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Optimal timing of partial outsourcing decisions

Abstract This article combines a real options approach to the optimal timing of outsourcing decisions with a linear programming technique for solving one-dimensional optimal stopping problems. We adopt a partial outsourcing model proposed by Y. Moon(2010) which assumes profit flows to follow a geometric Brownian motion and explicitly takes into account the benefits and costs of all efforts which a firm spends on the project prior to the outsourcing date. The problem of deciding when to outsource and how much effort to spend is solved when the underlying profit flows or index processes are modeled by general one-dimensional diffusion. Optimal outsourcing times are proved to be of threshold type, and sensitivity results regarding market volatility and other quantities are derived. The corresponding optimal stopping problems are reformulated in terms of infinite dimensional linear programs and nonlinear optimization problems. These reformulations are exploited to prove sensitivity results in a novel way. Specific management recommendations are provided.

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1. Introduction Over the past decades outsourcing has become an integral part of the globalized economy. Nowadays, to improve their competitiveness firms outsource various elements of their in-house operations to subcontractors, e.g. production, development, marketing, distribution and services (see e.g. [2], [10], or [18]). One obvious incentive for outsourcing operations - especially to low-wage regions like Asia or Eastern Europe - is to reduce costs (see [2] and [18]). But as shown in [19], transferring operations to a subcontractor may not only be motivated by cost reduction. For example, it is possible that a third party can provide services more efficiently or can offer specialized technologies; the IT sector provides prominent examples (see [4]). An additional reason for outsourcing is that by handing over some operations or part of an operation to subcontractors a firm is able to concentrate on its core competencies (see [19]).

This particular aspect of an outsourcing project highlights the fact that outsourcing does not always require transferring a whole operation to an outside provider. As the authors of [3] emphasize in the context of their

analysis of call center services, many firms hesitate to hand over critical parts of their operation to a third party. In a survey investigating IT outsourcing projects of 50 companies one can see that less than 30% of the companies outsourced more than 50% of their IT budget (see [4]).

In light of such observations, Shy and Stenbacka studied in [17] a model where a firm uses a finite number of inputs to produce some final good. For each input the firm can decide between producing the input by its own or subcontracting some specialized firm to manufacture the input instead. Considering cost reduction by outsourcing production lines on the one hand and additional monitoring costs on the other hand Shy and Stenbacka identify a profit maximizing fraction of inputs to be outsourced.

A more general model that allows deciding on the proportion to be outsourced and assumes this decision to be based on observable but uncertain market information has been proposed by Alvarez et al. in [2]. In [14], Moon looks at a similar setting but lifts the decision on the fraction of the operation to be kept in-house to a strategic level. In the present article we will consider the case of partial outsourcing when a given proportion α of an operation, $\alpha \in [0, 1]$, will be outsourced. Moreover, the survey results in [4] reveal that the success of an outsourcing project crucially depends on the effort a firm invests in the project, and Barthelemy lists four different kinds of [potentially high] hidden costs caused by such efforts: costs for vendor search, costs for contracting, costs for effort management and costs for the transition process. Furthermore, effort management is more than just identifying, evaluating and controlling costs since, e.g., choosing a qualified vendor and negotiating proper contracts usually affect the outcome of the project (see [4], [14], and [19]). Spending extra time and money on identifying qualified vendors will typically imply products to be of better quality and/or production costs to be lower. For this reason, aspects of effort and efficiency will be an integral part of the partial outsourcing decision model to be analyzed in this paper.

Following [2], [14] and many other references, we will adopt a real options approach to model a firm's outsourcing decision. This implies that based on some relevant market information which is modeled by a one dimensional stochastic diffusion process a firm tries to maximize its profit by outsourcing at the first time some optimal market value is achieved. Both, [2] as well as [14], use geometric Brownian motion (GBM) as the underlying state process which determines the profit flow of the firm. Although GBM is a popular model, in the context of outsourcing problems it has certain drawbacks. For example, if the drift parameter of GBM is larger than the discount rate, then the optimal value of the outsourcing problem to be studied below will be infinite. Furthermore, if the *diffusion* coefficient of GBM is "large" in comparison to the drift parameter, then all trajectories of future profit flows will converge to zero. An other disadvantage of the GBM model is revealed by carefully reviewing the analysis in [14]. To guarantee the existence of

an optimal outsourcing time a very technical condition on the relationship between the cost structure of the effort and the parameters of the profit flow process has to be imposed (see Section 2 for the precise statement). For other processes such a condition will be unnecessary (see Subsection 4.3). In the context of outsourcing models, future profit flows are usually expected to be increasing in the short or medium term, and might hover around some specific positive value in the longer run. We would also like to analyze the situation in which profit rates are allowed to take on negative values, at least for some (short) period of time. To consider both aspects we consider a model with a general underlying one-dimensional diffusion process and apply a linear programming method to analyze the case of GBM as well as other processes, f.i. mean-reverting processes. As mentioned in the previous paragraph, effort and efficiency shall be taken into account. To do so, we will adopt the model proposed in [14] and quantify effort by a number $\theta \geqslant 0$, not necessarily bounded from above. We will also investigate how an upper bound on effort, $0 \leqslant \theta \leqslant \overline{\theta}$, affects the results. All benefits associated with effort θ are supposed to be quantified by an expression which is proportional to θ times the profit flow at the time of outsourcing. A pay-off term of this form is appropriate if, for example, the firm's activities justify a discount on a cost-plus outsourcing contract or a gain-sharing one. All expenses related to an amount θ are assumed to be captured by a convex power function which is defined on the non-negative real line. It will turn out that the super-linearity of this function and the product form of the benefit expression are two essential features of the model.

In contrast to [2] we follow the reasoning in [14] and assume that the proportion α of the operation to be outsourced is exogenously given, and that it is not determined at the operational level but at the strategic level. In this article we thus consider an outsourcing model where the firm decides on the effort θ it spends on the project and on the time T when it hands over part of the operation to an outside provider. The total pay-off of a firm's outsourcing decision is the sum of

(i) the expected discounted profit before outsourcing,

(ii) the percentage of the expected value of discounted future profits to be generated by the operations being outsourced,

(iii) the profit of the operations that will be kept in-house and

(iv) the contribution (the optimal value of a simple maximization problem over all $\theta \geqslant 0$) related to the firm's effort.

Variational inequalities or the principle of "smooth fit" are classical approaches used to solve optimal stopping problems, which are equivalent to the optimal timing of the outsourcing problems under consideration. In [14]

the "smooth fit" technique is further used to derive sensitivity results for the optimal outsourcing time in the case of GBM. For example, Moon investigates, among other issues, the influence of market uncertainty on the timing of an outsourcing decision (i.e. the impact that changes of the magnitude of the diffusion coefficient and other parameters of the model have on the optimal outsourcing time). In our generalization of the partial outsourcing model studied in [14] we will consider GBM as a special case. We also rectify a trivial but crucial typo in [14] and point to some necessary changes of the original results. Furthermore, we will discuss a linear programming method and a formulation of a nonlinear optimization problem to derive similar results for much more general processes than GBM. The characterization of an optimal stopping time in terms of the arg-max of a specific nonlinear function offers management the possibility to easily determine the [optimal] time when outsourcing should take place and makes it possible to derive sensitivity results of quantities of interest in a novel way. Additional references which treat optimal stopping problems of one-dimensional homogeneous diffusion processes using fractional programming reformulations include [5], [8], [13] and [16].

To illustrate the general method we will analyze the case of Brownian motion with positive drift and the cases of two mean-reverting processes in addition to GBM. The first mean-reverting process we will analyze is the Cox-Ingersoll-Ross model (CIR). The CIR-model is well known in the context of interst rate modeling in financial mathematics (see e.g. [7]) and a prominent population model in other sciences (see e.g. [12]). The diffusion term of the CIR-process and its drift term differ from the ones of a GBM-process. Instead of σx, the diffusion expression is of the form $\sigma\sqrt{x}$ and the drift term is an affine function of the form $\mu(1-\gamma x)$; μ, γ and σ are positive constants while x represents a profit flow value. The second mean-reverting process studied in detail is a Geometric-Ornstein-Uhlenbeck process (GOU). Its drift term is identical to the one of the CIR-model while the diffusion term σx is the same as in the case of a GBM-model. Compared to more general diffusions, see Subsection 4.1, these two mean-reverting processes are special since characteristic quantities can be expressed in terms of analytical expressions involving confluent hyper-geometric functions (see Subsections 4.3 and 4.4). The case of drifted Brownian motion is even easier to analyze because the characteristics can be expressed in terms of elementary functions; this is similar to a GBM-model. For general processes, we will have to rely on numerical computations to determine these characteristics.

This paper is organized as follows. In Section 2 we review an extended version of the outsourcing model proposed by [14]. Due to a minor but crucial typo in the derivation of a fundamental formula in [14] some of the general statements therein need to be reformulated and the expression of the value function has to be adjusted (see Section 3 for details). In particular, we give

an example which shows that one of Moon's sensitivity results generally does not hold true. In Section 4, we extend results – proved for GBM in Section 3 – to more general processes such as special classes of mean-reverting processes. Using a (infinite dimensional) linear programming approach to stopping-time problems (cf. [11]) we can easily characterize optimal outsourcing decisions for a fairly large class of time homogeneous regular diffusion processes in terms of threshold policies. It will be shown that optimal threshold values can be computed by solving one dimensional non linear optimization problems (see 4.1 for details). In particular, see Theorem 4.2, we prove sensitivity results for general processes. We illustrate the results by analyzing Brownian motion with a constant positive drift as a simple example of a profit flow process. In Subsections 4.2 – 4.4 we apply the general theory of Subsection 4.1 to the important cases of GBM-, CIR- and GOU-models. Conclusions can be found in Section 6.

2. Model and Notation

Following [2] and [14], we formulate a firm's problem of how to optimally outsource parts of its in-house operation using a real options approach. Since a firm's [future] profit is uncertain, we assume the profit flow $P = (P_t)_{t\geqslant 0}$, or an economic index which influences the profit, to be a stochastic process such as a geometric Brownian Motion. The main difference between the model presented here and the ones in [2] and [14] is that the profit flow process is a general 1-dim diffusion process not necessarily geometric Brownian motion. This enables applications of the model to different kinds of profit or market structures.

Similar to [2], [14], [17] and [19], we assume that the firm has access to a market of outside providers/producers. These outside producers or vendors are either specialized in the activity which shall be outsourced or can operate with lower costs (labor, materials, tax, etc.) such that the firm can achieve a marginal cost advantage by outsourcing. This cost advantage is reflected by the constant factors $0 < a_I < a_O$ which are interpreted as follows: given a model of the index or of the profit flow P it is stipulated that a firm generates the random future profit flow $a_I P_t$ by in-house production at time t, and $a_O P_t$ by outsourced production. As discussed in the introduction we assume that $\alpha \in (0,1)$, the proportion of the operation which will be outsourced, is determined by strategic deliberations and thus is given exogenously. The value $(1-\alpha)$ specifies the fraction of the operation that is kept in-house.

Let $r > 0$ denote a discount rate. For any time $t \geqslant 0$, the expected future profit from in-house production conditioned on P_t is given by

$$\pi_I(P,t) = a_I \mathbb{E}\left[\int_t^\infty e^{-r(s-t)} P_s ds \,\middle|\, P_t\right];$$

analogously, the expected profit from the outsourced operation equals

$$\pi_O(P,t) = \mathbb{E}\left[\int_t^\infty e^{-r(s-t)} a_O P_s ds \,\middle|\, P_t\right] = a_O \frac{\pi_I(t)}{a_I}.$$

When outsourcing the proportion α of the operation at a future [random] time $T \geqslant 0$, a firm's total expected discounted profit generated by in-house production during the period $[0,T]$ equals $\pi_I(P,0) - e^{-rT}\pi_I(P,T)$. After outsourcing the fraction α of the operation at time T the firm expects its average discounted profit on (T,∞) to amount to $(1-\alpha)\pi_I(P,T)+\alpha\pi_O(P,T)$.

As mentioned in the Introduction, outsourcing in-house operations requires certain kinds of investments at the preparation stage. Some of the costs a firm is facing during such project, i.e. vendor searching costs, transition preparation costs and organizational costs, can be seen as a firm's efforts to increase efficiency of the project. In addition, there will be further costs as contract processing fees (see [4] and [14]). Following [14], we assume for the latter costs some fixed amount K_f. The firm's effort is measured by its investment $I(\theta)$, $\theta \geqslant 0$, per unit profit flow before outsourcing, which yields a marginal profit $\alpha \cdot \theta$ after outsourcing the fraction α of the former in-house operation. Thus, investing effort at the preparation stage the firm achieves a profit surplus of $\alpha \cdot \theta \cdot P_T$, where P_T denotes the profit flow at the outsourcing time T. The cost for the profit surplus are modeled by the convex function $I(\theta) = \frac{K}{q}\theta^q$, $K > 0$, where the parameter $q > 1$ is a measure for the efficiency of the investment (cf. [14]). Thus, for the profits resulting from the firm's effort we have $\alpha(\theta P_T - I(\theta))$.

