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D. M. Ramsey (Wrocław)

A secretary problem with missing observations

Abstract Two versions of a best choice problem in which an employer views a sequence of N applicants are considered. The employer can hire at most one applicant. Each applicant is available for interview (and, equivalently, for employment) with some probability p. The available applicants are interviewed in the order that they are observed and the availability of the i-th applicant is ascertained before the employer can observe the $(i+1)$-th applicant. The employer can rank an available applicant with respect to previously interviewed applicants. The employer has no information on the value of applicants who are unavailable for interview. Applicants appear in a random order. An employer can only offer a position to an applicant directly after the interview. If an available applicant is offered the position, then he will be hired. In the first version of the problem, the goal of the employer is to obtain the best of all the applicants. The form of the optimal strategy is derived. In the second version of the problem, the goal of the employer to obtain the best of the available applicants. It is proposed that the optimal strategy for this second version is of the same form as the form of the optimal strategy for the first version. Examples and the results of numerical calculations are given.

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1. Introduction We present two versions of a best choice problem in which an employer (referred to as she) views a sequence of N applicants (referred to as he) and can hire at most one of them. Each applicant is available for interview (and employment) with some probability p independently of the value of the applicant. The available applicants are interviewed in the order that they are observed and the availability of the i-th applicant is ascertained before the employer can observe the $(i+1)$-th applicant. The employer can rank an available applicant with respect to previously interviewed applicants. It is assumed that the employer has no information on the value of applicants who are unavailable for interview. Applicants appear in a random order (i.e. the rank of an applicant is independent of his position in the sequence). An employer can only offer a position to an applicant directly after the interview (i.e. no recall of previous applicants is possible). If an available applicant is offered the position, then he will accept it, in other words be hired. In the first version of the problem, the employer obtains a payoff of 1 if she hires the best of all the N applicants, otherwise she obtains a payoff of 0. In the second version of the problem, the employer obtains a payoff of 1 if she hires the best of the available applicants, otherwise she obtains a payoff of 0.

Best choice problems have a long history. Gilbert and Mosteller [9] presented two classical problems in this field, where the employer must choose one applicant from a sequence of N available applicants without recall of previous applicants. Her goal is to choose the best applicant. In the first version, the employer can rank each applicant with respect to previous applicants. In accordance with Ferguson [7], problems of this type will be referred to as secretary problems. Bruss [3] and Szajowski [23] consider some generalizations of such secretary problems. In the second version of this problem, the employer observes the value of each applicant. These values are assumed to be independent realizations of a continuous random variable. Without loss of generality, it may be assumed that the values of the applicants come from the uniform distribution on the interval $[0, 1]$. This problem will be referred to as the full-information best choice problem. Smith [20] considers a secretary problem in which each applicant is available with probability p. The employer observes the rank of all the applicants (even those unavailable) with respect to previous applicants. Her goal is to hire the best of all the applicants. The solution of this problem is independent of whether the availability of an applicant is ascertained before or after the employer can make an offer. Whenever an applicant is unavailable, the employer must continue searching.

As Tamaki [24] notes, a more reasonable goal might be to hire the best of the available applicants. In this case, the solution depends on whether the employer can ascertain the availability of an applicant before offering the position. Suppose she can only ascertain the availability of an applicant by offering the position. If a applicant who is unavailable is offered the position, then the position is not filled and the employer continues searching. The employer should remember the best applicant who was offered the position, but proved to be unavailable. When the employer can ascertain the availability of an applicant before an offer is made, she should remember the best available applicant seen so far. An offer should only be made to an available applicant if he has a higher rank than any of the previous available applicants. The optimal expected reward is greater in this second problem as the employer has more information (she knows the availability of all the previous applicants). This problem is obviously strongly related to the problems considered here. The main difference lies in the fact that in the problems considered by Tamaki, the employer can rank all of the applicants to have appeared so far. In fact, at the end of his paper, Tamaki proposes the second problem considered in this paper. In terms of the time at which availability is ascertained, the model presented here assumes that this occurs before an offer is made. Samuel-Cahn [18], [19] consider a similar problem in which a known number N of applicants appear in random order, but the decision maker is unable to select an applicant after a number M of applicants have appeared. It is assumed that M is a random variable with support $\{1, 2, \ldots, N\}$. The decision maker knows the distribution of M, but does not observe the realization of

M until she can no longer accept an applicant. Hence, a certain proportion of applicants (the last to appear) are unavailable.

Tamaki [25] also uses a similar approach to the full-information best choice problem. Ano *et al.* [1] extend this model to include a constraint on the number of offers that can be made.

