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Existence theory for the equations of inelastic material behavior of metals – Transformation of interior variables and energy estimates

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Introduction. The system of equations, which we study, consists of linear partial differential equations and of nonlinear ordinary differential equations for internal variables. The existence theory for such systems was studied first by the french mathematicians G. Duvaut and J.L. Lions [1]. Next we can find in the literature a work of C. Johnson [2] on a quasi-static problem for a special model. Then in the nineties we can find more works in the domain.

This work consists of two parts. In the first part we will classify constitutive equations and therefore we define constitutive equations of monotone type. Moreover by transformation of internal variables we will enlarge the class of constitutive equations, for which we can prove a global in time existence theorem for large initial data. But there exist models, which are not of monotone type and which we can not transform to monotone type. Therefore we must study such models with other methods. This is the second part of the work. We write about the energy method for the model of Bodner-Partom.

1. Formulation of the initial-boundary value problem for the inelastic material behavior of metals. Let $\Omega \subseteq \mathbb{R}^3$ be an open set with smooth boundary $\partial\Omega$. This set represents the masspoints of the body. Let $u(x,t) \in \mathbb{R}^3$ be the displacement of the masspoint $x \in \Omega$ at the time $t \geq 0$, let $\rho > 0$ be the density, which we assume to be constant in space and time,

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and let $T(x,t) \in \mathcal{S}^3$ (the set of symmetric 3×3 matrices) be the Cauchy stress tensor. We study the problem

$$\begin{aligned} \rho u_{tt}(x,t) &= \operatorname{div}_x T(x,t) \\ T(x,t) &= D\left(\varepsilon(\nabla_x u(x,t)) - \varepsilon_p(x,t)\right) \\ z_t(x,t) &= f\left(\varepsilon(\nabla_x u(x,t)), z(x,t)\right). \end{aligned} \tag{P}$$

Here the subscripts t and x denote the partial derivatives with respect to the variable t or x respectively,

$$\varepsilon(\nabla_x u(x,t)) = \frac{1}{2}\left(\nabla_x u(x,t) + (\nabla_x u(x,t))^T\right) \in \mathcal{S}^3$$

is the linear strain tensor,

$$\varepsilon_p(x,t) \in \mathcal{S}^3$$

is the plastic strain tensor, and

$$z(x,t) = (\varepsilon_p(x,t), \tilde{z}(x,t)) \in \mathcal{S}^3 \times \mathbb{R}^{N-9} \subseteq \mathbb{R}^N$$

is the vector of internal variables. $D : \mathcal{S}^3 \to \mathcal{S}^3$ denotes a symmetric linear mapping, which is called elasticity tensor. We assume that the mapping D is positive definite, and that D is independent of x and t. Finally

$$f : D(f) \subseteq \mathcal{S}^3 \times \mathbb{R}^N \to \mathbb{R}^N$$

is a given function.

We assume that one of the two homogeneous boundary conditions

$$\begin{aligned} &u(x,t)|_{\partial\Omega} = 0 \\ &\text{or} \\ &T(x,t)n(x)|_{\partial\Omega} = 0 \end{aligned} \tag{BC}$$

is satisfied, where $n(x)$ denotes the exterior unit normal vector to $\partial\Omega$ at $x \in \partial\Omega$. The initial conditions are

$$u(x,0) = u^{(0)}(x),\ u_t(x,0) = u^{(1)}(x),\ z(x,0) = z^{(0)}(x) \quad x \in \Omega \tag{IC}$$

with given functions

$$u^{(0)}, u^{(1)} : \Omega \to \mathbb{R}^3, \quad z^{(0)} : \Omega \to \mathbb{R}^N.$$

We also require that there exists a function $\psi : D(f) \subseteq \mathcal{S}^3 \times \mathbb{R}^N \to [0,\infty)$ with

$$\begin{aligned} \rho\nabla_\varepsilon \psi(\varepsilon, z) &= T \\ \rho\nabla_z \psi(\varepsilon, z) \cdot f(\varepsilon, z) &\le 0 \end{aligned} \tag{FC}$$

for all $(\varepsilon, z) \in D(f)$. The function ψ is called free energy and the last inequality is called dissipation inequality. The existence of such a function ψ is a consequence of the second law of thermodynamics. Therefore the constitutive equations are called thermodynamically admissible if a free energy exists satisfying (FC). From the constitutive equation (P2) and from the symmetry of D it follows that

