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Michał Marcin Krówczyński (Cracow, Poland)

On a definitional understanding of the root concept

Abstract: The paper reports on a study of student difficulties inherent in understanding of the root concept. The situation was analysed in the context of three distinguished components of understanding of mathematical concepts. The first, as a mechanism associated with concept formation in terms of didactic theories. The second, as a learner experience resulting from the way of teaching the concept. As a third component, the influence of the formal definition on the ways student understand the notion. A complete comprehension of the mathematical concepts that includes the formal definition is described in this study as *definitional understanding*. The results of a didactic research diagnosing a way of definitional understanding of the root concept by students in one of the Polish schools are presented. Some aspects corresponding to the theoretical foundations of the roots were related to the way of teaching the concept. Some symptoms of misunderstanding the root concept are described. An attempt was made to identify and characterize the prime sources of the errors in relation to mathematical, psychological and pedagogical contexts.

1 Prelude

The concept of root is one of the basic concepts of elementary mathematics. Nevertheless, very often students experience a lot of difficulties with the correct understanding of this notion. From the didactic point of view the process of concept formation is a challenge for both the student and the teacher. It seems, however, that the main source of problems related to the root lies somewhere out of this dialogue and may result from the very complex, complicated and ambiguous nature of the concept. Therefore, designing the educational process

Key words: definitional understanding, root, radical, procept.

associated with learning of the root concept requires careful and thoughtful steps.

The starting point for addressing this issue were difficulties with the correct understanding of the concept of root that were observed in didactic practice. Due to the often ambiguous origin of these misconceptions, various forms of their exteriorisation, as well as their continuous presence at various, even advanced stages of education, this problem is important from the point of view of didactic of mathematics.

There have been many scientific studies on the concept of root. Numerous, multi-aspect analyses of the issue in World literature are the reflection of its rank and complexity. In addition, the timeliness of research work may indicate that this subject is inexhaustible and still not fully understood by researchers. Among the scientific publications on the concept of root the following eight research matters can be distinguished.

(i) The assumptions in the formal definition of both rational exponent and root concepts that determine rules and properties associated with (Tirosh, Even, 1997; Goel, Robillard, 1997; Choi, Do, 2005).

(ii) Students' misconceptions and difficulties in simplifying and determining the expressions with roots, rational exponents as well as radical numbers (Pitta-Pantazi, Christou, Zachariades, 2007; Cangelosi et al., 2013; Roach, Gibson, Werber, 1994; Özkan, 2011; Özkan, Özkan, 2012).

(iii) Irrational numbers as the roots of some integers – understanding of the concept of irrationality, representation of the irrational numbers on a real line and geometric understanding of radical numbers (Sirotic, Zazkis, 2007; Scott, 2011; Pagni, 2007).

(iv) Variety approaches to the nth root concept, misconceptions about the vocabulary associated with roots, different meaning of the radical sign $\sqrt{\cdot}$ in the mathematics community (Gómez, Buhlea, 2009; Crisan, 2014; Kontorovich, 2016; Roach, Gipson, Weber, 1994).

(v) Inaccurate or ambiguous representation of the square root concept – inconsistencies and misuse of the radical symbol (Gómez, Buhlea, 2009; Crisan, 2012).

(vi) A critical role of notation in the development of students' conceptual understanding of roots (Cangelosi et al., 2013; Crisan, 2012).

(vii) The nth root as a bridging concept between secondary and tertiary mathematics with particular emphasis of the transfer of real and complex number domain – (Kontorovich, 2018).

(viii) The singularity of the formal definition of the nth root that can radically differ depending on the mathematical domain in which the concept is considered. The cross-curricular nature of a concept and the cognitive aspects of its polysemous approach was studied in (Kontorovich, 2017).

Extensive literature analyses the extremely complex nature of the root concept. It seems, however, that mere ability of finding simple roots may be insufficient for solving problems without a correct understanding of the notion.