Since the number of available applicants is random, the problems considered here are clearly related to work on secretary problems with a random number of applicants (see Presman and Sonin [14] and Petruccelli [10]). Porosiński [11] considers the full-information best choice problem with a random number of applicants. However, there is an important difference between models with uncertain availability, considered here, and models with a random number of applicants. In the problems considered here, by observing the availability (or unavailability) of an applicant, the employer obtains a richer source of information regarding the number of available applicants than under Presman and Sonin's model. For example, the number of available applicants under the models considered here has a binomial distribution. The employer should base her decisions on the conditional number of available applicants remaining given the number of previous applicants (both available and unavailable). Interpreting Presman and Sonin's model within the framework of this paper, the employer only has information on the number of available applicants that have appeared. The structure of the problems considered here is more related to the model considered by Cowan and Zabczyk [6]. According to this model, the employer can search for a deterministic length of time T and applicants appear according to a homogeneous Poisson process. Their model was extended by Bruss [4] to include imperfect knowledge of the rate of arrival of applicants and non-homogeneous processes. Szajowski [21] considers a game theoretic version of this problem with two employers.

Porosiński and Szajowski [12], [13] consider a version of the full-information problem in which the employer cannot observe the value of an applicant perfectly and adapt this problem to a game theoretic problem in which there are two employers. For other examples of game theoretic approaches to best choice games, see Ramsey and Szajowski [16] and Sakaguchi [17]. Another problem related to the secretary problem is the duration problem. In this case, the employer wishes to maximize the length of time for which the applicant hired is the best applicant seen so far (see Ferguson *et al.* [8]).

Section 2 presents the problem in which the goal of the employer is to hire the best of all the applicants. The optimality equation and the form of the optimal strategy are derived. An example is presented, together with numerical and asymptotic results. Section 3 presents the problem in which the goal of the employer is to hire the best of the available applicants. A proposition is formulated regarding the form of the optimal strategy. An example is presented, together with numerical and asymptotic results. Section

4 is a brief conclusion and proposes directions for further research.

2. Model One Suppose that an employer observes a sequence of N applicants. Her aim is to choose the best applicant of all N, i.e. her reward is 1 if she picks the best, otherwise her reward is 0. On receiving an application the employer invites the applicant for interview and it assumed that these interviews are carried out immediately. However, each of these applicants is available for interview with probability p, where $0 < p < 1$, and when unavailable an applicant cannot be employed or compared to any of those interviewed. The employer may rank the applicants she has already interviewed in order of preference. The decision on whether to accept an applicant must be made before the next interview. Once the employer has hired an applicant she stops searching. No recall of previous applicants is possible.

Using the interpretation of the secretary problem, each applicant may have found a position somewhere else and thus would decline the offer of an interview. The assumption of a fixed probability of availability may be interpreted as assuming the preferences of the employer in question are independent of the preferences of other employers on the market.

2.1. The Optimality Equation Consider the optimal expected reward from future search of an employer at some stage of the process. This reward clearly depends on the number of applicants remaining. Denote this number by i. Also, each available applicant gives the employer some information. Denote the number of previous applicants who were available by j. Since applicants appear in a random order, availability is independent of the value of an applicant and no information is gained from unavailable applicants, this is all the information required to define the optimal expected reward from future search. Denote $v_{i,j}$ to be the optimal expected reward of the employer when i applicants remain and j of the previous applicants were available, where $i \in \{0, 1, \ldots N\}$ and $j \in \{0, 1, \ldots N - i\}$. The optimal expected reward from search is by $v_{N,0}$.

We first define a recursive formula for the expected reward. Suppose there is only one applicant remaining. The employer can do no better than accept such an applicant. Since the availability of the applicant is independent of his rank, the probability that this applicant is both available and the best of all the applicants is $\frac{p}{N}$. Thus, we have

$$v_{1,j} = \frac{p}{N}, \qquad \forall j \in \{0, 1, 2, \ldots, N - 1\}.$$

Also, $v_{0,j} = 0, \forall j \in \{0, 1, 2, \ldots, N\}$.

It is clear that the employer should only offer a position to an available applicant who is the best seen so far. Such an applicant will be called a candidate. Suppose the $(N - i)$-th applicant is a candidate and j of the applicants seen so far (including the present one) were available, $j \in \{1, 2, \ldots, N - i\}$. The state at such a decision point can be described by the vector (i, j).