$$\rho\psi(\varepsilon, z) = \frac{1}{2}[D(\varepsilon - \varepsilon_p)] \cdot (\varepsilon - \varepsilon_p) + \psi_1(z)$$

with a suitable function $\psi_1 : D(\psi_1) \subseteq \mathbb{R}^N \to [0, \infty)$. The constitutive equation (P3) assigns to given initial data $z^{(0)}$ and a given function $\varepsilon(x, t)$ the function $z : \Omega \times [0, \infty) \to \mathbb{R}^N$, and thus defines the constitutive relation

$$z(x, t) = \underset{0 \leq \tau \leq t}{\mathcal{F}} \left(\varepsilon(x, \tau), z^{(0)}(x) \right). \tag{C}$$

DEFINITION 1. We call the constitutive relation (C) rate independent, if for all $z, \varepsilon, z^{(0)}$ satisfying (C) we have

$$z(x, \lambda t) = \underset{0 \leq \tau \leq t}{\mathcal{F}} \left(\varepsilon(x, \lambda\tau), z^{(0)}(x) \right)$$

for all $\lambda \geq 0$.

The differential equation (P3) is rate independent only in the degenerate case $f \equiv 0$. But (P3) can be considered to be a special case of the more general constitutive relation

$$z_t(x, t) \in f\left(\varepsilon(\nabla_x u(x, t)), z(x, t)\right)$$

which also includes rate indpendent constitutive models. Here

$$f : D(f) \subseteq \mathcal{S}^3 \times \mathbb{R}^N \to \mathcal{P}(\mathbb{R}^N)$$

is a mapping with values in the set $\mathcal{P}(\mathbb{R}^N)$ of all subsets of $\mathbb{R}^N$. From mathematical reasons it is important to generalize also equation (P2) and replace it by

$$T(x, t) = D\left(\varepsilon(\nabla_x u(x, t)) - Bz(x, t)\right),$$

where $B : \mathbb{R}^N \to \mathcal{S}^3$ is a given linear operator. For example we can choose for B the linear operator

$$z \to Bz = \frac{1}{2}(z_p + z_p^T)$$

where z_p denotes the 3×3 matrix formed out of the first nine components of z. The general initial-boundary value problem thus consists of the equations

$$\begin{aligned} \rho u_{tt} &= \operatorname{div}_x T \\ T &= D\left(\varepsilon(\nabla_x u) - Bz\right) \\ z_t &\in f\left(\varepsilon(\nabla_x u), z\right) \end{aligned} \tag{GP}$$

with a free energy ψ in the form

$$\rho\psi(\varepsilon, z) = \frac{1}{2}[D(\varepsilon - Bz)] \cdot (\varepsilon - Bz) + \psi_1(z)$$

satisfying the dissipation inequality

$$\rho \nabla_z \psi(\varepsilon, z) \cdot \xi \leq 0 \quad \text{for every } \xi \in f(\varepsilon, z)$$

and with the boundary conditions (BC) and the initial conditions (IC).

2. Energy estimate and constitutive equations of monotone type. Let (u, z) be a sufficiently smooth solution of (GP). Integration and application of the Divergence Theorem yields the energy equality

$$\int_\Omega \rho \left(\frac{1}{2} |u_t|^2 + \psi \right)_t dx = \int_{\partial\Omega} (Tn) \cdot u_t dS(x) + \int_\Omega \rho \nabla_z \psi \cdot f dx.$$

From the boundary conditions (BC) and from the dissipation inequality we obtain

$$\frac{d}{dt} \int_\Omega \rho \left(\frac{1}{2} |u_t|^2 + \psi \right) dx = \int_\Omega \rho \nabla_z \psi \cdot f dx \leq 0, \tag{EI}$$

which means that the sum of kinetic and free energy is decreasing.