As we have indicated in the Introduction, the optimal random outsourcing time T will be determined by a threshold value p^*. The value p^* depends on a firm's profit flow up to time T and the expected future profits after T. Thus, for any T and any p, considering the possible efforts described above, the first step of the outsourcing analysis requires maximizing (over all $\theta \geqslant 0$) the expression

$$\bar{F}(P,T,\theta) = (1-\alpha)\pi_I(P,T) + \alpha(\pi_O(P,T) + \theta P_T - I(\theta)) - K_f, \qquad (1)$$

where $I(\theta) = \frac{K}{q}\theta^q$. Let $G(P_T,\theta) := \alpha P_T \theta - \alpha\frac{K}{q}\theta^q$ denote the difference of the two terms of F which include θ. Maximizing G with respect to θ will yield the maximum of (1). To get a formula of the (unique) maximizer θ^* we take the derivative of G and set it equal to zero in the case when $P_T > 0$, and take $\theta^* = 0$ else. The trivial typo mentioned in the Introduction affected the derivative of G and the value of the optimal θ (see [14], p. 31). Checking second order conditions, the correct expression is given by

$$\theta^* = \left(\frac{P_T \vee 0}{K}\right)^{\frac{1}{q-1}}. \qquad (2)$$

Note, contrary to the statement in [14], θ^* depends on P, and

$$\begin{aligned} F(P,T) &:= \max_{\theta \geqslant 0} \bar{F}(P,T,\theta) \\ &= (1-\alpha)\pi_I(P,T) + \alpha\pi_O(P,T) - K_f + \alpha\frac{q-1}{q}K\left(\frac{P_T \vee 0}{K}\right)^{\frac{q}{q-1}} \quad (3). \end{aligned}$$

In the sequel, for nonnegative processes we will simply write P_T instead of $P_T \vee 0$. The second step of the outsourcing analysis requires a firm to find an optimal stopping time T (adapted to the profit/index process) so that the total expected profit is maximized should the firm outsource at time T and should it spend effort θ^*.

The value function of the outsourcing problem, when computed at the initial time $t = 0$ and for an initial value p_0, is given by

$$V(p_0) = \sup_T \left\{ \mathbb{E}_{p_0}\left[\int_0^T e^{-rt} a_I P_t dt \; + \; e^{-rT} F(P,T) \cdot \mathbb{1}_{\{T<\infty\}}\right]\right\}. \tag{4}$$

Expression (4) specifies the value function of an optimal stopping problem to be solved over all stopping times T. In special cases, the function $F(P,T)$ is independent of the time variable T; for an example see (9) below.

3. The case of Geometric Brownian Motion

Alvarez et al. as well as Moon (cf. [2] and [14]) assume the profit flow $(P_t)_{t\geqslant 0}$ to be Geometric Brownian Motion (GBM), i.e. $(P_t)_{t\geqslant 0}$ satisfies the stochastic differential equation

$$dP_t = \mu P_t dt + \sigma P_t dW_t, \; P_0 = p_0, \tag{5}$$

where $(W_t)_{t\geqslant 0}$ is a standard Brownian Motion, μ is the mean-growth rate and σ specifies how volatile the profit flow evolution will be. In [2] the solution to (5) is interpreted as a general economic index process. The authors define profit rates to be transformed values of the index process.

To ensure that the optimal stopping time problem (4) has a non trivial solution we assume the discount factor r to be larger than the mean growth rate μ, i.e. $r > \mu$. It is well known that for Geometric Brownian Motion the profit flow at time t, when compared with its value at time s, $s \geqslant t$, satisfies the identity

$$P_s = P_t e^{(\mu - \frac{\sigma^2}{2})(s-t) + \sigma(W_s - W_t)}. \tag{6}$$

It follows from (6) that at any time t the expected future profit from in-house activities can be written as

$$\begin{aligned} \pi_I(P,t) &= a_I \mathbb{E}\left[\int_t^\infty e^{-r(s-t)} P_s ds \,\middle|\, P_t\right] \\ &\overset{(6)}{=} a_I P_t \mathbb{E}\left[\int_t^\infty e^{(\mu-\frac{\sigma^2}{2})(s-t) - r(s-t) + \sigma(W_s - W_t)} \,\middle|\, P_t\right] \\ &= a_I P_t e^{t(r-\mu)} \int_t^\infty e^{-s(r-\mu)} ds \; = \; \frac{a_I P_t}{r-\mu}; \end{aligned} \tag{7}$$

similarily,

$$\pi_O(P,t) = \mathbb{E}\left[\int_t^\infty e^{-r(s-t)} a_O P_s ds \,\middle|\, P_t\right] = \frac{a_O P_t}{r-\mu}. \tag{8}$$

Hence, for Geometric Brownian Motion the function $F(P,T)$ only depends on the value of P_T. Thus, we will treat F as a real valued function defined on $\mathbb{R}_+$ and assume

$$F(p) := \frac{p}{r-\mu}\left[a_I + \alpha(a_O - a_I)\right] - K_f + \alpha\frac{(q-1)K}{q}\left(\frac{p}{K}\right)^{\frac{q}{q-1}}. \tag{9}$$

Due to the functional form of F outsourcing might never be optimal at a finite time. This possibility occurs whenever we are in a situation characterized by the following result.

Lemma 3.1 *Let $\zeta := \frac{q}{q-1}$, and assume the profit flow is a geometric Brownian motion with mean-growth rate μ and volatility σ. If λ,*

$$\lambda := \lambda(\sigma,\mu,q,r) := \frac{\sigma^2}{2}\zeta(\zeta-1) + \zeta\mu - r,$$

is positive, then no finite optimal outsourcing time exists and the optimal value of (4) is infinite.

Proof For any finite stopping time T,

$$\begin{aligned}
\mathbb{E}\left[\int_0^T a_I P_t dt + e^{-rT}F(P_T)\right] &\geqslant \mathbb{E}[e^{-rT}F(P_T)] \\
&\geqslant \mathbb{E}\left[-K_f + \underbrace{\alpha\frac{(q-1)K}{q}K^{-\frac{q}{q-1}}}_{c_F} e^{-rT}P_T^{\frac{q}{q-1}}\right] \\
&= c_F\mathbb{E}\left[e^{-rT}P_T^{\frac{q}{q-1}}\right] - K_f,
\end{aligned}$$

since $-K_f e^{-rT} \geqslant -K_f$ and $\frac{P_T}{r-\mu}[a_I + \alpha(a_O - a_I)] \geqslant 0$. Hence, if we can find a sequence of stopping times τ_n with $\mathbb{P}(\tau_n < \infty) = 1$ while $\mathbb{E}\left[e^{-r\tau_n}P_{\tau_n}^{q/(q-1)}\right] \to \infty$ if $n \to \infty$, then $V(P_0) = \infty$, and it is optimal to keep all of the production in-house. Let $\zeta := \frac{q}{q-1} > 1$. Identity (6) implies

$$\begin{aligned}
e^{-rt}P_t^\zeta &= P_0^\zeta \exp\left[(\zeta\mu - \zeta\sigma^2/2 - r)t + \zeta\sigma W_t\right] \\
&= \underbrace{P_0^\zeta \exp\left[\zeta\sigma W_t - \frac{\zeta^2\sigma^2}{2}t\right]}_{=:M_t} \exp\left[\left(\frac{\sigma^2}{2}\zeta(\zeta-1) + \zeta\mu - r\right)t\right] \qquad (10)
\end{aligned}$$

Since $(M_t)_{t\geqslant 0}$ is a martingale and $\mathbb{E}[M_t] = M_0 = P_0^\zeta$, taking expectations on both sides of (10) yields a lower bound of $V(P_0)$, namely, $0 \leqslant t < \infty$,

$$V(P_0) \geqslant \mathbb{E}\left[e^{-rt}F(P_t)\right] \geqslant c_F\mathbb{E}[M_t]e^{\lambda t} - K_f. \tag{11}$$

Thus, whenever the expression $\lambda = \lambda(\sigma, \mu, q, r)$ is positive, choosing $\tau_n :\equiv n$ implies

$$\mathbb{E}\left[\int_0^{\tau_n} a_I P_t dt + e^{-r\tau_n}F(P_{\tau_n})\right] \geqslant \mathbb{E}[e^{-rn}F(P_n)] \overset{n\to\infty}{\to} \infty,$$

and there is no optimal finite stopping time. Therefore, outsourcing will never take place. ■

Note, λ is independent of α, where α equals the fraction of the operation that will be outsourced. For the remainder of this section we shall assume that $\lambda(\sigma, \mu, q, r) \leqslant 0$. Let β denote the solution of the quadratic equation

$$\frac{1}{2}\sigma^2\beta(\beta - 1) + \mu\beta - r = 0 \tag{12}$$

which is bigger than 1. The definition of β implies that the property $\lambda \leqslant 0$ is equivalent to the assumption $1 < \zeta < \beta$.

The next two theorems are the main results of this section. Theorem 3.2 provides the correct formula of the value function in the case of GBM. The difference between formula (15) and the expression given in [14] is due to the typo referred to above. Theorem 3.3 summarizes the comparative statics; it is based on Theorem 3.2.

The proofs of the results in Section 3 follow the traditional approach of constructing the value function V using candidates of V on the continuation region and candidates of V on the stopping region and piecing them together at the boundary according to the principle of smooth fit. This way it will be easy to compare the original proof given by Moon with the proof below. The case of GBM will be revisited in Subsection 4.2 in the view of the LP-formulation mentioned in the introduction.

Theorem 3.2 *Assume $1 < \zeta < \beta$, $K_f > 0$ and $r > \mu$. Let*

$$c_0 := \frac{K_f}{\alpha}, \qquad c_1 := \frac{a_O - a_I}{r - \mu}\left(1 - \frac{1}{\beta}\right) \qquad \text{and} \qquad c_q := K^{-\frac{1}{q-1}}\left(\frac{1}{\beta} - \frac{q-1}{q}\right);$$

the constant c_q is negative while c_0, c_1 are positive numbers. The optimal outsourcing time T^* is determined by

$$T^* = \inf\{t \geqslant 0 \,|\, P_t \geqslant p^*\}, \tag{13}$$

where the optimal threshold p^* is (implicitly) given by the equation

$$c_1 p^* - c_0 = c_q (p^*)^{\frac{q}{q-1}}. \tag{14}$$

As a function of the initial value p_0 the optimal expected profit equals

$$V(p_0) = \begin{cases} F(p_0) & \text{if} \quad p_0 \geqslant p^* \\ Ap_0^\beta + \frac{a_I}{r-\mu} p_0 & \text{if} \quad p_0 \leqslant p^*, \end{cases} \tag{15}$$

where $F(p_0)$ is defined by (9), $A := (p^*)^{-\beta}\left(F(p^*) - \frac{a_I}{r-\mu}p^*\right)$ and $\beta > 1$ is specified by (12).

