dimensional version of Strassen's functional LIL (see Theorem 4.3 below).

Einmahl and Li (2008) (see [9, Theorem 2.2.]) found an extension of Theorem 2.1 to Banach space valued random variables which gives the following result when specialized to $\mathbb{R}^d$,

THEOREM 3.2 *Assume that $X, X_1, X_2, \dots$ are i.i.d random vectors such that $\sum_{n=1}^{\infty} \boldsymbol{P}\{|X| \geqslant c_n\} < \infty$, where c_n is a sequence of positive real numbers satisfying (1) and (2). Then we have with probability one,*

$$\limsup_{n\to\infty} |S_n|/c_n = \alpha_0,$$

where

$$\alpha_0 = \sup\left\{\alpha > 0 : \sum_{n=1}^{\infty} n^{-1} \exp\left(-\frac{\alpha^2 c_n^2}{2nH(c_n)}\right) = \infty\right\}.$$

The function H is defined by $H(t) = \sup_{|v|\leqslant 1} \boldsymbol{E}\langle X, v\rangle^2 \mathbb{1}_{\{|X|\leqslant t\}}$, $t \geqslant 0$.

Having this result it is easy to get extensions of all results in Section 2 to the d-dimensional case. As an example we look at the generalized law of the iterated logarithm. This reads as follows in the d-dimensional setting where we no longer distinguish between the two parameters λ and α_0.

THEOREM 3.3 (GENERAL LAW OF THE ITERATED LOGARITHM IN $\mathbb{R}^d$) *Let $X, X_1, X_2, \dots$ be i.i.d. random vectors. Then we have for $p \geqslant 1$ and $\alpha_0 \geqslant 0$ with probability one,*

$$\limsup_{n\to\infty} |S_n|/\sqrt{2n(LLn)^p} = \alpha_0$$

if and only if $\limsup_{x\to\infty} H(x)/(LLx)^{p-1} = \alpha_0^2$ *and* $\boldsymbol{E}[X] = 0$.

The determination of the cluster set $C(\{S_n/c_n : n \geqslant 1\})$, however, is less evident in the multidimensional setting. As in the 1-dimensional case this set is with probability one equal to a non-random set which we again denote by A. The above lim sup-result implies that A is a subset of $\alpha_0 U$, where U is the Euclidean unit ball. In the classical case (finite second moment) the set A is an ellipsoid which is determined by the covariance matrix, but in the infinite second moment case there is no covariance matrix and thus no reason why A should be an ellipsoid.

What one can show in general is

The cluster set has to be closed, symmetric and star-like with respect to 0.

Recall that a set M in a linear space is called star-like w.r.t to an element

$x_0 \in M$ if it contains for any element $x \in M$ the line segment connecting x and x_0. Obviously, convex sets M are star-like w.r.t to any point $x \in M$.

Note that the star-like sets in the 1-dimensional case are just the intervals so that the above result already implies that $A = [-\alpha_0, \alpha_0]$. In the multidimensional setting the situation becomes much more complex. Any union of (possibly degenerate) closed ellipsoids centered at 0 has the above properties and if $\alpha_0 < \infty$ it is possible to find distributions for which the cluster set is equal to a union of finitely many (possibly degenerate) closed ellipsoids contained in $\alpha_0 U$, where (at least) one of the ellipsoids must have a point with norm equal to α_0. A possibly choice for a degenerate ellipsoid is a line segment of the form $\{s\vec{v} : |s| \leqslant \alpha\}$, where $\vec{v}$ is a unit vector in $\mathbb{R}^d$. Any set which is star-like and symmetric with respect to 0 can be written as a closure of (at most) countably many line segments of this form. Using this observation Einmahl (1995) (see [3, Theorem 4]) was able to show the following (if $d = 2$),

Theorem 3.4 *Let $\alpha_0 > 0$ and suppose that $A \subset \mathbb{R}^2$ is closed, symmetric and star-like with respect to 0 and satisfies $\max_{x \in A} |x| = \alpha_0$. Then one can define i.i.d. mean zero random vectors $X_1, X_2, \ldots$ with $\boldsymbol{E}|X_1|^2 = \infty$ such that for a suitable sequence γ_n satisfying (1) and (2) with probability one,*

$$\limsup_{n\to\infty} |S_n|/\gamma_n = \alpha_0,$$

and

$$C(\{S_n/\gamma_n : n \geqslant 1\}) = A.$$

It is possible to extend this result to the general setting considered in Theorem 3.3 (general normalizing sequence $\{c_n\}$ and $d \geqslant 2$) so that we have an answer to the cluster set problem for Theorem 3.3.

