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DOCTORAL THESES

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Algorithmization as mathematical activity and skill*

Abstract: The article presents a concise description of the research presented in the doctoral dissertation. In my research I focus on improving the quality of teacher's preparation at the university level, and through it at school level in the future. I present the proposal of teaching mathematics with the use of algorithmization. The process of algorithmization of mathematical procedures is associated with various types of mathematical activities. This teaching proposition deals with the problem of algorithmic computation of purely mathematical problems as well as those which deal with the problems of applying mathematics in everyday life. I consider algorithmization as one of the forms of creating a mathematical model of a situation known from real world.

1 Introduction

According to Zofia Krygowska (1977, p. 3): "Development of mathematical activity of the student we think as one of the most important goal of mathematics teaching." Inspired by this statement, I decided to explore the issue of algorithmization in mathematical education, because this topic involves a whole series of mathematical activities and skills. It is good to remind what we understand by mathematical activity. Wanda Nowak (1989, p. 110) states that:

Key words: algorithmization, mathematical modelling, quality of teaching, higher education.

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"*Mathematical activity* of the pupil is a work of mind oriented to learning of concepts and on the reasoning of mathematical type, which is stimulated by the situations that lead to formulation and solving theoretical and practical problems." Therefore, my research concerns both pure mathematics issues and problems of application of mathematics to solve problems from everyday life.

When I explore the activity of teaching mathematics, I see many opportunities for algorithmization. It is possible, because mathematics has two faces: conceptual and algorithmic (Krygowska, 1977; Rams, 1982; Sysło, 2008). However, we cannot separate conceptual and algorithmic mathematics because these two aspects constantly permeate each other and they are equally important. In order to analyze conceptual elements, we need computational methods. On the other hand, algorithms treated separately from concepts are only automatic patterns for calculations. Taking into account both sides of mathematics, teachers should emphasize the role of algorithmic elements in mathematics and appreciate the importance of its conceptual elements at the same time. Students learning a new definition or a theorem must realize the functional and operational sense of this information, i.e. they need to know what they can do with it, what operations are possible and what calculations they can perform (Siwek, 2005). If this does not happen, the information will be useless. In addition to the knowledge of basic algorithms and ability to use them correctly, a broader understanding of the algorithm is necessary, which allows the student to produce an abstract object (concept) which is the result of numerical or algebraic calculation. A teacher who in his work consciously emphasizes both conceptual and algorithmic approach to mathematics allows students to learn this field of knowledge comprehensively. This encourages the development of students' mathematical thinking because theoretical teaching together with functional teaching enable intuitive and formal understanding of new problems.

M. Sysło (2008, p. 14) states that: "When reading school mathematics textbooks one will see few examples of integration of the conceptual and algorithmic approach." Furthermore, "Future teachers of mathematics do not meet with examples of integration of the conceptual approach with algorithmic approach in the modern understanding of an algorithm." I decided to explore this problem and look for its solution. I focused my research on the issues of algorithmization in the course of didactics of mathematics with regard to the skills of my students – future teachers of mathematics. In my approach to algorithmization, I do not deal with the problem of computability itself, which is a separate research area within mathematics.

2 Algorithmization in didactics of mathematics

There are many definitions of the algorithm, representing different levels of mathematical precision (Krygowska, 1977; Brookshear, 2003; Sysło, 2008). Their review led me to accept the following definition:

"*Algorithm* is an ordered set of unambiguous, executable steps that determine a finite process, which leads to the realization of a certain task." (Brookshear, 2003, p. 181).

According to this definition, every algorithm should have the following features: unambiguity, effectiveness, generality, elementariness of operations.

Unambiguity means that it should precisely define the sequence of operations leading to the result. Therefore, a student who has mastered basic operations is able to get to the solution of a complicated task by doing the step-by-step activities planned in the scheme. The *effectiveness* of an algorithm guarantees that the resulting outcome is the correct solution of the task after a finite number of steps, whereas the *generality* condition means that an algorithm should comprise the whole class of tasks by working on parameters, the specification of which defines a given task. Of course, an algorithm always works in the same way for the same initial data. The feature of unambiguity imposes this.

The *elementariness of operations* means that each operation appearing in the scheme is controlled by the student. This feature is relative and depends on the skills of the student at a given level of education because some operations that are not elementary at a certain stage may become elementary in the further course of study (Krygowska, 1977).

Regardless of how an algorithm is presented (a verbal description, a list of steps, a block diagram or a programming language), its features impose specific working methods, which has many advantages.

Mathematics teaching with the use of algorithmization can be done in many ways (Krygowska, 1977; Rams, 1982; Wójcicka, 2005), for example:

- teaching basic algorithms,
- execution of ready-made algorithms,
- comparing algorithms,
- algorithm analysis,
- algorithm supplementation.