Suppose the $(N-i)$-th applicant is available. He will be the best of those viewed (j applicants in total) with probability $\frac{1}{j}$. The probability that such a candidate is the best of all N applicants is $\frac{j}{N}$. From the optimality criterion, in state (i,j) the employer should offer a candidate a position if and only if $\frac{j}{N} \geqslant v_{i,j}$, i.e. if the expected reward from accepting such a candidate is greater than the expected reward from continued search. We thus obtain the optimality equation

$$v_{i,j} = (1-p)v_{i-1,j} + \frac{pjv_{i-1,j+1}}{j+1} + \frac{p}{j+1}\max\{v_{i-1,j+1}, \frac{j+1}{N}\}. \qquad (1)$$

The first term corresponds to the case of an unavailable applicant (this occurs with probability $1-p$). The employer must then continue searching, $i-1$ applicants remain and j of the applicants seen were available. The second term corresponds to the case in which an applicant is available, but is not a candidate [this occurs with probability $\frac{pj}{j+1}$]. In this case, the employer should continue searching, $i-1$ applicants remain and $j+1$ of the applicants seen were available. The final term corresponds to the case in which an applicant is a candidate.

2.2. Form of the Optimal Strategy It is simple to show that in the standard form of the secretary problem the expected reward from search after rejecting j applicants is non-increasing in j. It follows that the optimal strategy is of the following form: accept the first candidate to appear after the j^*-th applicant.

In the problem considered here, the expected reward from future search not only depends on the number of applicants remaining, but also on the number who were available when called for interview. For example, suppose that there are a large number of applicants. Intuitively, the employer should always reject early applicants. Assume the first k applicants should always be rejected. In this case, $v_{k,k} > v_{0,0}$, since after seeing all of the first k applicants, an employer has more information than she would have expected at the beginning and would have rejected the first k applicants anyway. The following two results regarding the form of the optimal expected payoff function lead directly to the main result of this section regarding the form of the optimal strategy.

THEOREM 2.1 *The payoff function satisfies* $v_{i+1,j+1} - v_{i,j} < \frac{1}{N}$, *for* $i = 0, 1, \ldots N$ *and* $j = 0, 1, 2, \ldots N-i-2$.

PROOF The proof is by induction in i. We have $v_{0,j} = 0$ for and $v_{1,j} = \frac{p}{N}$ for $j = 0, 1, 2, \ldots N-2$. It follows that the theorem holds for $i = 0$.

Suppose the theorem holds for $i = k-1$ and $j = 0, 1, 2, \ldots N-k-1$. It suffices to show that $v_{k+1,j+1} - v_{k,j} < \frac{1}{N}$ for $j = 0, 1, 2, \ldots N-k-2$. From the optimality equation, we obtain

$$v_{k,j} \geqslant (1-p)v_{k-1,j} + pv_{k-1,j+1} \tag{2}$$

$$v_{k,j} \geqslant (1-p)v_{k-1,j} + \frac{pjv_{k-1,j+1}}{j+1} + \frac{p}{N}. \tag{3}$$

Suppose $v_{k,j+2} \geqslant \frac{j+2}{N}$. It follows from the optimality equation that

$$v_{k+1,j+1} = (1-p)v_{k,j+1} + pv_{k,j+2}.$$

Condition (2), together with the induction hypothesis gives

$$v_{k+1,j+1} - v_{k,j} \leqslant (1-p)[v_{k,j+1} - v_{k-1,j}] + p[v_{k,j+2} - v_{k-1,j+1}] < \frac{1}{N}.$$

Now suppose $v_{k,j+2} < \frac{j+2}{N}$. It follows from the optimality equation that

$$v_{k+1,j+1} = (1-p)v_{k,j+1} + \frac{p(j+1)v_{k,j+2}}{j+2} + \frac{p}{N}.$$

Condition (3), together with the induction hypothesis gives

$$\begin{aligned} v_{k+1,j+1} - v_{k,j} &\leqslant (1-p)[v_{k,j+1} - v_{k-1,j}] + \frac{pj[v_{k,j+2} - v_{k-1,j+1}]}{j+1} \\ &\quad + pv_{k,j+2}\left[\frac{j+1}{j+2} - \frac{j}{j+1}\right] \\ &< \frac{1-p}{N} + \frac{pj}{(j+1)N} + \frac{p}{N(j+1)} = \frac{1}{N}. \end{aligned}$$

The theorem thus follows by induction. ■

THEOREM 2.2 *The payoff function satisfies $v_{i,j} - v_{i-1,j} \geqslant 0$, for $i = 0, 1, 2, \ldots N$ and $j = 0, 1, 2, \ldots N - i$.*

PROOF After $(N-i)$ applicants have passed by and j have been available, the employer can ensure herself a payoff of $v_{i-1,j}$ by ignoring the $(N-i+1)$-th applicant and thereafter using the optimal policy. ■

The following theorem regarding the form of the optimal strategy for this problem is the main theorem of this section. It is assumed that if the employer is indifferent between accepting and rejecting a candidate, then she will accept him.

