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Thermal Waves, Second Sound. Works of Witold Kosinski

Abstract This paper is dedicated to Witold Kosinski. Our contribution to this special issue will concentrate on the properties of thermal waves, one of many scientific interests of our friend and collaborator, and this article is dedicated to his memory. Working together with Witold was always an insightful and pleasant experience, and it benefited all of his coworkers including the authors of this note. His scope of research was broad, spanning many disciplines and applications. Here we focus on a few of those aspects to which he applied a deep knowledge of continuum thermodynamics and its mathematical foundations.

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1. Introduction In the classical theory of thermodynamics, heat conduction is viewed as a purely diffusive process, typically described using a Fourier law in which the heat flux is proportional to the gradient of the temperature distribution. This results in the associated evolution equations being of parabolic type and, as a consequence, completely absent of wavelike propagation. Though discovered initially at very low temperatures, experiments on diverse materials, however, long pointed to the existence of thermal waves, which travel at speeds related to the material and its temperature. These waves, second sound, were first detected in ^{3}He, and subsequently in high purity dielectric crystals of sodium fluoride, NaF, and bismuth, Bi.

The possibility of wavelike heat transport was also later discovered at moderate and even high temperatures in phenomena active over short time scales, for instance in the temperature distribution around propagating cracks and in trains of high frequency laser pulses. To account for such cases, Fourier's law required modification. One approach, in the case of rigid materials, was to modify the notion of heat flux, $\mathbf{q}$, using a memory functional to incorporate the history of the absolute temperature gradient, such as

$$\mathbf{q} = -\alpha(\vartheta) \int_{-\infty}^{t} e^{-b(t-s)} \nabla f_1(\vartheta)(\mathbf{x}, s)\, ds. \tag{1}$$

If for example $\alpha(\vartheta) = k_0/\tau$ and $b = 1/\tau$, where τ denoted a 'relaxation' time, then it followed simply from (1) that one obtained the earlier Maxwell–Cattaneo generalization of Fourier's law,

$$\tau \mathbf{q}_t + \mathbf{q} = -k_0 \nabla \vartheta, \quad \tau > 0, \tag{2}$$

a fundamental heuristic model, often used in motivating more thermodynamically consistent approaches.

In order to extend the idea behind (1), define

$$\beta = \int_{-\infty}^{t} e^{-b(t-s)} f_1(\vartheta)(\mathbf{x}, s) ds \tag{3}$$

(see [10]), so that (1) is equivalent to

$$\mathbf{q} = -\alpha(\vartheta) \nabla \beta, \tag{4}$$

$$\beta_t = -b\beta + f_1(\vartheta). \tag{5}$$

In 1989, W. Kosinski ([18]) introduced the idea of the material gradient of an internal, scalar, state variable as a contributing factor in the constitutive description of thermoelastic materials. In accordance with the restrictions imposed on constitutive functions by the second law of thermodynamics, he then proposed a relationship between the heat flux and the gradient of this state variable, or 'empirical' temperature, rather than between the heat flux and the gradient of temperature as originally in Fourier's law. The new 'temperature', β, assumed its rôle as an internal parameter by being the solution to an evolution equation related to the absolute temperature, ϑ. Formally, however, β was time-asymptotically, functionally equivalent to ϑ, and Fourier's law could eventually emerge. The physical interpretation of the new temperature scale was presented as being representative of the absolute temperature itself minus a frictional term due to internal Van der Waals related forces, and a demonstration of its compatibility with kinetic theory was given in [3] and [4].

We will not attempt to present all of Kosinki's work on the dynamics of deformable, heat-conducting continua, but we focus on some models for which wave structure and heat propagation are featured. The thermodynamic models introduced by Kosinski were generally governed by nonconservative quasilinear hyperbolic systems of partial differential equations. These systems allow shocks to be formed even in the presence of dissipation.

2. Thermomechanical Models In the introduction we gave a simple illustration to show how a constitutive equation for heat flux, which depends on the history of the absolute temperature gradient, can be prescribed by introducing a suitable internal state variable subject to its own dynamics. The evolution equation for β, (5), is unfortunately too simple to account for the properties of thermal pulses observed experimentally. In direct attempts to make the model more useful one can try to modify the history functional (1), or to introduce so called rate-type materials where the temperature rate $\dot{\vartheta}$ is one of the independent constitutive functions, however Kosinski's approach was instead to generalize (5) together with (4), implicitly retaining dependence on the history of the temperature gradient[1].