This is how we were taught since very little – said Jan, when asked why he had misused the radical symbol in the didactic research (Crisan, 2012). It is shown that the moment of first introducing the concept induces a source of convictions that become extremely difficult to modify in the future. Similarly, Özkan (2011) suggests that many errors derive from previous misconceptions that students have developed in prerequisite mathematical topics such as arithmetic operations and absolute value. Indeed, the practice of teaching provides many examples of how difficult it is to change some misconceptions about the root concept. In such cases, the attempts to take corrective actions and change the strong held beliefs in the students' minds may not have any effects. Accordingly, the issues associated with the formation and constitution of the notion should be considered locally in relation to a given country and the mathematics curricula applicable in it.

The first task of the paper is a theoretical analysis of how the root concept is formed in the didactic process in the Polish educational system, where students entering high school are familiar with the concepts of square root and cubic roots, at least on the procedural level. At the beginning, the ambiguous relationship between the dual nature of the notion and its formal definition is discussed. The moment of introducing the concept as well as subsequent stages of acquiring mathematical experience by a student is analysed. At each stage, the student's skills participating in the teaching process is listed. The importance of formal aspects related to the definition of the concept and the terminology in force in Poland is emphasized. Some of the students' difficulties with the correct understanding of the root concept that occur very often in teaching practice are analysed in terms of their possible origin.

Another aim of the paper is to introduce the four-part didactic research conducted among students aged 15–16 years (the high school level). The study highlights some symptoms that may indicate the incorrect understanding of the root concept. In the conclusion, the attempt is made to identify the genesis of the diagnosed difficulties. Additionally, some didactic guidelines are formulated to draw attention to particular elements of the educational process that are crucial for understanding the concept.

In this study the root concept is explored in the field of real numbers.

2 Procept and its definition

Deductive nature of mathematics sets the formal definition of a concept as a starting point for any reflections on it. However, not every mathematical notion has a formal definition, for example, primitive notions. Among the others concepts, one can also distinguish those that are explicitly present in school education, but they are never defined in a formal way, for example the area of figure or the concept of number. On the other hand, the vast majority of mathematical concepts are defined in school teaching by means of a formal definition. In the early stages of education the role of a definition is limited mainly to clarify and explain the meaning of term or word used. Its formal structure is not clearly exposed at that time. At the age of 13–14 students face the term "definition" for the first time when defining a rational number (Braun, et al., 2017, p. 13). From that point on the definition of a notion becomes an autonomous metamathematical concept, which task is not only to clarify the name of a notion but also, above all, to distinguish the set of abstract objects that designate the notion. This is done by establishing clear boundaries of such a set, specifying the rules of being a designate of a given notion. From the point of view of didactic process, this type of distinction between mental entities is a breakthrough. However, the students' awareness associated with the actual role of a definitional rigour in mathematics become gradually acquired at the subsequent stages of education. It should be emphasized that it is influenced by a way of displaying the formal definition in the elements of deductive reasoning experienced by the student during a mathematical activity.

Of course, mathematical objects are defined to make working with them easier. Perhaps this is why the current unified educational system assumes the way of verifying students' knowledge by controlling their ability to perform a set of procedures. Evaluating is carried out with tests or tasks, often very schematic. From the teacher's point of view, such a method is convenient, and from the student's perspective, very simple – just to learn to solve specified tasks and get a certain number of points. In the current system of education, adhering students to the requirements of knowing the formal definition of concepts is considered as an outdated method of teaching. To understand the concept is limited to the feasibility of the related procedure. Such ability is called the *operativeness* associated with a given concept.

The comprehension of a mathematical concept in accordance with current cognitive theories is a very complex issue. The theory of dual nature of the mathematical concept (Sfard, 1991) assumes that a given notion can be interpreted both as a dynamic operation or as a static mental structure. Similarly, in the *procept theory*, a concept (called procept) can be treated simultaneously as a process and object (Gray, Tall, 1994). From this angle, the root concept as a procept can be decomposed as in Figure 1 (see the first division).