THEOREM 2.3 (FORM OF THE OPTIMAL STRATEGY.) **1** *If the $(N-1)$-th applicant is a candidate, then he should always be employed.*

2 *The optimal strategy can be described by a set of m (where m is derived by the induction procedure) integer thresholds $\{i_j^*\}_{j=1}^m$. For $j > m$, if the j-th available applicant is a candidate, then he should always be accepted. For $j \leqslant m$, the employer should accept a candidate in state (i, j) (i.e. i applicants remain and the candidate is the j-th available applicant), if and only if $i \leqslant i_j^*$.*

3 *For $j < m$, $i_{j+1}^* > i_j^*$ (i.e. the greater the number of applicants that were available, the earlier an individual should be willing to accept a candidate).*

Note: We may think of m as the maximum number of applicants that should be automatically rejected when all previous applicants were available. The optimal strategy and payoff function, $v_{i,j}$, together with m, can be derived by induction.

PROOF Part 1 follows from the fact that the probability of any applicant being the best of all the objects is $\geqslant \frac{1}{N}$. Since the expected reward from future search (i.e. continuing on to the final applicant) is $\frac{p}{N}$, it is always optimal to take a candidate who is the $(N-1)$-th applicant.

Part 2 follows directly from Theorem 2. Suppose a candidate should be accepted in state (i, j), i.e. $v_{i,j} \leqslant j/N$. Since $v_{i-1,j} \leqslant v_{i,j}$, it follows that $v_{i-1,j} \leqslant j/N$. Hence, a candidate should be accepted in state $(i-1, j)$.

Part 3 follows from Theorem 1. Suppose the employer should accept a candidate in state (i, j), i.e. $\frac{j}{N} \geqslant v_{i,j}$, but should not accept one in state $(i+1, j)$. Since $v_{i+1,j+1} < v_{i,j} + \frac{1}{N}$, it follows that $\frac{j+1}{N} > v_{i+1,j+1}$ and thus a candidate should be accepted in state $(i+1, j+1)$. Hence, $i_j^* < i_{j+1}^*$. ■

2.3. Example Consider a problem in which $N = 5$ and $p = 0.75$. We have $v_{1,j} = \frac{p}{N} = 0.15$, for $j = 0, 1, 2, 3, 4$. A candidate who appears as the fourth applicant should always be accepted. From Equation (1) we obtain

$$v_{2,j} = 0.25v_{1,j} + \frac{0.75jv_{1,j+1}}{j+1} + \frac{0.75}{5}, \quad j = 0, 1, 2, 3.$$

It follows that $v_{2,3} = 0.271875$, $v_{2,2} = 0.2625$, $v_{2,1} = 0.24375$ and $v_{2,0} = 0.1875$. Based on these values, we can state whether the employer should accept a candidate in state $(2, j)$, $j = 1, 2, 3$. The employer should accept such a candidate if $\frac{j}{5} \geqslant v_{2,j}$. It is easy to check using the values given above that such a candidate should be accepted if $j = 2, 3$, but not if $j = 1$. Hence,

$$\begin{aligned} v_{3,j} &= 0.25v_{2,j} + \frac{0.75jv_{2,j+1}}{j+1} + \frac{0.75}{5}, \quad j = 1, 2 \\ v_{3,0} &= 0.25v_{2,0} + 0.75v_{2,1}. \end{aligned}$$

These equations give $v_{3,2} \approx 0.351563$, $v_{3,1} \approx 0.309375$, $v_{3,0} \approx 0.229688$. Suppose the second applicant is a candidate and the j-th available applicant ($j = 1, 2$). Such a candidate should be accepted if $\frac{j}{5} \geqslant v_{3,j}$. It follows that such a candidate should be accepted only if $j = 2$. Hence,

$$\begin{aligned} v_{4,1} &= 0.25v_{3,1} + \frac{0.75v_{3,2}}{2} + \frac{0.75}{5} \approx 0.359180 \\ v_{4,0} &= 0.25v_{3,0} + 0.75v_{3,1} \approx 0.289453. \end{aligned}$$

Finally, if the first applicant is available for interview, he is a candidate and should be accepted if $0.2 \geqslant v_{4,1}$. Hence, the first applicant should be rejected. It follows that the probability of employing the best applicant is

$$v_{5,0} = 0.25v_{4,0} + 0.75v_{4,1} \approx 0.341748.$$

The optimal strategy can be succinctly described by giving the set of thresholds $\{i_j^*\}_{j=1}^m$. The first available applicant should be accepted only if he is the penultimate or final applicant. It follows that $i_1^* = 1$. A candidate who is the second available applicant should always be accepted. It follows that $m = 1$.