We briefly present Kosinki's thermodynamic theory of non-deformable continua, which admit wavelike propagation of heat within certain temperature ranges. Our presentation is based on selected publications listed in the references, however a broad overview can be found in [11]. Beginning with the case of rigid heat conductors, more general thermodynamic theories of deformable continua admitting wavelike heat propagation will be discussed in the second part of this chapter.

We assume that a rigid heat conductor undergoes a thermodynamic process restricted by two thermodynamic laws,

$$(\rho\varepsilon)_t \ + \operatorname{div}\mathbf{q} = \rho r, \tag{6}$$

$$(\rho\eta)_t + \operatorname{div}(\mathbf{q}/\vartheta) \geq \rho r/\vartheta, \tag{7}$$

where r is the body heat supply per unit volume, ε the internal energy per unit volume, η is the entropy, and ρ the mass density.

The following sets of constitutive functions and the evolution equation for β are the most general that Kosinski considered in this context of his work ([15]). Simplifications of material functions were derived in accordance with the second law of thermodynamics (7), in particular that the free energy, $\psi = \varepsilon - \eta\vartheta$, is independent of $\nabla\vartheta$.

$$\psi = \psi(\vartheta, \beta, \nabla\beta), \qquad \eta = -\partial_\vartheta \psi(\vartheta, \beta, \nabla\beta), \tag{8}$$

$$\mathbf{q} = \mathbf{q}(\vartheta, \beta, \nabla\vartheta, \nabla\beta), \qquad \dot{\beta} = F(\vartheta, \beta, \nabla\vartheta, \nabla\beta). \tag{9}$$

The following is also required to hold,

$$\partial_{\nabla\beta}\psi \cdot \partial_{\nabla\vartheta}F = 0, \qquad \partial_{\nabla\beta}\psi \cdot \partial_{\nabla\beta}F = 0, \tag{10}$$

together with the residual internal dissipation inequality,

$$-\rho\partial_{\nabla\beta}\psi \cdot \partial_\beta F\nabla\beta - \rho\partial_\beta\psi F - \rho(\partial_{\nabla\beta}\psi\partial_\vartheta F + (\rho\vartheta)^{-1}\mathbf{q}) \cdot \nabla\vartheta \geq 0. \tag{11}$$

The orthogonality conditions (10) introduce constraints between β and ϑ. Kosinski carefully analyses and justifies a particular case of (8) and (9) in

[1] In early work involving internal parameters, a rigorous comparison of two purely mechanical models was made in [14], contrasting cases of simple materials where the Cauchy stress tensor depends either directly on the deformation history, or on corresponding internal parameters.

order to arrive at a useful model for which a knowledge of all material coefficients can be easily obtained from three experimental curves, namely specific heat, thermal conductivity and speed of thermal disturbances ('second sound'). This model is derived assuming that $\partial_{\nabla\vartheta}F = \partial_{\nabla\beta}F = \mathbf{0}$, in which case ψ and F trivially provide the orthogonality conditions. An immediate implication is that

$$\dot{\beta} = F(\vartheta, \beta), \tag{12}$$

while ψ may depend on $\nabla\beta$.

In the isotropic case $\mathbf{q}$ may depend linearly both on $\nabla\vartheta$ and $\nabla\beta$, as in

$$\mathbf{q} = -k(\vartheta)\nabla\vartheta - \alpha(\vartheta)\nabla\beta \tag{13}$$

where it is sufficient for k and α to be functions only of ϑ in this simplified setting.

Letting $k = 0$ gives a crucial relation between α and the function F,

$$\alpha(\vartheta) = \rho\vartheta\psi_2(\vartheta)F_1'(\vartheta), \quad F(\vartheta, \beta) = F_1(\vartheta) + F_2(\beta) \tag{14}$$

with

$$\psi = \psi_1(\vartheta) + \frac{1}{2}\psi_2(\vartheta)|\nabla\beta|^2 \tag{15}$$

(Note: Kosinski observed that the case $k \neq 0$ makes it possible to consider such diffusive effects as the observed broadening of traveling pulses of second sound with increasing temperature, as well as providing a physical selection mechanism analogous to that of the viscosity admissibility criterion using parabolic regularization found in gas dynamics (cf. [23])).

