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On bound for the solution of the unified algebraic Riccati equation*

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Abstract. In recent years, several bounds of eigenvalues, norms and determinants for solutions of the continuous and discrete Riccati equations have been separately investigated. In this paper, an upper bound for the solution of the unified algebraic Riccati equations is presented. In the limit case, the result is reduced to a new upper bound for the solution of discrete and continuous Riccati equation.

1. Introduction. The continuous and discrete algebraic Riccati equation are important in various areas of engineering system theory, particularly in control system theory and also in study of the stability of linear systems, see [6], [7]. Numerous papers have presented bounds of eigenvalues of the solution of continuous and discrete Riccati equation separately. A summary of results on this topic is given in [9].

The objective of this paper is to present an upper bound for sums and products of the eigenvalues of the solution of the unified algebraic Riccati equation. In the limit cases, we get some new bounds for the discrete and continuous Riccati equation. Similar results for lower bounds are obtained in [10].

Consider the unified-type algebraic Riccati equation

$$(1)\qquad -Q = A'P + PA + \Delta A'PA - (I_n + \Delta A)'PB(I_r + \Delta B'B'P)^{-1}B'(I_n + \Delta A)$$

$$(2)\qquad = (I_n + \Delta A)'\frac{P}{\Delta}(I_n + \Delta A) - \frac{P}{\Delta} - (I_n + \Delta A)'PB(I_r + \Delta B'PB)^{-1}B'P(I_n + \Delta A),$$

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where I_n is the identity matrix of order n, $\Delta \in R$, $\Delta > 0$, $A, P, Q \in R^{n\times n}$, and $B \in R^{n\times r}$, with $Q > 0$ and $P > 0$ is the solution. If (A, B) is stabilizable and (C, A), where $Q = C'C$, has no unobservable mode inside the stability boundary, then the equation (7) has a unique positive definite solution, see [7].

Let $\lambda_i(X)$, $i = 1, \dots, n$ be the ith eigenvalue of a nonegative definite matrix $X \in R^{n\times n}$. Since X has only real eigenvalues, they can be ordered in a nonincreasing order

$$\lambda_1(X) \geq \dots \geq \lambda_n(X).$$

REMARK. From the unified Riccati equation (1), we obtain in limit cases continuous and discrete Riccati equations, see [7]. In continuous case ($\Delta = 0$) from (1) we obtain the continuous algebraic Riccati equation. In discrete case and for $\Delta = 1$ by substituting A by $A - I_n$, we obtain from (2) the discrete algebraic Riccati equation.

Before formulating the main result of this paper we recall the following results.

THEOREM 1.1. *For any symmetric positive semidefinite matrices $R \in R^{n\times n}$ and $S \in R^{n\times n}$ following inequalities hold*:

$$\lambda_{i+j-n}(S+T) \geq \lambda_i(S) + \lambda_j(T) \quad \text{if } i+j \geq n+1 \tag{3}$$

$$\lambda_{i+j-1}(ST) \leq \lambda_i(S)\lambda_j(T) \quad \text{if } i+j \leq n+1 \tag{4}$$

$$\sum_{i=1}^{l} \lambda_i(X+Y) \leq \sum_{i=1}^{k} \lambda_i(X) + \sum_{i=1}^{k} \lambda_i(Y), \qquad k = 1, \dots, n. \tag{5}$$

Proof. Inequality (3) and (4) were proven in [1]; for (5), see [2].

THEOREM 1.2 [8]. *Let $x_1, \dots, x_n$ be nonnegative real numbers. Then*

$$\sqrt[n]{\prod_{i=1}^{n} x_i} \leq \frac{\sum_{i=1}^{n} x_i}{n}. \tag{6}$$

2. Main results. Main results of this paper are given by the following theorem and corollaries.

THEOREM 2.1. *The eigenvalues of the positive definite matrix solution P of* (1), *satisfy, for $k = 1, \dots, n$ following inequality*

$$(7) \quad \sum_{i=1}^{k} \lambda_i(P)$$
$$\leq \begin{cases} \dfrac{N + \sqrt{N^2 + 4k\lambda_n(BB')\sum_{i=1}^{k}\lambda_i(Q)}}{2\lambda_n(BB')} & \textit{if } \lambda_n(BB') \neq 0 \\ \dfrac{-\sum_{i=1}^{k}\lambda_i(Q)}{\lambda_1(A + A' + \Delta AA')} & \textit{if } \lambda_n(BB') = 0 \\ & \textit{and } \lambda_1(A + A' + \Delta AA') < 0 \end{cases}$$

where $N = k\lambda_1(A + A' + \Delta AA') + \Delta\lambda_n(BB')\sum_{i=1}^{k}\lambda_i(Q)$.