2.4. Asymptotics and Numerical Results It should be noted that in order to choose the best applicant, the employer must employ the best available applicant and this applicant must be better than any of the applicants who were unavailable. When the number of applicants is finite, given the information regarding all the available applicants, the probability that the best available applicant is better than any of the unavailable applicants is simply the proportion of applicants who were available (a random number about which the employer gains information during the search process). Let $N \to \infty$. From the law of large numbers, this conditional probability is p. It follows that by maximising the probability of choosing the best applicant out of the set of available applicants, the employer simultaneously maximises the probability of choosing the best applicant out of all the applicants.

Let t be the proportion of applicants that have passed by, $t \in [0, 1]$, t will be referred to as time. Suppose a candidate appears at time t. Using the law of large numbers, a proportion p of previous applicants will have been available for interview. Hence, the proportion of available applicants to have appeared by time t is also t. It follows from the above argument that this problem is equivalent to the asymptotic version of the standard secretary problem. Hence, the probability of accepting the best available applicant under the optimal policy is e^{-1} and it follows that the expected reward from search is pe^{-1}.

The table below gives the expected reward from search for various values of p and N. The standard secretary problem corresponds to $p = 1$.

3. Model Two This model is the same as the first model, except for the following: the payoff of the employer is 1 if she employs the best available

	$p = 0.25$	$p = 0.5$	$p = 0.75$	$p = 1$
$N = 5$	0.1525 (0.3520)	0.2422 (0.5590)	0.3417 (0.7886)	0.4333
$N = 10$	0.1173 (0.2942)	0.2117 (0.5310)	0.3058 (0.7670)	0.3987
$N = 50$	0.0967 (0.2583)	0.1892 (0.5055)	0.2817 (0.7526)	0.3743
$N = 100$	0.0943 (0.2542)	0.1865 (0.5027)	0.2788 (0.7515)	0.3710
$N = 500$	0.0924 (0.2508)	0.1845 (0.5005)	0.2765 (0.7503)	0.3685
$N = 1000$	0.0922 (0.2504)	0.1842 (0.5003)	0.2762 (0.7501)	0.3682
$N = 5000$	0.0920 (0.2501)	0.1840 (0.5001)	0.2760 (0.7500)	0.3679
$N \to \infty$	0.0920 (0.2500)	0.1839 (0.5000)	0.2759 (0.7500)	0.3679

Table 1: Expected reward from search for various realizations of the problem. The figures in brackets give the expected reward as a proportion of the expected reward in the corresponding standard secretary problem with N applicants.

applicant, otherwise her payoff is 0. Note that if none of the candidates are available, then the reward of the employer is defined to be 0.

3.1. The Optimality Equation Again, the employer should only accept an available applicant if he is the best of the available applicants observed so far. As before, an available applicant who is better than all the previous available applicants is called a candidate. Suppose that on the appearance of a candidate, i applicants have yet to appear and j of the previous applicants were available. As before, the state of the employer at such a decision point can be defined as (i, j). We have $i \in \{0, 1, 2, \ldots, N\}$, $j \in \{1, 2, \ldots N - i\}$. Denote by $r_{i,j}$ the probability that such a candidate is the best of all the available applicants. Let X_i be the number of the remaining applicants who prove to be available. It follows that $X_i \sim \mathrm{Bin}(i, p)$. For $i \geqslant 1$, we have

$$r_{i,j} = \mathbb{E}\left(\frac{j}{j + X_i}\right) > \frac{j}{j + ip},$$

where the inequality results from Jensen's inequality, since $f(x) = j/(j + x)$ is a convex function of x.

Let $v_{i,j}$ denote the expected reward of the employer after observing j available applicants and rejecting them when i applicants remain ($i, j \geqslant 0$, $i + j \leqslant N$). We have $v_{0,j} = 0$ for all $j \in \{0, 1, \ldots N\}$. A candidate should be accepted in state (i, j), if and only if $r_{i,j} \geqslant v_{i,j}$. The optimal reward from search, $v_{N,0}$, and the optimal strategy can be calculated inductively in i. Suppose just one applicant remains. Obviously, the employer should accept any available applicant. She obtains a payoff of 1 if this final applicant is the best of all the available applicants. Suppose j of the previous applicants were available. Given the final applicant is available, the probability that he is the best of all the available applicants is $1/(j+1)$. It follows that

$$v_{1,j} = \frac{p}{j+1}, \qquad j \in \{0, 1, 2, \ldots, N-1\}.$$

The derivation of the optimality equation is analogous to the derivation of Equation (1). The only difference is that under the first model the searcher should compare the probability that a candidate is the best of all the applicants to the expected future reward from search, under the second model the searcher should compare the probability that a candidate is the best of the **available** applicants to the expected future reward from search. Hence, we obtain

$$v_{i,j} = (1-p)v_{i-1,j} + \frac{pjv_{i-1,j+1}}{j+1} + \frac{p}{j+1}\max\{v_{i-1,j+1}, r_{i-1,j+1}\}. \qquad (4)$$