The dissipation inequality is satisfied naturally, if to the function $f : D(f) \subseteq \mathcal{S}^3 \times \mathbb{R}^N \to \mathbb{R}^N$ or to the function $f : D(f) \subseteq \mathcal{S}^3 \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^N)$ there exists a free energy $\psi : D(f) \to [0, \infty)$ satisfying (FC) and a monotone function

$$g : D(g) \subseteq \mathbb{R}^N \to \mathbb{R}^N \quad (g : D(g) \subseteq \mathbb{R}^N \to \mathcal{P}(\mathbb{R}^N))$$

with

$$g(0) = 0 \qquad (0 \in g(0))$$

such that

$$f(\varepsilon, z) = g(-\rho \nabla_z \psi(\varepsilon, z))$$

holds for all $(\varepsilon, z) \in D(f)$. To see that, we obtain from the monotonicity of g

$$\rho \nabla_z \psi(\varepsilon, z) \cdot f(\varepsilon, z) = -(-\rho \nabla_z \psi(\varepsilon, z) - 0) \cdot (g(-\rho \nabla_z \psi(\varepsilon, z)) - g(0)) \leq 0,$$

which is the dissipation inequality.

If an energy inequality (EI) also holds for the difference of two solutions of the problem (GP) then we can prove a uniqueness result of the problem. But to derive an energy estimate for the difference of two solutions we need in addition that the free energy ψ is a positive semi-definite quadratic form with $\psi(0) = 0$. Thus necessarily ψ must be of the form

$$\rho \psi(\varepsilon, z) = \frac{1}{2} [D(\varepsilon - Bz)] \cdot (\varepsilon - Bz) + \frac{1}{2} (Lz) \cdot z$$

with a symmetric, positive semi-definite $N \times N$ matrix L. Derivation of the above expression of ψ yields

$$-\rho \nabla_z \psi(\varepsilon, z) = B^T D(\varepsilon - Bz) - Lz = \overline{L}\varepsilon - Mz$$

with linear mappings $\overline{L} = B^T D$ and $M = B^T DB + L$. Now we are ready to define constitutive equations of monotone type.

DEFINITION 2. a) We say that a pair (f, B) of a function

$$f : D(f) \subseteq \mathcal{S}^3 \times \mathbb{R}^N \to \mathbb{R}^N \quad (f : D(f) \subseteq \mathcal{S}^3 \times \mathbb{R}^N \to \mathcal{P}(\mathbb{R}^N))$$

and a linear mapping $B : \mathbb{R}^N \to \mathcal{S}^3$ is of *pre-monotone type*, if there exists a symmetric, *positive definite* $N \times N$ matrix M with the property that the symmetric $N \times N$ matrix

$$L = M - B^T DB$$

is *positive semi-definite*, and if there exists a function

$$g : D(g) \subseteq \mathbb{R}^N \to \mathbb{R}^N \quad (g : D(g) \subseteq \mathbb{R}^N \to \mathcal{P}(\mathbb{R}^N))$$

such that

$$f(\varepsilon, z) = g(-\rho \nabla_z \psi(\varepsilon, z)) = g(B^T D\varepsilon - Mz)$$

for all $(\varepsilon, z) \in D(f)$, with the positive semi-definite quadratic form

$$\rho\psi(\varepsilon, z) = \frac{1}{2}[D(\varepsilon - Bz)] \cdot (\varepsilon - Bz) + \frac{1}{2}(Lz) \cdot z.$$

b) If the function g is the gradient (or subdifferential)

$$g = \nabla \chi \quad (g = \partial \chi)$$

of a function $\chi : D(\chi) \subseteq \mathbb{R}^N \to (-\infty, \infty]$, then we say that the pair (f, B) is of gradient type.

c) If the function g is monotone and satisfies

$$g(z) \cdot z \geq 0 \quad (w \cdot z \geq 0)$$

for all $z \in D(g)$ (and all $\omega \in g(z)$), then we say that the pair (f, B) is of *monotone type*. By $\mathcal{M}^*, G, \mathcal{M}$ we denote the sets of pairs (f, B), which are of pre-monotone, of gradient, or of monotone type, respectively.

For constitutive equations of monotone type the energy estimate (EI) can be generalized to prove uniqueness of solutions. The linearity of $\nabla\psi$ allows to derive an energy inequality for the difference of two smooth solutions (u, z) and $(\overline{u}, \overline{z})$ of the problem (GP):

$$\frac{d}{dt} \int_\Omega \rho\Big[\frac{1}{2}|u_t - \overline{u}_t|^2 + \psi(\varepsilon(\nabla_x(u - \overline{u})), z - \overline{z})\Big] dx$$
$$= \int_\Omega [-\rho \nabla_z \psi(\varepsilon, z) + \rho \nabla_z \psi(\overline{\varepsilon}, \overline{z})] \cdot [g(-\rho \nabla_z \psi(\varepsilon, z)) - g(-\rho \nabla_z \psi(\overline{\varepsilon}, \overline{z}))]\, dx \leq 0.$$