One of them is also *creating algorithms by students*. This method requires from them variety mathematics activities, which are discussed in more detail in the next paragraph.

Introduction of algorithmization in teaching brings many educational advantages, because the characteristics of an algorithm force the specific activity of the student. The didactic benefits of teaching mathematics with the use of algorithmization are the following (Krygowska, 1977; Rams, 1982; Wójcicka, 2005):

- knowledge of basic algorithms streamlines calculations, allows automatism,
- algorithm analysis allows you to see the precision and simplicity of a logical sequence of operations — method of operation,
- self-created algorithm:
  - requires logical thinking,
  - forces a very clear and unambiguous record solution plan,
  - develops reflective thinking — allows self-control;
- graphical writing allows:
  - non-verbal way of showing relationships and procedures,
  - comprehensive presentation of the method;
- algorithmization enables systematization of knowledge and it is an IT-oriented teaching.

Those skills are reflected in the goals of mathematics education included in the curriculum on every educational level, especially with regard to mathematical modeling. In this way mathematics, which is taught at school, forces the need to look at algorithms through the problem of mathematical modeling. This approach is supported by the fact that in school textbooks for mathematics elements of algorithmization appear sporadically (although they have different functions). However, by applying the elements of algorithmization many objectives of mathematical education can be achieved at every level of education. A teacher who uses the existing textbooks and who is aware of these benefits can, for example, use algorithms to develop students' logical and mathematical thinking, to revise the material or to develop students' habit of validating the results.

It is worth mentioning that there are proposals for integrated teaching of mathematics and computer science (Kąkol, Moszner, 2000; Sysło, 2008), but so far it has been rarely practiced. In the literature, more and more often benefits of interdisciplinary teaching are being emphasized, especially in combination with the use of modern technology. The use of computers in teaching mathematics creates great opportunities. Nevertheless, in my research I had to omit this aspect because of time constraints. However, this aspect should be explored in further studies.

2.1 Algorithmization as a form of mathematical modeling

Algorithmization is one of the forms of mathematical modeling. Blum and Borromeo Ferri (2009, p. 45) state that: "*Mathematical modeling* is the process of translating between the real world and mathematics in both directions." Mathematical model can be presented in the form of: formula, equation, equation system, function and **algorithm**.

Mathematical modeling is an important part of mathematics education, because one of the main objectives of this education is achieving by the students skills to solve problems encountered in everyday life. Moreover, mathematical modeling is one of the main groups of skills included in the teaching objectives defined in the Polish core curriculum of mathematics, and in similar documents all over the world (Lingefjärd, 2006; Siller, Greefrath, 2010; Bautista, Wilkerson-Jerde, Tobin, Brizuela, 2013).

In Poland, mathematical modeling is one of several groups of mathematical learning objectives included in the core curriculum. In accordance with this act, students should possess the ability of mathematical modeling as they finish the fourth stage of education (age of 19). For students on mathematical education at the advanced level, this requirement is formulated as follows: a student creates the mathematical model of a given situation, taking into account limitations and reservations (Core curriculum with commentary, 2009). However, in Polish schools, mathematical modeling is generally associated with the ability to use mathematical tools (e.g. systems of equations, linear functions) to solve the so-called word problems.

In popular mathematical journals other faces of mathematical modeling can be found because examples of real situations appear in them. We can distinguish here two types of work (Rybak, 2014; Zarzycki, 2009):

- searching for a model based on some specific data,
- creating a model from scratch, knowing only a problem from reality.

Despite numerous educational benefits of unassisted creating of a mathematical model, such approach is absent from school textbooks, not to mention even the attempts to create the general model by searching for and making a symbolic notation of relations within the considered problem.

At the university level, students can meet with mathematical modeling – this applies to certain fields of study. In Poland, they are mostly technical and economic fields. Unfortunately, modeling hardly ever appears in Polish mathematics courses, even in those with teaching specialization. In the scientific literature, one can find much information both about learning and teaching mathematical modeling (Blum, Borromeo Ferri, 2009; Lingefjärd, 2006; Maaß, 2006; Niss, 2012; Perrenet, Zwaneveld, 2012; Siller, Greefrath, 2010; Bautista, Wilkerson-Jerde, Tobin, Brizuela, 2013; Warwick, 2007). Scientists highlight the benefits of teaching mathematical modeling, that result from many mathematical activities performed during the modeling process.