3.2. Form of the Optimal Strategy The derivation of the general form of the optimal strategy is complicated by the fact that $r_{i,j}$ is not a linear function of either i or j (unlike the probability of a candidate being the best of all N applicants, as considered in model 1). For this reason, we only give a simple intuitive argument for the following proposition:

PROPOSITION 3.1 *If a candidate should be accepted in state (i, j), then a candidate should be accepted in state (k, m), where $k \leqslant i$ and $m \geqslant j$. It follows that the optimal strategy is of the following form:*

1 *Any candidate should be accepted at the final stage.*

2 *There exists a sequence $\{i_j^*\}_{j=1}^m$, such that if j available applicants have been seen, a candidate should be accepted if the number of applicants remaining is $\leqslant i_j^*$. For $j > m$, if a candidate is the j-th applicant seen, then he should always be accepted.*

3 *The sequence i_j^* is non-decreasing in j.*

This proposition states that the form of the optimal strategy is basically the same as the form of the optimal strategy for model 1.

ARGUMENT. Note that $r_{i,j}$ is decreasing in i for fixed j and increasing in j for fixed i. Suppose $j < m$. Intuitively, $v_{i,j} \geqslant v_{i,m}$, since the left hand side is the probability of obtaining an applicant better than the remaining available applicants and the previous j available applicants, while the right hand side is the probability of obtaining an applicant better than the remaining available applicants and the previous m available applicants (a more stringent condition). It follows that if a candidate should be accepted in state (i, j), then he should be accepted in state (i, m).

Numerical calculations show that $v_{i,j}$ is not in general monotonic in i for fixed j. This is due to two counteracting effects. Firstly, as i decreases the range of choice available to the employer decreases, causing $v_{i,j}$ to fall as i decreases. Secondly, given there is a sequence of unavailable applicants, the employer thus has to search for the best of the available applicants from a smaller set, causing $v_{i,j}$ to rise as i decreases. It seems reasonable that if a candidate should be accepted in state (i, j), then this state is in a region in which the first effect either balances or outweighs the second effect. If this is the case, then since $r_{i,j}$ is decreasing in i, a candidate should be accepted in state (k, j), where $k \leqslant i$. Given this, the proposition given above is true. ■

An algorithm was written to calculate the optimal expected reward and the optimal strategy for various values of p and N. The solution was derived by induction in i. It was not assumed that the solution is of the form given above. However, all the solutions found were of the appropriate form and $v_{i,j}$ was non-increasing in j for fixed i.

3.3. Example Suppose that $p = 0.75$ and $N = 5$. If no previous applicant has been accepted, then the final applicant should be accepted (if he is available). We have

$$v_{1,j} = \frac{0.75}{j+1}, \quad j \in \{0, 1, 2, 3, 4\}.$$

If the penultimate applicant is the j-th available applicant, he should be accepted if $r_{1,j} \geqslant v_{1,j}$, $j \in \{1, 2, 3, 4\}$, i.e.

$$r_{1,j} = \mathbf{E}\left(\frac{j}{j+X_1}\right) = (1-p) + \frac{pj}{j+1} \geqslant v_{1,j}.$$

It is simple to show that this inequality is satisfied for each possible j. Hence, the penultimate applicant should always be accepted if he is a candidate. It follows that

$$v_{2,j} = 0.25v_{2,j} + \frac{0.75jv_{1,j}}{j+1} + \frac{0.75r_{1,j+1}}{j+1}, \quad j \in \{0, 1, 2, 3\}.$$

It follows that $v_{2,0} = 0.65625$, $v_{2,1} = 0.46875$, $v_{2,2} = 0.359375$, $v_{2,3} = 0.290625$.

If the third applicant is the j-th available applicant, he should be accepted if $r_{2,j} \geqslant v_{2,j}$, $j \in \{1, 2, 3\}$. We have

$$r_{2,j} = \mathbf{E}\left(\frac{j}{j+X_2}\right) = 0.25^2 + \frac{2 \times 0.25 \times 0.75j}{j+1} + \frac{0.75^2 j}{j+2}.$$

It follows that $r_{2,1} = 0.4375$, $r_{2,2} = 0.59375$ and $r_{2,3} = 0.68125$. Hence, a candidate should be accepted at moment 3, if he is the second or third available applicant. We thus obtain

$$\begin{aligned} v_{3,0} &= 0.25v_{2,0} + 0.75v_{2,1} \\ v_{3,j} &= 0.25v_{2,j} + \frac{0.75jv_{2,j+1}}{j+1} + \frac{0.75r_{2,j+1}}{j+1}, \quad j \in \{1, 2\}. \end{aligned}$$

It follows that $v_{3,0} = 0.515625$, $v_{3,1} \approx 0.474609$ and $v_{3,2} \approx 0.405469$.