Combining (6), (13) with $k = 0$, and (14), a single second order nonlinear differential equation is derived in [11] for β (see, for example, (38) below). Using the above assumptions in inequality (11) together with the stability requirement $F_2'(\beta) < 0$ for the solution of the evolution equation, this equation is hyperbolic provided the heat capacity satisfies $c_\vartheta(\vartheta) = \epsilon'(\vartheta) \geq 0$. In the range of temperatures where second sound is detected, heat capacity approximately obeys Debye's law, $c_\vartheta(\vartheta) = c_0\vartheta^3$ which is non-negative for $c_0 \geq 0$. In order to satisfy Debye's law, then $\psi_2(\vartheta) = \psi_{20}\vartheta$, where $\psi_{20} = const.$

Two further experimental measurements which may be used to specify material functions are the steady-state conductivity, K, and the second sound speed, U_E. The steady-state ('quasi-equilibrated') relation between ϑ and β is a function $B(\vartheta)$ such that $F_1(\vartheta) = -F_2(B(\vartheta))$, obtained by letting $\dot{\beta} \to 0$ in (14). Setting $\beta = B(\vartheta)$ in (13) (for $k = 0$) first gives

$$\mathbf{q} = -K(\vartheta)\nabla\vartheta, \quad K(\vartheta) = \alpha(\vartheta)B'(\vartheta). \tag{16}$$

A formula corresponding to the speed of second sound propagating in an undisturbed region is next given by

$${U_E}^2 = \frac{\alpha(\vartheta)F_1'(\vartheta)}{\rho c_\vartheta(\vartheta)}. \tag{17}$$

It has been observed that in the ranges of absolute temperature at which experiments have been performed, the time of arrival of heat pulses sent through a specimen is an approximately linear function of the reference temperature. However, towards the upper limit of temperatures where experimental measurements detect second sound, the arrival time measured at the leading edge of heat pulses rises rapidly with increasing temperature. The latter suggests a very fast decay to zero of the second sound speed, U_E, with temperature, the same effect which occurs in superfluid helium at the well-known λ-point[2]. The *critical* temperature where $U_E = 0$ is denoted by ϑ_λ. A further phenomenon concerns the observed heat conductivity which shows a pronounced peak close to ϑ_λ, whose height depends on the purity of the specimen. Motivated by this, a later hypothesis was considered that the temperature of maximum heat conductivity coincides with ϑ_λ, below which second sound appears. Above this temperature heat, conduction becomes purely diffusive, obeying a general nonlinear Fourier law. To model the phenomenon, the functions F_1 and F_2 in the evolution equation for β were taken in the form ([25], [11]),

$$F_1(z) = a(|z|^{p-1}z)_-, \quad F_2(z) = -b|z|^{h-1}z \tag{18}$$

where the constants a, b are positive, $1 < p < 2$, and $h \geq p/(2-p)$. The subscript '$-$' means that when $z \geq 0$, F_1 is taken to zero, and in the case of F_1, z represents $\vartheta - \vartheta_\lambda$.

Analysis of weak and strong discontinuity waves below ϑ_λ established a further, *structural*, critical temperature, ϑ_m, separating two distinct ("hot" and "cold") classes of propagating discontinuity waves. The amplitude of weak discontinuities remains bounded in time and no shocks form if the temperature in front of the wave is equal to ϑ_m. Such a temperature had previously been discovered by Ruggeri *et al.* [24], within the context of extended thermodynamics (a comparison of the extended thermodynamics model with the internal state variable model appears in [6]). However, using Kosinski's approach made it possible to establish a direct relationship between ϑ_m and ϑ_λ, in particular confirming the necessary condition that $\vartheta_m < \vartheta_\lambda$.

For the case of deformable body, Kosinski was the first to introduce a formulation for an anisotropic thermoelastic material subject to finite strains, in which damaging effects can appear and all pulses, mechanical and thermal, are transmitted by waves. A rigorous thermodynamic procedure of derivation was used, in the framework of a gradient generalization of the theory with internal state variables, to achieve this goal. Details can be found for example in [11], [1] and [21]. In the finite strain case, the classical Fourier law can either imply that the actual heat flux vector $\mathbf{q}$ is proportional to the