Of course, one can also ask whether there are conditions which lead to "nicer" cluster sets. This problem was also investigated [3]. In this paper it is shown (see [3, Theorems 1 and 2]) that if $X = (X^{(1)}, X^{(2)})$ is a random vector with two independent components the cluster set $A = C(\{S_n/\gamma_n : n \geqslant 1\}$, where γ_n is as in Theorem 4.1, has a representation as the topological closure of a sequence of standard ellipses. Here we call any ellipse (possibly also degenerate) which is centered at zero where the main axes match the coordinate axes a standard ellipse. Moreover it is shown that all these sets occur also as cluster sets in this setting (random vectors with independent components). These sets are also among the cluster sets in Theorem 4.1, but the random vectors $X_1, X_2, \ldots$ in this result have not necessarily independent components.

4. Functional Strassen type LIL results As usual, we denote the set of all continuous functions $f : [0, 1] \to \mathbb{R}$ by $C[0, 1]$. Strassen (1964) proved a functional LIL which substantially improves the Hartman-Wintner LIL. It is natural now to ask whether there are also functional versions of the above results. It turned out that this is indeed the case. In Einmahl (2007) we have been able to show that whenever we have for the basic parameter in Theorem 2.1: $\boxed{0 < \alpha_0 < \infty}$, one also has a functional LIL with limit set $\alpha_0 \mathcal{K}$, where $\mathcal{K}$ is defined as in the Strassen LIL, i.e.

$$\mathcal{K} = \{f \in C[0,1] : f(t) = \int_0^t g(u)du, 0 \leqslant t \leqslant 1, \text{ where } \int_0^1 g^2(u)du \leqslant 1\}.$$

This means that whenever one has a "usual" LIL type result such as Theorem 2.2, one has automatically a functional version thereof.

To formulate our result we need some extra notation. Let $S_{(n)} : \Omega \to C[0,1]$ be the partial sum process of order n, that is,

$$S_{(n)}(t) = S_{[nt]} + (nt - [nt])X_{[nt]+1}, 0 \leqslant t \leqslant 1.$$

Theorem 4.1 (General functional LIL in $\mathbb{R}$)
Let $X, X_1, X_2, \dots$ be i.i.d. mean zero random variables.

Assume that $\sum_{n=1}^{\infty} \boldsymbol{P}\{|X| \geqslant c_n\} < \infty$, where c_n is a sequence of positive real numbers satisfying conditions (2.1) and (2.2). If $\alpha_0 < \infty$ we have with probability one,

$$\{S_{(n)}/c_n : n \geqslant 1\} \textit{ is relatively compact in } C[0,1] \tag{6}$$

and

$$C(\{S_{(n)}/c_n : n \geqslant 1\}) = \alpha_0 \mathcal{K}. \tag{7}$$

Note that this result implies for any continuous functional $\psi : C[0,1] \to \mathbb{R}^d$ that

$$\{\psi(S_{(n)}/c_n) : n \geqslant 1\} \text{ is relatively compact in } \mathbb{R}^d \text{ (=bounded)}$$

with cluster set equal to $\psi(\alpha_0 \mathcal{K})$.
Choosing $\psi(f) = f(1), f \in C[0,1]$ we can re-obtain Theorem 2.1 including the determination of the cluster set $C(\{S_n/c_n : n \geqslant 1\})$.
So in the 1-dimensional case the "story" is simple: whenever one has an LIL type result in the infinite variance case with normalizing sequence $\{c_n : n \geqslant 1\}$ satisfying conditions (2.1) and (2.2), one has also a functional LIL type result with canonical cluster set.
We now turn to the multi-dimensional setting where the problem becomes much more involved. First of all, one can also show that this (random) subset of $C_d[0,1]$ is with probability 1 equal to a (deterministic) function class in

$C_d[0,1]$), where $C_d[0,1]$ is the set of all continuous functions $f:[0,1]\to\mathbb{R}^d$ (see Proposition 3.1 in [7]).
In the previous section we have already seen that the cluster sets $A = C(\{S_n/c_n : n\geqslant 1\})$ can become very complicated and we cannot expect simple cluster sets for the corresponding functional LIL type results as we have $A=\{f(1): f\in\mathcal{A}\}$, where we set $\mathcal{A}=C(\{S_{(n)}/c_n : n\geqslant 1\})$.
It is quite remarkable that Strassen (1964) already proved the functional LIL for d-dimensional Brownian motion and combining this with a strong invariance principle for sums of independent d-dimensional random vectors due to Philipp (see [20]) one has under the classical moment assumptions,

