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On robust tests in some parametric models

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Summary. For the exponential-scale and Pareto-shape testing problems the unbiased tests based on order statistics are considered. Under the violations defined by hazards and of the contamination type the tests with the most robust (stable) power functions are explicitly constructed.

1. Introduction. Robustness of the test concerns the insensitivity of its characteristic (size, power, risk) to small deviations of the actual situation from the idealized theoretical model. In this paper we are interested in the approach to the robustness given by Zieliński ([11], [12]). This means, the purpose is to investigate the performance of the test characteristic when passing from the theoretical model to the supermodel, i.e. to the extended model described by the violation. Then the robustness of the test is identified with the stability of its characteristic under a given violation. Size and power robustness problems have been studied among others in [4], [7], [15] and [8], [9], [13], [14] respectively. In the paper the robustness of the test concerns the behaviour of the power function identified with its graph on the plane, and is measured by the area of the region in which the power function can oscillate under a given violation. We confine ourselves to the exponential and Pareto testing models. The above models are widely used in practical problems (see e.g. [5], [6], [10]).

2. Notation. Let $\underline{X}$ be a random sample of size n, $n \geq 2$, from the distribution function (d.f.) F with density f. Denote by $X_{1:n} \leq \ldots \leq X_{n:n}$ the order statistics of $\underline{X}$. Define $X_{0:n} = \sup\{x : F(x) = 0\}$, if it is finite. The corresponding random variables $D_{i:n} = (n - i + 1)(X_{i:n} - X_{i-1:n})$, $i =$

$1, \ldots, n$, are called normalized spacings and the d.f. of $\sum_{i=1}^{n} a_i D_{i:n}$ is denoted by $\widetilde{F}_{\underline{a}}$. The inverse and failure rate functions are defined by $F^{-1}(x) = \inf\{y : F(y) \geq x\}$ and $r_F(x) = f(x)/(1 - F(x))$ respectively. The notation F_λ, F^γ means $F_\lambda(x) = F(\lambda x)$, $F^\gamma(x) = F(x^\gamma)$, $x \geq 0$. Moreover, μ denotes the Lebesque measure on R^2. If r_F is nonincreasing then we write $F \in DFR$.

3. Exponential model. Let $\underline{X}$ be a sample of size n, $n \geq 2$, from the exponential d.f. $K_\lambda(x) = 1 - \exp(-\lambda x)$, $x \geq 0$, with an unknown scale parameter $\lambda > 0$. We consider the problem of testing the null hypethesis H_0: $\lambda \geq \lambda_0$ $(\lambda_0 > 0)$ against the alternative hypothesis H_1: $\lambda < \lambda_0$ by using the unbiased tests based on normalized spacings. Given $\alpha \in (0, 1)$, each considered test $\varphi_{\underline{a}}(\underline{X})$, $\underline{a} = (a_1, \ldots, a_n) \geq \underline{0}$ has the critical region of the form $\{\sum_{i=1}^{n} a_i D_{i:n} > c_{\underline{a},\alpha}\}$, where $\sup\limits_{\lambda \in H_0} E_{K_\lambda} \varphi_{\underline{a}}(\underline{X}) = \alpha$, and $E_F \varphi_{\underline{a}}(\underline{X})$ denotes the expectation of $\varphi_{\underline{a}}(\underline{X})$ if the sample comes from F.

Suppose that due to measurement errors the observations are slightly disturbed and come from an unknown d.f. F, where F belongs to $\pi(K_\lambda)$ – "some neighbourhood" of K_λ. Then

$$(1) \quad R_{\underline{a}} = \mu\{(\lambda, y) : \lambda > 0, y \in (\inf_{F \in \pi(K_\lambda)} E_F \varphi_{\underline{a}}(\underline{X}), \sup_{F \in \pi(K_\lambda)} E_F \varphi_{\underline{a}}(\underline{X}))\},$$

if it is finite, describes the dispersion of the power function (risk) under the violation π and gives us a measure of robustness of the test $\varphi_{\underline{a}}$.

In the set of life d.f.'s F (i.e. $F(0) = 0$), consider the violation of the hazard type, i.e.