[2]Motivated by Landau's two fluid model for liquid helium, Kosinski and Cimmelli also derived a model for a binary mixture of fluids using an internal state variables approach, which was valid below the λ-point, [2]

spatial gradient of the temperature $\nabla\vartheta$, or that the referential heat flux $\mathbf{q}_\kappa$ and the Lagrangian gradient of temperature $\mathrm{Grad}\,\vartheta$ are proportional. Kosinski assumes the first possibility holds in that case. His general assumption is that $\psi = \psi(\mathbf{F}, \vartheta, \nabla\vartheta, \nabla\beta)$, where $\mathbf{F}$ denotes the deformation gradient tensor. In a similar fashion to the case of rigid conductor, the following potential relations are consequences of the second law of thermodynamics,

$$\partial_{\nabla\vartheta}\psi = 0, \tag{19}$$

$$\eta = -\partial_\vartheta\psi, \tag{20}$$

$$\mathbf{S} = \rho\partial_{\mathbf{F}}\psi + (\vartheta\partial_\vartheta\mathbf{F})^{-1}(\nabla\beta\otimes\mathbf{q}_\kappa), \tag{21}$$

$$\mathbf{q}_\kappa = -\rho(\vartheta\partial_\vartheta\mathbf{F})(\partial_{\nabla\beta}\psi)\mathbf{F}^{-T}, \tag{22}$$

$$\tag{23}$$

together with the reduced dissipation inequality,

$$(\vartheta\partial_\vartheta\mathbf{F})^{-1}(\partial_\beta F)\mathbf{q}_\kappa\mathbf{F}^T\cdot\nabla\beta \geq 0, \tag{24}$$

where $\mathbf{S}$ is the Piola–Kirchhoff stress tensor and $\nabla\beta$ represents the history of the spatial gradient of ϑ. Here the material time derivative satisfies

$$\dot{\overline{\nabla\beta}} = \nabla F(\vartheta, \beta) - \nabla\beta\dot{\mathbf{F}}\mathbf{F}^{-1} \tag{25}$$

Note that the potential relation for the heat flux cannot be obtained in the classical theory of thermoelasticity.

In order to verify the hyperbolicity of the model, Kosinski examines weak discontinuities propagating in a thermoelastic isotropic body for which the equations of motion and the first law of thermodynamics are written in terms of the Cauchy stress tensor $\mathbf{T}$, $\mathbf{S} = J\mathbf{T}\mathbf{F}^{-T}$, $J = \det\mathbf{F}$, and the actual heat flux $\mathbf{q}$, $\mathbf{q}_\kappa = J\mathbf{q}\mathbf{F}^{-T}$.

The results presented below are obtained if the free energy function is taken to have the form

$$\psi = \psi_1(\mathbf{B}, \vartheta) + \frac{1}{2}\psi_2(\mathbf{B}, \vartheta)\,|\nabla\beta|^2 \qquad \psi_2(\mathbf{B}, \vartheta) = \psi_{21}(\vartheta)J, \tag{26}$$

where $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ is the left Cauchy–Green tensor, and the heat flux is given by

$$\mathbf{q} = -\alpha(\mathbf{B}, \vartheta)\nabla\beta, \quad \alpha(\mathbf{B}, \vartheta) = \rho(\vartheta\partial_\vartheta F)\psi_2(\mathbf{B}, \vartheta). \tag{27}$$

Independence of ψ on β leads to the evolution equation (14). Both linear and nonlinear forms of F_1 and F_2 have been discussed in [18] and [13].

The system of equations to be analyzed consists of (12), (14), conservation of energy, and the equations of motion in the spatial setting,

$$\rho\dot{\varepsilon} + \operatorname{div}\mathbf{q} - \mathbf{T}\cdot\nabla\mathbf{v} = \rho r, \tag{28}$$

$$\frac{\partial\rho\mathbf{v}}{\partial t} + \operatorname{div}(\rho\mathbf{v}\otimes\mathbf{v} - \mathbf{T}) = \rho\mathbf{b} \tag{29}$$

with **b** denoting an external body force.