Since by assumption ψ is positive semi-definite, we obtain

$$\int_\Omega |u_t - \overline{u}_t|^2 dx = 0,$$

hence $u_t = \overline{u}_t$ and therefore $u = \overline{u}$. Moreover we conclude that

$$0 = \int_\Omega \rho\psi(0, z-\overline{z})dx = \int_\Omega \frac{1}{2}[DB(z-\overline{z})]\cdot B(z-\overline{z}) + \frac{1}{2}[L(z-\overline{z})]\cdot(z-\overline{z})dx$$
$$= \int_\Omega \frac{1}{2}[M(z-\overline{z})]\cdot(z-\overline{z})dx$$

which implies that $z = \overline{z}$, since by assumption M is *positive definite*.

3. Existence of solutions for constitutive equations of monotone type. The existence theorem for (f, B) of monotone type can be proved using the theory of evolution equations to monotone operators. We can do this under the stronger hypotheses that the free energy is not only positive semi-definite, but *positive definite*, and that g is not only monotone, but *maximal monotone*.

To prove this we write the problem (GP) in the form

$$w_t \in Cw \quad w(0) \in D(C)$$

with

$$w = (u_t, \varepsilon, z) : \Omega \times [0,\infty) \to W = \mathbb{R}^3 \times \mathcal{S}^3 \times \mathbb{R}^N$$

and with an operator

$$C : D(C) \subseteq \mathbb{L}^2(\Omega; W) \to \mathcal{P}(\mathbb{L}^2(\Omega; W))$$

$$C(v, e, z) = \Big\{ w = (w_1, w_2, w_3) \in \mathbb{L}^2(W) \mid w_1 = \operatorname{div} \frac{1}{\rho} D(e - Bz),$$
$$w_2 = \varepsilon(\nabla v),\ w_3(x) \in g\left(-\rho\nabla_z\psi(e(x), z(z))\right) \text{ a.e. in } \Omega \Big\}.$$

By definition we have

$$D(C) = \{(v, e, z) \mid C(v, e, z) \neq \emptyset\}.$$

Additionally for the Dirichlet problem we assume that

$$v \in \mathbb{H}_1^\circ(\Omega, \mathbb{R}^3)$$

and for the Neumann problem we assume that the weak divergence $\operatorname{div} \frac{1}{\rho} D(e - Bz)$ satisfies

$$\left(\operatorname{div} \frac{1}{\rho} D(e - Bz), u\right) = -\left(\frac{1}{\rho} D(e - Bz), \nabla u\right) \quad \forall u \in \mathbb{H}_1(\Omega; \mathbb{R}^3).$$

THEOREM 1. *Let the quadratic form*

$$\rho\psi(\varepsilon, z) = \frac{1}{2}[D(e - Bz)](e - Bz) + \frac{1}{2}(Lz)\cdot z$$

be positive definite on $\mathcal{S}^3 \times \mathbb{R}^3$ *and let* $g : \mathbb{R}^N - \mathcal{P}(\mathbb{R}^N)$ *be maximal monotone with* $0 \in g(0)$. *Then the operator*

$$-C : \mathbb{L}^2(\Omega, W) \to \mathcal{P}(\mathbb{L}^2(\Omega, W))$$

is maximal monotone if the space $\mathbb{L}^2(\Omega, W)$ *is equipped with the scalar product*

$$\langle (v, e, z), (\overline{v}, \overline{e}, \overline{z}) \rangle = \int_\Omega \{\rho v \cdot \overline{v} + [D(e - Bz)] \cdot (\overline{e} - B\overline{z}) + (Lz) \cdot \overline{z}\} dx.$$

THEOREM 2. *Let the hypotheses of the preceding theorem be satisfied. Then to every* $w^{(0)} = \left(v^{(0)}, e^{(0)}, z^{(0)}\right) \in D(C)$ *there exists a unique function* $w : [0, \infty) \to \mathbb{L}^2(\Omega; W)$ *such that*

(i) $w(t) \in D(C)$ *for all* $t \geq 0$.
(ii) w *is Lipschitz-continuous on* $[0, \infty)$.
(iii) $w_t(t) \in C(w(t))$ *a.e. in* $[0, \infty)$.
(iv) $w(0) = w^{(0)}$.