$\pi_{G,H}(K_\lambda) = \{F : r_{G_\lambda} \leq r_F \leq r_{H_\lambda}\}$, where $G \neq H$, $r_G \leq r_K \leq r_H$ (see [2]),

and of Tukey type, i.e.

$$\pi_d^\varepsilon(K_\lambda) = \{(1 - \varepsilon)K_\lambda + \varepsilon K_{c\lambda} : d \leq c \leq 1\}, \quad \text{where} \quad 0 < \varepsilon, d < 1, \lambda > 0.$$

Given $\pi \in \{\pi_{G,H}, \pi_d^\varepsilon\}$, we find the test $\varphi_{\underline{b}}$ such that $R_{\underline{b}} = \min\{R_{\underline{a}} : \underline{a} \geq \underline{0}\}$.

We prove first some auxiliary lemmas. Let F and G be the life d.f.'s.

LEMMA 1. *If F or $G \in DFR$ and $r_G \leq r_F$ then $r_G G^{-1} \leq r_F F^{-1}$.*

The proof follows directly from the definition of DFR class.

LEMMA 2. *If $r_G G^{-1} \leq r_F F^{-1}$ and $\underline{a} \geq \underline{0}$ then $\widetilde{G}_{\underline{a}} \leq \widetilde{F}_{\underline{a}}$.*

Proof. By assumption, $G^{-1}F(x) - x$ is nondecreasing. Let $\underline{X}$ be a sample from F and $\underline{a} \geq \underline{0}$. Obviously,

$$\sum_{i=1}^{n} a_i(n - i + 1)(X_{i:n} - X_{i-1:n}) \leq$$

$$\leq \sum_{i=1}^{n} a_i(n-i+1)(G^{-1}F(X_{i:n}) - G^{-1}F(X_{i-1:n})).$$

From the fact that $G^{-1}F(X_i)$, $i = 1, \ldots, n$, are i.i.d. random variables from G, the proof is complete.

LEMMA 3. *Let* $F = (1-\varepsilon)K + \varepsilon K_c$, $G = (1-\varepsilon)K + \varepsilon K_d$. *If* $0 < \varepsilon < 1$, $d \leq c \leq 1$ *then* $r_G G^{-1} \leq r_F F^{-1}$.

Proof. By assumption, $dG^{-1} \leq cF^{-1}$. Consequently for $0 < x < 1$

$$r_F F^{-1}(x) - r_G G^{-1}(x) = (1-x)^{-1}\varepsilon((1-d)(1-K(dG^{-1}(x))) - (1-c)(1-K(cF^{-1}(x)))) \geq 0,$$

which completes the proof.

Let $\pi \in \{\pi_{G,H}, \pi_d^\varepsilon\}$, $E_G X$ be finite and $G, H \in DFR$. We introduce the following notation. Define $p_i = E_G D_{i:n} - E_H D_{i:n}$, $i = 1, \ldots, n$, where in the case $\pi = \pi_d^\varepsilon$ we put $H = K$, $G = (1-\varepsilon)K + \varepsilon K_d$. Note that under the definition of π, $p_i > 0$ for each i. Let S_n be the permutation group on $\{1, \ldots, n\}$. Given $\underline{a} = (a_1, \ldots, a_n)$, $\tau \in S_n$, let $\underline{a}(\tau) = (a_{\tau(1)}, \ldots, a_{\tau(n)})$, $a_{1:n} \leq \ldots \leq a_{n:n}$ be n ordered coordinates of $\underline{a}$ and $N(i)$ be the rank of p_i in $(p_{n:n}, \ldots, p_{1:n})$. If $p_i = p_j$ then $N(i) < N(j)$ for $i < j$. Let $\underline{a}^0 = (a_{N(1):n}, \ldots, a_{N(n):n})$ and $\underline{e} = (1, 0, \ldots, 0)$.

THEOREM 1. *If* $0 < \alpha < 1$, $\underline{a} \geq \underline{0}$, $\tau \in S_n$ *then* $R_{\underline{a}^0} \leq R_{\underline{a}(\tau)}$.