To this end, a one-dimensional numerical method was developed using a modification of the Lax–Wendroff scheme in order to examine solutions to the initial-boundary problem for the above system in the case of thermal waves in a sodium-fluoride crystal, [22]. A further analysis was undertaken using acceleration waves. Weak discontinuity waves (acceleration waves) are defined for a smooth two dimensional surface (wave front) propagating through space, across which second derivatives of displacement $\mathbf{u}$ and β are discontinuous. The corresponding amplitudes consist of a mechanical term, the vector-valued function $\mathbf{s}$, and a thermal term, the scalar function w, given by

$$[\nabla \mathbf{v}\, \mathbf{n}] = -U\mathbf{s}, \tag{30}$$

$$[\nabla\nabla\beta\, \mathbf{n}] = w\, \mathbf{n}, \tag{31}$$

$$U := u_n - \mathbf{v}\cdot\mathbf{n}. \tag{32}$$

Here $[\cdot]$ represents a jump discontinuity across the surface, $\mathbf{v}$ is the particle velocity, u_n is the normal speed of propagation of the acceleration wave, and $\mathbf{n}$ is a unit normal to the wave front. Using the classical kinematic and geometrical compatibility conditions, Kosinski derives a system of four equations for these amplitudes. A nontrivial solution to this system exists if the determinant of the system vanishes, this condition giving a dispersion relation between the speed U, normal $\mathbf{n}$, and values of the state variable at the wave front. In a special case where the wave propagates into a state for which $\nabla\beta = \mathbf{0}$ the characteristic equation becomes simpler and reduces to

$$\det(\rho U^2\mathbf{I} - \mathbf{Q})\,(\rho U^2 c_\vartheta \tau - \alpha) = 0, \;\; \tau(\vartheta) := 1/F_1'(\vartheta) \tag{33}$$

Thus, there are three symmetric mechanical wave speeds if the acoustic tensor $\mathbf{Q}$, which is related to the derivative of Cauchy stress with respect the deformation gradient, is positive definite, and one symmetric thermal wave speed provided that the specific heat c_ϑ is positive. In general, $\nabla\beta \neq \mathbf{0}$ and one may have coupled thermomechanical wave speeds.

Some one-dimensional examples of thermal and mechanical waves are discussed in [12] and [13]. Investigation of three-dimensional thermal weak discontinuity waves can be found in [20] and Rankine-Hugoniot conditions for shock wave propagation in of thermoelastic materials were considered in [1]. Together with numerical simulation of traveling waves in finite domains, the case of elasto-viscoplastic materials, where stress additionally depends on irreversible deformations accumulated in the past, was demonstrated in [9].

3. Mathematical Foundations Over time, Kosinski added considerable mathematical foundation to thermodynamics involving internal state variables. We describe several of the results below, representing some of his work involving gradient catastrophes, admissibility and uniqueness of weak solutions, weak solutions to thermodynamic balance laws, local well-posedness of strong solutions, and geometric optics. Although all of these methods were connected to questions of thermodynamics, here we pick out some of the more mathematical aspects and refer the reader to the previous section and the original sources for full details of the modeling.

In early work ([16]) Kosinski demonstrated that discontinuities arise from smooth initial data for the system

$$\mathbf{u}_t + \mathbf{A}(\mathbf{u})\mathbf{u}_x + \mathbf{B}(\mathbf{u}) = \mathbf{0}. \tag{34}$$

The formation of singularities, either as finite time blow-up of the solution itself, or in one or more derivative becoming unbounded as in the case of a gradient catastrophe, had been a known feature found in conservative systems for which $\mathbf{B} = \mathbf{0}$, in the first case indicating material instability due to constitutive assumptions, or demonstrating the potential for shock wave formation in the second. In the work of [16] on generally non-conservative, balance-law derived systems, the inclusion of a non-vanishing vector field $\mathbf{B}(\mathbf{u})$ extended the class of applications to systems involving internal state variables and other applications. As a result of assuming $\mathbf{A(u)}$ to be a symmetric $n \times n$ matrix, genuine nonlinearity, *i.e.* $\frac{d}{ds}\lambda_{(i)}(\mathbf{u}+s\mathbf{r^i})|_{s=0} \neq 0$ for each eigenvalue $\lambda_{(i)}(\mathbf{u})$ of $\mathbf{A}(\mathbf{u})$ with corresponding eigenvector $\mathbf{r}^i$, and strict hyperbolicity, $\lambda_{(i)} \neq \lambda_{(j)}$ for $i \neq j$, Kosinski derived a Bernoulli equation for $\mathbf{u}_x \cdot \mathbf{l}_i, 1 \leq i \leq n$, along each corresponding characteristic. This leads directly to conditions on initial data for a finite time gradient catastrophe to appear. It is further shown that supplementary hypotheses on the derivatives of $\mathbf{A}(\mathbf{u}), \mathbf{B}(\mathbf{u})$, and initial data provide conditions for solutions to decay to zero or have specified finite, or infinite, limits over bounded and unbounded time intervals.