Moreover

(v) w *has everywhere a derivative from the right.*
(vi) *If* w *and* $\overline{w}$ *are two solutions satisfying* (*i*) – (*iii*), *then*

$$\|w(t) - \overline{w}(t)\| \leq \|w(0) - \overline{w}(0)\|$$

for all $t \geq 0$, *where* $\|\cdot\|$ *denotes the norm of* $\mathbb{L}^2(\Omega, W)$.

4. Transformation of interior variables. We will enlarge the scope of the existence theorem to a class of constitutive equations, which are not of monotone type, by transformation of interior variables. We study the transformations of interior variables in the system

$$\begin{aligned} \rho u_{tt} &= \operatorname{div}_x T \\ T &= D(\varepsilon(\nabla_x u) - Bz) \\ z_t &= f(\varepsilon(\nabla_x u), z). \end{aligned} \tag{US}$$

Let $H : \mathbb{R}^N \to \mathbb{R}^N$ be a continuously differentiable vector field and let

$$h(x, t) = H(z(x, t))$$

be transformed interior variables.

LEMMA. *If* (u, z) *is a solution of* (*US*) *then the function* (u, h) *satisfies the transformed system*

$$\begin{aligned} \rho u_{tt} &= \operatorname{div}_x T \\ T &= D\left(\varepsilon(\nabla_x u) - BH^{-1}(h)\right) \\ h_t &= f_H(\varepsilon(\nabla_x u), h) \end{aligned} \tag{TS}$$

with the transformed function $f_H : D(f_H) \subseteq \mathcal{S}^3 \times \mathbb{R}^N \to \mathbb{R}^N$ *defined by*

$$f_H(\varepsilon, h) = H'(H^{-1}(h)) f(\varepsilon, H^{-1}(h)).$$

DEFINITION 3. Let $\mathcal{TM}^*, \mathcal{TG}, \mathcal{TM}$ respectively, denote the set of pairs (f, B) of functions $f : D(f) \subseteq \mathcal{S}^3 \times \mathbb{R}^N \to \mathbb{R}^N$ and linear mappings $B : \mathbb{R}^N \to \mathcal{S}^3$ with the property that the constitutive equations

$$T = D(\varepsilon - Bz)$$
$$z_t = f(\varepsilon, z)$$

can be transformed to pre-monotone type, gradient type or to monotone type, respectively.

For a class of constitutive equations it is possible to choose a transformation field such that (TS) is of monotone type. We illustrate this by an example.

5. Model of Bodner–Partom. We discuss applicability of the theory of transformation of interior variables for the constitutive equations of Bodner-Partom. In the model of Bodner-Partom the vector z of internal variables consists of the nine components of the plastic strain tensor ε_p, and of one positive variable y, which describes isotropic hardening. Thus $N = 10$, and

$$z(x,t) = (\varepsilon_p(x,t), y(x,t)) \in \mathbb{R}^{10}$$

with $y(x,t) > 0$. The constitutive equations are of the form

$$\frac{\partial}{\partial t}\varepsilon_p = F\left(\frac{|\sigma|}{y}\right)\frac{\sigma}{|\sigma|}$$
$$\frac{\partial}{\partial t} y = \gamma(y) F\left(\frac{|\sigma|}{y}\right)|\sigma| - \delta(y)$$

where $F : \mathbb{R}_+ \to \mathbb{R}_+$, $\gamma : D(\gamma) \subseteq \mathbb{R}_+ \to \mathbb{R}_+$, $\delta : D(\delta) \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ are given functions and $\sigma = T - \frac{1}{3}(\operatorname{tr} T)\cdot I$. The constitutive equations of Bodner-Partom are not of monotone type and the applicability of the transformation theory depends strongly on the form of the function F. Only for $F(s) = ds^n$ (a homogeneous polynomial) we can apply this theory. But Bodner and Partom have proposed functions F, γ, δ in the form

$$F(s) = d\exp\left(-\alpha\left(\frac{1}{s}\right)^n\right) \qquad \alpha = \frac{n/2+1}{n} \quad .$$
$$\gamma(y) = n(y_1 - y) \qquad \delta(y) = Ay_1\left(\frac{y - y_2}{y_1}\right)^r$$

where $n, r > 1$, $d, m > 0$, $A \geq 0$, $y_1 > y_2 > 0$ are constants depending on the material considered. For this reason we must study the problem with other methods. The method, often applicable to nonlinear partial differential equations is the energy method. The function F proposed by Bodner and

Partom is bounded and therefore we can try and prove an existence result only with the energy estimates of the first order (for the first derivatives of the functions $v = u_t, T$ and y.)