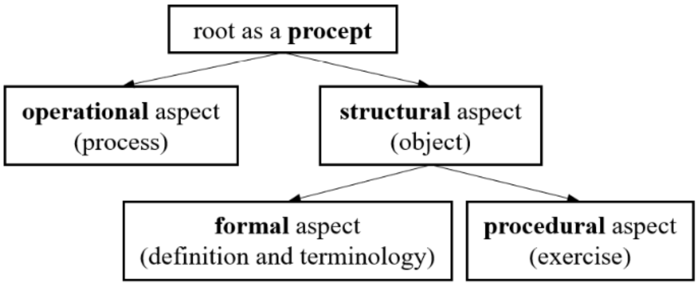

Figure 1. The concept of root as a procept.

Indeed, by $\sqrt{2}$ one has to think of the sentence "find the number whose square is 2" (operation) or of the irrational number $\sqrt{2}$ as an object. It is difficult to clearly distinguish between the two interpretation approaches, they constitute a coherent dual whole of conceiving the concept. However, the operational aspect of the concept most often associated with the procedures being performed appears in the student's consciousness as the first (Sfard, 1989). The comprehension of the concept as an abstract object occurs only with time as the student acquires experiences related to a given notion.

The building of an abstract conceptual structure associated with the given procept is the task of the so-called *encapsulation process* (Tall et al., 1999). The process of operational-structural transition itself, due to its strong interior character, is extremely difficult to control. The correctness of its course may only be verified by means of the assessment of students' activities focused on the relevant part of a structure related to the given concept. It should be emphasized that the procept theory does not specify the role of the formal definition in the encapsulation process. The conceptual structure is mainly built up by interiorization of procedures performed by the student (Sfard, 1991). Because of such a pronounced activity, it is necessary to design procedures in which the student also has to refer to the formal aspect of the mathematical concept (see Figure 1 – the second division). Otherwise, the final shape of the concept structure may not coincide with its formal definition. Any attempt to specify proper comprehension of a given mathematical concept should consist in describing a conceptual structure that takes into account both the formal and procedural aspects. Such a complete understanding of a given mathematical concept for the needs of this paper will be called its *definitional understan-*

ding. It should be emphasized that the schematic distinction into formal and procedural aspects presented in Figure 1 is not really that sharp. In fact, it is rather dual and simultaneous. In other words, the definitional understanding of a concept is not limited solely to indicating the notion in the formal structure of mathematics nor to the knowledge and comprehension of its definition. This term should be also understood in a broader sense – as the student's awareness of the formal aspect when performing any activities related to a given concept. This way of understanding assumes an equilibrium between procedures which are often associated with certain habits and a defined mathematical object.

3 The concept of root in the mathematics curriculum in the Polish system of education

3.1 Previous concepts

From the first years of school, students perform calculations on numbers using four basic arithmetic operations: addition, subtraction, multiplication and division. Each of them is a certain two-argument function the output of which is the result of the given operation. However, the number of independent variables of such a function may be mentally reduced. For example, in the product $3 \cdot 4$ the first number can be mentally fixed whilst the second number is the input (or vice versa), so that the operation is called "multiplication by 3". In this approach, each of the basic arithmetic operations can be understood as the function of one variable. Hence, the following pairs of operations "addition of the number a" – "subtraction of the number a" ($a \in \mathbb{R}$), "multiplication by the number a" – "division by the number a" ($a \neq 0$) stand for mutually inverse functions. The natural association of an operation with single-valued and invertible function is strongly established in the students' consciousness at the beginning of mathematics education.

In this context, the exponentiation a^n, ($n \in \mathbb{N}$) that students learn for the first time at the age of 10-11 (primary school) is an operation of another type. In reference to the one-to-one correspondence it may be a non-invertible function when the exponent n is even (see Figure 2). Whereas in the case of an odd power the operation is invertible. It is worth noting such a clear distinction between the nature of objects within the same concept. Of course students do not realize that at the very beginning, since they raise to the second or third power only positive numbers. However, over time this awareness is essential and crucial for later understanding of the root concept.

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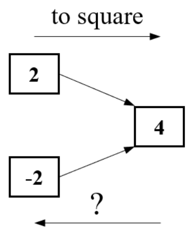

Figure 2. How to understand the inverse operation of squaring?

At the moment of introducing the exponentiation, the students' ability to evaluate mathematical expressions using the rule for order of operations is developed, so that the new operation in spite of the different nature is compared to previous well-known arithmetic operations.