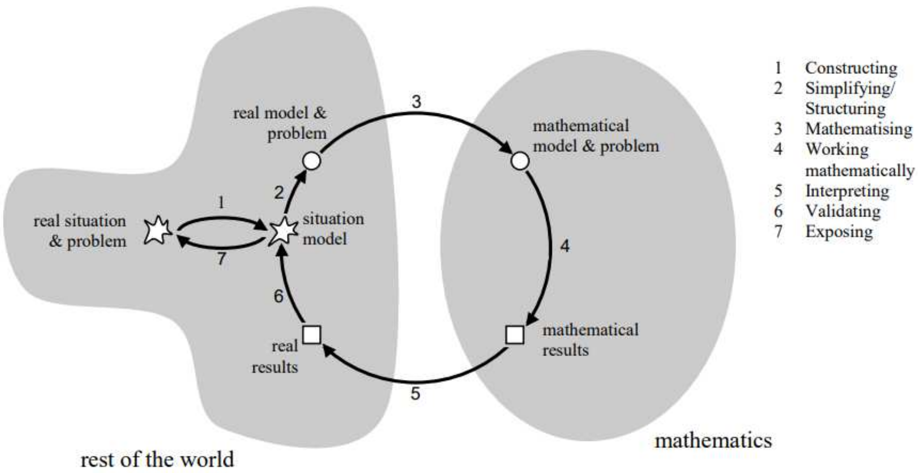

Figure 1: Mathematical modeling process (Blum, Leiß, 2006).

Activities related to mathematical modeling are connected with the activities involved in creating an algorithm to solve the problem (Broda, Smołucha, 2011; Perrenet, Zwaneveld, 2012). That are (Blum, Borromeo Ferri, 2009):

- constructing,
- simplifying,
- mathematizing (recording mathematical relations between considered variables),
- working mathematically,
- interpreting,
- validating.

These activities are important not only in mathematics, but also in the real world. In figure 1 we can see that mathematical modeling process is immersed in mathematics and in the real world. It is also important that it is a looping process. It is not enough to just do it once, but it should be repeated so as to verify and correct the first solution of the problem. It should be repeated to return to the mentioned activities.

The issue of mathematical modeling is very extensive. In my research I confined myself to showing a mathematical model in the form of an algorithm. I am aware that this is narrowing of the topic. I do that, because in the course of mathematics didactics there is no special place for algorithmization and such an approach allows to use mathematical activities related to the algorithmization in mathematics classes.

3 Research methodology

3.1 Research questions, group, course of research

In my research, I tested the concept of teaching mathematics using mathematical modeling, in particular algorithmization. I had a few courses with students of mathematics (future teachers) during which I checked their knowledge and skills related to mathematical modeling (in particular algorithmization) and I tested whether introducing interdisciplinary teaching method would improve the students' mathematical competences. I was looking for answers to the following research questions:

- Do future teachers have competences related to mathematical modeling?
- Can students create algorithms for solving typical mathematical problems?
- What effects will algorithmization bring in the perspective of mathematical modeling?
    - To what extent is the proposed concept possible to be realized within existing realities?
    - Did the classes significantly increase the competences of the students?
- Will the application of algorithmization to teaching mathematics improve students' mathematical skills?

The last question was the most important for me.

Looking for answers to research questions I worked with 16 university students of mathematics with specialization in teaching, future teachers of mathematics (case study). I chose this group because they are people who should theoretically have the knowledge and skills needed to solve basic mathematical problems and problems encountered in everyday life that require mathematical modeling. What is more, they should be familiar with the issues of algorithmization during IT (information technology) classes.

I conducted my research in 2016, but first there were several stages of initial research. Then I tested a variety of tasks, which allowed me to choose the types of problems, which I used in my research.

The research was based mainly on testing different activities, in which students created algorithms by themselves. I put a lot emphasis on students' reflection, verification of their solutions and drawing conclusions concerning the correctness of their work.

The research consisted of three parts. In the first part, students created algorithms for solving problems based on their own knowledge and skills, which they gained during previous education. At the time students did an exercise 1a, exercise 1b and exercise 3 (see: research tool). In the second part of the research I had classes with students during which they learned how to correctly create algorithms that help to solve problems and how to use mathematical modeling (particularly algorithmization) in teaching. The respondents built models together, analyzed their solutions, corrected mistakes. They compared different algorithms (solutions) and evaluated their correctness. Students were forced to reflect on the activities they performed because they had to write down the activities they had applied during the exercise. Then they got acquainted with the list of competences needed for modeling (see: Maaß, 2006). This list was the basis for creating by me a research tool, that made it possible to evaluate the correctness of algorithms. Following the list of competences, given by Maaß, I have distinguished a list of positive features that should include models created by students. When assessing students' answers, I recorded cases when a given feature appeared in the work (see: table 1). The students got to know the list of competencies so that they would be aware of which activities they need to train and what should be improved in their further work.

The third part of the research consisted in checking the effectiveness of the lessons. Students solved control exercises by themselves – among other, exercise 2 and exercise 3. Comparison of the characteristics of the work done at the beginning and end of the course allowed for drawing some conclusions. This research lasted three weeks.