5.1. Existence theorem for the model of Bodner-Partom. We can prove an existence result in the case $A = 0$. Thus the Bodner-Partom system of equations has the form

$$\begin{aligned} \rho u_{tt} &= \operatorname{div}_x T \\ T &= D(\varepsilon(\nabla_x u) - \varepsilon_p) \\ \sigma &= T - \frac{1}{3}(\operatorname{tr} T)\cdot I \\ \frac{\partial}{\partial t}\varepsilon_p &= F\left(\frac{|\sigma|}{y}\right)\frac{\sigma}{|\sigma|} \\ \frac{\partial}{\partial t} y &= m(y_1 - y) F\left(\frac{|\sigma|}{y}\right)|\sigma|. \end{aligned} \tag{BP}$$

We assume that the homogeneous Dirichlet boundary condition and initial conditions (IC) hold. For the initial data $y^{(0)}$ we assume that

$$\exists_{y^*>0}\,\forall_{x\in\Omega}\quad y^* \le y^{(0)}(x) \le y_1. \tag{A1}$$

By (A1) the hardening function y is a positive $\mathbb{L}^\infty$-function with

$$y^* \le y(x,t) \le y_1 \quad \text{for all } (x,t)\in \Omega\times[0,\infty).$$

Moreover, we assume that the function F satisfies:

1. $F(s)$ and $F(s)/s$ are $\mathbb{C}^\infty(\mathbb{R}_+,\mathbb{R}_+)$ functions, (A2)
2. F is bounded, (A3)
3. $F' \ge 0$, (A4)
4. the function $F'(s)s^2$ is bounded. (A5)

An easy computation shows that the function F proposed by Bodner and Partom satisfies all assumptions (A2) – (A5). As free energy we choose only the potential energy

$$\psi(\varepsilon, z) = [D(\varepsilon - \varepsilon_p)]\cdot(\varepsilon - \varepsilon_p).$$

Therefore the constitutive equations of Bodner-Partom with the free energy are thermodynamically admissible if and only if F is a nonnegative function.

THEOREM 3. *Assume that the initial data satisfy*:

$$u^{(0)} \in \mathbb{H}_2(\Omega;\mathbb{R}^3),\quad u^{(1)} \in \mathbb{H}_1(\Omega;\mathbb{R}^3)$$
$$z^{(0)} = (\varepsilon_p^{(0)}, y^{(0)}) \in \mathbb{H}_1(\Omega; \mathcal{S}^3\times\mathbb{R}_+),$$

with $\operatorname{tr}\varepsilon_p^{(0)}(x) = 0$ *for almost all* $x\in\Omega$. *Moreover suppose that the assumptions (A1) – (A5) and compatibility conditions*

$$u^{(0)}, u^{(1)}|_{\partial\Omega} = 0$$

hold. Then for all $t > 0$ there exists a solution

$$(u, \varepsilon_p, y) : \Omega \times [0, t] \to \mathbb{R}^3 \times \mathcal{S}^3 \times \mathbb{R}_+$$

of the problem (BP) with

$$u \in \mathbb{H}_2(\Omega \times [0, t]; \mathbb{R}^3), \varepsilon_p \in \mathbb{H}_1(\Omega \times [0, t); \mathcal{S}^3), y \in \mathbb{H}_1(\Omega \times [0, t]; \mathbb{R}_+).$$