In subsequent years of education students gradually acquire skills of raising any rational number to the power with a positive integer exponent. They actively use the general rules for exponents such as zero-exponent rule, power rule, product rule, quotient rule. Initially, numerous errors of the type: $3^2 = 3 \cdot 2 = 6$ involving the change of operations occur less and less frequently. As the students are at the age of 13–14, the concept of exponent becomes clearly structured in their minds. Then there is the breakthrough resulting from the introduction of the new concept of root.

3.2 The definition of the root concept

The defining of the root concept takes place three times in the education process. In the real number domain it occurs in primary school and later in high school. At the university level, the root concept in the domain of complex number is defined, which however is beyond the scope of this study. It should be emphasized that in each of the mentioned moments of defining there are some changes regarding the concept and its symbol. Modifications relate to the classification of roots, terminology and, especially in the last stage, the nature of definiendum.

In primary school students learn about two types of root – *square root* and *cube root*. In the textbook (Braun et al., 2017, p. 133, p. 150) the introduction of these notions takes place in the following way.

Remember. *A square root of a nonnegative number a is a nonnegative number b such that $b^2 = a$. We denote $b = \sqrt{a}$.*

Remember. *A cube root of a nonnegative number a is a number b such that* $b^3 = a$*. We denote* $b = \sqrt[3]{a}$*.*

It seems that the formal definition is not sufficiently emphasized and respected in this case. For example, in a manner inconsistent with the definition above in some textbooks there are occurrences of cube roots of negative numbers afterwards. This type of expansion or free interpretation of formal definitions of mathematical concepts may affect the downgrading of formal structures in later stages of education. In addition, especially in the case of the root concept, it should be noted that, with accordance to a number of scientific studies, reliance on the students' intuition in any activities related to this notion can be highly improper. Therefore, it seems necessary to maintain great caution and precision which contributes to minimizing the risk of an incorrect course of the process of defining the concept.

With the first applications of the root concept in primary school students acquire experience in the arithmetic and geometry tasks. They are expected to perform arithmetic operations with radicals using rules for radicals such as: product rule, quotient rule, move a number from inside the radical to outside the radical (and vice versa). In geometry tasks the root concept occurs mainly in the application of the Pythagorean theorem. Subsequently, most often based on the example of $\sqrt{2}$, the presence of non-terminating and non-repeating decimals is shown and consequently irrational numbers are defined. Afterwards, students construct segments whose length are equal to surd numbers. In the arithmetic tasks students simplify radical terms, perform operations of addition, subtraction, multiplication, and division on like and unlike radical terms and express the result in the simplest form. They also estimate the value of a square root by integers and rationalize the denominator of a fraction.

At the beginning of high school (students aged 16–17) the terminology related to the root concept (the square and cube root) that was introduced earlier is not generalized or even recalled. Instead, the formal structure of the concept is built anew. The term "root" refers to three different concepts: an *arithmetic root*, a *root of an odd degree of a negative number* and an *algebraic root*. The differences between various roots, which has already depended on the degree of the root, become determined in a completely different way. At the defining stage it may create a lot of confusion.

The following definitions have been adopted on the basis of the textbook (Kurczab, Kurczab, Świda, 2015).

Definition 1 *(arithmetic root of a degree n of a number* $a \geqslant 0$*)*
An arithmetic root of a degree $n \in \mathbb{N}, n \geqslant 2$ *of a nonnegative number a is a number b such that* $b^n = a$*. We denote* $b = \sqrt[n]{a}$*.*

Definition 2 *(root of an odd degree n of a number $a < 0$)*
A root of a degree $n = 2k + 1, k \in \mathbb{N}_+$ of a negative number a is a number $b = -\sqrt[n]{-a}$. We denote $b = \sqrt[n]{a}$.

It should be emphasized that by the definition the arithmetic root is always a nonnegative number. The root of an odd degree of a negative number is defined by means of an arithmetic root $(-\sqrt[n]{-a})$ and stands for its generalization if only the degree n is an odd number.