In subsequent work ([17], [19]) he examined weak solutions to

$$\mathbf{u}_t + \nabla \cdot \mathbf{f}(\mathbf{u}) = \mathbf{B}(\mathbf{u}) \tag{35}$$

constrained by the second law of thermodynamics,

$$\eta(\mathbf{u})_t + \nabla \cdot \mathbf{k}(\mathbf{u}) \geq \mathbf{r}_\eta(\mathbf{u}), \tag{36}$$

having a strictly concave entropy function, η. Here, working in the class BV of functions of bounded variation in the sense of Tonelli–Cesari, Kosinski defined weak solutions, then considered a parabolic version of the system in order to define admissible weak solutions to the hyperbolic initial value problem, as limits of Lipschitz continuous solutions to the corresponding parabolic problem. He also established uniqueness of these solutions in BV, with much of this work inspired by contemporary results of C. Dafermos and R. J. DiPerna.

Working together with Cimmelli in [5], Kosinski considered local well-posedness of the Cauchy problem for an isotropic rigid heat conductor subject to the effects of an internal state variable. Upon adopting certain constitutive assumptions in the general model, the absolute to semi-empirical relationship was reduced to

$$\vartheta = \vartheta^0 \exp\left(-\frac{\tau_0\beta_t + (\beta - \beta^0)}{\vartheta^0}\right), \quad \tau_0 > 0, \tag{37}$$

and the associated second order, quasi-linear, hyperbolic equation for β read, essentially, as

$$\tau_0(\vartheta\beta_{tt} - \nabla\beta \cdot \nabla\beta_t) + \vartheta\beta_t - \vartheta^0\Delta\beta = 0. \tag{38}$$

Here we have normalized several coefficients in order to emphasize the principal features, and we also neglected any body heat supply. Equation (38) can be converted into a system by letting $\mathbf{u} = (\beta, \dot{\beta})$ and setting

$$\mathbf{A}(\mathbf{u}) = -\begin{pmatrix} 0 & 1 \\ A_1 & A_2 \end{pmatrix} \tag{39}$$

where $A_1, A_2(\mathbf{u})$ (again normalized) represent the second order partial differential operators $A_1 = \delta_{ij}\dfrac{\partial^2}{\partial x_i \partial x_j}$, $A_2 = \delta_{ij}\dfrac{\partial\beta}{\partial x_i}\dfrac{\partial}{\partial x_j}$.

Based on fundamental work by T. Kato, the above formulation, together with suitable conditions on the initial data for ϑ and β provided a basis for obtaining local existence, uniqueness and continuous data dependence of solutions belonging to the class

$$(\beta(t,x) - \beta^0, \vartheta(x,t) - \vartheta^0) \in C^p([0,T), H^{s+1-p}(\mathbb{R}^n, \mathbb{R}) \times H^{s-p}(\mathbb{R}^n, \mathbb{R})),$$

$s > n/2$, $0 \le p \le s$, for some time $T > 0$.

Later employing the fact that the initial-value problem for symmetric, hyperbolic, quasi-linear systems is locally well-posed, Domanski, Jablonski and Kosinski revisited local well-posedness via Friedrichs symmetrization, [7]. In so doing, they first expressed the equations for energy balance and the evolution of β using as dependent variables β, internal energy ε, and heat flux $\mathbf{q}$, and then derived the form of a convex entropy function whose gradient components could be used as a further replacement of these dependent variables for the system of balance laws to become symmetric.

Taking a geometric optics approach to (34) in [8], Kosinski and Domanski considered the initial value problem

$$\mathbf{u}^\epsilon{}_t + \mathbf{A}(\mathbf{u}^\epsilon)\mathbf{u}^\epsilon{}_x + \mathbf{B}(\mathbf{u}^\epsilon) = \mathbf{0}, \tag{40}$$

$$\mathbf{u}^\epsilon(\mathbf{x}, 0) = \mathbf{u_0} + \epsilon\mathbf{u_1}(\mathbf{x}, \mathbf{x}/\epsilon) + O(\epsilon^2), \tag{41}$$

and examined weakly nonlinear asymptotic solutions of the form

$$\mathbf{u}^\epsilon(x,t) = \mathbf{u}_0 + \epsilon\sum_{j=1}^{n}\sigma_j\left(x, t, \frac{x - \lambda_j t}{\epsilon}\right)\mathbf{r}_j + O(\epsilon^2). \tag{42}$$

Here the amplitudes σ_i satisfied a set of uncoupled, asymptotic evolution equations,

$$\sigma_{i,t} + \lambda_i \sigma_{i,x} + \frac{1}{2}\Gamma_i \sigma_{i,\eta}^2 + \Lambda_i \sigma_i = 0, 1 \le i \le n. \tag{43}$$

where $\Gamma_i = \mathbf{l}_i \cdot (\nabla_{\mathbf{u}} \mathbf{A})|_{\mathbf{u}=\mathbf{u_0}} |\mathbf{r}_i|^2$ and $\Lambda_i = \mathbf{l}_i \cdot (\nabla_{\mathbf{u}} \mathbf{B})|_{\mathbf{u}=\mathbf{u}_0} \mathbf{r}_i$, and the eigenvalues, $\{\lambda_i \, 1 \le i \le n\}$, left and right eigenvectors, $\mathbf{l}_i$ and $\mathbf{r}_i$, satisfy

$$(\mathbf{A}(\mathbf{u_0}) - \lambda_\mathbf{i} \mathbf{I})\, \mathbf{r_i} = \mathbf{0},\ \mathbf{l_i}\,(\mathbf{A}(\mathbf{u_0}) - \lambda_\mathbf{i} \mathbf{I}) = \mathbf{0}, \quad \mathbf{l_i} \cdot \mathbf{r_j} = \delta_{\mathbf{ij}}. \tag{44}$$

This approach was used to compare Kosinski's internal state variable model for rigid conductors with another model developed by Morro and Ruggeri ([24]) in the framework of extended, or non-equilibrium, thermodynamics. Both models describe heat propagation over a limited range of very low temperatures where experimental measurements of specific heat, heat conductivity and second sound speed allow the identification of needed constitutive functions. The asymptotic forms of the evolution equations for the two models are shown both to be of type (43) and, with suitable identification of the coefficients, deliver identical solutions.

4. *Addendum*

All those who knew Witold greatly enjoyed his company, humour and kindness. His enthusiasm was contagious and his work inspired many, for which we are grateful.

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Fale termiczne, Drugi dźwięk. Wyniki prac Witolda Kosińskiego

Katarzyna Saxton, Ralph Saxton

Streszczenie Niniejsza praca jest poświęcona pamięci naszego przyjaciela Witolda Kosińskego. Chcielibyśmy przedstawić jego najważniejsze osiągnięcia w dziedzinie propagacji fal termicznych, termodynamiki i teorii hiperbolicznych układów różniczkowych. Zakres badań Kosińskiego był bardzo bogaty. Obejmował wiele dyscyplin na pograniczu mechaniki, matematyki i teorii komputerowych. Współpraca z Witoldem była owocna, zawsze wypełniona entuzjazmem i wzajemnym szacunkiem. Autorzy tego artykułu jak i inni współpracownicy Witolda korzystali z jego wiedzy i zawodowego doświadczenia. Będzie nam go bardzo brakowało.

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Słowa kluczowe: mechanika ośrodków ciągłych, fale nieliniowe, osobliwości, prawa zachowania.

Katarzyna Saxton received her M.Sc. in 1972 at the Department of Mathematics, University of Warsaw. She obtained her doctorate in 1979 under the direction of Professor P. Perzyna at the Institute of Fundamental Technological Research, Polish Academy of Science. Her interests lie in applied mathematics where she has studied problems from hyperbolic equations and continuum thermodynamics. Her selected publications, reported in MathSciNet, are under ID:322513, ID:566548, ID:548138, ID:548139.

Ralph Saxton obtained his doctorate in mathematics in 1981 under the direction of Professor Robin Knops at Heriot–Watt University, near Edinburgh, Scotland. His interests lie in applied mathematics in which he has studied problems of continuum mechanics such as well-posedness in incompressible elasticity, micro-dynamics of liquid crystals, global existence and finite-time breakdown of solutions in fluids and solids. His selected publications reported in MathSciNet are ID:155280.

Figure 1: Visit in New Orlenas, March 1998. From the left Ralph Saxton, Ludwika Szmit, Katarzyna Saxton, Ewa Kosinska and Witold Kosinski

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