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BOOK REVIEW
Entire solutions of semilinear elliptic equations
by I. Kuzin, S. Pohozaev

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The book reviewed is devoted to study of elliptic semilinear equations of the type

$$-\Delta u + f(x,u) = 0 \text{ on } \mathbb{R}^n. \tag{1}$$

Such equations have many applications. Consider for instance the well known equation in quantum physics

$$u_{tt} - \Delta u + f(|u|)u = 0 \text{ on } \mathbb{R} \times \mathbb{R}^N \tag{2}$$

Suppose that we want to find solutions of (2) in the form of traveling waves $u = v(x - ct)$, where c is some vector in $\mathbb{R}^N$, $|c| < 1$. Let v be a real valued function. It may be shown that, after a Lorenz transformation of coordinates, v satifies an equation of form (1).

Equations of type (1) in bounded domains have been studied extensively. It is sufficient to refer to classical monographs for example Gilbarg and Trudinger (1983), and Ladyzhenskaya and Ural'ceva (1968). The main goal of Kuzin and Pohozaev book is the a description of methods for studying semilinear elliptic problems on $\Omega = \mathbb{R}^N$.

Throughout the book the boundary problems of the form

$$\begin{gathered} -\Delta u + f(x,u) = 0 \text{ on } \mathbb{R}^N \\ u \to 0 \text{ as } |x| \to +\infty \end{gathered}$$

are studied.

The book is composed of five chapters and appendix. Chapter 1 entitled "Classical variational method" concerns the following topics:

1. The classical method: absolute minimum.
2. Approximation by bounded domains.
3. Approximation for problems on an absolute minimum
4. The monotonicity method.

Several specific examples are given. For instance it is shown that the problem

$$-\Delta u + \lambda_1|u|^{p-2} + \lambda_2|u|^{q-2}u = h_1(x) + h_2(x) \text{ in } \mathbb{R}^N$$

has a generalized solution $u \in \mathcal{E}_{p,q}$ for any $N, 1 < p \leq q < +\infty, \lambda_1 > 0, \lambda_2 > 0, h_1 \in L_{p/(p-1)}(\mathbb{R}^N), h_2 \in L_{q/(q-1)}$ where $\mathcal{E}_{p,q}$ is the closure of the set of infinitely differentiable functions with compact support and the norm $\|\nabla u\|_2 + \|u\|_p + \|u\|_q$.

Chapter 2, entitles "Variational methods for eigenvalue problems" is devoted among other things to equations of the forms:

1. $-\Delta u + a(x)|u|^{p-2}u - \lambda b(x)|u|^{q-2}u = 0,$
2. $-\Delta u + \lambda f(u) = 0,$
3. $-\Delta u - \lambda|u|^{p-2}u - b(x)|u|^{q-2} = 0$

In 2. the results of Berestycki and Lions and in 3. some ideas of Rother and Stuart are used. Chapter 2 contains also some theorems about radial solutions of some elliptic problems and first example of the compactification method proposed by Lions, which is known as the concentration compactness method.

Chapter 3 entitles "Special variational methods" deals with more complicated methods. The authors start with the simple application of the mountain pass theorem by Ambrosetti and Rabinovitz to noncoercive problems. Then they show how the mountain pass method together with the concentration compactness method can be applied to noncoercive elliptic problems on $\mathbb{R}^N$. The next part of chapter 3 provides some theorems about nonradial solutions of radial equations and an existence result for the problems

$$-\Delta u + u - b(|x|)|u|^{q-2}u = 0 \text{ on } \mathbb{R}^N$$

where $N > 2, 2 < q < 2M/(N-2)$ and $b : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative nontrivial locally Hölder continuous function.

Chapter 4 entitled "The ODE method" contains the method of ordinary differential equations which, of course, can be applied only to the radial solutions. Among other thing the results of Keller, Berestycki Lions in this area are presented. The last part of chapter 4 concern to a simplified version of the paper by Berestycki, Lions and Peletier.

Chapter 5 is devoted to other modern methods. First part of chapter 5 (§25) yields the approach to upper and lower solutions together with simple examples. Paragraph 26 is an illustration of the Leray-Schauder method applied to the specific problems arising on $\mathbb{R}^N$. Paragraph 27 contains the

Pohozaev identities and their application to elliptic problems on $\mathbb{R}^N$. Paragraph 28 is aimed at presentation of the fibration method.

The last part of chapter 5 provides examples of nonexistence theorems. To sum up, the book "Entire solutions of semilinear elliptic equations" presents a lot of modern methods to solve semilinear elliptic problems. It is very carefuly edited (though one can find some small misprints) and presents up-to-date state in the area of semilinear elliptic equations on $\mathbb{R}^N$.