Why, then, such a generalization of an arithmetic root of an even degree is not possible without departing from the set of real numbers? Because the equation $x^n = a$ is inconsistent whenever the right hand side is a negative number. The reasoning behind this fact is trivial – the square of any real number cannot be negative. In spite of this, very often the students' solutions involve a root of an even degree of a negative number. For example, it happens that when solving the equation $x^2 = -4$ the student does not notice the inconsistency and includes the square root of (-4) in the solution as regards the context of real numbers. It can often be observed that, despite a simple justification, students are not aware of the essence of the assumption in the definitions above.

Gómez and Buhlea (2009) points out the ambiguity of the radical sign as it changes the meaning from arithmetic to algebra. It is stated that the radical sign holds different interpretations in algebra and arithmetic. Therefore, the name of an arithmetic root becomes very accurate. The same as the name of an algebraic root concept, which is also defined in school education in the following way.

Definition 3 *(algebraic root of a degree n of a number a)*
An algebraic root of a degree $n \in \mathbb{N}, n \geqslant 2$ of a real number a is a set of solution to the equation $x^n = a$ with the unknown x.

The above definition comes from (Kurczab, Kurczab, Świda, 2015). An algebraic root, regardless of the parity of its degree, is the full solution to the equation $x^n = a$. This concept does not have its own symbol. In some British textbooks the symbol $\pm\sqrt{}$ is used as the representation of the similar concept to the algebraic root for $n = 2$. However, Crisan (2012) points out that this type of symbolism generates a lot of inconsistencies and problems with understanding the concept by students.

3.3 Common student difficulties related to the root concept

According to the Definitions (1-2), the number $\sqrt[n]{a}$ is obtained by means of the mental solution to the following power equation

$$x^n = a \tag{1}$$

and then the necessary selection of its solution, if only n is even and $a > 0$. Very often, in the students' solution the radical sign $\sqrt[n]{\cdot}$ indicates the operation being performed on both sides of the power equation, for example in the following way.

$$x^2 = 4 \quad /\sqrt{} \tag{2}$$

This proves that the process of solving the equation is often strongly associated with the radical sign. Such a dependence could be shaped by the definition of the root concept. However, in the context of finding the complete solution to the equation (2) (i.e. the algebraic root of 4), the use of the $\sqrt{}$ symbol is inconsistent with the definitional meaning of the concept which represents (an arithmetic root).

The association "equation (1) – symbol $\sqrt{}$" can also be shaped by the students' experiences related to the use of Pythagorean theorem in geometry tasks. For example in a typical task of finding the third side of a right triangle (Braun et al., 2017, p. 261).

Exercise. *Find the hypotenuse of the right triangle whose legs are 9 and 12.*

The lengths of the legs are denoted as $a = 9$, $b = 12$ whereas the c stands for the length of the hypotenuse. Based on Pythagorean theorem the following calculations are performed in the cited textbook.

$$c^2 = a^2 + b^2 = 81 + 144 = 225$$

The next step in the presented solution is $c = \sqrt{225} = 15$. The attention is drawn to the symbol $\sqrt{}$ just below the quadratic equation. In addition to the tasks in the margin the following note is given:

The equation $c^2 = 225$ has in fact two solutions: $c = 15$ and $c = -15$. However, in this task the c is the length of the side of a triangle so it must be positive.

In primary school students solve many such tasks and over time they reflexively ignore the negative solution. The one-time remark regarding the full solution to this type of equation is extremely valuable, but it may not be sufficient to avoid such activities in the future. A lack of attention to particular circumstances such as the positivity of the solution, can build long-lasting associations in the students' minds. Indeed, omitting one of the solutions to the equation (1), where n is even and $a > 0$, is a very common error as seen in the practice of teaching.

Another factor that can affect the frequency of its occurrence results from the very process of solving equations. From this purpose, students most often transform both sides of the equation to the equivalent form. This procedure uses pairs of arithmetic operations as mutually inverse functions. Solving the equation (1) students have the need to perform the inverse operation of exponentiation on both its sides. It so happens that the root operation is erroneously presented in textbooks as the inverse of exponentiation (Braun et al., 2017, p. 135). However, this applies only to specific circumstances – whenever the exponent is an odd number. While on the case of an even exponent (see Figure 2) this sentence without the necessary explanation of how the term "inverse operation" should be understood, can be misinterpreted. For example, by solving the equation (2) an algebraic root of the number 4 is searched for, which in fact has two opposite values. The operation to be performed on both sides of the equation is certainly not a unique function. Indeed, such an operational character of solving equations may also determine the false conviction that square root and exponent eliminate each other.