Proof. Using the matrix identity

$$(I_n + ST)^{-1} = I_n - S(I_r + TS)^{-1}T,$$

where $S = PB$ and $T = B'$, (1) can be transformed to the form:

$$(8) \qquad P = (I_n + \Delta A)'(P^{-1} + \Delta BB')^{-1}(I_n + \Delta A) + \Delta Q.$$

From (8) by using (5) and taking into acount that

$$\lambda_i((I_n + \Delta A)'(P^{-1} + \Delta BB')^{-1}(I_n + \Delta A)) = \lambda_i((I_n + \Delta A)(I_n + \Delta A)'(P^{-1} + \Delta BB')^{-1})$$

we obtain

$$(9) \quad \sum_{i=1}^{k}\lambda_i(P) \leq \sum_{i=1}^{k}\lambda_i((I_n+\Delta A)(I_n+\Delta A)'(P^{-1}+\Delta BB')^{-1})+\sum_{i=1}^{k}\lambda_i(\Delta Q).$$

From (4) it follows that

$$(10) \quad \lambda_i((I_n + \Delta A)(I_n + \Delta A)'(P^{-1} + \Delta BB')^{-1}) \leq \lambda_1((I_n + \Delta A)(I_n + \Delta A)')\lambda_i((P^{-1} + \Delta BB')^{-1}).$$

Since

$$(11) \qquad \lambda_i((P^{-1} + \Delta BB')^{-1}) = \lambda_{n-i+1}^{-1}(P^{-1} + \Delta BB')$$

and using (3) $(i = n, j = n - i + 1)$, we rearange (10) to obtain

$$(12) \quad \lambda_i((I_n + \Delta A)(I_n + \Delta A)'(P^{-1} + \Delta BB')^{-1}) \leq \lambda_1((I_n + \Delta A)(I_n + \Delta A)')\frac{\lambda_i(P)}{1 + \lambda_i(P)\lambda_n(\Delta BB')}.$$

Applying (12) on the right hand side of (9), we have

$$(13) \quad \sum_{i=1}^{k}\lambda_i(P) \leq \lambda_1((I_n + \Delta A)(I_n + \Delta A)')\sum_{i=1}^{k}\frac{\lambda_i(P)}{1 + \lambda_n(\Delta BB')\lambda_i(P)} + \sum_{i=1}^{k}\lambda_i(\Delta Q).$$

We conclude from (13) that the theorem is true in the case of $\lambda_n(BB') = 0$. Assume now that $\lambda_n(BB') \neq 0$. For any convex function $f : [a,b] \to R$ and real numbers $x_i \in [a,b]$, $\alpha_i > 0$, $i = 1,\dots,k$ we have

$$f\left(\frac{1}{S}\sum_{i=1}^{n}\alpha_i x_i\right) \leq \frac{1}{S}\sum_{i=1}^{n}\alpha_i f(x_i), \tag{14}$$

where $S = \sum_{i=1}^{k}\alpha_i$. Application of inequality (14) with function $f(x) = -\frac{x}{1+\lambda_n(\Delta BB')x}$ and $x_i = \lambda_i(P), \alpha_i = \frac{1}{k}, i = 1,\dots,k$ to (13) gives

$$\begin{aligned}\sum_{i=1}^{k}\lambda_i(P) \\ \leq \lambda_1((I_n + \Delta A)(I_n + \Delta A)')\frac{\sum_{i=1}^{k}\lambda_i(P)}{k + \lambda_n(\Delta BB')\sum_{i=1}^{k}\lambda_i(P)} + \sum_{i=1}^{k}\lambda_i(\Delta Q).\end{aligned} \tag{15}$$

Rearranging (15) we get

$$\lambda_n(BB')\Big(\sum_{i=1}^{k}\lambda_i(P)\Big)^2 - N\sum_{i=1}^{k}\lambda_i(P) - k\sum_{i=1}^{k}\lambda_i(\Delta Q) \leq 0. \tag{16}$$

From (16), (17) follows immediately.