Proof. Let $\alpha \in (0,1)$ be a given significance level. By the definition of the test $\varphi_{\underline{a}}$ we obtain $c_{\underline{a},\alpha} = [\widetilde{K}_{\underline{a}}]^{-1}(1-\alpha)/\lambda_0$, and hence $E_F \varphi_{\underline{a}}(\underline{X}) = 1 - \widetilde{F}_{\underline{a}}([\widetilde{K}_{\underline{a}}]^{-1}(1-\alpha)/\lambda_0)$.

By using Lemmas 1, 2, if $\pi = \pi_{G,H}$ or Lemmas 2, 3, if $\pi = \pi_d^\varepsilon$, (1) gives

$$R_{\underline{a}} = \lambda_0 \sum_{i=1}^{n} a_i p_i / [\widetilde{K}_{\underline{a}}]^{-1}(1-\alpha). \tag{2}$$

Given $\tau \in S_n$, we have

$$\sum_{i=1}^{n} a_{N(i):n} p_i \leq \sum_{i=1}^{n} a_{\tau(i)} p_i, \qquad \widetilde{K}_{\underline{a}} = \widetilde{K}_{\underline{a}(\tau)},$$

which completes the proof.

THEOREM 2. *Let* $0 < \alpha < 1$ *and* $\underline{a} \geq \underline{0}$ *be such that* $a_{n:n}/a_{k:n} < (p_{n-k+1:n}/p_{1:n})^{1/2}$ *for some* k. *Then for the vector* $\underline{b}$ *with coordinates*

$$b_i = a_{i:n} \quad \text{for} \quad i \neq k, n, \qquad b_i = q a_{n-i+k:n}/p_{n-i+1:n} \quad \text{for} \quad i = k, n, \tag{3}$$

where $q = (a_{k:n} p_{n-k+1:n} + a_{n:n} p_{1:n})/(a_{k:n} + a_{n:n})$ *it holds* $R_{\underline{b}^0} < R_{\underline{a}^0}$.

Proof. Let $\alpha \in (0,1)$. Without loss of generality we assume that $a_i > 0$ for $i = 1,\ldots,n$. Let $\underline{b}$ be defined by (3). Then

$$a_{k:n}p_{n-k+1:n} + a_{n:n}p_{1:n} = b_k p_{n-k+1:n} + b_n p_{1:n},$$
$$1/b_k + 1/b_n = 1/a_{k:n} + 1/a_{n:n} \text{ and } b_k < a_{k:n}.$$

From the fact that the function $\widetilde{K}_{\underline{a}}(t)$, $t \ge 0$, is strictly Schur-concave in $(1/a_1,\ldots,1/a_n)$ (see e.g. [3]) we have $[\widetilde{K}_{\underline{a}}]^{-1}(1-\alpha) < [\widetilde{K}_{\underline{b}}]^{-1}(1-\alpha)$, and the result follows by (2) and Theorem 1.

Corollary. *Let $0 < \alpha < 1, p_i \ne p_j$ for some i,j. Then for the vector $\underline{b}$ such that $b_1 = (p_{1:n} + p_{n:n})/2p_{n:n}$, $b_n = (p_{1:n} + p_{n:n})/2p_{1:n}$, $b_i = 1$ for $i \ne 1,n$, it holds $R_{\underline{b}^0} < R_{(1,1,\ldots,1)}$.*

This means, the test $\varphi_{\underline{b}^0}$ is more robust than the uniformly most powerful test.