If the second applicant is the j-th available applicant, he should be accepted if $r_{3,j} \geqslant v_{3,j}$, $j \in \{1, 2\}$. We have

$$r_{3,j} = \mathbf{E}\left(\frac{j}{j+X_3}\right) = 0.25^3 + \frac{3 \times 0.25^2 \times 0.75j}{j+1} + \frac{3 \times 0.25 \times 0.75^2 j}{j+2} + \frac{0.75^3}{j+3}.$$

It follows that $r_{3,1} \approx 0.33203$ and $r_{3,2} \approx 0.489063$. Hence, a candidate should be accepted at moment 2, only if he is the second available applicant. We thus obtain

$$\begin{aligned} v_{4,0} &= 0.25v_{3,0} + 0.75v_{3,1} \approx 0.484863 \\ v_{4,1} &= 0.25v_{3,1} + \frac{0.75v_{3,2}}{2} + \frac{0.75r_{3,2}}{2} \approx 0.454102. \end{aligned}$$

If the first applicant is available, he is a candidate. He should be accepted if $r_{4,1} \geqslant v_{4,1}$. We have

$$r_{4,1} = \mathbf{E}\left(\frac{1}{1+X_4}\right) \approx 0.266406.$$

It follows that the first applicant should not be accepted. The optimal expected reward of the employer is

$$v_{5,0} = 0.25v_{4,0} + 0.75v_{4,1} = 0.461792.$$

Hence, the optimal strategy is

a Reject the first applicant.

b Accept a candidate at moment 2 or 3, if he is at least the second available applicant.

c Accept any candidate at moment 4 or 5.

It should be noted that this is also the optimal strategy in the example considered for model 1. However, this is not true in general.

3.4. Numerical Results and Asymptotic Solution

Suppose $N \to \infty$. Let t be the proportion of applicants that have passed by, $t \in [0, 1]$, t will be referred to as time. Suppose a candidate appears at time t. Using the law of large numbers, a proportion p of previous applicants will have been available for interview. Hence, the proportion of available applicants to have appeared by time t is also t. It follows that this problem is equivalent to the asymptotic version of the standard secretary problem. Hence, the probability of accepting the best available applicant under the optimal policy is e^{-1}.

It should be noted that for large N there are numerical problems associated with the calculation of $r_{i,j}$. This is due to underflow for many of the probabilities associated with realizations of X_i. In order to deal with this, for $i \geqslant 150$ we used the following approximation:

$$\mathbf{E}(f(X)) \approx f(\mathbf{E}(X)) + \frac{Var(X)f''(\mathbf{E}(X))}{2}.$$

It follows that

$$\mathbf{E}\left(\frac{j}{j+X_i}\right) \approx \frac{j}{j+ip} + \frac{jip(1-p)}{(j+ip)^3}.$$

For $i = 150$, the error obtained using this approximation compared to the standard numerical summation was at most of order 10^{-5}. The table below gives the optimal expected reward from search for various values of p and N. The standard secretary problem corresponds to $p = 1$. This table only gives expected rewards for $N \leqslant 100$, since the approximation used above means that comparison of similar payoffs may well be misleading. The optimal solution in each case (also for all the values of N considered for Model 1) was of the form given by Proposition 1.

Table 2: Expected reward from search for various realisations of Model 2. The figures in brackets give the expected reward as a proportion of the expected reward in the corresponding standard secretary problem with N applicants.

	$p = 0.25$	$p = 0.5$	$p = 0.75$	$p = 1$
$N = 5$	0.5605 (1.2935)	0.4948 (1.1418)	0.4618 (1.0657)	0.4333
$N = 10$	0.4818 (1.2085)	0.4273 (1.0719)	0.4086 (1.0249)	0.3987
$N = 50$	0.3875 (1.0352)	0.3785 (1.0112)	0.3756 (1.0037)	0.3743
$N = 100$	0.3773 (1.0170)	0.3731 (1.0056)	0.3717 (1.0019)	0.3710

4. Conclusion This paper has considered a secretary problem with uncertain employment in which the value of unavailable applicants cannot be compared to value of available applicants. It is interesting to note that the asymptotic form of the optimal strategy and the probability of obtaining the best available applicant are independent of the probability of an applicant being available. Tamaki [25] obtains similar results for a similar full-information best choice problem. For large N, the optimal strategy is to automatically reject the first Ne^{-1} applicants and then offer the position to the next available individual who is better than all the previously interviewed applicants. Bruss [3] shows that this solution holds for a class of similar problems.