3.2 Research tool

I present the research tool on five problems, which represent two types of issues investigated in my research – pure mathematics and from the real world. In the research ten tasks were used, but due to the limits of this article I show five of them.

3.2.1 Problems from the real world

Problems from the real world force to create a mathematical model of the situation. In such a case, it is important to:

- identify the factors that influence the considered problem,
- search and save relationships between them,
- verify the results obtained with regard to reality.

Exercise 1a. *Every year a school organizes a "Green trip". Create an algorithm that will allow to calculate the cost of participation of pupils in that tour when you know that:*

- *the tour includes six nights;*
- *one adult can take care of a maximum of 15 pupils;*
- *arrival is by train;*
- *there is a visit to a local museum in the tour program.*

The algorithm should be so clear and understandable that it can be used by another organizer.

Exercise 1b. *Create an algorithm that answers the question: How much water can a family drink in the week?*

Exercise 2. *Mr. Kowalski is an indoor painter and he doesn't like to do calculations. Construct an algorithm for him that determines the amount of paint needed to paint a room in cuboid shape.*

3.2.2 Pure mathematic problems

Exercise 3. *Create an algorithm that solves square equations* $ax^2+bx+c=0$.

Exercise 4. *Create an algorithm that solves square inequalities* $ax^2+bx+c>0$.

These tasks

- require comprehensive understanding of methods, understanding of mathematical issues,
- require considering all possible solutions to the problem,
- need verifying the knowledge.

3.3 Example of correct solution of two tasks

The models presented here are my proposals.

Exercise 2. Mr Kowalski's problem.

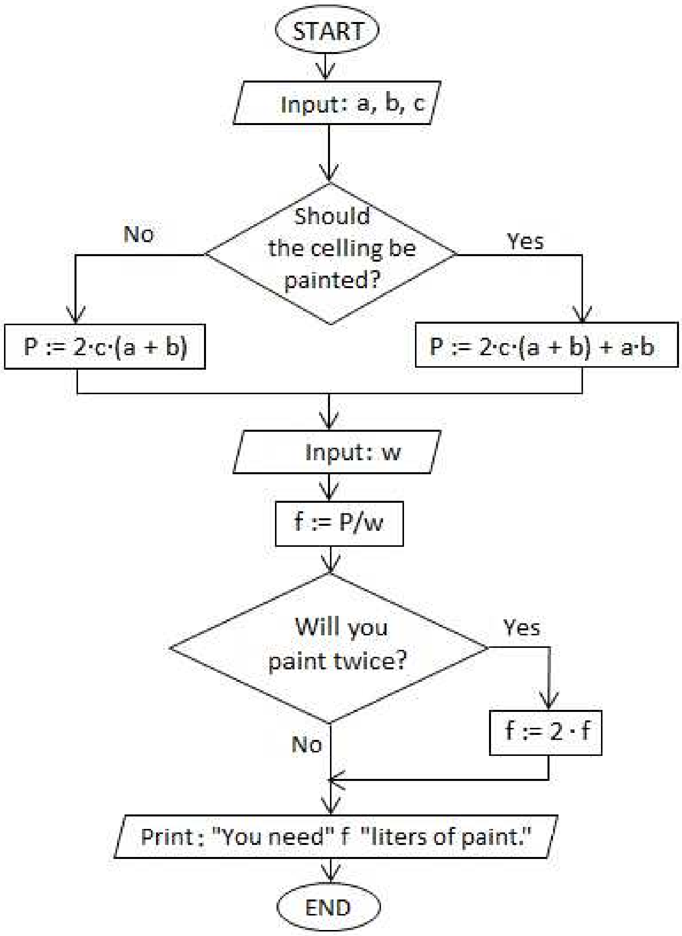

Figure 2: Algorithm determining the amount of paint needed to paint the room (exercise 2).

I will present an example of an algorithm, that determines the amount of paint needed to paint the room in cuboid shape, in block diagram form. The algorithm loads information about the dimensions of the room and performance of the paint used, which determines the surface possible to paint with 1 liter of paint. I assume that the data for the room will be given in meters and the performance in m^2/l. Model shown in the figure 2 allows to decide whether the calculated amount of paint is also used to paint the ceiling. The possibility of double painting is also considered, if you paint the selected type of paint required. The algorithm provides information that determines the amount of paint needed.

Algorithm specification:
Algorithm problem: Determining the amount of paint needed to paint a room in cuboid shape.
Input data: $a, b, c, w \in \mathbb{R}_+$,
Output data: the amount of paint – $f \in \mathbb{R}, f \geqslant 0$.
Auxiliary variables: $P \in \mathbb{R}_+$.
Designations of variables:
a – length of the room,
b – width of the room,
c – height of the room,
w – paint performance, it means how many m^2 surface can be painted using 1 liter of paint,
f – the amount of paint (in liters) needed to paint the room.

The algorithm presented here is quite general. At the same time this model is not too complicated. Larger details of this algorithm would allow, for example, the deduction of the surface of windows and doors or different colors (types) of paint. On the other hand, any additionally considered aspect of this situation would expand the algorithm, which would make it difficult to execute. Regardless of the level of detail, the algorithm should be clear (unambiguity), general and effective. The proposed solution to the problem presented here meets these requirements.

Exercise 4. Square inequalities.

An exemplary algorithm for solving square inequality $ax^2 + bx + c > 0$ has been shown in block diagram form. It was assumed that the coefficient a is different from zero. Therefore, the algorithm does not check the correctness of the data being loaded, because it was assumed that the data provided would meet the conditions specified in the specification of the algorithm.

Algorithm specification:
Algorithm problem: Solution of square inequalities $ax^2 + bx + c > 0$.
Input data: $a, b, c \in \mathbb{R}$, $a \neq 0$.
Output data: Set of solutions of inequality (interval, sum of intervals, set of real numbers or set of real numbers without a single point) or information about the lack of solutions.

This algorithm considers all possible solutions of given square inequalities. The solution is given in the form of unambiguously defined intervals, and in the case of an empty set, it issues the information "No solutions". The proposal is an optimal algorithm that preserves generality, uniqueness and effectiveness.

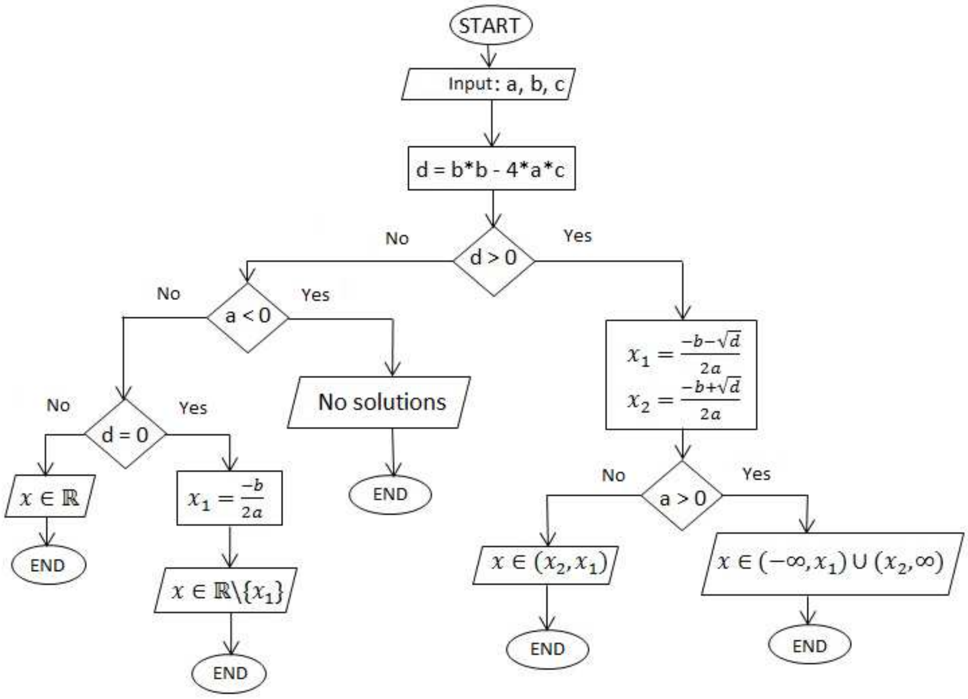

Figure 3: Algorithm which solves square inequalities (exercise 4).

4 Errors of beginners

Research has shown that the mathematics students had many problems with constructing algorithms which solved the problems encountered in everyday life and the mathematical problems of everyday school life. Theoretically, students could solve tasks used in the research (as related to the problem familiar to them), but in spite of this they made many errors. Prospective teachers, despite the fact that they theoretically should have tested skills, can't apply them (especially in open situations). Students do not know the concept of mathematical modeling. They only have some of the competencies related to mathematical modeling. That is why (among other things) there are numerous errors in their works.

The most common mistakes are:

- they do not consider all possible solutions to the problem (in particular, they do not cover all solutions of the equation),
- they do not verify the obtained results,
- they use unauthorized constants in place of variables,
- they do not implement knowledge from different branches, nor knowledge from everyday life.

A significant problem is the fact that students do not test, do not check correctness of the solution. Students do not verify the correctness of their solution – they focus on giving "result" fast. If there is some result, the activity ends. No verification of the correctness of the solution implies another significant failure, namely students often do not consider all possible solutions to the problem. This demonstrates the misunderstanding of the fully discussed mathematical issues.