Idea of the proof: First we regularize the singular behavior of the right hand side of the equations (BP4) and (BP5) at the point $\sigma = 0$. We define for all $k \in \mathbb{N}$ a new constitutive function

$$F_k(s) = \chi(ks)F(s)$$

with a cut-off function $\chi \in \mathbb{C}^\infty(\mathbb{R})$, $\chi = 0$ for $s < \frac{1}{2}$, $\chi = 1$ for $s > 1$, $0 \le \chi \le 1$, $\chi' \ge 0$ and $s\chi'(s) \le \chi_0$. The functions F_k satisfy the assumptions (A2) – (A5). Now we construct a Faedo-Galerkin approximate sequence $(u^k, \varepsilon_p^k, y^k)$ where

$$u^k(x, t) = \sum_{j=0}^{k} \alpha_{kj}(t) w_j(x)$$

and w_j are eigenfunctions of the operator

$$-\operatorname{div} D\varepsilon(\nabla u(x))$$

with the Dirichlet boundary condition $w|_{\partial\Omega} = 0$. We can assume that the set $\{w_j\}_{j=0}^{\infty}$ is an orthonormal basis in the space $\mathbb{L}^2(\Omega; \mathbb{R}^3)$. The coefficients $\alpha_{kj}(t)$ and other components of the approximate sequence satisfy the following system of ordinary differential equations:

$$\begin{aligned}
\rho \sum_{l=1}^{k} (w_\ell, w_j) \frac{d^2}{dt^2} \alpha_{kj}(t) = & - \sum_{l=1}^{k} (D\varepsilon(\nabla w_i), \nabla w_j) \alpha_{k\ell}(t) \\
& + (D\varepsilon_p^k(t), \nabla w_j) \quad j = 1, \dots, k \\
\frac{d}{dt} \varepsilon_p^k(t) = & \, F_k\left(\frac{|\sigma^k|}{y^k}\right) \frac{\sigma^k}{|\sigma^k|} \\
\frac{d}{dt} y^k(t) = & \, m(y_1 - y^k)|\sigma^k| F_k\left(\frac{|\sigma^k|}{y^k}\right)
\end{aligned}$$

where $\sigma^k = T^k - \frac{1}{3}(\operatorname{tr} T^k) I$ and

$$T^k(t) = \sum_{l=1}^{k} D\varepsilon(\nabla w_\ell) \alpha_{k\ell}(t) - D\varepsilon_p^k(t).$$

The theory of ordinary differential equations implies that there exists a unique solution $(\alpha_{kj}, \varepsilon_p^k, y^k)$ of the above system. The next step is the energy estimation. Let us define

$$\begin{aligned}\mathcal{E}_k(t) = &\frac{1}{2}(\rho u_t^k, u_t^k) + \frac{1}{2}\left(T^k, D(\varepsilon(\nabla u^k) - \varepsilon_p^k)\right) \\ &+ \frac{1}{2}(\rho u_{tt}^k, u_{tt}^k) + \frac{1}{2}\left(T_t^k, D(\varepsilon(\nabla u_t^k) - \partial_t \varepsilon_p^k)\right) \\ &+ \sum_{i=1}^{3} \frac{1}{2}(\rho u_{tx_i}^k, u_{tx_i}^k) + \frac{1}{2}\left(T_{x_i}^k, D(\varepsilon(\nabla u_x^k) - \partial_{x_i} \varepsilon_p^k)\right).\end{aligned}$$

The main result in the proof is the following inequality

$$\forall_{t^* \geq 0}\, \exists_{C_{t^*} > 0}\, \forall_{t \leq t^*}\, \forall_{k \in \mathbb{N}} \quad \mathcal{E}_k(t) \leq C_{t^*}. \tag{EE}$$

By (EE) the sequence of approximate solutions is bounded in the space $(u^k, \varepsilon_p^k, y^k) \in \mathbb{H}_2(\Omega \times [0,t]; \mathbb{R}^3) \times \mathbb{H}_1(\Omega \times [0,t]; \mathcal{S}^3 \times \mathbb{R}_+)$. Thus the sequence $\{(u^k, \varepsilon_p^k, y^k)\}_{k=1}^{\infty}$ contains a subsequence $\{(u^{k_j}, \varepsilon_p^{k_j}, y^{k_j})\}_{j=1}^{\infty}$ which converges to (u, ε_p, y) in the space $\mathbb{H}_1(\Omega \times [0,t]; \mathbb{R}^3) \times L_2(\Omega \times [0,t]; \mathcal{S}^3 \times \mathbb{R}_+)$ and weakly in the space $\mathbb{H}_2(\Omega \times [0,t]; \mathbb{R}^3) \times \mathbb{H}_1(\Omega \times [0,t]; \mathcal{S}^3 \times \mathbb{R}_+)$. Moreover, we can assume that the subsequence $(u^{k_j}, \varepsilon_p^{k_j}, y^{k_j})$ converges almost everywhere pointwise to (u, ε_p, y). By the boundedness of the function F we can show that (u, ε_p, y) is a solution of the problem (BP).