Often students disbelieve the following equality

$$\sqrt{x^2} = |x|, \quad x \in \mathbb{R}. \tag{3}$$

Specifically, it refers to the absolute value that appears in the above expression. This equality occurs in all textbooks at the stages of education preceding high school, for example in (Braun et al., 2017). Once, a student of the first class of high school expressed her indignation

"We counted differently in the previous school!"

Similar problems highlights Crisan (2008) *"my first year undergraduates need a lot of convincing before they understand why, for all real numbers:* $\sqrt{x^2} = |x|$*"*. The studies of Özkan (2011) and Özkan, Özkan (2012) suggest that mathematics students in Turkey commonly misuse properties when simplifying radical expressions, especially properties which involve an absolute value. Make students aware of the correctness of the equality (3) and development of its applicability is extremely difficult. This is done against strong held beliefs and intuitions that may be established as a result of not entirely correct mathematical experiences.

The introduction of a new classification and the extension of terminology associated with the root concept that takes place at the high school stage of education may also cause students' perplexity. The ordering of a nomenclature together with a set of definitional forms is important due to the deductive nature of a mathematical theory. At the high school level there is a tendency to increase the importance of formal structures, such as: definition, theorem, proof and symbol. Also, the skills of precise coding and decoding mathematical text are developed. The introduction of new terminology related to the root concept at this stage is connected with the necessity of linking new objects with the concepts defined in previous stages of education. It is also necessary to provide an exhaustive explanation of how the radical sign $\sqrt[n]{a}$ should be understood. In can denote an arithmetic root if only a is nonnegative, or a root of an odd degree of a negative number whenever $a < 0$ and n is odd. The newly introduced terminology should also appear explicitly in subsequent stages of education, otherwise it seems to have no deeper meaning. Especially in the context of increasing the importance of formal definition, this type of activity can be inadvisable.

In definitions (1-2) it is also worth highlighting the aspect related to the clear separation of the nature of roots of odd and even degrees. The assumption concerning the nonnegativity of a radicand in the Definition 1 is obligatory if the degree of a root is an even number. Otherwise, the assumption is a just ordening since a root can also be calculated regardless of the sign of a radicand. In that case, only its name is different. Perhaps for this reason, in teaching practice, there is a much higher frequency of mistakes made by students when they use square root rather than cube root. It should be emphasized that the operative use of a root of even degree requires a much deeper interiorization of the concept than if the degree is odd. This is also influenced by other aspects such as a complex definition of the concept, the necessity to refer to rules and properties of exponentiation, and to the solvability of the power equation. A root of an odd degree is somewhat "safer" to use. This is due to the significant reduction of mathematical context to which students must simultaneously refer. It seems that often errors appearing in the case of square roots are caused by the adaptation of certain properties of a cube root, in particular: $\sqrt[3]{-a} = -\sqrt[3]{a}$ and $\sqrt[3]{a^3} = a$, where $a \in \mathbb{R}$. Such natural transfer of properties which results from a certain "economy" of thinking may seem apparently justified since both a square root and a cube root are classified into the same concept – "root". The mentioned irregularities within the same mathematical notion, as in the case of the root concept, may be an obstacle in its correct understanding.

4 Didactic research

4.1 Objectives

The starting points for this research were, observed in the practice of teaching at the high school level, students' difficulties associated with an incorrect definitional understanding of the root concept together with a root operation. Such commonly occurring problems may be derived from previous misconceptions that students have developed in prerequisite mathematical topics. In addition, these difficulties are so severe that in some students they are not defeated despite teachers' active actions.