Proof Following the proof of Theorem 1 in [14] (see also Chapter 5 in [15] and Section 4 of this article) outsourcing *takes place* when the profit flow reaches a threshold p^*. The value function of the optimal stopping problem satisfies the ordinary differential equation

$$0 = \frac{1}{2}\sigma^2 p^2 V''(p) + \mu p V'(p) - rV(p) + a_I p, \tag{16}$$

whenever p belongs to $(0, p^*)$. The optimal threshold p^* is characterized by the smooth-pasting conditions

$$V(p^*) = \frac{p^*}{r-\mu}\left[a_I + \alpha(a_O - a_I)\right] - K_f + \alpha\frac{(q-1)K}{q}\left(\frac{p^*}{K}\right)^{\frac{q}{q-1}}, \tag{17}$$

and

$$V'(p^*) = \frac{1}{r-\mu}\left[a_I + \alpha(a_O - a_I)\right] + \alpha\left(\frac{p^*}{K}\right)^{\frac{1}{q-1}}. \tag{18}$$

A general solution of the differential equation (16) which satisfies (17) and (18) is given by

$$V(p) = \begin{cases} F(p) & \text{if} \quad p \geqslant p^* \\ Ap^\beta + a_I\mathcal{R}(p) & \text{if} \quad p < p^* \end{cases},$$

where $\beta > 1$ satisfies (12) and

$$\mathcal{R}(p) := \mathbb{E}\left[\int_0^\infty e^{-rt}P_t dt \,\middle|\, P_0 = p\right] = \frac{p}{r-\mu}. \tag{19}$$

To determine the optimal threshold p^* and the constant A, we consider equations (17), (18) and expression (15). These equations imply the non-linear system

$$\begin{aligned} A(p^*)^\beta &= \frac{p^*}{r-\mu}\left[\alpha(a_O - a_I)\right] - K_f + \alpha\frac{(q-1)K}{q}\left(\frac{p^*}{K}\right)^{\frac{q}{q-1}} \\ \beta A(p^*)^{\beta-1} &= \alpha\frac{a_O - a_I}{r-\mu} + \alpha\left(\frac{p^*}{K}\right)^{\frac{1}{q-1}}. \end{aligned} \tag{20}$$

Dividing the left-hand side, right-hand side resp., of the first equation of (20) by the left-hand side, right-hand side resp., of the second equation yields the identity

$$\beta^{-1}p^* = \frac{\alpha\frac{a_O-a_I}{r-\mu}p^* - K_f + \alpha\frac{(q-1)}{q}K^{-\frac{1}{q-1}}(p^*)^{\frac{q}{q-1}}}{\alpha\frac{a_O-a_I}{r-\mu} + \alpha K^{-\frac{1}{q-1}}(p^*)^{\frac{1}{q-1}}}.$$

Thus, an optimal threshold p^* satisfies the implicit equation

$$\underbrace{\left(\frac{a_O-a_I}{r-\mu}\right)\left(1-\frac{1}{\beta}\right)}_{=:c_1}p^* = \underbrace{K^{-\frac{1}{q-1}}\left(\frac{1}{\beta}-\frac{q-1}{q}\right)}_{=:c_q}(p^*)^{\frac{q}{q-1}} + \underbrace{\frac{K_f}{\alpha}}_{=:c_0}.$$

Since $\beta > 1$ and $K_f > 0$, it is obvious that c_0 and c_1 are positive. The fact that c_q is negative follows from the inequality

$$\beta > \zeta = \frac{q}{q-1} \iff c_q = \left(\frac{1}{\beta}-\frac{q-1}{q}\right) < 0.$$

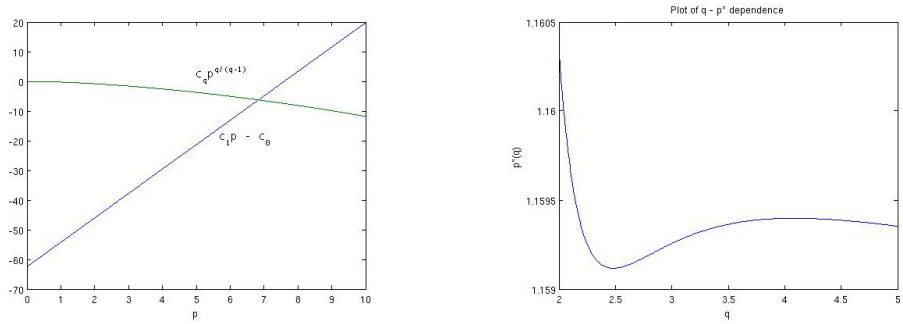
Basic calculus shows that for positive c_0, c_1 and negative c_q there always exists a unique positive solution p^* of equation (14); Figure 1 (a) (see Section 3) illustrates this fact. ■

Window (a) of Figure 1 illustrates that the equation (14) always has a unique solution. Hence, a unique outsourcing date exists whenever the assumptions of Theorem 3.2 hold. The way the optimal threshold value p^* depends on various model parameters is described in the next theorem. Window (b) of Figure 1 shows the dependence of p^* on q; it provides a counterexample to the claim in [14] that p^* is an increasing function of the exponent q.

THEOREM 3.3 *Let $1 < \zeta < \beta$, and assume the profit flow process to be GBM.*

1. *The optimal threshold p^* is a decreasing function of α, i.e. the higher the outsourcing percentage the lower the threshold value.*
2. *p^* is a decreasing function of $(a_O - a_I)$, i.e. the more profitable outsourcing is compared to keeping the operation in-house the sooner the outsourcing decision will be taken.*
3. *If market volatility σ increases then p^* increases as well.*
4. *The trigger value p^* is increasing in K_f, i.e. higher fixed costs imply later outsourcing times.*

PROOF Define the function $h(p) := c_1 p - c_0 - c_q p^{q/(q-1)}$. Assume the parameters $\sigma, \mu, \alpha, r, q, K, K_f$ are fixed and the inequality $\beta > q/(q-1)$ holds. The function h is strictly monotone increasing (in p) and the optimal threshold p^* is the unique root of h. Furthermore, $h(0) = -c_0$ is negative. For the rest of the proof we shall use the notation h_α, h_σ, etc. to highlight the parameter that will be varied.



(a) The two sides of equation (14) as functions of p.

(b) The optimal threshold p^* as a function of q.

Figure 1: Graphical descriptions of equation (14), and of the optimal threshold p^* as a function of q. The common parameters of both plots are: $\mu = 0.01$, $\sigma = 0.1$, $r = 0.06$, $a_O - a_I = 0.8$, $K = 0.3$ and $\alpha = 0.8$. For Window (a), the fixed cost K_f equals 50 and q equals 2.5; for Window (b), the fixed cost K_f equals 10.8 and the parameter q runs from 2 to 5 in steps of 0.01.

Ad 1: Whenever α increases the constant part $c_0 = K_f/\alpha$ decreases. The other two constants c_1 and c_q do not depend on α. Thus, for any pair $\bar{\alpha} > \alpha$ it follows that $h_{\bar{\alpha}}$ strictly dominates h_α, i.e. $h_{\bar{\alpha}}(\cdot) > h_\alpha(\cdot)$. Hence, $p^*(\bar{\alpha}) < p^*(\alpha)$ follows.

Ad 2: Whenever the difference $(a_I - a_O)$ increases the coefficients c_0 and c_q do not change. But the graph of the function $h_{a_O - a_I}(\cdot)$ rotates counterclockwise. Hence, p^* will move to the left and the firm will outsource sooner.

Ad 3: It is shown in [14] (see proof of Theorem 3) that $\frac{\partial \beta}{\partial \sigma} < 0$. For $\bar{\sigma} > \sigma$ such that $\beta(\bar{\sigma}) > q/(q-1)$, the fact that $\beta(\sigma)$ is strictly monotone decreasing combined with the definitions of c_1 and c_q imply the two inequalities

$$c_1(\bar{\sigma}) < c_1(\sigma) \quad \text{and} \quad c_q(\bar{\sigma}) > c_q(\sigma)$$

to hold. Hence, $h_{\bar{\sigma}} > h_\sigma$, and with increased market volatility $\bar{\sigma}$ the optimal outsourcing time is determined by a threshold value $p^*(\bar{\sigma})$ which is larger than $p^*(\sigma)$.

Ad. 4 Whenever K_f increases the constant part $c_0 = K_f/\alpha$ increases too. The other two constants c_1 and c_q do not depend on K_f. Thus, for any pair $\bar{K_f} > K_f$ it follows that h_{K_f} strictly dominates $h_{\bar{K_f}}$ (i.e. $h_{K_f}(\cdot) > h_{\bar{K_f}}(\cdot)$). Hence, $p^*(\bar{K_f}) > p^*(K_f)$ follows. ■

4. General profit flow processes

In this section we shall extend the previous analysis to a larger class of profit flow processes. The process GBM, if $\mu > \sigma^2/2$, is the prototypical example of a random profit flow process for which the average profit flow grows exponentially over time. To ensure the outsourcing problem is non-trivial (see Section 3) the restriction $r > \mu$ needs to be imposed. Typically, this condition implies that the average growth rate of a GBM-model needs to be fairly small. Actually, the fact that μ is less than the discount rate r leads to the question why anybody would invest in this kind of business in the first place. This observation is one of our motivations for analyzing the outsourcing problem for processes other than GBM. There are many business sectors and companies which promise substantial profits and growth over a short or medium time period but experience profit rates to stabilize and fluctuate around some level in the long run. The prototypical model which reflects such behavior is the first mean-reverting process to be studied below. When analyzing this process and other diffusion processes, we have the opportunity to present a nonlinear optimization formulation for the outsourcing problem. This formulation provides an alternative to the smooth pasting technique used in Section 3. The alternative formulation offers a different way to compute optimal threshold values p^* and to extend the comparative statics to more general situations. The approach is a special case of the method proposed in [11] to characterize the value function of general 1-dimensional time homogeneous stopping problem. First, we will explain this method in a general setting. To illustrate the method and point out differences between the classical smooth pasting technique and the LP-method, we shall revisit the case of GBM (see Subsection 4.2) before analyzing in greater detail the cases when the profit processes are assumed to be mean-reverting processes (see Subsections 4.3 and 4.4).

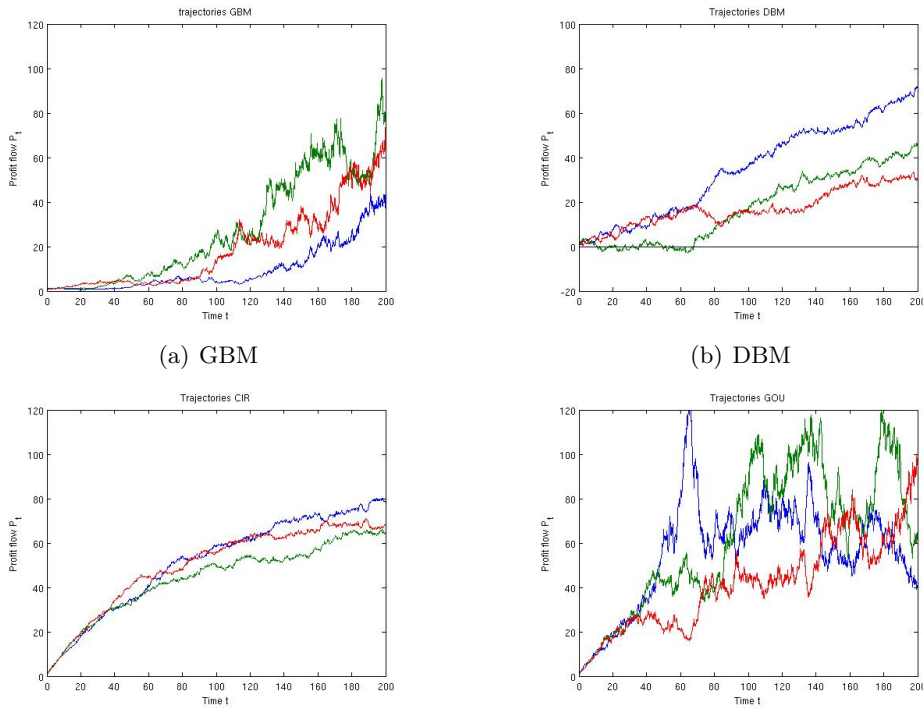
Window (a) of Figure 2 illustrates the (expected) exponential growth of GBM trajectories if the condition $\mu > \sigma^2/2$ holds; $\mu = 0.025$, $\sigma = 0.1$. Window (b) shows trajectories of a drifted BM; $\mu = 0.25$, $\sigma = 1$. One of the three trajectories illustrates the possibility that the profit flow can become negative. Windows (c) and (d) show trajectories of a CIR-model and a GOU-model; $\mu = 1$, $\sigma = 0.1$, $1/\gamma = 80$. The equilibrium point $80 = 1/\gamma$ is the same in both cases. The graphs illustrate the fact that realizations of these two mean-reverting processes approach $1/\gamma$ quite differently and that fluctuations about $1/\gamma$ are very different as well.

4.1. A class of general index processes

In general, we assume that an index process $(P_t)_{t\geqslant 0}$ satisfies the stochastic differential equation

$$dP_t \;=\; \mu(P_t)dt \;+\; \sigma(P_t)dW_t, \quad P_0 = p_0, \tag{21}$$

where $(W_t)_{t\geqslant 0}$ is a standard Brownian motion and μ, σ are continuous func-



(a) GBM (b) DBM

(c) CIR (d) GOU

Figure 2: Three sample trajectories of four different profit/index models.

tions on a subset J of the real line. Additionally, we assume that for the functions μ and σ there exists a unique strong solution $P = (P_t)_{t \geqslant 0}$ of (21) living on J (see [6], III.3). The specific processes analyzed below satisfy the sufficient conditions for existence (see [6]). To illustrate the techniques presented in this subsection we will consider, along the general case, the special case when μ and σ are constant functions, i.e. $\mu(p) \equiv \mu > 0$, $\sigma(p) \equiv \sigma > 0$. In the constant case, we immediately see that the process is from the form

$$P_t \;=\; p_0 + \mu t + \sigma W_t; \tag{22}$$

the process (22) represents Brownian motion with positive drift (DBM). Let $\mathcal{A}$ denote the generator of a general (time-homogeneous) diffusion process $(P_t)_{t \geqslant 0}$ (i.e. for smooth test functions f),

$$\mathcal{A}f(p) \;=\; \mu(p)\frac{\partial f}{\partial p}(p) \;+\; \frac{\sigma(p)^2}{2}\frac{\partial^2 f}{\partial p^2}(p). \tag{23}$$

In Section 3, the objective function of the decision maker was expressed by (4). There are, however, different yet equivalent ways to express the objective

function of the outsourcing problem (see [2]). To this end, let $\mathcal{R}(p)$ denote the expected net present value of future profits when the flow starts in p. If $p \in J$, let

$$\mathcal{R}(p) := \mathbb{E}_p \left[\int_0^\infty e^{-rt} P_t \, dt \right]. \tag{24}$$

Thus, the net present value of a company which will never outsource equals $a_I \cdot \mathcal{R}(p)$. The function $\mathcal{R}$ is the image of the resolvent operator R_r applied to the identity function $p \mapsto p$ (see [6], I.7). The family of operators R_r, $r > 0$, is intimately related to the operator $\mathcal{A}$. In the case of DBM it immediately follows from (22) that $\mathcal{R}$ equals $\mathcal{R}_{DBM}$,

$$\mathcal{R}_{DBM}(p) := \frac{p}{r} + \frac{\mu}{r^2}. \tag{25}$$

Throughout, the following conditions will be imposed on processes P:

Assumption 1

(i) The process P is regular (see [6], II.3), i.e. for any two values $p, \tilde{p} \in J$, where p lies in the interior of J, the probability of hitting $\tilde{p}$, when starting in p, is positive.

(ii) $\forall\, p \in \mathbb{R}:\ \mathcal{R}(p)$ is well defined and finite.

(iii) $\mathcal{R}(p)$ is a monotone increasing function.

The specific examples considered in Subsections 4.2 – 4.4 as well as DBM-models all satisfy Assumption 1. It follows from the definition of a resolvent operator that for GBM (cf. (7) – (9)) but also for more general time-homogeneous 1-dimensional diffusions the objective function of the outsourcing problem can be written as

$$\mathbb{E}_{p_0} \left[e^{-r\tau} g(P_\tau) \mathbb{1}_{\{\tau < \infty\}} \right] + a_I \cdot \mathcal{R}(p_0), \tag{26}$$

where

$$g(p) := \alpha \frac{q-1}{q} K \left(\frac{p \vee 0}{K} \right)^{\frac{q}{q-1}} + \alpha(a_0 - a_I)\mathcal{R}(p) - K_f \tag{27}$$

denotes the payoff function. In the case of a GBM-model the function g equals $F(p) - a_I \mathcal{R}(p)$, where F was defined by (9). Assumption 1 (ii) implies that the function g (see (27)) is always a monotone increasing function.

For most processes P the function $\mathcal{R}$ satisfies the ordinary differential equation

$$\mathcal{A}\mathcal{R}(p) - r\mathcal{R}(p) = -p, \tag{28}$$

together with specific growth conditions. This observation can be exploited to compute $\mathcal{R}$ for many functions $\mu(p)$ and $\sigma(p)$. It is easy to verify that the

function $\mathcal{R}_{DBM}$, cf. (25), satisfies the ODE (28) (see (23) for a definition of $\mathcal{A}$) when $\mu(p) \equiv \mu > 0$, $\sigma(p) \equiv \sigma > 0$ and $p \in J = \mathbb{R}$.

To reformulate the optimal stopping problem as a 1-dimensional optimization problem we have to consider a further differential equation. From now on we shall assume that any generator $\mathcal{A}$ is such that there is a non negative increasing solution ψ of the homogeneous ODE

$$\mathcal{A}\psi(p) - r\psi(p) = 0. \tag{29}$$

In the case of the DBM-model such a solution is of the form $\psi(p) = const \cdot e^{\delta p}$, $const > 0$, where δ is the positive solution to the quadratic equation

$$\frac{\sigma^2}{2}\delta^2 + \mu\delta - r = 0.$$

It follows from [11] that for the stopping time problem (4), and the more general problems to be considered in this section, an optimal outsourcing date is a hitting time τ_p (i.e. for a threshold p),

$$\tau_p := \inf\{t \,|\, P_t = p, \, where \, P_0 = p_0\}.$$

It is well known that for regular diffusions, and whenever $p_0 \leqslant p$,

$$\mathbb{E}\left[e^{-r\tau_p} g(X_{\tau_p}) \mathbb{1}_{\{\tau_p < \infty\}}\right] = g(p)\mathbb{E}\left[e^{-r\tau_p} \mathbb{1}_{\{\tau_p < \infty\}}\right] = g(p)\mathbb{E}\left[e^{-r\tau_p}\right] = g(p)\frac{\psi(p_0)}{\psi(p)}. \tag{30}$$

For drifted Brownian motion, (30) yields

$$\mathbb{E}\left[e^{-r\tau_p} g(X_{\tau_p}) \mathbb{1}_{\{\tau_p < \infty\}}\right] = g(p) \cdot e^{-\delta(p-p_0)}.$$

Since g is monotone increasing, an optimal outsourcing time will be a hitting time of the form τ_p, where $p > p_0$ or $p = p_0$. Hence (see also [11]) we shall consider the nonlinear optimization problem

$$\max_{p \geqslant p_0} \left\{ \frac{g(p)}{\psi(p)} \right\}. \tag{31}$$

Let p^* be a solution of (31). In general, it is not at all clear that such a solution p^* exists. In the case of GBM the condition $\beta > \zeta$ ensures the existence of such a value. In the case of DBM – no matter what the values of the parameters are – there is always a solution to this maximization problem. The proof of this claim is very similar to the proof of the claim in the case of GBM (see Subsection 4.2). The following result is an extension of Theorem 3.2.

THEOREM 4.1 *Let $(P_t)_{t\geqslant 0}$ be a general index process (or a profit flow process) which satisfies (21). Assume that the non-linear optimization problem (31) has at least one solution $p^* \in \mathbb{R}$. Let*

$$p^* = argmax_{p\geqslant p_0}\left\{\frac{g(p)}{\psi(p)}\right\}$$

be the smallest maximizer of (31). The optimal policy recommends outsourcing at the first moment when the process hits the (optimal) threshold p^*. For any initial value $p_0 \leqslant p^*$, the maximal (expected) profit reaped by the outsourcing company equals

$$\psi(p_0)\frac{g(p^*)}{\psi(p^*)} + a_I\mathcal{R}(p_0); \tag{32}$$

more generally, the value function V is given by

$$V(p_0) = \begin{cases} \psi(p_0)\frac{g(p^*)}{\psi(p^*)} + a_I\mathcal{R}(p_0) & , \quad p_0 < p^* \\ g(p_0) + a_I\mathcal{R}(p_0) & , \quad p_0 \geqslant p^*. \end{cases} \tag{33}$$

Theorem 3.3 is a collection of sensitivity results for the case when the homogeneous diffusion process is GBM (see [14]). To prove similar results for general diffusion processes, we need to impose conditions on ψ, $\mathcal{R}$ and g. If such conditions hold (see below) such sensitivity results can be proved for general processes. Checking these sufficient conditions is typically a non-trivial matter (see the proofs of Lemmas 4.3, 4.4 and 4.5).

THEOREM 4.2 *Let $(P_t)_{t\geqslant 0}$ denote a general index process (or a profit flow process) which satisfies (21) and all other conditions specified above. Assume that for any optimization problem (31) considered in the sequel a maximizer exists.*

1. *The optimal threshold p^* is a decreasing function of α (i.e. the higher the outsourcing percentage is going to be the lower the threshold value will become). Furthermore, p^* is an increasing function of the fixed cost K_f.*

2. *Let $\mathcal{R}$ be such that the function $p \mapsto g(p)/\mathcal{R}(p)$ is monotone increasing. The optimal threshold p^* is a decreasing function of $\Delta = a_O - a_I$.*

3. *Let two processes with identical drift functions $\mu(p)$ and different diffusion functions $\sigma_i(p)$ be given, $i = 1, 2$. Let $\sigma_1(p) < \sigma_2(p)$, $p \in \mathbb{R}$, and ψ_1, ψ_2 denote the corresponding ψ functions. If the quotient ψ_1/ψ_2 is monotone increasing, and if the function $\mathcal{R}$ is independent of the diffusion function $\sigma(p)$ then $p_1^* < p_2^*$, where p_i^* is optimal threshold corresponding to the model characterized by σ_i, $i = 1, 2$.*

Proof Ad 1: Let $0 \leqslant \alpha_1 < \alpha_2 \leqslant 1$; introduce the constant $h := K_f/\alpha_1 - K_f/\alpha_2 > 0$. Define functions

$$\hat{g}_i(p) := \frac{g_i(p)}{\alpha_i} = \frac{K}{\zeta}\left(\frac{p}{K}\right)^{\zeta} + (a_O - a_I)\mathcal{R}(p) - \frac{K_f}{\alpha_i}, \quad i = 1, 2.$$

For each α_i, the problem of maximizing $\frac{g_i}{\psi}$ can be rewritten as

$$\max_{p \geqslant p_0} \frac{g_i(p)}{\psi(p)} = \alpha_i \max_{p \geqslant p_0} \frac{\hat{g}_i(p)}{\psi(p)}.$$

By definition, $\hat{g}_2 = \hat{g}_1 + h$. Let p_1 and p_2 be the optimal profit values associated with the parameters α_1 and α_2, i.e. $\hat{g}_i(p_i)/\psi(p_i) \geqslant \hat{g}_i(p)/\psi(p)$ for all $p \geqslant p_0$. It follows that either $g_1(p_2) > 0$ or $g_2(p_1) > 0$. Without loss of generality we assume $g_1(p_2) > 0$. By construction,

$$\frac{\hat{g}_2(p_2)}{\psi(p_2)} = \frac{\hat{g}_1(p_2) + h}{\psi(p_2)} = \frac{\hat{g}_1(p_2)}{\psi(p_2)}\left[1 + \frac{h}{\hat{g}_1(p_2)}\right] \tag{34}$$

and

$$\frac{\hat{g}_2(p_2)}{\psi(p_2)} \geqslant \frac{\hat{g}_2(p_1)}{\psi(p_1)} = \frac{\hat{g}_1(p_1)}{\psi(p_1)}\left[1 + \frac{h}{\hat{g}_1(p_1)}\right]. \tag{35}$$

By taking the logarithm on both sides of equation (34) and of inequality (35), the transformed expressions imply the inequality

$$\log\left(\frac{\hat{g}_1(p_1)}{\psi(p_1)}\right) - \log\left(\frac{\hat{g}_1(p_2)}{\psi(p_2)}\right) \leqslant \log\left(1 + \frac{h}{\hat{g}_1(p_2)}\right) - \log\left(1 + \frac{h}{\hat{g}_1(p_1)}\right). \tag{36}$$

Since the left hand side of (36) is non negative the inequality $\hat{g}_1(p_1) \geqslant \hat{g}_1(p_2)$ follows. If $\mathcal{R}$ is monotone increasing then $\hat{g}_1$ is strictly increasing and we obtain $p_1 > p_2$. The sensitivity of $p^*(K_f)$ on variations of K_f can be shown in the same manner.

Ad 2: Let Δ denote the difference $a_O - a_I$; assume $\Delta_2 > \Delta_1$. For $i = 1, 2$, define the function g_i as the payoff function associated the parameter Δ_i, i.e.

$$g_i(p) := \frac{1}{\zeta} K \left(\frac{\alpha}{K}\right)^{\zeta} p^{\zeta} - K_f + \alpha \Delta_i \mathcal{R}(p).$$

We can express g_2 in terms of g_1 and some function h, viz.

$$g_2(p) = g_1(p) + \underbrace{\alpha(\Delta_2 - \Delta_1)\mathcal{R}(p)}_{=:h(p)}. \tag{37}$$

The maximizers p_1 and p_2, the first one associated with the parameter Δ_1 and the second one determined by Δ_2, satisfy $\frac{g_i(p_i)}{\psi(p_i)} = \max_p \frac{g_i(p)}{\psi(p)}$.

Similar to the first part of the proof, we derive inequalities similar to (34), (35) and (36) but we replace the constant h by the function $h(p)$ (cf. (37)). Hence, we obtain the inequality $h(p_2)/g_1(p_2) \geqslant h(p_1)/g_1(p_1)$, which is equivalent to the inequality

$$\frac{g_1(p_2)}{\mathcal{R}(p_2)} \leqslant \frac{g_1(p_1)}{\mathcal{R}(p_1)}.$$

Since the function $p \mapsto g_1(p)/\mathcal{R}(p)$ is assumed to be monotone increasing, the inequality $p_2 \leqslant p_1$ follows.

Ad 3: Let p_i denote the optimal threshold corresponding to σ_i. For any $p \geqslant p_0$,

$$\frac{\psi_i(p_i)}{g(p_i)} \leqslant \frac{\psi_i(p)}{g(p)}.$$

Let $h(p) := (\psi_1 - \psi_2)(p)$. The reciprocal ψ_i/g satisfies the following two conditions:

$$\frac{\psi_1(p_1)}{g(p_1)} = \frac{\psi_2(p_1) + h(p_1)}{g(p_1)} = \frac{\psi_2(p_1)}{g(p_1)}\left[1 + \frac{h(p_1)}{\psi_2(p_1)}\right]$$

and

$$\frac{\psi_1(p_1)}{g(p_1)} \leqslant \frac{\psi_1(p_2)}{g(p_2)} = \frac{\psi_2(p_2)}{g(p_2)}\left[1 + \frac{h(p_2)}{\psi_2(p_2)}\right].$$

Thus,

$$\frac{\psi_2}{g}(p_1) \cdot \frac{g}{\psi_2}(p_2) \leqslant \frac{1 + \frac{h}{\psi_2}(p_2)}{1 + \frac{h}{\psi_2}(p_2)}. \tag{38}$$

By taking the logarithm on both sides of (38) and using the property that p_2 minimizes the reciprocal ψ_2/g, we obtain

$$\frac{\psi_1}{\psi_2}(p_1) - 1 = \frac{h}{\psi_2}(p_1) \leqslant \frac{h}{\psi_2}(p_2) = \frac{\psi_1}{\psi_2}(p_2) - 1.$$

Since ψ_1/ψ_2 is assumed to be monotone increasing, the inequality $p_1 \leqslant p_2$ follows. ■

In the following subsections we will exploit the last two theorems and compute and analyze the optimal threshold values and the value functions if $P = (P_t)_t$ is a GBM-, a CIR- or a GOU-process.

Remark. For a GBM-model, a DBM-one, and for both mean-reverting models (see below) the expected net present value of future profits $\mathcal{R}$ satisfies the first two conditions of the last theorem. The conditions required in part 3 of Theorem 4.2 are satisfied for a GBM-, DBM- and one of the mean-reverting processes considered below (see Subsections 4.2, 4.3 and 4.4). In the case of a

GBM-model, the diffusion function is of the form $\sigma(p) = \bar{\sigma} \cdot p$. If we consider two GBM-processes with diffusions $\sigma_i(p) := \bar{\sigma}_i \cdot p$, $i = 1, 2$, $\bar{\sigma}_1 < \bar{\sigma}_2$, then part 3 of Theorem 4.2 can be applied. Obviously, the sufficient condition of part 3 of Theorem 4.2 is satisfied in the case of a DBM-model.

4.2. GBM revisited

It is well known that the operator

$$\mathcal{A}f(p) := \frac{\sigma^2}{2} p^2 f''(p) + \mu p f'(p), \tag{39}$$

acting on smooth functions f, is the generator of geometric Brownian motion. In the case of a GBM-model, the net present value function $\mathcal{R}$ is $\mathcal{R}_{GBM}(p) = \frac{p}{r-\mu}$. For simplicity, in Sections 4.2 – 4.4 we shall always call the process P a profit flow process. The function $\mathcal{R}_{GBM}$ satisfies the ordinary differential equation (ODE)

$$\mathcal{A}\mathcal{R}_{GBM}(p) - r\mathcal{R}_{GBM}(p) = -p.$$

The payoff function g, cf. (27), equals

$$g_{GBM}(p) = \alpha \frac{q-1}{q} K \left(\frac{p}{K}\right)^{\frac{q}{q-1}} + \alpha(a_0 - a_I)\frac{p}{r-\mu} - K_f. \tag{40}$$

Non negative increasing solutions of the homogeneous ODE

$$\mathcal{A}\psi_{GBM}(p) - r\psi_{GBM}(p) = 0$$

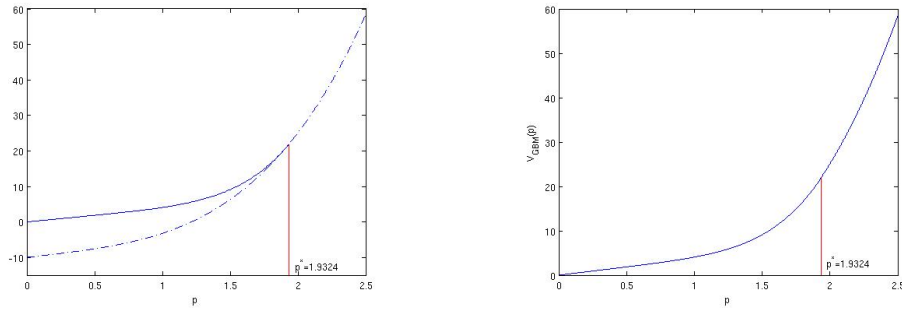
are given by $\psi_{GBM}(p) = const \cdot p^{\beta}$, $const > 0$, where β is defined by (12). Without loss of generality (cf. (30)) let $const = 1$. If the condition $\beta > \zeta = \frac{q}{q-1}$ holds (c.f. Theorem 3.2 in Section 3) elementary calculus shows the existence of the optimal threshold level $p^* > 0$, which is the solution of the nonlinear optimization problem

$$\max_{p \geqslant p_0} \left\{ \frac{g_{GBM}(p)}{\psi_{GBM}(p)} \right\}. \tag{41}$$

The following arguments justify this claim: If $\beta > \frac{q}{q-1}$, then $\psi_{GBM}(p)$ dominates $g_{GBM}(p)$ for large values of p. Since $K_f > 0$, the function $g_{GBM}(p)$ is negative if p is close to 0. Furthermore, $g_{GBM}(p)$ is positive for large values of p. Therefore, it suffices to analyze the maximization problem (41) on a compact subset of $(0, \infty)$.

Let $p^* := argmax \left\{ \frac{g_{GBM}}{\psi_{GBM}} \right\}$, and let p_0 denote the initial value of the profit flow. It follows from Theorem 4.1 that the value function is given by the expression

$$V_{GBM}(p_0) = \begin{cases} \left(\frac{p_0}{p^*}\right)^{\beta} g_{GBM}(p^*) + a_I \frac{p_0}{r-\mu} & , \quad p_0 < p^* \\ \\ g_{GBM}(p_0) + a_I \frac{p_0}{r-\mu} & , \quad p_0 \geqslant p^* \end{cases} . \tag{42}$$



(a) (b)

Figure 3: Panel (a) shows both pieces of V_{GBM} as functions of p. The solid line represents $p \mapsto const \cdot p^{\beta} + a_I p/(r-\mu)$; $const = g_{GBM}(p^*)/(p^*)^{\beta}$. The dashed line depicts $p \mapsto g_{GBM}(p) + a_I p/(r-\mu)$. Panel (b) shows the value function $V_{GBM}(p)$. The parameters are: $r = 0.3$, $\alpha = 0.4$, $\mu = 0.03$, $\sigma = 0.1$, $a_I = 1$, $a_O = 1.5$, $Kf = 10$, $K = 0.3$ and $q = 1.4$.

This is the result stated in Theorem 3.2. In Section 3, however, the value function was derived using the smooth pasting technique.

Figure 4.2 (a) shows the graphs of both components of V_{GBM} as functions of p on the interval $[0\,,\,2.5]$ (see (42)); the interval includes $p^* = 1.9324$. To the left of p^* the solid line represents the value function: V is the sum of an affine function plus a power function with exponent β. To the right of p^* the value function equals the sum of an affine function and a power function with exponent $q/(q-1)$. Panel (b) of Figure 4.2 shows the plot of V_{GBM}. While the dashed line is negative for a while, $g_{GBM}(0) = -K_f$, the value function V_{GBM} is always non-negative.

4.3. The Cox-Ingersoll-Ross process

Let μ, γ and σ be positive parameters, and let $p_0 > 0$ denote the initial value of the mean-reverting process $(P_t)_t$ defined by the stochastic differential equation

$$dP_t = \mu(1-\gamma P_t)dt + \sigma\sqrt{P_t}\,dW_t, \quad P_0 = p_0. \tag{43}$$

The value $1/\gamma$ represents a long term profit flow level of a firm. Whenever $p_0 < 1/\gamma$, following an initial (random) growth period the firm's profit flow will fluctuate around $1/\gamma$. The size of the fluctuations depend on σ, while μ determines (i) how quickly rates grow towards $1/\gamma$ and (ii) how strongly deviating trajectories are pulled back towards the long term equilibrium level. Figure 2 (b) shows sample paths of this mean-reverting process.

The operator

$$\mathcal{A}f(p) = \frac{\sigma^2}{2} p f''(p) + \mu(1-\gamma p) f'(p) \tag{44}$$

is associated with the stochastic differential equation (43).

When $(P_t)_t$ is a CIR-process defined by (43) the function $\mathcal{R}_{CIR}(p) := \mathcal{R}(p)$ (see (24)) is given by

$$\mathcal{R}_{CIR}(p) = \frac{1}{r+\gamma\mu}\left(p + \frac{\mu}{r}\right).$$

Let $\psi_{CIR}(p)$ denote the non negative increasing solution of the homogeneous ODE

$$\mathcal{A}\psi_{CIR}(p) - r\psi_{CIR}(p) = 0,$$

and $\psi_{CIR}(1) = 1$. It is well known that the function $\psi_{CIR}(p)$ can be expressed by the Kummer M-function

$$\mathcal{K}_M(a,b,z) = 1 \; + \; \frac{az}{b} \; + \; \frac{(a)_2}{(b)_2}\frac{z^2}{2!} \; + \; \frac{(a)_3}{(b)_3}\frac{z^3}{3!} \; + ...,$$

where $(c)_n := c(c+1)\cdot ... \cdot (c+n-1)$, $(c)_0 := 1$, denotes the n-th rising factorial of a real number c (see [1]). Given parameters μ, σ, γ and r, the value $\psi_{CIR}(p)$ can be expressed as

$$\psi_{CIR}(p) = \mathcal{K}_M\left(\frac{r}{\gamma\mu}, \frac{2\mu}{\sigma^2}, \frac{2\gamma\mu}{\sigma^2}p\right).$$

This representation follows from the fact that $\mathcal{K}_M(a,b,z)$ solves the differential equation

$$zf''(z) + (b-z)f'(z) - af(z) = 0. \tag{45}$$

Hence, the optimization problem (31) becomes

$$\max_{p \geqslant p_0}\left\{\frac{g_{CIR}(p)}{\psi_{CIR}(p)}\right\}, \tag{46}$$

where

$$g_{CIR}(p) = \alpha\frac{q-1}{q}K\left(\frac{p}{K}\right)^{\frac{q}{q-1}} - K_f + \alpha\frac{a_O - a_I}{r+\gamma\mu}\left(p + \frac{\mu}{r}\right). \tag{47}$$

If a firm's future profit flow is faithfully modeled by a CIR-process then the next result reveals that waiting is optimal whenever an outsourcing decision involves a non-zero fixed cost K_f.

Lemma 4.3 *Let $K_f > 0$. For any $p_0 \geqslant 0$, there exists a (smallest) maximizer p^*, $0 < p^* < \infty$, of (46), i.e.*

$$\frac{g_{CIR}(p^*)}{\psi_{CIR}(p^*)} = \max_{p \geqslant p_0}\left\{\frac{g_{CIR}(p)}{\psi_{CIR}(p)}\right\}.$$

Proof For large values of p the function g_{CIR} is positive. It follows from the representation of ψ_{CIR} as an infinite power series with positive coefficients that ψ_{CIR} dominates g_{CIR} when p tends to infinity. Hence,

$$\lim_{p\to\infty} \frac{g_{CIR}(p)}{\psi_{CIR}(p)} = +0.$$

In contrast to a GBM-model, we do not necessarily know that g_{CIR} is negative close to 0; the constant part of g_{CIR} (see (47)) does not have to be negative. To see that g_{CIR}/ψ_{CIR} attains a positive maximum in the interior of a compact subset of $(0,\infty)$, we will prove that the derivative $(g/\psi)'$ is positive at 0. By definition,

$$g(p) = \underbrace{\alpha\frac{q-1}{q}K^{-\frac{1}{q-1}}}_{=:c_q} p^{\frac{q}{q-1}} + \underbrace{\alpha\frac{a_O-a_I}{r+\mu\gamma}}_{=:c_1} p + \underbrace{\alpha\frac{a_O-a_I}{r+\mu\gamma}\frac{\mu}{r} - K_f}_{=:c_0},$$

and

$$\psi(p) = K_M(a,b,c\cdot p),$$

where $a = r/(\gamma\mu)$, $b = 2\mu/(\sigma^2)$ and $c = 2\gamma\mu/(\sigma^2)$. Thus, $g(0) = c_0$, $g'(0) = c_1$ and $\psi(0) = 1$. Let $b_i = 2\mu/(\sigma_i^2)$ and $z_i = b_i\gamma p$, $i = 1,2$. Since

$$z\frac{dK_M}{dz}(z) = a(K_M(a+1,b,z) - K_M(a,b,z)),$$

cf. [1], and since ψ is continuously differentiable we get

$$p\frac{d\psi_i}{dp}(p) = z_i\frac{dK_M}{dz}(z_i) = a(K_M(a+1,b_i,z_i) - K_M(a,b_i,z_i)). \tag{48}$$

For positive p,

$$\begin{aligned}
\psi'(p) &= \frac{a}{p}\left[K_M(a+1,b,c\cdot p) - K_M(a,b,c\cdot p)\right] \\
&= ac\left[\frac{1}{b} + \sum_{n=1}^{\infty}\frac{(c\cdot p)^n}{(b)_{n+1}\cdot(n+1)!}((a+1)_{n+1} - (a)_{n+1})\right] \\
&\xrightarrow{p\to 0} \frac{ac}{b}.
\end{aligned}$$

Thus, $\psi'(0) = \frac{r}{\mu}$, and

$$g'(0)\psi(0) - g(0)\psi'(0) = c_1\cdot 1 - c_0\cdot\frac{r}{\mu} = K_f\cdot\frac{r}{\mu}$$

is positive. ∎

The associated value function is given by the expression

$$V_{CIR}(p_0) = \begin{cases} \psi_{CIR}(p_0)\frac{g_{CIR}(p^*)}{\psi_{CIR}(p^*)} + a_I\frac{p_0+\frac{\mu}{r}}{r+\gamma\mu} & , \quad p_0 < p^* \\ \\ g_{CIR}(p_0) + a_I\frac{p_0+\frac{\mu}{r}}{r+\gamma\mu} & , \quad p_0 \geqslant p^* \,. \end{cases} \tag{49}$$

Theorem 4.2 can be employed when sensitivity results of CIR-models are derived. To this end, we shall verify that the sufficient conditions assumed in Theorem 4.2 are satisfied. Lemma 4.3 ensures optimal thresholds to exist whenever $K_f > 0$. Obviously, $g_{CIR}(p)$ is a monotone increasing function (see (47)). Hence, the optimal threshold decreases should the fixed proportion α increases (i.e. outsourcing takes place sooner). To see that g_{CIR}/ψ_{CIR} is monotone increasing (in p) (cf. part 2 of Theorem 4.2) the function $g'_{CIR}\mathcal{R}_{CIR} - g_{CIR}\mathcal{R}'_{CIR}$ has to be non-negative. For better reading, we shall drop the subscript CIR for the next couple of lines. By definition,

$$\begin{aligned} g'(p)\mathcal{R}(p) - g(p)\mathcal{R}(p)' &= \alpha K^{-\frac{1}{q-1}} p^{\frac{1}{q-1}} \left[\mathcal{R}(p) - \frac{q-1}{q} p\mathcal{R}'(p)\right] - K_f\mathcal{R}'(p) \\ &= \frac{\alpha}{r+\gamma\mu}\left(\frac{p}{K}\right)^{\frac{1}{q-1}} \left[\frac{1}{q}p + \frac{\mu}{r}\right] + \frac{K_f}{r+\gamma\mu}; \end{aligned}$$

this expression is positive for all $p \geqslant 0$. Hence, should the difference $(a_O - a_I)$ of the profit contribution increases the optimal threshold decreases. Finally, the following lemma ensures the sufficient condition of part 3 of Theorem 4.2 holds.

Lemma 4.4 *Let positive parameters μ, γ and r be given. Let $0 < \sigma_1 < \sigma_2$, and consider the ψ-functions of both CIR-models, i.e.*

$$\psi_i(p) = K_M\left(\frac{r}{\gamma\mu}, \frac{2\mu}{\sigma_i^2}, \frac{2\mu\gamma}{\sigma_i^2}p\right), \quad i = 1, 2.$$

The quotient $\frac{\psi_1}{\psi_2}$ is a strictly monotone increasing function on $(0, \infty)$.

Proof Let $a := r/(\gamma\mu)$, $b_i := 2\mu/\sigma_i^2$ and $z_i := b_i\gamma p$, $i = 1, 2$. Thus, $\psi_i(p) = K_M(a, b_i, z_i)$. To show that the derivative of ψ_1/ψ_2 is monotone increasing we shall verify that

$$f(p) := p \cdot (\psi_1'(p)\psi_2(p) - \psi_1(p)\psi_2'(p)) \tag{50}$$

is a positive function on $(0, \infty)$. Using (48) and simplifying terms we get

$$\frac{f(p)}{p} = K_M(a+1, b_1, z_1)\cdot K_M(a, b_2, z_2) - K_M(a, b_1, z_1)\cdot K_M(a+1, b_2, z_2). \tag{51}$$

Since the Kummer function $K_M(a,b,z)$ can be represented by the infinite series

$$K_M(a,b,z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!},$$

the two products in (51) are Cauchy products. For $i,j \in \{1,2\}$, $i \neq j$, we express these products as

$$K_M(a+1,b_i,z_i) \cdot K_M(a,b_j,z_j) = \sum_{n=0}^{\infty} c_n^{(i,j)}.$$

By definition,

$$c_n^{(1,2)} = \sum_{k=0}^{n} \frac{(a+1)_k}{(b_1)_k} \frac{z_1^k}{k!} \cdot \frac{(a)_{n-k}}{(b_2)_{n-k}} \frac{z_2^{n-k}}{(n-k)!} = \frac{z_2^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \frac{(a+1)_k (a)_{n-k}}{(b_1)_k (b_2)_{n-k}} \left(\frac{z_1}{z_2}\right)^k$$

and

$$c_n^{(2,1)} = \sum_{k=0}^{n} \frac{(a)_k}{(b_1)_k} \frac{z_1^k}{k!} \cdot \frac{(a+1)_{n-k}}{(b_2)_{n-k}} \frac{z_2^{n-k}}{(n-k)!} = \frac{z_2^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \frac{(a)_k (a+1)_{n-k}}{(b_1)_k (b_2)_{n-k}} \left(\frac{z_1}{z_2}\right)^k.$$

Let $y := z_1/z_2 = \sigma_2^2/\sigma_1^2$; note, y is positive. To prove $f(p)$ is positive if $p > 0$, it is sufficient to show that for all $n \in \mathbb{N}$,

$$\frac{n!}{z_n^2} \cdot \left(c_n^{(1,2)} - c_n^{(2,1)}\right) = \sum_{k=0}^{n} \underbrace{\frac{(a+1)_k (a)_{n-k} - (a)_k (a+1)_{n-k}}{(b_1)_k (b_2)_{n-k}} \binom{n}{k} y^k}_{=:d_{k,n}} \quad > \quad 0. \tag{52}$$

The coefficients $(a)_k$ satisfy the recursions $(a+1)_k = (a)_k \frac{a+k}{a}$, $k > 0$, and $(a+1)_{n-l} = (a)_{n-l} \frac{a+n-l}{a}$, $0 < l < n$. Thus, (see (52))

$$\begin{aligned} d_{0,n} &= -\frac{(a)_n \left(\frac{a+n}{a} - 1\right)}{(b_2)_n} &&= -\frac{n(a)_n}{a} \frac{1}{(b_2)_n}, \\ d_{n,n} &= \frac{(a)_n \left(\frac{a+n}{a} - 1\right)}{(b_1)_n} y^n &&= \frac{n(a)_n}{a} \frac{y^n}{(b_1)_n}, \end{aligned}$$

and

$$d_{k,n} = \frac{2k-n}{a} \frac{(a)_k (a)_{n-k}}{(b_1)_k (b_2)_{n-k}} \binom{n}{k} y^k.$$

Next, we will compare d_k with d_{n-k}. If n is odd, the index k runs from 0 to $\frac{n-1}{2}$; if n is even, k runs from 0 to $\frac{n}{2} - 1$. Note, $d_{n/2} = 0$ if n is even. For $k = 0$ and $\sigma_1 < \sigma_2$, we have

$$\begin{aligned} \frac{y^n}{(b_1)_n} - \frac{1}{(b_2)_n} &= \sigma_2^{2n} \left[\frac{1}{\sigma_1^{2n} (b_1)_n} - \frac{1}{\sigma_2^{2n} (b_2)_n}\right] \\ &= \frac{\sigma_2^{2n}}{2\mu} \left[\frac{1}{(2\mu + \sigma_1^2) \cdot \ldots \cdot (2\mu + (n-1)\sigma_1^2)}\right. \\ &\qquad \left. - \frac{1}{(2\mu + \sigma_2^2) \cdot \ldots \cdot (2\mu + (n-1)\sigma_2^2)}\right] \end{aligned}$$

is positive. Hence, $d_{0,n} + d_{n,n} > 0$. To prove the sum $d_k + d_{n-k}$ to be positive if $k > 0$, it is enough to show that

$$\frac{y^{n-k}}{(b_1)_{n-k}(b_2)_k} > \frac{y^k}{(b_1)_k(b_2)_{n-k}} \Leftrightarrow \frac{(b_1)_k}{(b_1)_{n-k}} \frac{\sigma_1^{2k}}{\sigma_1^{2(n-k)}} > \frac{(b_2)_k}{(b_2)_{n-k}} \frac{\sigma_2^{2k}}{\sigma_2^{2(n-k)}}. \quad (53)$$

We know

$$(b_i)_k \sigma_i^{2k} = (2\mu + \sigma_i^2)(2\mu + 2\sigma_i^2) \cdot ... \cdot (2\mu + (k-1)\sigma_i^2)$$

and

$$(b_i)_{n-k} \sigma_i^{2(n-k)} = (2\mu + \sigma_i^2)(2\mu + 2\sigma_i^2) \cdot ... \cdot (2\mu + (n-k-1)\sigma_i^2).$$

Thus, (53) is equivalent to

$$\frac{1}{(2\mu + k\sigma_1^2) \cdot ... \cdot (2\mu + (n-k-1)\sigma_1^2)} > \frac{1}{(2\mu + k\sigma_2^2) \cdot ... \cdot (2\mu + (n-k-1)\sigma_2^2)},$$

and this strict inequality holds since $\sigma_1 < \sigma_2$. Hence, $d_k + d_{n-k}$ is positive for all $k = 0, ..., \frac{n-1}{2}$, or $k = 0, ..., \frac{n}{2} - 1$. The positivity of the sum of any two of these terms implies the expression (52) to be positive. ■

4.4. The Geometric-Ornstein-Uhlenbeck process

In this subsection, we are going to analyze a second mean-reverting process – the GOU-process – in detail. We shall demonstrate that for a GOU-process not all sufficient conditions of Theorem 4.2 hold.

A GOU-process satisfies the stochastic differential equation

$$dP_t = \mu(1 - \gamma P_t)dt \; + \; \sigma P_t dW_t, \quad P_0 = p_0, \quad (54)$$

(cf. [9]). The expected flow of a CIR-process and the expected flow of a GOU-process both converge to $1/\gamma$ if t tends to infinity. The two processes, however, fluctuate around $1/\gamma$ quite differently, as figure 2 (c) and figure 2 (d) illustrate. The difference in the size of the fluctuations is due to the fact that for large values of P_t the diffusion expression in the case of a GOU-model, σP_t, is much larger than the diffusion expression of a CIR-model, $\sigma\sqrt{P_t}$.

The generator of a GOU-process is given by

$$\mathcal{A}f(p) = \frac{1}{2}\sigma^2 p^2 f''(p) \; + \; \mu(1 - \gamma p)f'(p). \quad (55)$$

The function $\mathcal{R}$ (see (24)) is the same as the $\mathcal{R}$-function of a CIR-model (i.e. $\mathcal{R}_{GOU}(p) = \frac{1}{r+\gamma\mu}\,(p + \frac{\mu}{r})$). Let a, b and c be defined as

$$c := \sqrt{\sigma^4 \; + \; (4\mu\gamma + 8r)\sigma^2 + 4\mu^2\gamma^2}, \quad a := \frac{c - \sigma^2 - 2\mu\gamma}{2\sigma^2} \quad \text{and} \quad b := 1 + \frac{c}{\sigma^2};$$

note, $(b-a-1)>1$. Simple analysis shows that

$$\psi_{GOU}(p) := p^{-a}\cdot K_U\left(a,b,\frac{2\mu}{\sigma^2 p}\right)$$

is a positive solution to the differential equation (29), where K_U denotes the Kummer-U function

$$K_U(a,b,z)=\frac{\pi}{sin(\pi b)}\left[\frac{K_M(a,b,z)}{\Gamma(b)\Gamma(1+a-b)}+z^{1-b}\frac{K_M(1+a-b,2-b,z)}{\Gamma(a)\Gamma(2-b)}\right].$$

The Kummer-U function is a second solution to the differential equation (45) (cf. [1]), which is linearly independent of K_M. The $\mathcal{R}$ functions of both models are the same. Thus, the g-function of a GOU-model coincides with the g-function of a CIR-model, should the parameters of both models be identical (see (27) and (47)). The optimization problem (31) becomes

$$\max_{p\geqslant p_0}\left\{\frac{g_{GOU}(p)}{\psi_{GOU}(p)}\right\}. \tag{56}$$

Whenever $K_f>0$, the optimization problem (46) has an optimal solution p^*. In the case of a GOU-process, an additional condition needs to be imposed to guarantee that the outsourcing problem is a non-trivial one; cf. the condition $q/(q-1)<\beta$ in the case of a GBM-model.

Lemma 4.5 *Let $K_f>0$, and assume $\frac{q}{q-1}<(b-a-1)$. For any $p_0\geqslant 0$ there exists a (smallest) maximizer p^* of (56), $0<p^*<\infty$, i.e.*

$$\frac{g_{GOU}(p^*)}{\psi_{GOU}(p^*)}=\max_{p\geqslant p_0}\left\{\frac{g_{GOU}(p)}{\psi_{GOU}(p)}\right\}.$$

Proof Let a,b,c as above. In the case of a GOU-model the ψ function is given by

$$\psi_{GOU}(p) := p^{-a}\cdot K_U\left(a,b,\frac{2\mu}{\sigma^2 p}\right),$$

while the g function equals

$$g_{GOU}(p)=\underbrace{\alpha\frac{q-1}{q}K^{-\frac{1}{q-1}}}_{=:c_q}p^{\frac{q}{q-1}}+\underbrace{\alpha\frac{a_O-a_I}{r+\mu\gamma}}_{=:c_1}p+\underbrace{\alpha\frac{a_O-a_I}{r+\mu\gamma}\frac{\mu}{r}-K_f}_{=:c_0}. \tag{57}$$

For the rest of the proof we will drop the subscript GOU. The Kummer-U function K_U can be expressed in terms of the Kummer-M function:

$$K_U(a,b,z)=\frac{\pi}{sin(\pi b)}\left[\frac{K_M(a,b,z)}{\Gamma(b)\Gamma(1+a-b)}+z^{1-b}\frac{K_M(1+a-b,2-b,z)}{\Gamma(a)\Gamma(2-b)}\right].$$

Using the fact $K_M(a,b,z) \to 1$, as $z \to 0$, it follows that the values of $K_U(a,b,z)$ grow like z^{1-b} if z converges to zero. Let $z(p) := \frac{2\mu}{\sigma^2 p}$, then ψ can be written as

$$\psi(p) = \left(\frac{\sigma^2}{2\mu}\right)^a z(p)^a K_U(a,b,z(p)).$$

Thus, if p tends to infinity the values of $\psi(p)$ grow like p^{b-a-1}. Since the highest exponent of g is $\frac{q}{q-1}$, the condition $\frac{q}{q-1} < b-a-1$ ensures that

$$\frac{g}{\psi}(p) \overset{p\to\infty}{\longrightarrow} 0.$$

Hence, the global maximum of g/ψ is attained at a finite value p^*. For a number $R \in \mathbb{N}$ formula 13.5.2 in [1] yields

$$z^a K_U(a,b,z) = \sum_{n=0}^{R-1} \frac{(a)_n (1+a-b)_n}{n!} (-z)^{-n} \ + \ O(|z|^{-R}). \tag{58}$$

Thus, for $R = 1$ we get

$$\lim_{p\to 0} \psi(p) = \lim_{p\to 0} \left(\frac{\sigma^2}{2\mu}\right)^a z(p)^a K_U(a,b,z(p)) = \left(\frac{\sigma^2}{2\mu}\right)^a$$

is positive. The final step of the proof requires verifying that the maximum is not attained at $p = 0$. To this end, we will show that the derivative of the quotient function is positive at the origin. It is sufficient to prove that

$$\lim_{p\to 0} \ g'(0)\psi(p) - g(0)\psi'(p) \ = \ \lim_{p\to 0} \ c_1\psi(p) - c_0\psi'(p) \tag{59}$$

is positive, where c_1 and c_0 are defined in (57). Since

$$\frac{dK_u}{dz}(a,b,z) = -aK_U(a+1,b+1,z),$$

see [1], and writing z instead of $z(p)$, the chain rule implies that the term $c_1\psi(p) - c_0\psi'(p)$ can be written as

$$\left(\frac{\sigma^2}{2\mu}\right)^a \left[c_1 z^a K_U(a,b,z) + c_0 a p^{-1}\left[z^a K_U(a,b,z) - z^{a+1} K_U(a+1,b+1,z)\right]\right].$$

As p tends to zero, the term $c_1 z^a K_U(a,b,z)$ tends to c_1. To analyze the second term of the sum we will use formula (58) for $R = 2$; we obtain

$$z^a K_U(a,b,z) = 1 + \frac{a(b-a-1)}{z} + O(|z|^{-2}) \tag{60}$$

and

$$z^{a+1} K_U(a+1,b+1,z) = 1 + \frac{(a+1)(b-a-1)}{z} + O(|z|^{-2}). \tag{61}$$

Subtracting (61) from (60) and using the definition of $z(p)$ we obtain

$$\begin{aligned} p^{-1}\left[z^a K_U(a,b,z) - z^{a+1}K_U(a+1,b+1,z)\right] &= \frac{\sigma^2}{2\mu}(b-a-1) + O(|z|^{-1}) \\ &\xrightarrow{p\to 0} \frac{\sigma^2}{2\mu}(b-a-1). \end{aligned}$$

Thus, the limit in (59) equals

$$c_1\psi(0) - c_0\psi'(0) = \left(\frac{\sigma^2}{2\mu}\right)^a \left[c_1 - c_0\frac{\sigma^2}{2\mu}a(b-a-1)\right]. \tag{62}$$

Simple algebra yields

$$c_1 = (c_0 + K_f)\frac{r}{\mu} \qquad \text{and} \qquad \frac{\sigma^2}{2\mu}a(b-a-1) = \frac{r}{\mu}.$$

Plugging these two identities into (62) we finally get

$$c_1\psi(0) - c_0\psi'(0) = \frac{r}{\mu}K_f + c_0\left[\frac{r}{\mu} - \frac{\sigma^2}{2\mu}a(b-a-1)\right] = \frac{r}{\mu}K_f > 0.$$

■

Since the $\mathcal{R}$-function of a GOU-model is the same as the $\mathcal{R}$-function of a CIR-model and if both models have identical parameters μ, γ and σ, the first two statements of Theorem 4.2 are true. The third part of Theorem 4.2 is not applicable in the case of a GOU-model. The assumption that the quotient ψ_1/ψ_2 is supposed to be monotone increasing if $\sigma_1 < \sigma_2$ does not generally hold. The parameters $\mu = 1/10$, $\gamma = 1/10$, $r = 1/10$ and $\sigma_1 = 2/10 < \sigma_2 = 3/10$ provide an example. In the case of a GOU-model, the open-ended question is whether or not the optimal outsourcing threshold is a monotone increasing function of the diffusion coefficient σ.

5. Comparison and sensitivity studies

In this section, we shall illustrate how the objective value and the outsourcing date depend on the process $(P_t)_t$. We numerically compute the optimal threshold value p^* for specific parameter settings of the four models analyzed in Section 4 (i.e. GBM-, DBM-, CIR- and GOU-processes). The parameters are chosen with the view towards highlighting two facts: first, while threshold values p^* can be very similar for different models, the average outsourcing dates can be quite different; second, if $(P_t)_t$ is a GBM-model such that $\mu > \sigma^2/2$ then the expected profit V^* of the outsourcing project is typically much higher than for other models (see Table 1). For DBM- and GBM-processes there are well known formulas of the mean hitting times τ_{p^*} when starting at p_0, $p^* \geqslant p_0$, viz.

$$\mathbb{E}_{p_0}\left[\tau_{p^*_{DBM}}\right] = \frac{p^*_{DBM} - p_0}{\mu}, \qquad \text{if} \quad \mu > 0, \tag{63}$$

and, if $2\mu > \sigma^2$,

$$\mathbb{E}_{p_0}\left[\tau_{p^*_{GBM}}\right] = \frac{2}{2\mu - \sigma^2} \log\left(\frac{p^*_{GBM}}{p_0}\right). \tag{64}$$

While the mean hitting time of GBM depends on μ and σ, the mean hitting time of a DBM process is independent of the diffusion coefficient σ. Table 1 shows that small parameter variations might cause fairly large changes of $\mathbb{E}[\tau_{p^*}]$. For example, in the case of GBM when changing σ from 0.18 to 0.2 the expected waiting time $\mathbb{E}[\tau^*_p]$ increases by more than 5 times.

The mean hitting times of the mean reverting processes introduced in 4.3 and 4.4 can only be numerically computed. Numerical values can be given by solving a boundary value problem or by evaluating the well known integral expressions of average hitting times of such processes (for example, see [12], 15.3).

Remark 5.1 (Bounded effort) We shall briefly comment on necessary changes of formulas and results in the situation when a firm's effort θ is bounded from above. Let $\theta \in [0, \bar{\theta}]$. The optimal (bounded) effort θ^*_b, cf. (2), equals

$$\theta^*_b = \theta^* \wedge \bar{\theta} = \left(\frac{p \vee 0}{K}\right)^{\frac{1}{q-1}} \wedge \bar{\theta}. \tag{65}$$

Basic algebra implies $\theta^*_b = \bar{\theta}$, whenever $p \geqslant K\bar{\theta}^{q-1}$. If (65) holds, function F (see (3) in Section 2) and functions g (see (27), (40) and (47) in Section 4) have to be adjusted. In all these cases we substitute

$$\alpha \frac{(q-1)K}{q}\left(\frac{p \vee 0}{K}\right)^{\frac{q}{q-1}} \qquad \text{by} \qquad \alpha\left[p\,\theta^*_b - \frac{K}{q}\,(\theta^*_b)^q\right].$$

The two expressions are identical if $p \leqslant K\bar{\theta}^{q-1}$. The restriction θ to be bounded only effects the g-functions. In all these cases, the g-functions of the processes considered in Section 4 grow linearly in p (if $p \mapsto \infty$). This fact allows us to drop the condition $\beta > q/(q-1)$ in subsection 4.2, and the condition $(b-a-1) > q/(q-1)$ in subsection 4.4. All statements of Section 4 remain true in the case of bounded effort.

6. Conclusion

This study describes methods for computing optimal outsourcing times. It is assumed that benefits and costs of a firm's effort, which it invests in the outsourcing project, are explicitly taken into account. The following contributions are made:

- The study extends results by Moon to general profit flow or index processes.

Model	**Parameters**					p^*	V^*	$\mathbb{E}[\tau^*]$
	μ	γ	σ	α	a_O			
GBM	0.03	-	0.18	0.35	1.2	1.0195	52.9	1.39
GBM	0.03	-	0.20	0.35	1.2	1.0939	52.9	8.60
GBM	0.03	-	0.18	0.30	1.2	1.1855	52.4	12.48
GBM	0.03	-	0.18	0.35	1.18	1.1304	52.6	8.86
DBM	0.03	0.1	0.18	0.35	1.2	1.1026	33.6	3.42
DBM	0.03	0.1	0.20	0.35	1.2	1.1641	33.7	5.47
DBM	0.03	0.1	0.18	0.30	1.2	1.1954	33.3	6.51
DBM	0.03	0.1	0.18	0.35	1.18	1.1669	33.4	5.56
CIR	0.03	0.1	0.18	0.35	1.2	1.0412	31.7	1.47
CIR	0.03	0.1	0.20	0.35	1.2	1.1011	31.7	3.52
CIR	0.03	0.1	0.18	0.30	1.2	1.1638	31.4	5.85
CIR	0.03	0.1	0.18	0.35	1.18	1.1245	31.5	4.44
GOU	0.03	0.1	0.18	0.35	1.2	0.9873	31.7	0.00
GOU	0.03	0.1	0.20	0.35	1.2	1.0452	31.7	1.62
GOU	0.03	0.1	0.18	0.30	1.2	1.1395	31.4	5.02
GOU	0.03	0.1	0.18	0.35	1.18	1.0899	31.5	3.23

Table 1: Values of the optimal threshold p^*, the optimal value function $V^*(p_0)$ and the expected outsourcing time τ^* for four different models and different parameters. All processes start at $p_0 = 1$. The fixed parameters are: $r = 0.05,\ K_f = 1,\ K = 0.3,\ q = 4$ and $a_I = 1$.