Another problem, which appears in students' works, especially when they were creating algorithm for everyday situation, is the use of unauthorized constants. Some students present the algorithm with specific data, make some calculations and that is all. They do not see differences between example and general method. What's more, several people have created "algorithms" in which there are only constant values, not variables. Such solution is an exemplary calculation for a particular situation (specific case) – this is not an algorithm for solving a task that should be general.

A big and common problem is that students do not implement knowledge from different subjects and also knowledge from everyday life. Many students think in the following way: "What is in the school, that is in the school; what is in the real, that is in in the real — we cannot connect them!". This is clearly shown in the work related to exercise 1a (school trip). In one of the student work, the algorithm which calculates the cost of participation in school trip, contains an information like: "We have to make a list of children who will go on the trip.", "We have to choose teachers who will go on the trip.", and so on. These are practical tips for organizing a school trip, however, they should not appear in the algorithm which calculates the cost of the trip. This kind of mistakes students made mostly in tasks related to the situation in everyday life, however, they also occur in purely mathematical tasks.

In the students' works there were more detailed errors that are largely related to the correctness of representation of the algorithm, however, in this publication I will focus only on the main problems appearing in the students' works.

5 The results of the course

Observations collected during the first phase of the research allowed me to determine basic problems related to algorithmic skills. They also gave me the opportunity to test a variety of tasks that could be helpful in overcoming these problems. I used this experience by designing my course for academic students.

Students of mathematics (future teachers) took part in a short course. During this, students' skills related to mathematical modeling were developed, in particular related to the algorithm. The main emphasis was on the students' mathematical activities that were shown when building algorithms, which solve the problems in question.

Improving student skills can be easily seen when analyzing their work done at the end of the course. The characteristics of the approaches presented by participants in the study, together with the number of occurrences are presented in table 1. The table shows a list of skills (a list of positive features of models created by students) that I have created by myself to evaluate students' competencies. The distinguished features of the solutions are strongly related to the competencies concerning mathematical modeling. The numbers shown in the table indicate the number of occurrences of a given characteristic among 16 student works (the number of students who have applied a given property in their solution). The research analysis of students' solutions consisted in assessing each work for each property. Columns with exercise 1a, 1b and 3 show students' results at the beginning of the course, and columns with exercise 2 and 4 – at the end. Let me remind that ex. 1a, ex. 1b and ex. 2 are problems from the real world and ex. 3 and ex. 4 are pure mathematics problems. Comparing the results from exercise 1a and exercise 1b to exercise 2, and the results from exercise 3 to those in exercise 4, we can see that after the course students improved their skills and their competencies. In most cases, there is a clear increase in the occurrence of the desired features, e.g., in most final works a description of the algorithm specification appeared.

The data presented in table 1 clearly show that during the classes mathematical modeling competencies of the respondents significantly increased, especially algorithmic. Different skill areas have improved with varying degrees, however, a significant increase is noticeable. Regardless of whether we compare similarly themed tasks, or whether we take them all into consideration, in general, we notice a significant improvement in students' skills. Nevertheless, there are still areas that require further work, because such a short didactic block proved to be insufficient to fully master the desired skills (competences).

Characteristics of student work	Ex. 1a	Ex. 1b	**Ex. 2**	Ex. 3	**Ex. 4**
Algorithm specification	0	0	**13**	1	**12**
Written input	1	3	**14**	5	**13**
Data signs were given	12	10	**16**	5	**13**
All data signs were given	10	7	**9**	0	**3**
Specified data type	0	0	**9**	2	**9**
Specified all data type	0	0	**2**	2	**3**
The assumptions are written	8	6	**7**	4	**8**
The reflections are presented	3	1	**8**	2	**8**
There is a drawing illustrating the deliberations	0	0	**9**	0	**4**
Comments are added	5	4	**4**	1	**6**
The algorithm is tested	0	0	**1**	0	**4**
There are some calculations	0	0	**2**	0	**10**
The result is verified with regard to reality	0	0	**1**	-	**-**
The algorithm is improved (II version)	0	0	**1**	0	**4**
Everything is correct	1	0	**4**	1	**2**
Properly shown algorithm	1	3	**9**	3	**4**
Technically the algorithm works	5	0	**6**	4	**2**
Logically shown dependencies	6	5	**7**	7	**5**
Unauthorized constants were introduced	5	5	**4**	0	**0**
All possibilities were considered	-	-	**-**	8	**7**
Data is loaded	4	7	**16**	12	**16**
There is the start and the end of the algorithm	1	2	**14**	13	**16**
Conditional statements were used	8	2	**5**	14	**16**
Loops were used	0	3	**2**	0	**2**

Table 1: Characteristics of student work – students skills

Comparing the performance of five tasks, it can be seen clearly, that the ability to save used data (variables) has improved significantly. In exercise 2 and 4, data signs were given 16 and 13 times, initially (in ex. 1a, 1b and 3) only 12, 10 and 5 times respectively. Initially only 2 people specified data type, while 9 people did it by doing exercise 2, as well as doing 4. It is worth emphasizing that, at the end of the course, all students correctly loaded data into algorithms. Furthermore, students' competences in graphical representation of the algorithm have clearly improved. It should also be emphasized that the use of the form of the algorithm in solving tasks has contributed to make students use such elements as conditional statement and loop, which determine the approach to problem solving.

The obvious advantage of acquainting students with the mathematical modeling process (and with the creation of algorithms) is the fact, that students more often analyzed the problem before giving solutions — they wrote their reflections and sketched aids (drawings) to illustrate the observed problem. The reflections (testifying to the problem analysis) are presented 8 times in both task 2 and 4, while in tasks 1a, 1b and 3 they were 3, 1 and 2 times respectively. In exercises 2 and 4 there was a drawing illustrating the deliberations 9 and 4 times, in the initial works no auxiliary drawings appeared. Previously students did not analyze the correctness of the algorithm – no corrected version of the first model appeared in any work. It is worth emphasizing that as a result of the experience gained in the 10 solutions of task 4, calculations were used to control the algorithm.

A very important positive effect of the course is that there are works in which the presented algorithm is verified. In exercises 1a, 1b and 3 no one tested the solution presented by them by running the algorithm for sample data, but in exercises 2 and 4 there are some calculations in over 10 works.

Compilation of data contained in Table 1 clearly show an increase in the competence of the students participating in the survey. Nevertheless, the compilation shows that some activities still need improvement. It is optimistic that even in these areas there is a slight improvement.

6 Conclusions

The studies discussed here have shown that many mathematical activities can be stimulated through the use of algorithmization in teaching mathematics. Many problems (purely mathematical and those from everyday life) can be solved by building mathematical models of the situation. Showing models in the form of algorithms brings additional educational benefits. Preliminary studies have shown numerous deficiencies in future teachers' competences in the field of mathematical modeling. It is comforting that these competences can be complemented by the introduction of algorithmization in the area of the teaching of mathematics. Unfortunately, it turned out that conducting a short course was not sufficient to make up for all deficiencies. However, it is positive that in almost every area students' skills have increased (more or less).

In the summary, we can list several specific positive effects of the study:

- much improved data representation,
- students remembered to write the specification of the algorithm – they wrote the variables used and their markings,

- most people do not use unauthorized constant values,
- the form of the algorithm was improved visibly,
- students deal a bit better with extra-mathematical aspects of the tasks.

There are areas of competence that still need improvement. Perhaps a longer course and more time spent on the algorithm would be better to reduce students' shortcomings. This aspect remains a question to be explored. However, there is a visible positive influence of the research on students' competences.

The main conclusions that can be given are as follows:

- Competences of students in mathematical modeling are insufficient – introductory classes should be introduced for their development.
- Participation of students in the discussed classes significantly increased their competence in mathematical modeling, especially in algorithmization.
- Algorithmization can be taught and shown in a broader perspective.

The main result of my doctoral dissertation is a ready didactic proposal of preparing students for teaching faculties to implement the problem of mathematical modeling, in which the algorithmization is playing a fundamental role.

The research carried out showed a gap in the preparation of mathematics students for the teaching profession. Teaching proposal, which is the result of the dissertation, is an offer to fill this gap. However, it is worth conducting further research on the issue of algorithmization and mathematical modeling. Examples of further research directions are:

- A thorough examination of the mathematical modeling problem among pupils, students and teachers.
- Development of a broad database of didactic materials concerning the teaching of mathematical modeling.
- Development and verification of a long-term course for students.
- Examining the results of students' work using computers to the problems discussed.
- Finding the reasons for low integration of mathematical and IT knowledge and possible solutions to this problem.

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Algorytmizacja jako aktywność i umiejętność matematyczna

S t r e s z c z e n i e

Myślą przewodnią mojej rozprawy były słowa Z. Krygowskiej (1977, s. 3): „Rozwój aktywności matematycznej ucznia uważamy za jeden z najważniejszych celów nauczania matematyki.”. Za aktywność matematyczną rozumiem (zgodnie z określeniem W. Nowak) „pracę umysłu ukierunkowaną na kształcenie pojęć i rozumowań typu matematycznego, stymulowaną przez sytuacje prowadzące do formułowania i rozwiązywania problemów teoretycznych i praktycznych”. Zofia Krygowska już w 1977 roku pisała o dydaktycznych korzyściach płynących z wykorzystania umiejętności algorytmizacji w kształceniu matematycznym. Pomimo upływu lat jej postulaty nie utraciły na aktualności. Wręcz przeciwnie – mogą być wykorzystywane w szerszej perspektywie w odniesieniu do zagadnienia modelowania matematycznego, które w ostatnich

dekadach znacznie zyskuje na znaczeniu w gronie naukowców podejmujących problem zastosowań matematyki.

Modelowanie matematyczne jest jedną z głównych grup umiejętności zawartych w celach kształcenia matematycznego na wszystkich etapach nauczania. Mimo to przyszli nauczyciele matematyki nie są zapoznawani z tym zagadnieniem. Także algorytm jest pojęciem często pomijanym w szkolnym nauczaniu matematyki. Warto, aby uległo to zmianie, gdyż algorytmizacja obejmuje szereg aktywności pozwalających rozwijać różnorodne kompetencje. Poznawanie matematyki może być wspomagane m.in. przez samodzielne konstruowanie czy analizowanie algorytmów. Dzięki swej jednoznaczności algorytmy przyzwyczajają ucznia do bardzo precyzyjnego i uporządkowanego myślenia dedukcyjnego, co związane jest z rozwojem rozumowania formalnego, głównego sposobu myślenia w matematyce. Dzięki zastosowaniu elementów algorytmizacji, na każdym poziomie nauczania można realizować wiele celów kształcenia matematycznego. Wykorzystując istniejące podręczniki, nauczyciel, który ma świadomość tych korzyści, może używać algorytmów np. do rozwijania u uczniów umiejętności wyrażania myśli w precyzyjnym matematycznym języku, porządkowania i pogłębiania treści matematycznych, czy wytworzenia nawyku krytycznego analizowania poprawności rozumowania. Warto zatem podejmować działania służące zwiększaniu świadomości studentów matematyki kierunków nauczycielskich dotyczącej modelowania matematycznego, w szczególności algorytmizacji.

Zainspirowana tymi poglądami postanowiłam przestudiować problem algorytmizacji rozpatrywanej jako aktywność i umiejętność matematyczna, zaś wyniki analizy uczynić podstawą do zaproponowania konkretnych rozwiązań, czego wynikiem jest rozprawa doktorska. Głównym celem rozprawy była weryfikacja koncepcji dydaktycznej dotyczącej wprowadzania studentów kierunków nauczycielskich w problematykę wykorzystania modelowania matematycznego (w szczególności algorytmizacji) w nauczaniu matematyki.

Koncepcja ta zakładała zestawienie dwóch sytuacji dotyczących algorytmizacji – algorytmizacji w obrębie samej matematyki oraz algorytmizacji w szerszym kontekście związanej z modelowaniem matematycznym. Przeplatanie tych dwóch podejść miało przyczynić się do zwiększenia świadomości studentów dotyczącej rozumienia istoty procesu algorytmizacji – zwrócić ich uwagę na aktywności (umiejętności) z nim związane. Jednocześnie studenci konfrontowali swoje własne działania z zaleceniami teoretycznymi, co było formą dydaktycznej refleksji.

Przeprowadzone badania wykazały, iż nie można oczekiwać, że pewne (mogłoby się wydawać oczywiste) kompetencje przyszli nauczyciele zdobędą samoistnie w toku kształcenia matematycznego. Wyniki badań jednoznacznie wskazują, iż tak się nie dzieje. Cykl przeprowadzonych badań ukierunkowanych na te matematyczne aktywności oraz analiza zebranego materiału badawczego pozwoliły wyciągnąć m.in. takie wnioski:

- Studenci matematyki o specjalizacji nauczycielskiej „matematyka z informatyką" mają liczne braki kompetencji z zakresu modelowania matematycznego, w szczególności algorytmizacji, pomimo, iż teoretycznie powinni je posiadać.
- Umiejętności modelowania matematycznego (w tym algorytmizacji) można skutecznie nauczać. Jest to potwierdzenie tezy zawartej w literaturze dydaktycznej.
- Przeprowadzone badania ukazują, że pomimo znaczącej poprawy umiejętności studentów dotyczących modelowania matematycznego (w szczególności algorytmizacji) nadal badani mają pewne problemy z niektórymi z aktywności towarzyszących algorytmizacji.

Zaobserwowane prawidłowości pokazują potrzebę kształcenia studentów matematyki mające na celu zwiększanie ich kompetencji związanych z modelowaniem matematycznym (w tym z algorytmizacją). Pokazują również, że niezbędne jest prowadzenie dalszych badań związanych z tą tematyką (np. w kierunku wykorzystania nowoczesnych technologii do budowy algorytmów rozwiązania problemów), gdyż ta rozprawa nie mogła w pełni wyczerpać poruszanego problemu.

Wynikiem całej pracy jest gotowa propozycja dydaktyczna przygotowania studentów kierunków nauczycielskich do realizacji zagadnienia modelowania matematycznego, w której to propozycji algorytmizowanie pełni zasadniczą rolę.

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