Theorem 3. *Let $t_2 = \exp(-3/2)$, $t_n = \exp(-(n+1))$ for $n > 2$. Then there exists $t \in (t_n, 1]$ such that for every $0 < \alpha \le t$ and all $\underline{a} \ge 0$ $R_{\underline{\varepsilon}^0} \le R_{\underline{a}}$.*

Proof. It is easily seen that we may restrict our attention to the vectors $\underline{a} \ge \underline{0}$ with $\sum_{i=1}^{n} a_i = 1$. Based on the results due to Bock et al. [3], $\widetilde{K}_{\underline{a}}(t)$, $t \ge (-\log t_n)\sum_{i=1}^{n} a_i$, is Schur-concave in $\underline{a} > \underline{0}$. Consequently, if $\underline{a} \ge \underline{0}$, $\sum_{i=1}^{n} a_i = 1$ then $K(t) \le \widetilde{K}_{\underline{a}}(t)$ for $t \ge -\log t_n$, and hence $[\widetilde{K}_{\underline{a}}]^{-1}(1-\alpha) \le K^{-1}(1-\alpha)$ for $\alpha \le t_n$. On the other hand $p_{1:n} \le \sum_{i=1}^{n} a_i p_i$.

By (2) the proof is complete.

If $G = K$ ($H = K$, as in π_d^ε) then by the stochastic monotonicity of the normalized spacings (see [1]) one can obtain that p_i is decreasing (increasing) in i. Consequently, the most robust is the test based on $D_{n:n}$ ($D_{1:n}$).

4. Pareto model. Let $\underline{X}$ be a sample of size n, $n \ge 2$, from the Pareto d.f. $P^\gamma(x) = 1 - x^{-\gamma}$, $x \ge 1$, with an unknown shape parameter $\gamma > 0$. To verify the hypothesis H_0': $\gamma \ge \gamma_0$ ($\gamma_0 > 0$) versus H_1': $\gamma < \gamma_0$, the unbiased α-level tests $\psi_{\underline{a}}(\underline{X})$, $\underline{a} = (a_1,\ldots,a_n) \ge \underline{0}$, with the critical regions $\{\sum_{i=1}^{n} a_i(n-i+1)\log(X_{i:n}/X_{i-1:n}) > d_{\underline{a},\alpha}\}$, where $\sup_{\gamma\in H_0'} E_{P^\gamma}\psi_{\underline{a}}(\underline{X}) = \alpha$, are considered.

In the set of d.f.'s F such that $F(1) = 0$, consider the violations

$$\widetilde{\pi}_{G,H}(P^\gamma) = \{F : r_{G^\gamma} \le r_F \le r_{H^\gamma}\}, \quad G \ne H,\ r_G \le r_P \le r_H,$$
$$\widetilde{\pi}_d^\varepsilon(P^\gamma) = \{(1-\varepsilon)P^\gamma + \varepsilon P^{c\gamma} : d \le c \le 1\}, \quad 0 < \varepsilon,\ d < 1,\ \gamma > 0.$$

Let $\pi \in \{\widetilde{\pi}_{G,H}, \widetilde{\pi}_d^\varepsilon\}$. Then the robustness of the test $\psi_{\underline{a}}$ under the violation π is measured by the quantity

$$\widetilde{R}_{\underline{a}} = \mu\{(\gamma, y) : \gamma > 0, y \in (\inf_{F\in\pi(P^\gamma)} E_F\psi_{\underline{a}}(\underline{X}), \sup_{F\in\pi(P^\gamma)} E_F\psi_{\underline{a}}(\underline{X}))\}.$$

Assume that $E_G \log X$ is finite, r_H/r_P, r_G/r_P are decreasing functions and write $\widetilde{p}_i = (n-i+1)(E_G \log(X_{i:n}/X_{i-1:n}) - E_H \log(X_{i:n}/X_{i-1:n}))$, $i = 1, \ldots, n$, where for the case $\pi = \widetilde{\pi}_d^\varepsilon$ we put $H = P$ and $G = (1-\varepsilon)P + \varepsilon P^d$. Since the Pareto d.f. is closely connected with the exponential d.f. through the relation $K(t) = P(\exp(t))$, $t \geq 0$, it follows that all results of the previous section remain valid if p_i and $R_{\underline{a}}$ are replaced by $\widetilde{p}_i$ and $\widetilde{R}_{\underline{a}}$ respectively.

Moreover, if $G = P$ ($H = P$, as in $\widetilde{\pi}_d^\varepsilon$) then the most robust is the test based on $\log(X_{n:n}/X_{n-1:n})$, $(n \log X_{1:n})$.

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