COROLLARY 2.1. *The eigenvalues of the positive definite matrix solution P of* (1), *satisfy for $k = 1,\dots,n$ following inequality*

$$\begin{aligned}&\prod_{i=1}^{k}\lambda_i(P) \\ &\leq \begin{cases}\left(\dfrac{N + \sqrt{N^2 + 4k\lambda_n(BB')\sum_{i=1}^{k}\lambda_i(Q)}}{2k\lambda_n(BB')}\right)^k & \text{if } \lambda_n(BB') \neq 0 \\ \left(\dfrac{-\sum_{i=1}^{k}\lambda_i(Q)}{k\lambda_1(A + A' + \Delta AA')}\right)^k & \text{if } \lambda_n(BB') = 0 \text{ and} \\ & \lambda_1(A + A' + \Delta AA') < 0,\end{cases}\end{aligned} \tag{13}$$

where $N = k\lambda_1(A + A' + \Delta AA') + \Delta\lambda_n(BB')\sum_{i=1}^{k}\lambda_i(Q)$.

Proof. Apply the arithmetic-mean geometric-mean inequality (6) to (7).

The following two Corollaries, follow from (8), (13) and Remark.

COROLLARY 2.2. *The eigenvalues of the positive definite matrix solution P of continuous Riccati equation*

$$-Q = A'P + PA - PBB'P, \tag{14}$$

satisfy for $k = 1, \ldots, n$ following inequalities

$$(15)\qquad \sum_{i=1}^{k} \lambda_i(P) \leq \begin{cases} \dfrac{k\lambda_1(A+A') + \sqrt{(k\lambda_1(A+A'))^2 + 4k\lambda_n(BB')\sum_{i=1}^{k}\lambda_i(Q)}}{2\lambda_n(BB')} & \\ \qquad\qquad \text{if } \lambda_n(BB') \neq 0 & \\ \dfrac{-\sum_{i=1}^{k}\lambda_i(Q)}{\lambda_1(A+A')} \qquad \text{if } \lambda_n(BB') = 0 \text{ and } \lambda_1(A+A') < 0 & \end{cases}$$

and

$$(16)\qquad \prod_{i=1}^{k} \lambda_i(P) \leq \begin{cases} \left(\dfrac{\lambda_1(A+A') + \sqrt{\lambda_1^2(A+A') + 4\lambda_1(A+A')\frac{\lambda_n(BB')}{k}\sum_{i=1}^{k}\lambda_i(Q)}}{2\lambda_n(BB')}\right)^k & \\ \qquad\qquad \text{if } \lambda_n(BB') \neq 0 & \\ \left(\dfrac{-\sum_{i=1}^{k}\lambda_i(Q)}{k\lambda_1(A+A')}\right)^k \qquad \text{if } \lambda_n(BB') = 0 \text{ and } \lambda_1(A+A') < 0. & \end{cases}$$

The bound (15) under the assumption that $\lambda_n(BB') \neq 0$ was derived in [11] by a different method in the case when $k = n$ and, in general, in [5]. The second part of bound (15) and the bound (16) are new.

COROLLARY 2.3. *The eigenvalues of the positive definite matrix solution P of discrete Riccati equation*

$$(17)\qquad P = A'PA - A'PB(I_r + B'PB)^{-1}B'PA + Q,$$

satisfy for $k - 1, \ldots, n$ the following inequalities

$$(18)\qquad \sum_{i=1}^{k} \lambda_i(P) \leq \begin{cases} \dfrac{M + \sqrt{M^2 + 4k\lambda_n(BB')\sum_{i=1}^{k}\lambda_i(Q)}}{2\lambda_n(BB')} & \text{if } \lambda_n(BB') \neq 0 \\ \dfrac{\sum_{i=1}^{k}\lambda_i(Q)}{1-\lambda_1(AA')} \qquad \text{if } \lambda_n(BB') = 0 \text{ and } \lambda_1(AA') < 1 & \end{cases}$$