5.2. Uniqueness problem for the model of Bodner-Partom

THEOREM 4. *Assume that for initial data $u^{(0)}, u^{(1)}, \varepsilon_p^{(0)}$ satisfying the assumptions of Theorem 3 the following inequality*

$$\forall_{t^* \geq 0}\, \exists_{C^*(t^*)} \quad \sup_{t \leq t^*} \|\sigma(t)\|_{L^\infty(\Omega; \mathcal{S}^3)} \leq C^*(t^*)$$

holds. Then for these initial data the problem (BP) has a unique solution.

Note that for the one-dimensional (BP) problem the energy inequality (EE) yields an $\mathbb{L}^\infty$–estimation for the stress deviator σ. Thus by Theorem 4 the one-dimensional (BP) problem has a unique solution. For dimensions greater than one the uniqueness problem is more difficult.

For the quasi-static Bodner-Partom problem

$$\begin{aligned}&\operatorname{div}_x T = f \\ &T = D(\varepsilon(\nabla_x u) - \varepsilon_p) \\ &\sigma = T - \frac{1}{3}(\operatorname{tr} T) I \\ &\frac{\partial}{\partial t}\varepsilon_p = F\left(\frac{|\sigma|}{y}\right)\frac{\sigma}{|\sigma|} \\ &\frac{\partial}{\partial t} y = m(y_1 - y) F\left(\frac{|\sigma|}{y}\right)|\sigma|,\end{aligned} \tag{QSBP}$$

where f is a given function $f : \Omega \times [0, \infty) \to \mathbb{R}^3$ the uniqueness problem is independent of the dimension. First, similarly as for (BP), we can prove for (QSBP) Theorem 3 and Theorem 4. Moreover, the following Lemma holds.

LEMMA. *Assume that the initial data* $(\varepsilon_p^{(0)}, y^{(0)})$ *satisfy*

$$\varepsilon_p^{(0)} \in W^{1,p}(\Omega; \mathcal{S}^3), \quad y^{(0)} \in W^{1,p}(\Omega; \mathbb{R}_+) \quad (p > n)$$

with $\operatorname{tr} \varepsilon_p^{(0)} = 0$ *for all* $x \in \Omega$. *Moreover, suppose that the assumptions (A1) - (A5) hold. Then a solution of the problem (QSBP) satisfies the following estimation*

$$\forall_{t^* \geq 0}\ \exists_{C^*(t^*)} \quad \sup_{t \leq t^*} \|T(t)\|_{W^{1,p}(\Omega; \mathbb{R}^3)} \leq C^*(t^*).$$

Thus by the Sobolev theorem and by Theorem 4 the (QSBP) problem has a unique solution.

For the dynamical (BP) system we can prove an $\mathbb{L}^\infty$–estimation for the stress deviator only in the two-dimensional case ($\Omega \subseteq \mathbb{R}^2$). Moreover we must assume that additionally the function F satisfies

(i) $\frac{F(s)}{s^2} \in \mathbb{C}^\infty(\mathbb{R}_+, \mathbb{R}_+)$, (A6)
(ii) $F''(s)s^3$ is bounded. (A7)

LEMMA. *Assume that the initial data satisfy*

$$u^{(0)} \in \mathbb{H}_3(\Omega; \mathbb{R}^3), \quad u^{(1)} \in \mathbb{H}_2(\Omega; \mathbb{R}^3)$$
$$\varepsilon_p^{(0)} \in \mathbb{H}_2(\Omega; \mathcal{S}^3), \quad y^{(0)} \in \mathbb{H}_2(\Omega; \mathbb{R}_+).$$

Moreover suppose that (A1) - (A7) hold. Then a solution of the problem (BP) satisfies the following estimation

$$\forall_{t^* \geq 0}\ \exists_{C^*(t^*)}\ \forall_{t \leq t^*} \quad \sup_{t \leq t^*} \|T(t)\|_{\mathbb{H}^p(\Omega; \mathcal{S}^3)} \leq C^*(t^*).$$

Thus again by the Sobolev theorem and by Theorem 4 the problem (BP) has a unique solution in the two-dimensional case.

The uniqueness problem for the (BP) system in the three-dimensional case is an open problem.

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