- We demonstrate that finding an optimal outsourcing time can be reduced to solving a 1-dimensional fractional optimization problem.
- Optimal outsourcing policies are of threshold type. The sensitivity of the optimal threshold p^* on changes of model parameters are analyzed. Structural properties of p^* as a function of model parameters are proved under sufficient conditions. For special diffusions, we show these conditions to hold.
- Numerical studies illustrate that optimal outsourcing times are quite sensitive with respect to variations of the fraction of the operation that will be outsourced and the times critically depend on the characteristics of the profit rate model which will be adopted.

Mean reverting profit models usually imply less optimistic predictions about the pay-off of outsourcing decisions than a GBM-model does. We offer numerical values supporting this claim. A critical issue is the relationship between different [optimal] trigger points – as suggested by different models – and the average waiting time until [part of] the in-house operation should be outsourced. The waiting time depends on the characteristics of the process. Simulation packages are useful tools when analyzing these relationships. In addition, integral expressions of expected hitting times exist and the evaluation of these integrals is an alternative to numerically solving the boundary value problems which are associated with the hitting-time problems.

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Optymalne momenty zlecania zadań do specjalizowanych wykonawców

Kurt Helmes & Torsten Templin

Streszczenie W pracy przedstawiono połączenie metody analizy opcji rzeczywistych do wyznaczania optymalnych momentów zleceń firmom zewnętrznym (ZFZ[1]) zadań będących częścią realizowanego projektu z techniką programowania liniowego do rozwiązywania jednowymiarowych zadań optymalnego zatrzymania procesów stochastycznych. Przyjęto model częściowego ZFZ zaproponowany przez Y. Moona(2010), w którym zakładamy, że proces zysków jest geometrycznym ruchem Browna i uwzględnia bezpośrednio korzyści i koszty wszystkich działań, które firma doznała przed chwilą decyzji o ZFZ. Problem ustalenia kiedy zastosować ZFZ oraz wielkość tego zadania jest ustalana na podstawie związanych z tą operacją procesów przepływów zysków lub indeksów modelujących które z założenia są jednowymiarowymi procesami dyfuzji. Otrzymane optymalne czasy zleceń są momentami zatrzymania typu progowego. Przeprowadzono analizę wrażliwości na zmienność rynku oraz inne parametry modelu. Odpowiednie problemy optymalnego zatrzymania są przeformułowane na nieskończenie wymiarowe zadania programowania liniowego i nieliniowe zadania optymalizacji. To pozwala na nowe podejście do analizy wrażliwości. Wynikają z tego szczegółowe zalecenia dotyczące zarządzania projektami z wykorzystaniem ZFZ.

[1] outsourcing

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Słowa kluczowe: korzystanie z zewnętrznych źródeł, optymalne zatrzymanie, procesy powracające do średniej, analiza wrażliwości, programowanie liniowe.

Wprowadzenie Współcześnie zamawianie realizacji części projektu czy produkcji u podmiotów zewnętrznych (ZFZ) stało się integralną częścią globalnej gospodarki. W celu poprawy konkurencyjności firmy zlecają różne elementy swoich działalności podwykonawcom, np. produkcję podzespołów, zadania rozpoznania rynku, dystrybucji i usług (zob. np. [2], [10], lub [18]). Oczywistym bodźcem do takich działań – zwłaszcza w regionach o taniej sile roboczej, takich jak Azja czy Europa Wschodnia – jest zmniejszenie kosztów (patrz: [2] i [18]). Jak pokazano w [19], przenoszenie części zadań na podwykonawcę nie mogą być spowodowane jedynie redukcją kosztów. Przykładowo, możliwe jest, iż osoba trzecia może bardziej efektywnie świadczyć usługi lub oferują wyspecjalizowane technologie. W sektor technologii informacyjnych pokazują to znane przykłady (patrz [4]). Dodatkowym powodem takich działań jest to, że przez przekazanie niektórych operacji lub części operacji do podwykonawców firma jest w stanie skoncentrować się na działaniach, w zakresie których ma najwyższe kompetencje (patrz [19]).

Szczególną cechą zarządzania projektem przez ZFZ jest możliwość realizacji w tym trybie również części podzadania. Askin i inni [3]pokazują to na przykładzie usług centrów telefonicznych (*call center*), gdzie wiele firm waha się przed oddaniem istotnej części tej działalności osobom trzecim. W ankiecie badającej ZFZ w 50-ciu przedsiębiorstwach okazało się, że mniej niż 30 % firm lokuje w zadaniach zewnętrznych więcej niż 50 % budżetu technologii informacyjnych (patrz [4]).

Te obserwacje doprowadziły Shy i Stenbacka [17] do wprowadzenia modelu, w którym firma na etapie planowania wyróżnia pewną liczbę zadań w procesie produkcji określonego produktu końcowego. Każde takie wyróżnione zadanie może wykonać we własnym zakresie lub u podwykonawcy, którym jest wyspecjalizowana firmy zewnętrzna. Biorąc pod uwagę redukcję kosztów poprzez „wyprowadzenie” linii produkcyjnej z jednej strony oraz dodatkowe monitorowanie kosztów z drugiej, Shy i Stenbacka zidentyfikowali część zysku wynikającą ze zlecania zadań poza firmę.

Bardziej ogólny model, który umożliwia podejmowanie decyzji w oparciu o proporcję zlecanych na zewnątrz zadań i zakłada, że dysponujemy pewnymi, niekompletnymi informacjami rynkowymi został zaproponowany przez Alvareza i in. [2]. Moon [14] natomiast, przyjmuje podobny model, ale kontroluje część zadań wykonywanych samodzielnie na poziomie strategicznym. W rozważaniach przeprowadzonych w tym artykule analizowane jest częściowe ZFZ, gdy poza firmę przekazywana jest ustalona część zadań α, $\alpha \in [0, 1]$. Ponadto, wyniki badań ankietowych zamieszczone u Barthelema [4] pokazują,

że sukces z ZFZ zależy od nakładów poniesionych w taki sposób zarządzania projektem. Barthelema wymienia cztery różne rodzaje (potencjalnie wysokich) ukrytych kosztów ponoszonych na ZFZ: koszty na poszukiwanie wykonawcy (dostawcy usługi), koszty zamawiającego, koszty zarządzania tymi zleceniami oraz koszty procesu transformacji projektu. Ponadto, koszty zarządzania to nie tylko identyfikacja, ocena i kontrola kosztów. Przykładowo, wybierając specjalizowanego dostawcę usługi i negocjowanie odpowiednich umów zazwyczaj wpływ na efekty projektu (patrz: [4], [14] oraz [19]). Poświęcenie dodatkowego czasu i środków na wyszukanie wyspecjalizowanych dostawców zwykle oznacza, że produkt końcowy jest lepszej jakości i/lub koszty produkcji są niższe. Z tego powodu uwzględnienie kosztów i efektów ZFZ jest integralną częścią analizy modelu zarządzania z wykorzystaniem ZFZ przeprowadzonego w tej pracy.

Wzorując sie na [2], [14] i innymi opracowaniami, w pracy przyjmuje się modelowanie ZFZ podobne jak przy analizie opcji rzeczywistych. Oznacza to, że na podstawie pewnych istotnych informacji rynkowych, modelowanych przez jednowymiarowy proces dyfuzji stochastycznej, firma usiłuje maksymalizować zysk poprzez wyprowadzanie do podmiotów poza firmą części zadań w chwili, gdy osiągane są optymalne wartości rynkowe. Podobnie jak [2], a także [14], wykorzystywany jest geometryczny ruch Browna (GBM) jako podstawowy procesu modelujący zysk firmy. Chociaż GBM jest popularnym modelem w kontekście zarządzania przez dzielenie zadań z podmiotami zewnętrznymi to ma on pewne wady. Na przykład, jeśli parametr dryfu GBM jest większy niż stopa dyskontowa, to optymalna wartość analizowanego problemu zarządzania przez dzielenie zadań z podmiotami zewnętrznymi, jest nieskończona. Ponadto, jeśli *współczynnik dyfuzji* GBM jest „duży” w stosunku do parametru dryfu, to trajektorie zysków będą dążyć do zera. Inną wadę modelu GBM pokazuje i starannie analizuje Moon [14]. Dla istnienia optymalnego momentu ZFZ trzeba przyjąć techniczny warunek na temat relacji między strukturą kosztów przygotowania zlecenia zewnętrznego i parametrów procesu zysków. Przy innych modelach procesu zysków takie założenia nie są konieczne o czym szerzej w podrozdziale 4.3. W kontekście modeli dopuszczających ZFZ przyjmuje się, że procesy zysku maja tendencję rosnącą przy horyzoncie krótkim i średnim, a w dłuższej perspektywie powinny oscylować wokół jakiejś konkretnej wartości dodatniej. Analizowany jest również przypadek ujemnej stopy zwrotu, co najmniej gdy ma to miejsce przez pewien (krótki) czas. W celu uwzględnienia tych aspektów rozważany jest ogólny procesu jednowymiarowej dyfuzji do modelowania zysków oraz stosuje się metody programowania liniowego do analizy przypadku GBM, jak również innych procesów, przykładowo procesów powracających do średniej. Jak już wspomniano należy brać pod uwagę zarówno koszty jaki i zyski wynikające ze zlecania zadań projektu podmiotom zewnętrznym. W tym celu w pracy przyjmuje się model Moon'a [14], w którym przez $\theta \geqslant 0$ oznacza na-

kłady (ograniczoność z góry nie jest zakładana). Badany jest również wpływ ograniczenia z góry nakładów, $0 \leqslant \theta \leqslant \overline{\theta}$ na wyniki. Przyjmuje się, że wszystkie korzyści związane z nakładem θ są ujęte ilościowo w wyrażeniu, które jest proporcjonalne do θ i strumienia zysków w czasie, gdy ZFZ zastosowano. Termin *wypłata* dla takiej formuły jest uzasadniony w pewnych przypadkach, jak np. gdy przyjęte rozwiązania ZFZ obniżają koszty lub dzielą zysk z podzadania. Wszystkie nakłady θ wchodzą do modelu jako wypukła funkcja potęgowa na półprostej dodatniej. Okazuje się, że ograniczenie z dołu tej funkcji przez funkcje liniową i multiplikatywna forma uwzględnienia zysków, to dwie znaczące cechy tego modelu.

Inaczej niz u Alvareza [2], w niniejszym opracowaniu zastosowano podejście Moon'a [14] zakładając, że część projektu, którą przeznaczamy do wykonania na zewnątrz firmy α jest zadana i nie wynika z przesłanek operacyjnych, ale jest wielkością statyczną. Oznacza to, że w tym artykule rozważamy model zarządzania projektem dopuszczający zlecanie części zadań na zewnątrz w ustalonej części nakładów *theta* w ustalonym czasie T. Łączna wypłata z decyzji ZFZ zależy od:

(i) oczekiwanego zdyskontowanego zysku bez ZFZ;

(ii) procentowy udział oczekiwanych zysków zdyskontowanych generowanych przez ZFZ;

(iii) zysków z zadań pozostawionych do wykonania w firmie;

(iv) część wypłaty (optymalna wartość wynikająca z zadania maksymalizacji po wszystkich $\theta \geqslant 0$) związanej z nakładami firmy na realizację części zadań przez ZFZ.

Nierówności wariacyjne lub zasada *gładkiego dopasowania*, to klasyczne metody stosowane do rozwiązywania problemów optymalnego zatrzymywania procesów. Problem wyznaczenia optymalnego momentu ZFZ, rozważany w tym artykule, jest równoważny problemowi optymalnego zatrzymania procesu. U Moon'a [14] technika *gładkiego dopasowania* służy do analizy wrażliwości modelu ZFZ gdy procesu zysku jest modelowany geometrycznym ruchem Browna. Bada on, między innymi, wpływ niepewności na rynku na optymalny termin ZFZ (czyli wpływu zmiany współczynnika dyfuzji i innych parametrów modelu na optymalny moment takiego zlecenia). W analizie częściowego zlecania podmiotom zewnętrznym części projektu przeprowadzonej w artykule przypadek rozważany przez Moon'a cite moon z geometrycznym ruchem Browna jest przypadkiem szczególnym. Dzięki temu udało się wykryć nieścisłość w pracy Moon'a [14] i wprowadzić korektę do jego wyników. Co więcej, wprowadzone metody programowania liniowego i sformułowania nieliniowych problemów optymalizacji pozwalają uzyskań analogiczne rezultaty dla znacznie ogólniejszych procesów zysku niż geometryczny proces Browna. Charakteryzacja optymalnego czasu zatrzymania w kategoriach

arg-max pewnej określonej funkcji nieliniowej daje możliwość łatwego, deterministycznego, wyznaczania optymalnych momentów ZFZ. To też daje możliwość analizy wrażliwości w nowy, efektywny sposób.

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