The overarching goal of the four-part research was to examine of how students comprehend the root concept taking into account the definitional understanding. Emphasis was placed on the aspect of operativeness related to the root concept with simultaneous consideration of its formal definition. Motivation to extend the analysis with additional formal issues results from the observations of global trends in school mathematics education, in which a student's knowledge is often assessed through the prism of the set of skills that he or she must be able to perform. These abilities are often limited to repetitive schemes or procedures. Definitional understanding, in addition to abovementioned activities, takes into account the structural aspect of mathematical objects as an important component of conscious knowledge.

Each part of the study concerned a certain distinguished aspects associated with the analysed notion. The tasks used in the research instrument were designed in a way that enables a solving participant to refer either to the formal definition of the concept or to the general rules related to it. In part I, the evaluation of students' ability to use the definitional condition of arithmetic root such as its sign in routine procedures was examined. Part II includes tasks requiring an operative calculation of expressions using the rules for exponents and radicals. The other aspects analysed in the study were directed mainly towards the operational layer of the concept i.e. the root operation. Students' ability to solve equations of the form $(x-b)^2 = a$, where $a, b \in \mathbb{R}$ was studied. An evaluation of the students' correctness was made with regard to the need to find more than one solution to the quadratic equation of the form $(x-b)^2 = a$, where $b \in \mathbb{R}$ and $a > 0$ (part III). An attempt was also made to analyse the students' conscious actions towards the quadratic equation with no solution in the form $(x-b)^2 = a$, where $b \in \mathbb{R}$ and $a < 0$ (part IV). The tasks given in parts III and IV were posed as word problems. The study distinguished two ways of solving an equation corresponding to a content of a task: mental and algebraical.

4.2 Methodology of the research

The study was conducted among high school first year students (aged 15–16) in the first month of their education at this level before the root concept was redefined and classified in the way described in the paragraph 3.2. Therefore, the participants' knowledge associated with such a notion was limited to the concepts of square root and cube root. Students had defined these notions for the first time two years before the study. Since then, they actively solved many tasks related to these concepts, many of which can be described as routine procedures. The moment of the research was chosen in such a way to assess the image of the concept definition remaining in the student's consciousness after performing many activities related to the concept, that were mostly directed to operability and procedurality.

The study involved 63 students attending two first-year classes with an advance high school mathematics curriculum.

The research method was based on the analysis of students' solutions. In a paper and pencil test they had to solve a set of four tasks that corresponded to the specified parts of the study. Participants were divided into two groups A and B with 30 and 33 students respectively. Group assignments were done randomly. Both group were compatible in terms of educational levels and abilities. Individual conversations with randomly selected students were also carried out. Personal interviews consisted of answering questions during resolving tasks using the "think aloud" method.

In the following paragraphs research tools and quantitative results of each part are presented. The qualitative analysis for the most important outcomes and conclusions from each part of the study are jointly collected and summarized in the paragraph "Research summary".

4.3 Part I: participation of the roots definition in routine procedures

The specific objective of the first part of the study was to determine whether and how students apply the assumptions of the definition of root concept associated with the sign of the number under the root. The presence of the definitional conditions is dictated by a parity of the root's degree. The analysis takes into account the conscious of such property when simplifying the expressions containing radicals. The tasks to be carried out by the students concerned cube root (group A) and square root (group B).

Exercise (IA) *Perform the following calculations*

$$\sqrt[3]{\left(\frac{2}{5}-\frac{3}{7}\right)^3}+\left(\sqrt[3]{-\frac{2}{7}}\right)^3=$$

Exercise (IB) *Perform the following calculations*

$$\sqrt{\left(\frac{2}{5}-\frac{3}{7}\right)^2}+\left(\sqrt{\frac{2}{7}}\right)^2=$$

The quantification of the correctness of solved Exercise IA and IB is provided in the Table 1.

	A	B
Number of participants	30	33
Lack of or incomplete solutions	3	5
Correct solutions	27	4
Incorrect solutions	0	24

Table 1. Part I – the quantitative results.

In both groups the radicand $\frac{2}{5}-\frac{3}{7}=-\frac{1}{35}$ is negative. Among solutions, the method of simplifying the exponent of power and the root prevailed. Despite the small population, there is a large disproportion between the number of correct solutions in both groups. