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Mechanics of sailing ship motion

Abstract The author considers the rules defining the efficiency of a sailing ship in steady motion and derives a number of mathematical equations that have proved in practice to be accurate. The equation for the sailing efficiency in a downwind sector is similar to the curves from a Wulff grid, while the equation for a windward sector, previously unpublished, is similar to a fourth degree curve. The article presents a description of the assumptions, a method for deriving the formulas, as well as conclusions and suggestions for new definitions of terms to be used in the nautical vocabulary.

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1. Present state of knowledge. The intensive development of sailing, both high-performance and for pleasure, on the sea and internal waters implies the engagement of important research into the subject of sailing ship motion. This problem is however exceptionally difficult, because this motion takes place on the boundary between two different media, water and air, and their parameters relative to a ship's motion are subject to considerable probabilistic components, which vary over time.

The basic source for the analysis of sailing ship motion is without doubt the well-known opus of Czesław Marchaj, published firstly in English (1964), then in Polish (1970) and later in German too, and completed by a series of handbooks and papers [12], [13], [14], [15].

Marchaj presented a full analysis of a sail's aerodynamics, based mainly on his own measurements and research in wind tunnels. He presented efficient mathematical methods for estimating a sailing ship's velocity as a function of the parameters of its components.

Among the research presented in recent years, it is worth mentioning here the review by J. Milgram [1]. R.M. Wilson presents an overview of current theory [2], giving a complete picture of the physical phenomena related to the motion of a sailing ship. An interesting complementary view may be found in

Hoffman and Johnson's paper [3], where the authors attempt to describe the complex phenomena of wind and water flow more clearly to non-specialists.

On the other hand, research has been carried out for some decades on the application of computational techniques in the field of hydro- and aeromechanics. They use the experience obtained in CFD, Computational Fluid Dynamics, associated with a continually growing data base containing empirically measured aerodynamic coefficients of sails. With this support, useful software has been created, both for producers and for general use (e.g. VPP — Velocity Prediction Program), even in the form of a MATLAB library (gvpp). However, these general use programs require access to the data base of aerodynamic coefficients of sails and general knowledge of the parameters of sailing ship design and motion. They involve an iterative procedure based on correcting initial data.

Recently, a new innovation in this field was published [6]. As the result of an extensive multi-year study, historical data derived from tank test results have been combined with new research using CFD tools to create this new formulation. However, considerable computing power is necessary.

The analysis presented below has the following goals:

— firstly, to obtain a simple and elegant classification of types of sailing ships (see formulas 7 to 11a and Table 1 in the text),

— secondly, using globe graphs and shell graphs, obtain equations and diagrams of the basis curve describing sailing ship efficiency (see Fig. 9 to 12).

1.1. Notation. In the text, the following notation is used — unless otherwise specified:

a — Vector of the aerodynamic force, the projection of the resultant of all the forces created by the air flowing around the sailing ship body onto the water's surface. The parameters describing the air are specified in the index.

a_v — Projection of the aerodynamic force vector in the direction of the ship's movement.

A — Angle between the resultant aerodynamic force and the apparent wind vector. Its maximum value in degrees is called the angular characteristic of the ship.

A_z — Angle A in the sector of full winds, a variable.

B — Transitional course, (in other words the back stay), dividing the sector of full winds from the sector of acute winds.

c — Chord of the inner circle, connecting the points of this circle common with the tangents to lines 1 and 2.

C — Coefficient of drag.

E — Angle between the vectors a, v.

F — Downwind course, where the vectors r, v are superposed.

G — Boundary course, dividing the dead and sailing zones.

h — Vector of hydrodynamic force, projection of the resultant of all the forces created by the water flowing around the ship's body onto the water's surface. The parameters describing the water are specified in the index.

h_v — Projection of the hydrodynamic force vector h in the direction of the ship's movement.

K — Non-dimensional characteristic of the ship, called the coefficient of draft.

m — Dead zone; the set of directions of motion such that the ship cannot go towards the target along a straight line.

M — Course, along which the ship's velocity is maximal within the sailing zone.

n — Efficiency of the ship, i.e. the ratio of its velocity to the velocity of the true wind.

o — Sector of windward sailing, where the vector of the aerodynamic force cannot be superposed onto the vector of the ship's motion.

O — Course, along which the projection of the velocity vector of the ship in the direction straight against the wind takes its maximum value.

p — Projection — onto the water's surface — of the vector of the apparent wind velocity, the vector sum of the velocities of the true wind and induced wind (opposite to the direction of motion with the same magnitude).

P — Course of the ship relative to the true wind direction, i.e. angle between the vectors r, v, the pole for polar coordinates and the apex of the triangle of wind velocities.

r — Projection of the vector of the true wind velocity, relative to the sea bed, onto the water's surface.

R — Course of the ship relative to the true wind direction, i.e. angle between the vectors p and v (for the meaning of notions: p, P, r, R — see Fig. 1, 2 below).

S_a — Surface of the sail.

S_h — Surface of the midship section below the water line (LW).

v — Projection of the vector of the ship's velocity relative to the sea bed onto the water's surface.

w — Projection of the vector of the induced wind velocity, onto the water's surface.

z — Sector of sailing in full wind, where the direction of the vector of aerodynamic force is superposed onto the ship's motion.

X — Course, along which the projection of the ship's velocity vector in the direction of the wind reaches its maximum value.

1.2. Definitions. Here, we formulate some definitions related to the mechanics of sailing ship motion discussed above, which are not outlined in the current literature or sailing practice, or defined in a different manner.

The polar curve of the efficiency graph is constructed according to the true wind direction, and not as is usual in the literature — according to the opposite direction.

The *course of the ship* is the angle between the true wind direction and the ship's motion; this course is different from the usual notion in the literature by 180°.

All the possible directions of a ship's motion on the open sea form a full angle, which may be divided into two zones: *the dead zone* and *the sailing zone*. In the first, the ship cannot sail towards the target along a straight line, while in the second – the sailing zone – the ship can reach the target along a straight line. The sailing zone is divided into two sectors: the *downwind motion sector*, in which the underwater part of the hull may have a vertical plane of symmetry — the heel angle is close to zero, and the *windward motion sector,* where the heel angle is appreciable and the hull must have a keel — fixed or drop.

Basic courses:

downwind course — along the axis with angular measure 0°,

transitional course, or back stay *course*, dividing the *downwind sector* from the *windward sector*,

boundary course, dividing the dead zone from the sailing zone.

Specified courses:

Tacking downwind course, optimal course towards a target in the downwind sector,

Tacking windward course, optimal course to a target situated in the dead zone, in the windward sector.

A sailing ship (ship) is a vessel that floats at the boundary of two fluids: air and water, due to the effect of the difference between their densities, and it may be put into motion due to the action of the difference between the movement velocities of these fluids. The two main parts of a sailing ship are: the dry one — above the water surface, and the wet one, below. There are four main *forces* acting on a ship. Two of them are static: *gravity* acting on the center of gravity of a ship, directed vertically downward, *buoyancy* acting on the center of the submerged part of the hull, directed vertically upward. The two other forces are dynamic: *aerodynamic force* generated by the airflow through the sail and other dry elements of the ship, *hydrodynamic force*, generated by the water flow on the wet part of the ship.

2. Introduction. Let us consider the uniform, linear motion of a sailing ship on quiet water, when the sum of all the four forces (gravity, buoyancy, aerodynamic and hydrodynamic forces) acting on a ship is zero. These static forces (the first two) determine the vertical plane. The resultant of the dynamic forces (the last two) must lie in the same plane. They form the heeling moment, which is equilibrated by the righting moment of the static forces. The vertical component of the dynamic forces is equilibrated by buoyancy. Both moments generate twisting tension in the ship's body.

In the analysis which follows we will only consider projections of forces and velocities onto the horizontal plane representing the water's surface. In this case, the projection of the aerodynamic force is equal in magnitude to the projection of the hydrodynamic force and lies along the same line, but has the opposite direction.

The main feature of a ship's quality is its efficiency, or the ratio of the attainable velocity to the wind velocity. The ship's efficiency is usually presented graphically in *polar coordinates*. The vector of the true wind r (oriented directly upwards) defines the *polar axis*, and its beginning lies at the origin of the system of coordinates, constituting its *pole*. The vector v of the ship's velocity originates from the *pole*. The end of this vector describes a curve of which the vector is its *leading radius*. The angle between the vectors r and v is the *argument* of the vector v and represents the course of the ship P. If the vector r represents the true wind velocity, then the vector v describes the curve of the ship's velocity. If the vector r has modulus 1, then the vector v describes the efficiency curve. Since a sailing ship is almost a symmetric object, their efficiency graphs are presented in the literature using angles between 0 and 180° , while in this article these graphs are extended to angles between 0 and 360°.

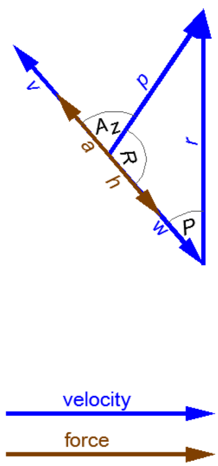

Figure 1: Vectors in the downwind sector

To simplify the notation of the parameters, their indexes are placed at

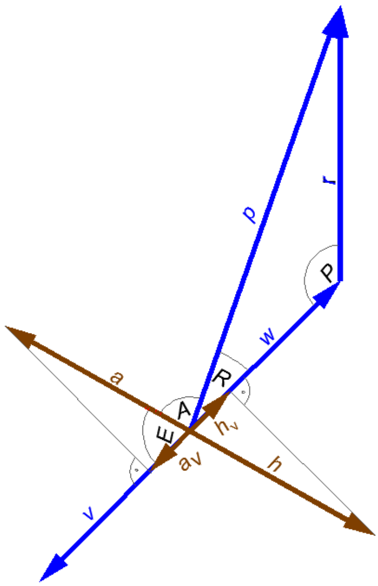

Figure 2: Vectors in the windward sector

fixed positions:

$$^{K}_{A}X_{P},$$

where X denotes one of the parameters described in the text above. Particular values of P, K and A may be used for e.g. designated courses or lower case letters to denote whole sectors. For example:

$$^{0.125}_{80}X_{90}$$

denotes: the efficiency of a ship with angular characteristic 80° and draft coefficient 0.125 at course 90°.

2.1. Assumptions.

1. The ship is in uniform, linear motion.
2. The water surface is quiet (no current, no waves).
3. The sails work optimally (with maximal efficiency).
4. The ship flows on an even keel, without heeling or tilt.
5. The air movement is an uniform vectorial field.
6. The coefficient of aerodynamic propulsive force C_a is constant under any course P.
7. The coefficient of hydrodynamic resistance C_h is constant under any course P.

Assumptions 1–4 may be fulfilled in appropriate sailing conditions. Assumption number 5 is not fulfilled in any case. Assumption 6 is fulfilled only in the windward sector. Assumption 7 is fulfilled only in the downwind sector.

3. General equation for the efficiency of a sailing ship. The efficiency n of a sailing ship is defined to be the ratio of the ship's velocity to the true wind velocity:

$$n = \frac{v}{r} \tag{1}$$

or

$$n = \frac{w}{r} \tag{2}$$

where w is the induced wind velocity, equal to v, but with the opposite orientation.

Let us consider the projections of the velocity vectors and the hydro- and aerodynamic forces onto the water's surface. (see Fig. 2). The force of hydrodynamic resistance, generated on the wet part of the hull, is described by the well-known formula:

$$h_v = 0.5 \cdot \rho_h \cdot S_h \cdot C_h \cdot v^2 \tag{3}$$

where: ρ — fluid density, and the force of aerodynamic resistance generated on the dry part of the ship:

$$a = 0.5 \cdot \rho_a \cdot S_a \cdot C_a \cdot p^2. \tag{4}$$

This aerodynamic force is created due to the air flow through the sail[1] by the apparent wind. The vector of this force may be derived from the vector of apparent wind when the angle $A < 90°$. The propulsive force of the ship, a_v, is the projection of the whole aerodynamic force a in the direction of motion:

$$a_v = a \cdot \cos E \tag{5}$$

and should be equal in magnitude to the resistance force from the water:

$$a_v = h_v \tag{6}$$

From the formulas (3), (4), (5) we obtain:

$$0.5 \cdot \rho_a \cdot S_a \cdot C_a \cdot p^2 \cdot \cos E = 0.5 \cdot \rho_h \cdot S_h \cdot C_h \cdot v^2.$$

Hence:

$$p^2 = \frac{\rho_h \cdot S_h \cdot C_h \cdot v^2}{\rho_a \cdot S_a \cdot C_a \cdot p^2 \cdot \cos E}. \tag{7}$$

[1] For simplification, the aerodynamic drag generated on the dry part of a ship — apart from the sails — has been omitted.

or:

$$p^2 = \frac{\rho_h}{\rho_a} \cdot \frac{S_h}{S_a} \cdot \frac{C_h}{C_a} \cdot \frac{v^2}{\cos E}. \tag{8}$$

For succinctness, let us introduce the symbols:

$$K_A = \frac{\rho_h}{\rho_a}, \tag{9}$$

$$K_B = \frac{S_h}{S_a}. \tag{10}$$

$$K_C = \frac{C_h}{C_a}, \tag{11}$$

$$K = K_A \cdot K_B \cdot K_C, \tag{12}$$

where:
K_A — non-dimensional characteristic of the sailing **A**rea,
K_B — non-dimensional characteristic of the ship's **B**ody,
K_C — non-dimensional characteristic of the ratio of the force **C**oefficients.

All the three numbers K_A, K_B, K_C are positive, so their product K must also be positive.

Since the domain of the possible values of K is infinite, specific choices of its values were made for the purposes of comparison. Let t be the number describing a specific type of sailing ship, limited to 11 example types. Let the relation between t and K be of the form presented in Formula (13). The types specified in this way are described in Table 1.

$$K = 2^{t-6}. \tag{13}$$

The only reason to introduce this kind of division is to obtain a visible and uniform variation in a graphical analysis . We call the series of ships, for which the number K is determined by (11a), as *types*. A ship of type $t = 6$, $K = 1$, is called *standard*. Ships with $K < 1$ will be called *light*, while those with $K > 1$ will be called *heavy*. The calculations for the *standard* type ($K = 1$) are presented just below Table 1. Moreover, for more detailed analysis, presented in the following text, two other types of ship were chosen: a *light ship* with $t = 3$, $K = 0.125$ and a *heavy ship* with $t = 9$, $K = 8$.

An example calculation of K for a standard ship:

For K_A: salt water density $\rho_h = 1020$ kg/m^2, air density $\rho_a = 1.25$ kg/m^3 [8, p. 19], $K_A = \rho_h/\rho_a = 1020/1.25 = 816$

For K_B: for a big, square rigged vessel, one can find in the bibliography, [9, p. 616], $K_B = S_h/S_a = 1/30 = 0.0333$.

Table 1: Partition of sailing ships by types

Types	t	K	Max. efficiency with angular coeficient A				Representatives	Coeff. A deg	Max. veloc. knots
			85°	60°	45°	0°			
Light	1	0.03125	8.442	1.867	1.333	0.878	Ice-boats	85	135
	2	0.0625	6.690	1.754	1.266	0.824	Windsurfing boards	85	50
	3	0.125	4.758	1.573	1.158	0.825	Katamarans, trimarans	85	40
	4	0.25	3.087	1.325	1.009	0.676	Race boats — keelers	85	30
	5	0.5	1.924	1.050	0.838	0.631	Drop keel race boats	88	20
Std.	6	1	1.212	0.795	0.664	0.500	Tourist yachts — keelers	80	15
Heavy	7	2	0.788	0.587	0.513	0.414	Drop keel tourist yachts	80	10
	8	4	0.528	0.430	0.389	0.333	Sailing ships, reefed sails	45	7
	9	8	0.362	0.313	0.291	0.261	Sailing ships in a storm	7	2
	10	16	0.252	0.227	0.216	0.200	Rafts with a sail	30	1.5
	11	32	0.177	0.164	0.158	0.150	Rescue rafts, buoys	0	0.5

For K_C: from [8, Table.8.1, entry 18 and 22] one may read: $C_h = 0.0508$, $C_a = 1.42$, hence:

$$K_C = C_h/C_a = 0.0508/1.42 = 0.0358$$

and

$$K = K_A \cdot K_B \cdot K_C = 0.974 \approx 1.$$

The mean values for the parameters were used above, giving a good approximation of the standard value of K as a result. For every ship, the number K will vary to some extent, but may be determined with sufficient precision by taking into account the influence of external factors such as: weather, sea, wind, amount of ballast, surface area of sails, roughness of the wet hull, heel and tilt angles.

Going back to Equation (8) in its succinct form:

$$p^2 = K \cdot \frac{v^2}{\cos E}. \tag{14}$$

From the velocity triangle p, r, w (Fig. 2), using the cosine formula, we obtain:

$$p^2 = r^2 + v^2 - 2rv\cos P. \tag{15}$$

Comparing the right hand sides of Eqns. (14) and (15) and dividing by r^2, then using Eqn. (1), we obtain the **general equation of sailing ship motion**:

$$\left(\frac{K}{\cos E} = 1\right) \cdot n^2 + 2n\cos P - 1 = 0. \tag{16}$$

The real positive root of this equation gives the formula for the efficiency n:

$$n = \frac{1}{\cos P + \sqrt{\cos^2 P - 1 + \dfrac{K}{\cos E}}}. \tag{17}$$

This is the general expression for the efficiency of a ship with respect to the true wind velocity at course P.

4. The efficiency of a sailing ship on downwind courses. The full wind sector is the segment of the sailing zone bounded by the following courses: downwind and back stay:

$$F \leqslant P_z < B.$$

Sailing in this sector is characterized by the fact that the orientation of the aerodynamic force a accords to the orientation of the ship's motion v (see Fig. 1), or:

$$E_z = 0; \quad \cos E_z = 1. \tag{18}$$

Eqn. (16) represents the motion of the ship in the downwind sector:

$$(K - 1) \cdot n_z^2 + 2\cos P_z \cdot n_z - 1 = 0. \tag{19}$$

The efficiency of a ship in the downwind sector is the root of Eqn. (19) and takes the form:

$$n = \frac{1}{\cos P + \sqrt{\cos^2 P - 1 + K}}. \tag{20}$$

For a downwind course $P_z = 0$; $\cos P_z = 1$ and the efficiency equals:

$$n_F = \frac{1}{1 + \sqrt{K}}. \tag{21}$$

From Eqn. (21), we can calculate the coefficient of draft on a downwind course:

$$K_F = \left(\frac{1}{n_F} - 1\right)^2 \tag{22}$$

or based on velocities which may be measured directly:

$$K_F = \left(\frac{p_F}{v_F}\right)^2. \tag{23}$$

For example on a downwind course with $p_F = 8$, $v_F = 4$, one obtains $K_F = 4$.

5. Globe diagram. In polar coordinates, the roots of Eqn. (19), (n, P_z), lie on a second degree curve. This curve is an arc of a circle whose center is located on the polar axis, and its equation is of the form:

$$n^2 - 2nm\cos P + m^2 = u_2^2, \tag{24}$$

where (see Fig. 3)

u_2 — radius of the circle,

m — distance from the center of the circle to the origin of the coordinate system.

For $P_z = 0$, Eqn. (19) has canonic form:

$$n^2 - 2n\frac{1}{1-K} + \frac{1}{1-K} = 0. \tag{25}$$

Comparing Eqns. (24) and (25), we see that:

$$m = \frac{1}{1-K} \tag{26}$$

and:

$$m^2 - u_2^2 = \frac{1}{1-K} \tag{27}$$

or:

$$u_2 = \frac{\sqrt{K}}{1-K}. \tag{28}$$

Hence, we conclude that the curve described by (19) is a circle with radius u_2 given by (28). The distance between the center of this circle and the origin of the coordinate system is equal to m, given by Eqn. (26). Negative values indicate that the vector from the origin to the center of the circle is orientated in the opposite direction to the polar axis. Substituting into Equation (20) the values of K for typical sailing ships from Tab. 1, we obtain a set of lines composed of two clusters of arcs, one for $K < 1$ and a second one for $K > 1$. These arcs are fragments of circles, whose centers are placed on the polar axis $P = 0$ at a distance m from the origin. The polar axis will be called the *zero meridian*, with the point $(0, 0°)$ being the *South Pole* **P**, and the point $(1, 0°)$ *the North Pole* **W**. The arcs are symmetric with respect to the line perpendicular to the axis and pass through the *zero meridian*. This segment is called the *equator* and is the graph of the equation for the specific case $K = 1$.

The arcs below the *equator* will be called *south parallels*, and those above — *north parallels*.

The south parallels (for $K > 1$) are eccentric with respect to the south pole, and the north parallels (for $K < 1$) are eccentric with respect to the north pole.

The intersection of these arcs with the zero meridian occur at a certain distance from the south pole, which simultaneously gives the efficiency of a ship on a downwind course (29).

$$n_F = \frac{1-\sqrt{K}}{1-K}. \tag{29}$$

If K is any positive number, then n_z in relation to the *equator* circumscribes two symmetric arcs, with radius u_2 equal to the radius given by the appropriate function of K, as follows:
for $K < 1$

$$u_2 = \frac{\sqrt{K}}{1-K}. \tag{30}$$

for $K > 1$

$$u_2 = \frac{\sqrt{\frac{1}{K}}}{1-\frac{1}{K}}. \tag{31}$$

Note that:

$$^{K}u_2 = {}^{1/K}u_2. \tag{32}$$

The angle of the deviation of the aerodynamic force from the apparent wind vector cannot be bigger than the angular characteristic A of the ship, and for this reason the range of the parallels will be determined by meridians. Meridians are segments of circles, which have the polar axis as a common chord. The centers of the circles lie on the equator and their radii are determined by the formula

$$u_1 = \frac{1}{2\sin A}. \tag{33}$$

The centers of the circles are placed at the point $O_1(n, P)$ with coordinates:

$$n = \frac{1}{2\sin A}; \quad P = 90^\circ - A. \tag{34}$$

The distances of the centers of the circles of these meridians from the zero meridian are:

$$O_1S = 2\operatorname{ctg} A. \tag{35}$$

Fig. 3 illustrates the dependency between the wind triangle (vectors p, r, w) and the efficiency of an example light ship with parameters $A = 75°$ and $K = 0.125$ in the downwind sector. The respective segments represent:

NR — efficiency curve in the downwind sector;

PW $= r = l$ — polar axis, zero meridian, chord of meridian;

PR $= n_B = c$ — efficiency on course B, the chord of the inner circle;

WR $= p$ — apparent wind vector;

PRW — the wind triangle;

PN $= n_F$ — efficiency on a downwind course;

O$_1$R $= u_1$ — radius of the meridian;

O$_1$S — distance from the centers of the meridian to the zero meridian along the equator;

O$_2$R $= u_2$ — radius of the parallel;

O$_3$R $= u_3$ — radius of the inner circle;

O$_2$P $= m$ — distance from the center of the parallel circle to the pole.

The radius of the circle u_1 at point R is tangential to the circle with radius u_2, which means that those radii are perpendicular to each other. Therefore, both circles intersect at point R at right angles, which is the condition *sine qua non* of the *Wulff grid*[2].

Knowing the draught coefficient K is sufficient to determine the curve of efficiency in the downwind sector. Similarly, knowledge of the angular characteristic A of a ship is sufficient to determine the domain where it is possible to sail downwind, i.e. the meridian.

Let us write the equation of the meridian in polar coordinates. For a circle passing through the pole (PRW in Fig. 3), the general equation for a circle gives

$$\rho^2 - 2\rho \cdot \rho_0 \cdot \cos(\varphi - \varphi_0) + \varphi_0^2 = R^2, \tag{36}$$

[2] According to [10, p. 618], Wulff grids [16] are applied to solve problems in spherical trigonometry, in astronavigation, under the name of an azimuthal grid — to transform equatorial coordinates to horizontal ones, in crystallography — to the mapping of crystals.

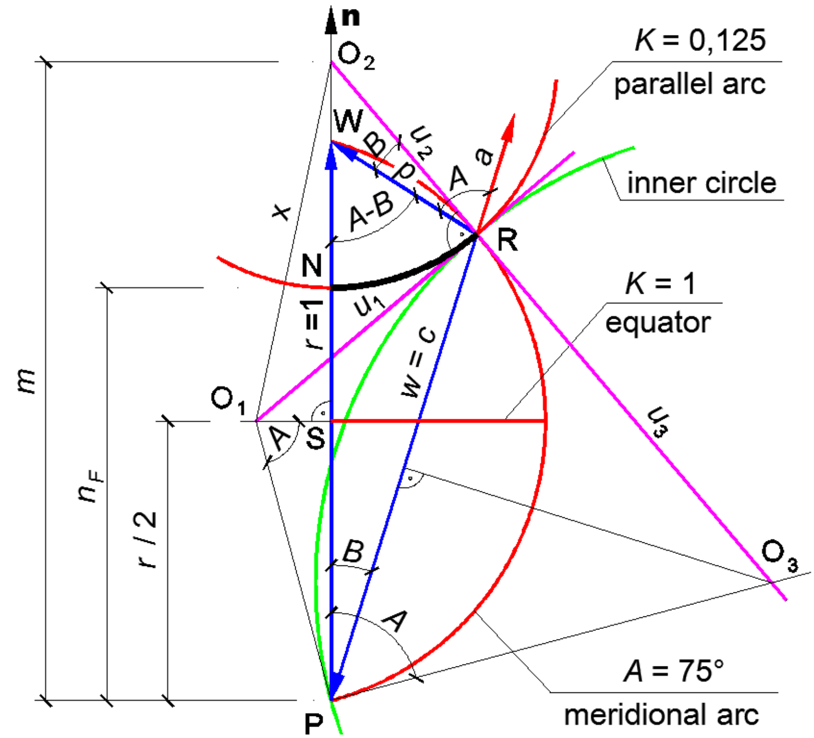

Figure 3: Elements of the globe curve

where from Fig. 3

$\rho = \mathrm{PR} = n_\lambda$ — leading radius of the meridian PRW.

$\varphi = B$ — angle made by leading radius

$\rho_0 = \mathrm{O_1R} = \frac{1}{2\sin A}$ — leading radius of the center of the circle PRW

$\varphi_0 = 90° = A$ — angle made by the center of circle PRW.

Hence:

$$\cos(\varphi - \varphi_0) = \sin(A + P_z).$$

Substituting this into Eqn. (36), we obtain:

$$n_\lambda^2 - n_\lambda \frac{\sin(A + P_z)}{\sin A} = 0.$$

The real positive root with $P_z < 90°$ is given by:

$$n_\lambda = \frac{\sin(A + P_z)}{\sin A} \tag{37}$$

or equivalently by:

$$n_\lambda = \cos P_z + \sin P_z \cdot \mathrm{ctg}\, A. \tag{38}$$

Formulas (37), (38) are the equations of the meridian arcs, which limit the domain of the parallel arcs determined by Eqn. (20).

Fig. 4 illustrates ship efficiency in the downwind sector for a starboard or port (right or left) tack, for various A and all the ship types given in Tab. 1. This diagram looks similar to a globe; hence it is normally called a *globe diagram*. Fig. 3 and 4 show the efficiency curve NR for an example light ship with parameters $A = 75°$ and $K = 0.125$. The segment of a straight line PR represents the efficiency on that course, which is determined by the angle NPR. This angle appears to be a maximal downwind course for the ship analyzed, i.e. the transitional course between the downwind and windward sectors.

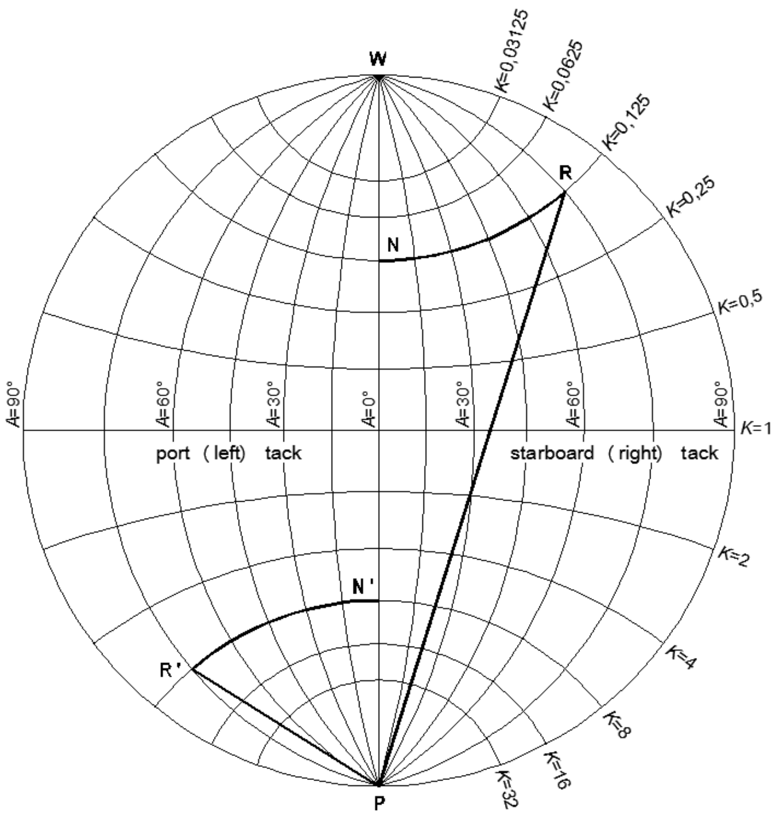

Figure 4: The parallel curves and meridians for various K and A

A globe diagram is similar to a Wulff grid except for the dimensions of parallels, which are described by draught coefficients instead of geographic latitude (in degrees).

A globe diagram may be transformed into a Wulff diagram by applying a substitution in Eqn. (20) so that the distances between parallels along the polar axis are equal to the distances according to the Wulff grid.

We obtain a cluster of circles representing the parallels by substituting the following expression into Eqn. (20):

$$K = \operatorname{tg}\frac{90^\circ \pm \varphi}{2}, \tag{39}$$

where in the northern hemisphere $0 \leqslant \varphi \leqslant 90^\circ$, and in the southern hemisphere — $90^\circ \leqslant \varphi \leqslant 0$.

Therefore,

$$n_\varphi = \frac{1}{\cos P + \sqrt{\operatorname{tg}\dfrac{90^\circ + \varphi}{2} - \sin^2 P}}. \tag{40}$$

The cluster of circles representing the meridians can be obtained by substituting $A = \lambda$ into Eqn. (36).

5.1. Features of globe curves.

1. Globe curves in polar coordinates are sets of points with coordinates (n_z, P_z).
2. They have domain: $-B < P < B$, $0 \leqslant A_z < A_{\max}$, $K > 0$.
3. They are composed of two pairs of clusters of circular arcs. One pair is called parallel and the other pair called meridional. They intersect each other at right angles.
4. Both pairs of clusters are symmetric in relation to two mutually perpendicular segments: the polar axis, called the zero meridian, and the symmetric polar axis called the equator.
5. The polar axis is a common chord of meridians. The centers of the arcs of these circles lie on the equator. Their distances from the zero meridian are given by Eqn. (35). Their radii are given by Eqn. (33).
6. The centers of the parallel arcs are placed on a straight polar line at a distance from the pole given by Eqn. (26). Their radii are given by Eqn. (28).
7. The arcs of the parallel circles depend on the parameter K.
8. The arcs of the meridians depend on the parameter A.
9. The segment of a parallel arc corresponding to parameter K between meridians corresponding to parameter A constitutes the graph of the efficiency of a sailing ship with parameters K and A in the downwind sector.

6. Summary of the results.

Efficiency in the downwind sector: $$n_z = \frac{1}{\cos P_z + \sqrt{K - \sin^2 P_z}}$$

Efficiency on a full wind course: $$n_F = \frac{1}{\sqrt{K} + 1}$$

The radius of a meridian: $$u_1 = \frac{1}{2 \sin A}$$

Coordinates of the center of a meridian, port (left) tack: $$\left(\frac{1}{2 \sin A},\ 90^\circ - A\right)$$

Coordinates of the center of a meridian, starboard (right) tack: $$\left(\frac{1}{2 \sin A},\ 270^\circ + A\right)$$

The radius of the parallel circle: $$u_2 = \left|\frac{\sqrt{K}}{1 - K}\right|$$

Coordinates of the center of the parallel circle: $$\left(\frac{1}{1 - K},\ 0^\circ\right)$$

7. Efficiency of a sailing ship on a transitional course. On a transitional course, the angle between the apparent wind vector p and the aerodynamic force a reaches its maximum value A. After any further increase in the course angle (to luff, closer to the wind), the direction of the propulsive force a will not correspond to the direction of the ship's motion v and these two vectors will deviate at angle E (see Fig. 2). Simultaneously, the efficiency graph described by Eqn. (16) will change to the graph described by Eqn. (49). This change of equations, *i.e.* change in the motion from one type to another, requires passing over the boundary between these two states. This boundary, called the transitional course, is denoted by the symbol B.

In Fig. 3, the angle $\mathrm{WRO_2} = B$, the angle between the chord RW and the tangent $\mathrm{O_2R}$ is equal to the angle RPW. The angle $\mathrm{O_2RP} = 180^\circ - (A - B)$, thus:

$$\sin \mathrm{PRW} = \sin(A - B).$$

Applying the Snellius theorem to $\triangle$PRW, we get:

$$\frac{n_E}{\sin(A - B)} = \frac{1}{\sin A}.$$

Hence:

$$n_E = \cos B - \sin B \cdot \mathrm{ctg}\, A. \tag{41}$$

Further, from the Snellius theorem for $\triangle$MPR:

$$\frac{u_2}{\sin B} = \frac{m}{\sin(A - B)}$$

Hence,

$$\frac{\dfrac{\sqrt{K}}{1-K}}{\sin B} = \frac{\dfrac{1}{1-K}}{\sin(A-B)}$$

Rearranging this equation, we obtain:

$$\operatorname{ctg} B = \frac{1+\sqrt{K}\cos A}{\sqrt{K}\sin A} \tag{42}$$

$$\sin B = \frac{\sqrt{K}\sin A}{\sqrt{1+2\sqrt{K}\cos A + K}} \tag{43}$$

$$\cos B = \frac{\sqrt{K}\cos A + 1}{\sqrt{1+2\sqrt{K}\cos A + K}} \tag{44}$$

Substituting the formulas (42), (43), (44) into Eqn. (41) and rearranging, we get:

$$n_B = \frac{1}{\sqrt{1+2\sqrt{K}\cos A + K}} = c. \tag{45}$$

Therefore

$$\sin B = c\sqrt{K}\sin A \tag{46}$$

$$\cos B = c(\sqrt{K}\cos A + 1). \tag{47}$$

The efficiency on course B, i.e. n_B, is presented in Fig. 3 as the length of the side PR in the triangle PRW (denoted by the letter w). This side is a chord of the inner circle, which connects the origin of the coordinate system P to the point R. The point R is the point of tangency of the inner circle with the parallel circle NR.

8. Efficiency of a ship on windward courses. In the windward sector, E is the angle between the projection of the vector of aerodynamic force onto the water's surface and the projection of the ship's motion (Fig. 2). Its cosine takes the following form:

$$\cos E = \frac{\cos(P-A) - n\cos A}{\sqrt{n^2 - 2n\cos P + 1}}. \tag{48}$$

Substituting Eqn. (35) into Eqn. (14) and rearranging, we obtain:

$$K^2 n^4 - (n^2 - 2n\cdot\cos P + 1)\cdot[n\cdot\cos A - \cos(P-A)]^2 = 0. \tag{49}$$

Eqn. (49) is called the **shell equation**.

The real positive roots of the shell equation satisfy the inequality:

$$n_{\text{in}} \leqslant n \leqslant n_{\text{out}} \tag{50}$$

where:
n_{in} — leading radius of the inner circle,
n_{out} — leading radius of the outer circle.

$$n_{\text{in}} = \frac{\cos(P-A)}{\sqrt{K}+\cos A} \tag{51}$$

$$n_{\text{out}} = \frac{\cos(P-A)}{\cos A}. \tag{52}$$

Eqn. (51) gives a description of the inner circle, which has diameter:

$$d_{\text{in}} = \frac{1}{\sqrt{K}+\cos A} \tag{53}$$

while Eqn. (52) gives a description of the outer circle, which has diameter:

$$d_{\text{out}} = \frac{1}{\cos A}. \tag{54}$$

Using Eqns. (51) and (52), we may determine the real positive root of Eqn. (49) for any given values P, A, K to the required precision by any suitable method, e.g. iteratively.

9. Shell diagram. Consider the set of points(n, P) in polar coordinates, where n are the real positive roots of Eqn. (49), and P are the angles in the domain: $90° - A \leqslant P \leqslant 90° + A$. Hence, taking n as the leading radius and P as the argument, we obtain a continuous and closed curve in two dimensions, which will be called a **shell curve**. The shell curves for chosen values of A and K are presented in Figs. 5, 6, 7, 8. The shell curve is a fourth degree polynomial function, and is different from well-known functions, such as Pascal's snail, cardioids, Cassini's oval or lemniscates.

10. Features of shell curves.

1. Shell curves in polar coordinates are sets of points (n, P) which are given by the real positive roots of Eqn. (49). Here n is the leading radius and P is the argument.
2. The domain is: $A - 90° \leqslant P \leqslant A + 90°$; $0 \leqslant A \leqslant 90°$; $K > 0$.
3. They are situated inside the eccentric ring created by the inner and outer circles.
4. The inner and outer circles have a common point at the pole with the first tangent having argument $P = A \pm 90°$.

5. The centers of both circles lie on the half-line with $P = A$.
6. The parallel curve and second tangent intersect with the inner circle at the point (c, B).
7. They are convex, except in cases where $K < 0.125$ in the sector $0 < P < 20°$, see Fig. 6.
8. The bisector of the chord c is the bisector of the angle between tangents *1* and *2*.
9. The center of the inner circle lies on the bisector of the chord c.
10. The segment $(0, 0°)$; $(1, 0°)$ of the polar axis is a chord of the outer circle.
11. For $A = 0°$, the curve is symmetric with respect to the polar axis.
12. For $A = 90°$, the curve is tangent to the polar axis.
13. ${}^{0}n_F = 1$.
14. ${}^{\infty}n_F = 0$.
15. ${}^{1}B = 0.5A$.

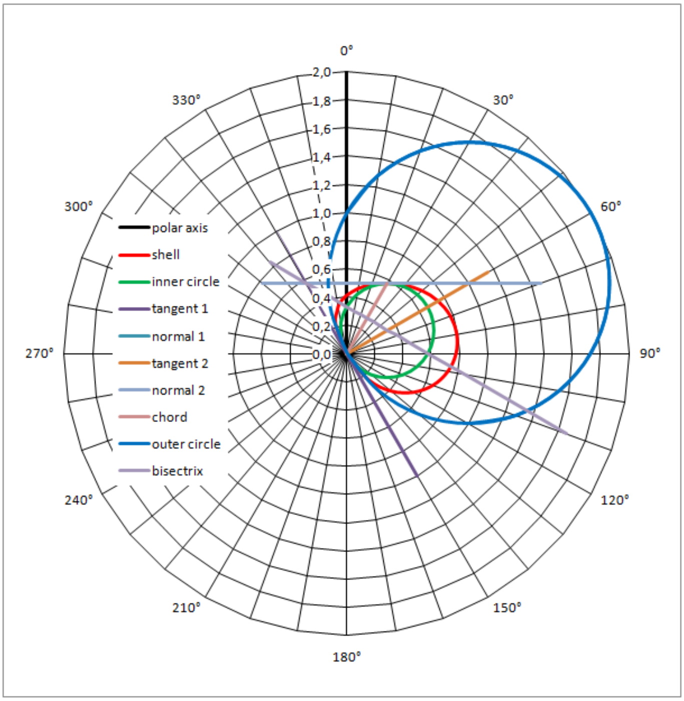

Figure 5: Shell curve for a standard sailing ship $(K = 1)$ at $A = 60°$

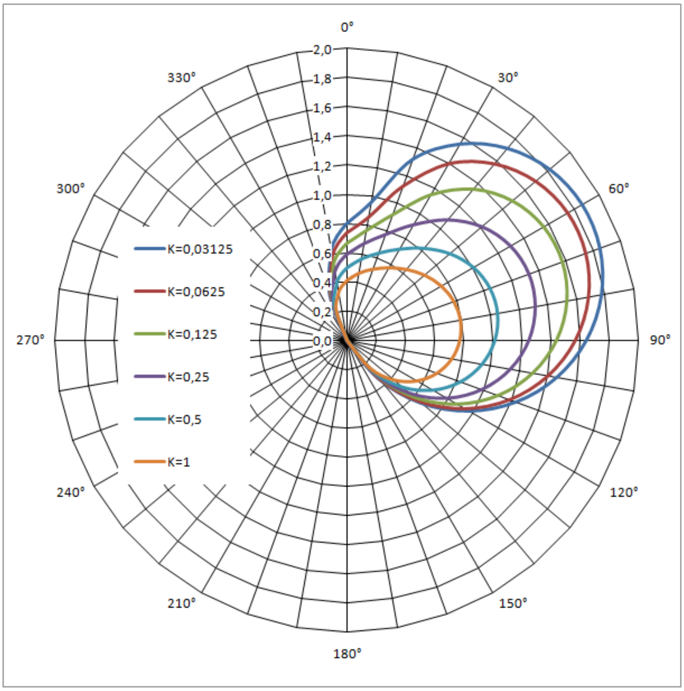

Figure 6: The shell diagram for light sailing ships at $A = 60°$

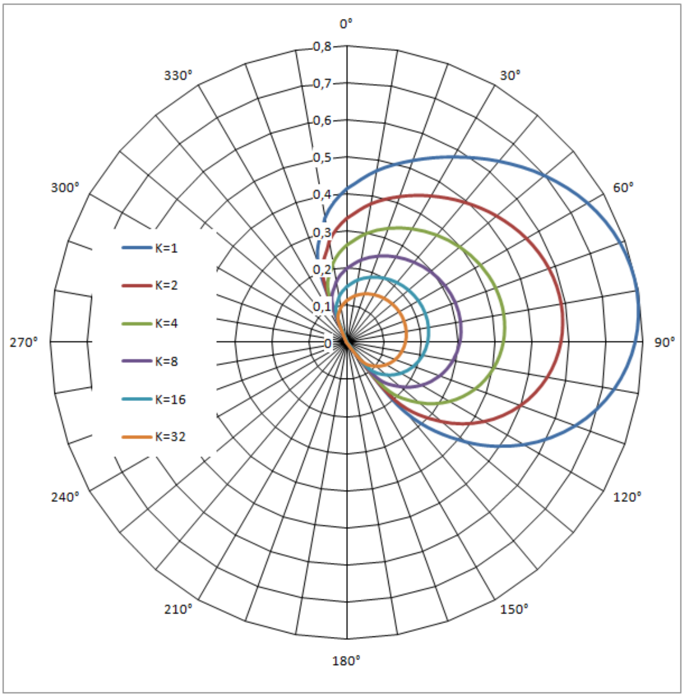

Figure 7: The shell diagram for heavy sailing ships at $A = 60°$

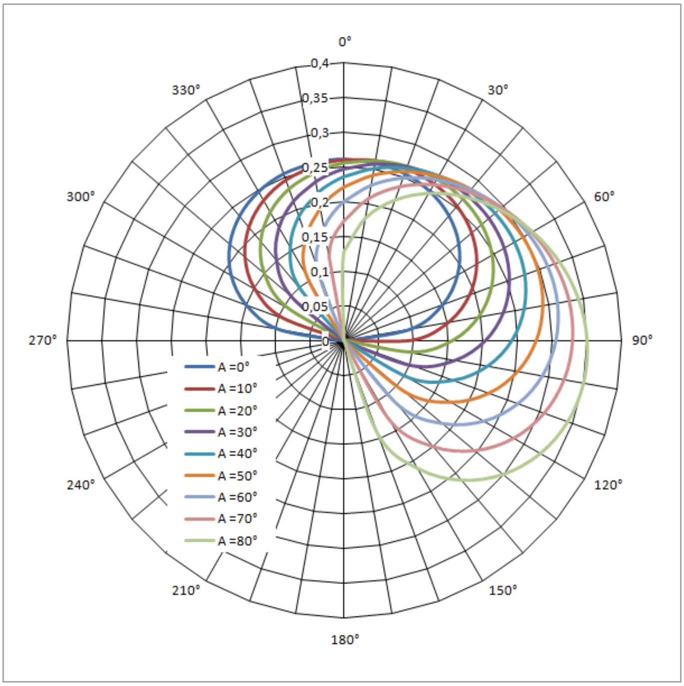

Figure 8: The shell diagram for a typical heavy sailing ship ($K = 8$)

11. Summary of the Results.

Leading radius of the outer circle:	$n_{\text{out}} = \dfrac{\cos(P-A)}{\cos A}$
Diameter of the outer circle:	$d_{\text{out}} = \dfrac{1}{\cos A}$
Leading radius of the inner circle:	$n_{\text{in}} = \dfrac{\cos(P-A)}{\sqrt{K}+\cos A}$
Leading radius of the inner circle:	$d_{\text{in}} = \dfrac{1}{\sqrt{K}+\cos A}$
Length of the chord of the outer circle, origin at the pole:	$n = 1$
Argument of the chord of the outer circle:	$P = 0$
Bisector of the chord of the outer circle:	$n_{\text{dout}} = \dfrac{1}{2\cos P}$
Length of the chord of the inner circle, origin at the pole:	$n_B = \dfrac{1}{\sqrt{1+2\sqrt{K}\cos A+K}} = c$

Argument of the chord of the inner circle:	$\cos B = c(\sqrt{K}\cos A + 1)$
Bisector of the inner circle:	$n_{\mathrm{din}} = \dfrac{c}{2\cos(P - B)}$
Line on which the center of the two circles lies:	$P = A$
First tangent to both circles at the pole:	$P = A \pm 90°$
Second tangent at the point (c, B):	$n_{\mathrm{dz}} = \dfrac{c \cdot \cos(A - B)}{\cos(P + A - 2B)}$
Point of tangency of the second tangent with the inner circle:	$n_B = c$; $P = B$
Distance between the second tangent and the pole:	$d = c \cdot \cos(A - B)$
Angle between the tangent and the bisector:	$\arccos(c \cdot \sin A)$

12. Basis equation of sailing ship efficiency. The basis equation of ship efficiency covers the entire sailing sector. It is a non-elementary function and is defined separately for two sectors:
— for the downwind sector, by Eqn. (19) in the domain $F \leqslant P_z < B$,
— for the windward sector, by Eqn. (49) in the domain $B < P_z < G$.

13. Basis diagram of ship efficiency. The basis diagram of ship efficiency is created by combining the efficiency diagram in the downwind sector, i.e. the globe diagram, with the efficiency diagram in the windward sector, i.e. the shell diagram. These diagrams intersect at the point of transition (c, B). Both of them were discussed above.

The basis curve has been created under the assumption, that in all the sailing sector $F \leqslant P \leqslant G$ the magnitudes A and K are constant. In reality, both A and K are variables and depend on many factors, among others the magnitude of the angle P. In order to obtain the true curve of ship efficiency, it is necessary to correct the basis curve by using the appropriate correctional coefficients, whose values can be either measured or calculated.

The most important corrections are:

1. The impact of the velocity and angle of attack on the coefficient C.
2. The influence of the aerodynamic drag on the course P.
3. Impact of the heeling angle on the aerodynamic force and the roll moment.
4. Impact of currents and waves.

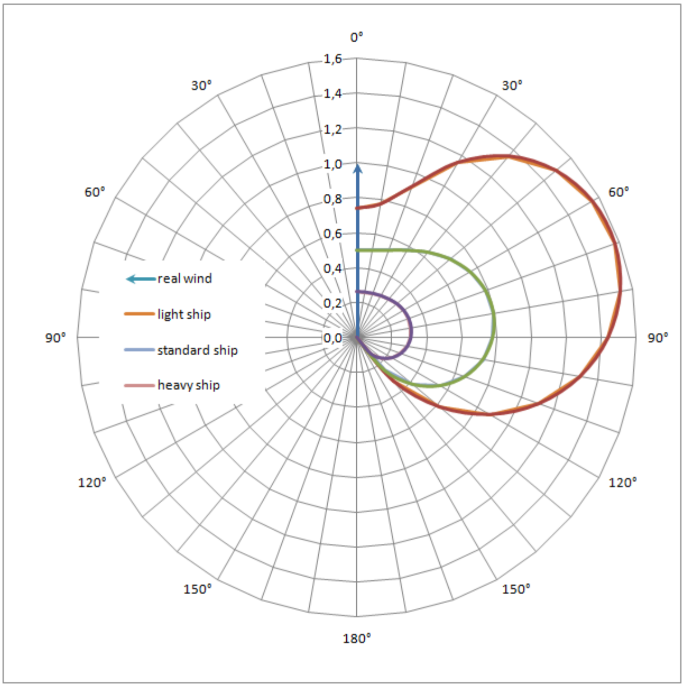

Figure 9: Basis curves for typical sailing ships on a starboard (right) tack, at $A = 60°$

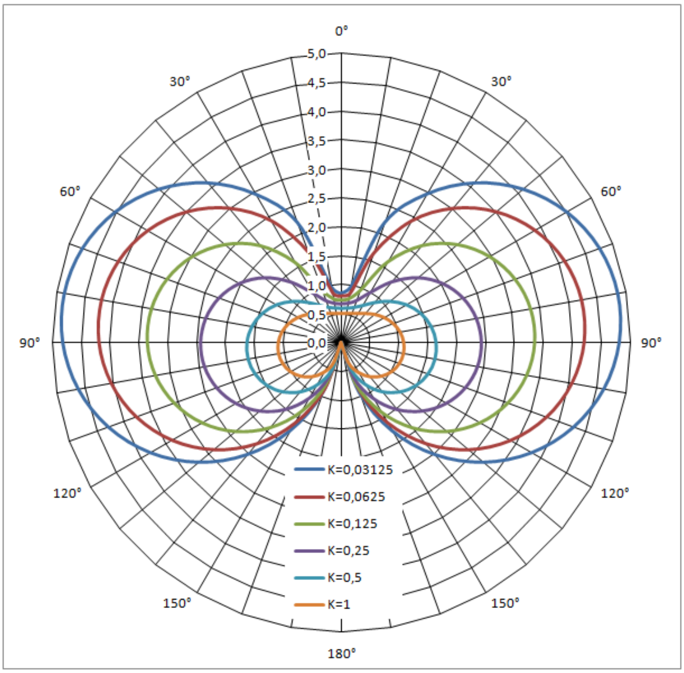

Figure 10: Basis efficiency curves for light sailing ships at at $A = 80°$

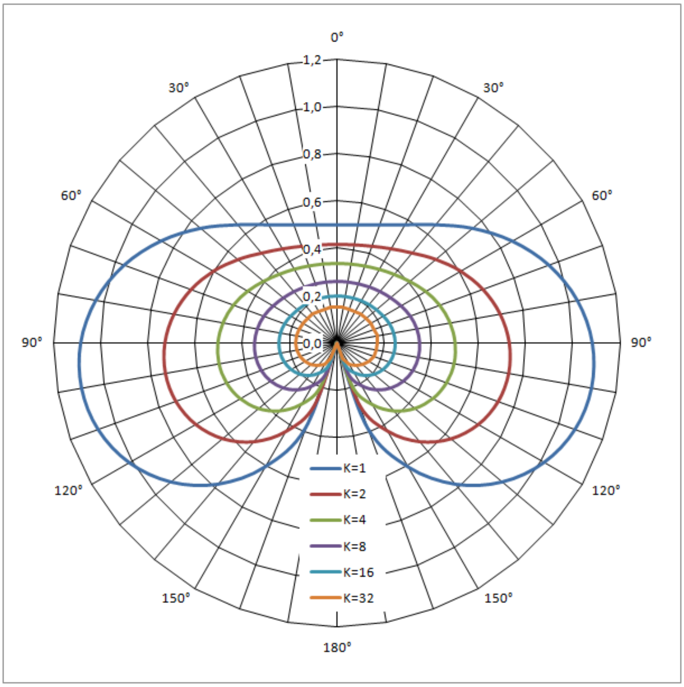

Figure 11: Basis efficiency curves for heavy sailing ships at at $A = 80°$

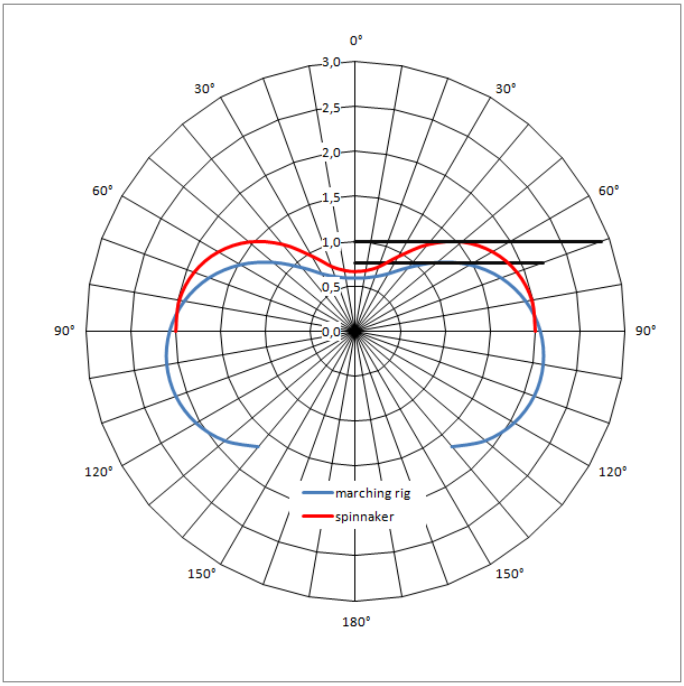

Figure 12: Efficiency curves for a ship with $K = 0.5; A = 88°$ and after adding a spinnaker $K = 0.35; A = 75°$

The corrected curve may be used to determine the optimal course, as well as at the stage of ship construction to calculate roll moments, design the surfaces of the hydrofoils, determine where the sails should be placed, and other elements influencing a ship's efficiency. The magnitude of the difference between the true and basis curves depends on the general quality of the ship and its crew and also on the correctness of the corrections applied.

From Fig. 9 we may observe that in the downwind sector the following relations exist:

for light ships: $Z > B$

for standard ships: $Z = n_z \cdot P_z$

for heavy ships: $Z = F$,

which means that to reach a target lying on a downwind course in the shortest possible time, a light ship should tack downwind, while a heavy ship should go straight to the target on the course $P = 0°$.

An understanding of the algorithms which describe a ship's efficiency on specific courses with respect to the wind direction allows us to determine the results of any decisions taken, while relying on common sense may lead to the wrong decisions. As an example, let us consider the motion of a light ship (race boat) both with and without a spinnaker. Compare Diagram 12 with the graph given in [11, page 432, graph 1]. It results from this last graph that when there is a spinnaker, the optimal downwind course will be $0°$, so the ship would be converted from being of the light type to being of the heavy type. On the other hand, it results from the above analysis that a light ship with say the parameters $K = 0.5$, $A = 88°$, after the addition of a spinnaker with the surface area of a normal sail, will change its parameters to: $K = 0.25$, $A = 75°$. The ship's efficiency will increase as shown by the curve in Fig. 12. The most rapid course Z before using the spinnaker is tacking at courses $\pm 53°$. However, with the spinnaker tacking at $\pm 49°$ will be a better course.

14. Applications. The principles, presented above, concerning the motion of a solid object on the boundary between two fluids with different densities, does not have to be limited to air and water, but could surely find applications in sailing at different levels, such as:

1. The design of sailing ships.
2. Comparing the current velocity with the maximal one.
3. Formulating precise rules for handicapping.
4. Formulating a training system for race tactics.
5. Choosing a speedy and safe route.
6. Designing games based on sailing.
7. Sea rescue.

References

[1] J.H. Milgram. Fluid Mechanics For Sailing Vessel Design, *Annual Review of Fluid Mechanics*, 30, 613–653, (1998). doi: 10.1146/annurev.fluid.30.1.613

[2] R.M. Wilson. The Physics of Sailing , 2010.

[3] J. Hoffman, C. Johnson. Mathematical Theory of Sailing, 2009.

[4] J. Hoffman, C. Johnson. The mathematical secret of flight, Normat, Vol.57, 145–169, 2009.

[5] J. Hoffman, C. Johnson. Resolution of d'Alembert Paradox, J. Math. Fluid Mech., Vol.12(3), 321–334, 2010.

[6] ORC announce Velocity Prediction Program improvements by Offshore Racing Congress Nov 6, 2012 Sail-World.

[7] ORC Sailor Service ORC VPP - Designer's version, 2013.

[8] J. Bukowski. Mechanika płynów [3]. PWN, Poznań 1959

[9] Poradnik okrętowca. Tom 2: Teoria okrętu [4], tom 2. Ed. *J.W. Doerffer i inni.* Wydawnictwo Morskie, Gdynia1960.

[10] Poradnik fizyko-chemiczny [5]. Ed. *Z. Bańkowski et al.*. WNT, Warszawa 1962

[11] Encyklopedia żeglarstwa [6]. Ed. *J. Czajewski.* PWN, Warszawa 1996.

[12] C. A. Marchaj. *Sailing Theory and Practice.* Adlard Coles Ltd, 1964.

[13] C.A. Marchaj. *Aero-hydrodynamics of sailing.* Adlard Coles Nautical, 3-rd edition, 2000.

[14] C. A. Marchaj. *Sail performance: techniques to maximize sail power.* Adlard Coles Nautical, 2003.

[15] C. A. Marchaj. *Seaworthiness: the forgotten factor.* International Marine Publishers, 1986.

[16] Wulff, George. Untersuchungen im Gebiete der optischen Eigenschaften isomorpher Kristalle. *Zeits. Krist.*, Vol. 36, 1-28, 1902.

[17] Lin Xiao, Jerome Jouffroy. On Motion Planning for Point-to-Point Maneuvers for a Class of Sailing Vehicles. *J. of Control and Engineering*, Vol. 2011, Article ID 402017, 2011. doi: 10.1155/2011/402017

[3] *Mechanics of fluids*
[4] Ship engineer handbook. vol. 2: Theory of a ship
[5] Physical-chemical Handbook
[6] Sailing Encyclopaedia

Mechanika ruchu żaglowca

Marek Kozłowski

Streszczenie W pracy, dociekając reguł określających sprawność żaglowca w ruchu jednostajnym prostoliniowym, otrzymano interesujące zależności matematyczne znajdujące potwierdzenie w praktyce. Równanie sprawności żaglowca w sektorze wiatrów pełnych jest tożsame z krzywymi z siatki Wulffa, które dotąd nie jest opisane w literaturze, a w sektorze wiatrów ostrych jest nie publikowaną dotąd krzywą stopnia czwartego. Artykuł przedstawia szczegóły przyjętych założeń, sposób wyprowadzania wzorów oraz wnioski i propozycje nowych definicji pojęć stosowanych w żeglarstwie.

Klasyfikacja tematyczna AMS (2010): 14H45; 76V05.

Słowa kluczowe: kurs baksztag (przejściowy), sprawność żaglowca, krzywa globusowa, krzywa muszli, krzywa bazowa, sprawności żaglowca.

Marek Kozłowski ur. 1940 r. w Krotoszynie. Ukończył liceum Ogólnokształcące im. Sowińskiego w Warszawie. Absolwent Politechniki Warszawskiej, Wydział Inżynierii Sanitarnej i Wodnej, specjalizacja Urządzenia Cieplne i Zdrowotne. Jako magister inżynier kierował robotami w Warszawie. W Szczecinie w Miejskim Przedsiębiorstwie Gospodarki Cieplnej zatrudniony na stanowisku dyrektora ds. Eksploatacji wdrożył ETO do sezonowej regulacji sieci cieplnej za co Przedsiębiorstwo uzyskało pierwsze miejsce w „Ogólnopolskim Współzawodnictwie Pracy w 1972 roku”. W Wojewódzkim Przedsiębiorstwie Energetyki Cieplnej jako Główny Specjalista wdrożył system rozliczenia zużycia opału w kotłowniach zaakceptowany przez GIGE, oraz doprowadził do realizacji swojego wniosku racjonalizatorskiego, co budżetowi miasta przyniosło kilkaset milionów złotych oszczędności. Po powrocie do Warszawy jako projektant instalacji wentylacji i klimatyzacji w BP „Instalprojekt” delegowany do pracy w Berlinie, Dreźnie i Wiedniu. W 1990 r. założył Przedsiębiorstwo Produkcyjno-Usługowe „San-Remo” sp. z o. o., które wykonało szereg instalacji w fazie projektowania i realizacji na terenie całego kraju. Od 1956 r. uprawia czynnie żeglarstwo, od 1976 r. w stopniu sternika morskiego, posiada patent do prowadzenia żaglowców o powierzchni do 80 m^2 żagla bez ograniczeń.

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