There are various directions in which the ideas presented in this paper can be extended. Individuals may use simple comparative rules when faced with a sequential decision process. However, in reality, it is likely that they derive utility not from obtaining the best possible object, but from the intrinsic value of the object. Todd [26] considers a problem in which an individual uses a rank-based selection rule, but the utility derived is based on the value of the object chosen and not its rank. Bearden [2] gives a more formal definition of such a problem and derives the asymptotic form of the optimal solution. Szajowski [22], [23] extends this model to incorporate search costs.

The models considered by Tamaki [24], [25] assume that the searcher can rank an applicant relative to all the previous applicants. The model considered here takes, in some ways, a diametrically opposite approach by assuming that no information is available on the value of an unavailable applicant. If we consider the standard procedure for applying for a job, an employer will gain some information from an applicant's application and some from an interview. This could be modeled using fuzzy algebra (see e.g. Chudjakow and Riedel [5]).

Finally, models involving partial information in best choice problems could be applied to game theoretic versions of such problems. For example, Ramsey [15] considers a game theoretic best choice problem in which one employer sees an applicant before the other employer has chance to see him. If the first employer wishes to hire an applicant, then the second employer does not see him. In order to solve such a game, we need to solve a set of auxiliary best choice problems in which the second employer is searching alone and has incomplete information regarding the sequence of applicants.

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O pewnym problemie sekretarki z brakującymi obserwacjami

David M. Ramsey

Streszczenie Artykuł rozważa dwie wersje problemu wyboru najlepszego obiektu w których pracodawca obserwuje sekwencyjnie N pracobiorców. Każdy pracobiorca jest wolny z prawdopodbieństwem p, a gdy nie jest wolny pracodawca nie może ani prowadzić rozmowy z nim, ani go zatrudnić. Rozmowa o pracy z i-tym pracobiorcą (o ile ma miejsce) jest przeprowadzana zanim pojawi się $(i+1)$-szy pracobiorca. Pracodawca może tylko porównać pracobiorców, czyli sporządzić ranking tych, z którymi już prowadził rozmowę. Nie posiada on żadnych informacji dotyczących wartości pracobiorców, którzy nie są wolni. Pracobiorcy pojawiają się w losowej kolejności. Pracodawca może zatrudnić tylko jednego pracobiorcę i to tylko zaraz po rozmowie z nim. Wolny pracobiorca zawsze przyjmuje ofertę pracy. Według pierwszego modelu, celem pracodawcy jest zatrudnienie najlepszego z wszystkich pracobiorców. Wyznaczono postać optymalnej strategii.

Według drugiego modelu, celem pracodawcy jest zatrudnienie najlepszego z wolnych pracobiorców. Podane jest uzasadnienie, iż postać optymalnej strategii jest taka sama jak przy pierwszym modelu. Rozważono parę przykładów i podano wyniki obliczeń numerycznych.

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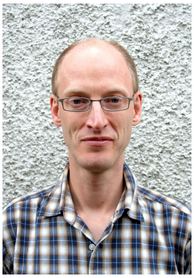

David M. Ramsey was born in Cardiff, Wales, and graduated from the University of Bristol in mathematics in 1990, specializing in applied mathematics. He gained his doctorate in statistics from the same university in 1994 on the basis of his thesis 'On Evolution, Learning and Interaction in Sequential Decision Processes'. This subject became the core of David's later research interests, which may be summarized as decision theory and, in particular, applications of sequential decision processes and dynamic games to economics and biology. In 1995, David obtained a grant from the Polish Cultural Institute to continue his research at the then Institute of Mathematics at Wrocław University of Technology for a year and afterwards worked in the same institute until 2006. From 2006 to 2012 he worked as a statistics lecturer at the University of Limerick, Ireland. In 2012, he returned to Wrocław and since then has been working as a lecturer in statistics and mathematical economics in the Faculty of Computer Science and Management, Wrocław University of Technology. In 2015, he obtained his habilitation in economics from Warsaw School of Economics on the basis of a series of articles on game theoretical models of the employment market. As a stereotypical Welshman, David enjoys rugby, choir singing and hill walking, as well as cycling. A part of his research paper are mentioned in MathSciNet under ID:668074

D. M. Ramsey
Wrocław University of Technology
Department of Operations Research
Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland
E-mail: david.ramsey@pwr.edu.pl

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