and

$$\text{(19)}\quad \prod_{i=1}^{k}\lambda_i(P) \le \begin{cases} \left(\dfrac{M+\sqrt{M^2+4k\lambda_n(BB')\sum_{i=1}^{k}\lambda_i(Q)}}{2k\lambda_n(BB')}\right)^k & \textit{if } \lambda_n(BB')\neq 0\\ \left(\dfrac{\sum_{i=1}^{k}\lambda_i(Q)}{k-k\lambda_1(AA')}\right) & \textit{if } \lambda_n(BB')=0 \textit{ and } \lambda_1(AA')<1, \end{cases}$$

where $M = k\lambda_1(AA') - k + \lambda_n(BB')\sum_{i=1}^{k}\lambda_i(Q)$.

The following bound was derived in [4]:

$$\text{(20)}\qquad \sum_{i=1}^{k}\lambda_i(P) \le \frac{a+\sqrt{a^2+4k\lambda_n(BB')\sum_{i=1}^{k}\lambda_i(Q)}}{2\lambda_n(BB')}$$

where $a = k(\lambda_1(AA') - 1 + \lambda_1(Q)\lambda_n(BB'))$ and $\lambda_n(BB')\neq 0$.

Our bound (18) is stronger than (20), because $\lambda_1(Q) \ge \frac{\sum_{i=1}^{k}\lambda_i(Q)}{k}$ and, therefore, $M \le a$ and

$$\frac{M+\sqrt{M^2+4k\lambda_n(BB')\sum_{i=1}^{k}\lambda_i(Q)}}{2\lambda_n(BB')} \le \frac{a+\sqrt{a^2+4k\lambda_n(BB')\sum_{i=1}^{k}\lambda_i(Q)}}{2\lambda_n(BB')}.$$

Our bound (18) is stronger than

$$\text{(21)}\qquad \sum_{i=1}^{k}\lambda_i(P) \le \sum_{i=1}^{k}\lambda_i(Q)\left[1-\frac{\lambda_1(AA')}{1+\lambda_k(Q)\lambda_n(BB')}\right]^{-1}$$

given in [3], where

$$\text{(22)}\qquad 1-\frac{\lambda_1(AA')}{1+\lambda_k(Q)\lambda_n(BB')} > 0.$$

Indeed, denoting the right-hand side of (18) and (21) by R_1 and R_2, respectively we have

$$\text{(23)}\qquad R_1 = \frac{k\lambda_1(AA')R_1}{k+\lambda_n(BB')R_1} + \sum_{i=1}^{k}\lambda_i(Q)$$

$$\text{(24)}\qquad R_2 = \frac{\lambda_1(AA')R_2}{1+\lambda_n(BB')\lambda_k(Q)} + \sum_{i=1}^{k}\lambda_i(Q).$$

The equations (11) and (12) imply (23) and (24) follows from (21). For the positive definite matrix solution P of discrete Riccati equation (15) we have

$$\lambda_k(Q) \leq \lambda_k(P), \tag{25}$$

see [3]. Using (23), (24) and (25) we can find the bound for the difference $R_2 - R_1$

$$\begin{aligned} R_2 - R_1 &= \frac{\lambda_1(AA')R_2}{1+\lambda_n(BB')\lambda_k(Q)} - \frac{k\lambda_1(AA')R_1}{k+\lambda_n(BB')R_1} \\ &\geq \frac{\lambda_1(AA')R_2}{1+\lambda_n(BB')\lambda_k(Q)} - \frac{\lambda_1(AA')R_1}{1+\lambda_n(BB')\lambda_k(Q)} \\ &= \frac{\lambda_1(AA')}{1+\lambda_n(BB')\lambda_k(Q)}(R_2 - R_1). \end{aligned} \tag{26}$$

In view of (26) and (21) we have $R_1 \leq R_2$.

The bound (19) is new.

4. Conclusion. Upper bound for the sums and products of the eigenvalues of the solution of the unified-type Riccati equation is presented in this note. Some new bounds for the discrete and continuous Riccati equation are obtained using this unified approach. A comparison with existing upper bounds for discrete Riccati equation was made.

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