Theorem 4.2 *Let $X, X_1, X_2, \ldots$ be i.i.d. random vectors and assume that $\boldsymbol{E}|X|^2<\infty$ and $\boldsymbol{E}X=0$. Then we have with probability one,*

$$\{S_{(n)}/c_n : n\geqslant 1\} \textit{ is relatively compact in } C_d[0,1] \tag{8}$$

and

$$C(\{S_{(n)}/c_n : n\geqslant 1\}) = \mathcal{K}_\Sigma, \tag{9}$$

where Σ is the covariance matrix of X,

$$\mathcal{K}_\Sigma = \{\Sigma^{1/2} f : f\in\mathcal{K}_d\},$$

and $\mathcal{K}_d=\{f=(f_1,\ldots,f_d)\in\mathcal{K}^d : \sum_{i=1}^d \|f_i'\|_2^2\leqslant 1\}$.

Note that Theorem 4.3 applied with the functional $\psi(f)=f(1), f\in C_d[0,1]$ implies Theorem 3.1.

If $\boxed{\boldsymbol{E}|X|^2=\infty, \boldsymbol{E}X=0}$ and c_n is a sequence satisfying conditions (1) and (2), it follows from a general result of Einmahl and Kuelbs (2014) (for random variables with values in a separable Banach space) that the first half of Strassen's LIL holds also in this setting whenever $\alpha_0<\infty$, that is, we have then with probability one,

$$\{S_{(n)}/c_n : n\geqslant 1\} \text{ is relatively compact in } C_d[0,1]. \tag{10}$$

From (10) it is obvious that then the functional cluster set $\mathcal{A}$ has to be a compact subset of $C_d[0,1]$. It is also quite easy to see that $\mathcal{A}$ has to be star-like and symmetric with respect to 0. But in this case we cannot get any compact subset of $C_d[0,1]$ which is star-like and symmetric with respect to 0 as a cluster set. To see that just apply Theorem 4.2 to the components of $X=(X^{(1)},\ldots,X^{(d)})$. Then it is easy to see that we must have

$$\mathcal{A}\subset \alpha_1\mathcal{K}\times\ldots\times\alpha_d\mathcal{K},$$

where $\alpha_i=\limsup_{n\to\infty}|S_n^{(i)}|/c_n\leqslant\alpha_0<\infty, 1\leqslant i\leqslant d$.

The right-hand set is in general still somewhat too big, as can be seen by calculating the set

$$\{f(1) : f \in \alpha_1\mathcal{K} \times \ldots \times \alpha_d\mathcal{K}\}$$

which is equal to the rectangular set $[-\alpha_1, \alpha_1] \times \ldots \times [-\alpha_d, \alpha_d]$ which is in many cases much bigger than the cluster set $A = C(\{S_n/c_n : n \geqslant 1\})$. Recall that $A = \{f(1) : f \in \mathcal{A}\}$.
Note also that this rectangular set is not among the cluster sets of S_n/c_n in the independent component case as it is impossible to obtain such a rectangular set as a closure of a countable union of standard ellipsoids.
Interestingly it turned out in this case that one can determine the functional cluster set $\mathcal{A}$ via the cluster set A (see [7, Theorem 2.2.]).

THEOREM 4.3 *Let $X = (X^{(1)}, \ldots, X^{(d)}) : \Omega \to \mathbb{R}^d$ be a mean zero random vector with independent components. Assume that $\sum_{n=1}^{\infty} \boldsymbol{P}\{|X| \geqslant c_n\} < \infty$, where c_n is a sequence satisfying conditions (2.1) and (2.2). If $\alpha_0 < \infty$ we have with probability one,*

$$\mathcal{A} = \{(x_1 f_1, \ldots, x_d f_d) : x = (x_1, \ldots, x_d) \in A, f_i \in \mathcal{K}, 1 \leqslant i \leqslant d\},$$

where A is the cluster set of S_n/c_n.

So in this case there is a $1-1$ correspondence between the two cluster sets A and $\mathcal{A}$. One might wonder whether this is a general principle, but it turned out that this is not true. We now look at the general case where we have the following result (see [7, Theorems 2.1 and 2.3]).

THEOREM 4.4 *Let $X : \Omega \to \mathbb{R}^d$ be a mean zero random vector such that $\sum_{n=1}^{\infty} \boldsymbol{P}\{|X| \geqslant c_n\} < \infty$, where c_n is a sequence satisfying conditions (2.1) and (2.2). If $\alpha_0 < \infty$ we have with probability one,*

(i) $\mathcal{A} \supset \{(x_1 f, \ldots, x_d f) : x = (x_1, \ldots, x_d) \in A, f \in \mathcal{K}\}$.

(ii) $\mathcal{A} \subset \{(x_1 f_1, \ldots, x_d f_d) : x = (x_1, \ldots, x_d) \in A, f_i \in \mathcal{K}, 1 \leqslant i \leqslant d\}$ *if* $d = 2$.

It is still an open problem whether the inclusion in (ii) holds if $d \geqslant 3$. As far is (i) is concerned, this is a sharp result as can be seen from the following theorem (see [7, Theorem 8.1]),

THEOREM 4.5 *Given any $p > 1$ and any fixed closed subset A of the Euclidean unit ball which satisfies*

(i) $\max_{x \in A} |x| = 1$

(ii) A is symmetric and star-like with respect to zero

one can find a d-dimensional distribution Q_p such that we have for $X_1, X_2, \ldots$ independent and Q_p-distributed with probability one,

(a) $\limsup_{n\to\infty} |S_n|/\sqrt{2n(LLn)^p} = 1$

(b) $C(\{S_n/\sqrt{2n(LLn)^p} : n \geqslant 1\}) = A.$

(c) $C(\{S_{(n)}/\sqrt{2n(LLn)^p} : n \geqslant 1\}) = \{(x_1 f, \ldots, x_d f) : f \in \mathcal{K}, x \in A\}.$

Applying this last result with A being the Euclidean unit ball, we obtain a much smaller functional cluster set than in the independent component case where one can construct a mean zero random vector X' with independent components so that (a) and (b) above are satisfied for the sums $S'_n = \sum_{j=1}^n X'_j, n \geqslant 1$, where the X'_n are independent copies of X', but the corresponding functional cluster set $C(\{S'_{(n)}/\sqrt{2n(LLn)^p})$ is equal to $\{(x_1 f_1, \ldots, x_d f_d) : x = (x_1, \ldots, x_d) \in A, f_i \in \mathcal{K}, 1 \leqslant i \leqslant d\}$.
The conclusion is:
There is no longer a 1-1-correspondence between the cluster sets $C(\{S_n : n \geqslant 1\})$ and the functional cluster sets $C(\{S_{(n)}/c_n : n \geqslant 1\})$.

In the final section we will try to explain this phenomenon and discuss some work in progress.

5. Final remarks As in the original work of Strassen (1964) we use strong approximation techniques to reduce the problems considered to problems on Gaussian random vectors.
In the sequel $\{W(t) : t \geqslant 0\}$ will be a standard d-dimensional Brownian motion. We set $W_{(n)}(t) := W(nt), 0 \leqslant t \leqslant 1$ and consider this as a sequence of random elements in $C_d[0,1]$. Then we have the following strong invariance principle (see [5, Theorem 2.1]),

THEOREM 5.1 *Let $X, X_1, X_2, \ldots$ be i.i.d. mean zero random vectors in $\mathbb{R}^d$ and assume that*
$\sum_{n=1}^\infty \boldsymbol{P}\{|X| \geqslant c_n\} < \infty$, where c_n is a sequence satisfying conditions (2.1) and (2.2). One can construct a d-dimensional standard Brownian motion $\{W(t), t \geqslant 0\}$ such that with probability one,

$$\|S_{(n)} - \Gamma_n \cdot W_{(n)}\|_\infty = o(c_n) \text{ as } n \to \infty,$$

where Γ_n is the increasing sequence of non-negative definite, symmetric matrices determined by

$$\Gamma_n^2 = \left(\boldsymbol{E}\left[X^{(i)} X^{(j)} \mathbb{1}_{\{|X| \leqslant c_n\}}\right]\right)_{1 \leqslant i,j \leqslant d}.$$

This result makes it possible for us to replace $S_{(n)}/c_n$ by $\Gamma_n W_{(n)}/c_n = \Gamma'_n W_{(n)}/\sqrt{2nLLn}$, where

$$\Gamma'_n := \Gamma_n \sqrt{2nLLn}/c_n.$$

Consequently, we have that

$$C(\{S_{(n)}/c_n : n \geqslant 1\}) = C(\{\Gamma'_n W_{(n)}/\sqrt{2nLLn} : n \geqslant 1\})$$

and we know that $C(\{W_{(n)}/\sqrt{2nLLn} : n \geqslant 1\}) = \mathcal{K}_d$.
Look now at the d-dimensional version of the generalized law of the iterated logarithm (= Theorem 3.4). Recalling the definition of the H-function it is easy to see that $\|\Gamma'_n\| = \sqrt{H(c_n)}$, where $\|\cdot\|$ denotes the Euclidean matrix norm, that is $\|C\| := \sup_{|x|\leqslant 1} |Cx|$ for any (d,d) matrix C. So in this case the matrix sequence Γ'_n is bounded and thus relatively compact in the space of all non-negative definite symmetric (d,d)-matrices.
If Γ'_n converges to a fixed non-negative definite symmetric matrix B it follows directly from Strassen's functional LIL for Brownian motion that $C(\{S_{(n)}/c_n : n \geqslant 1\}) = C(\{BW_{(n)}/\sqrt{2nLLn}\}) = \mathcal{K}_\Sigma$, where $\Sigma = B^2$.
In general, it can happen that the sequence Γ'_n has more than one limit point and one obtains more complicated functional cluster sets. Using the separability of the set of all non-negative definite symmetric (d,d)-matrices one should be able to find a representation of the general functional cluster set via an at most countable subclass of limit points B_n of the sequence $\{\Gamma'_n : n \geqslant 1\}$ and we have the following

Conjecture 1 If $\alpha_0 < \infty$, the functional cluster set $\mathcal{A}$ has a representation as $\mathrm{cl}(\cup_{n=1}^\infty \mathcal{K}_{\Sigma_n})$, where Σ_n is a sequence of non-negative definite symmetric matrices satisfying $\limsup_{n\to\infty} \|\Sigma_n\| = \alpha_0^2$.

Here $\mathrm{cl}(M)$ stands for the topological closure of a set M in $C_d[0,1]$. In the independent component case all the limiting matrices of the sequence Γ'_n would have to be diagonal matrices which leads to a representation via standard ellipsoids for the cluster set A of S_n/c_n if one uses once more the fact that $A = \{f(1) : f \in \mathcal{A}\}$. It also explains why one has a 1-1 correspondence between A and $\mathcal{A}$ in this case (see Theorem 4.4).

Finally, this could also explain why one obtains in some cases (see Theorem 5.1) minimal cluster sets as in Theorem 4.5 (ii). The reason is that all the limit points of the matrix sequence Γ'_n have rank 1.

We finally believe that all sets as described in Conjecture 1 actually occur as functional cluster sets and this even for the generalized law of the iterated logarithm.

Conjecture 2 Given $0 < \alpha_0 < \infty$, $1 < p < \infty$, and a sequence of non-negative definite symmetric matrices Σ_n satisfying $\limsup_{n\to\infty} \|\Sigma_n\| = \alpha_0^2$, one can find a d-dimensional distribution Q'_p such that if $X_1, X_2, \ldots$ are i.i.d. random vectors with distribution Q'_p we have with probability one,

$$\limsup_{n\to\infty} |S_n|/\sqrt{2n(LLn)^p} = \alpha_0,$$

and
$$C(\{S_{(n)}/\sqrt{2n(LLn)^p} : n \geqslant 1)\} = \mathrm{cl}(\cup_{n=1}^{\infty}\mathcal{K}_{\Sigma_n}).$$

We close by mentioning some related work. One can ask whether there are also related results on cluster sets in the infinite-dimensional case. Cluster sets for partial sums have been considered in [6] where among other things an extension of Theorem 4.1 has been established. As for the functional cluster sets much less seems to be known. Some interesting results in this direction can be found in [15]. Another interesting question is whether there are one-sided versions of Theorem 2.2. For some results in this direction refer to [17]. Finally, there is also a version of Theorem 2.1 for independent, but not necessarily identically distributed random variables in separable Banach spaces (see [18]).

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Prawa typu iterowanego logarytmu dla wektorów losowych z nieskończonym drugim momentem.

Uwe Einmahl

Streszczenie Ten tekst jest rozszerzoną wersją prezentacji autora na konferencji *A path through probability in honour of F. Thomas BRUSS* które odbyła się na *Unversité Libre de Bruxelles* w Brukseli w dniach 9-11 września 2015 roku. W pierwszej części przedstawiam niektóre wyniki uogólniając klasyczne prawo Hartmana-Wintnera iterowanego logarytmu dla zmiennych 1-wymiarowych z nieskończonym drugim momentem, a następnie pokazuję, jak te wyniki mogą być rozszerzone do zagadnień d-wymiarowych. Wywody kończę na ogólnym funkcjonalnym prawie tego typu.

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Słowa kluczowe: PIL Hartmana-Wintnera, funkcjonalne PIL, nieskończona wariancja, zachowania typu PIL, funkcje wolno zmieniające się, zbiory skupień, zasada silnej niezmienniczości.

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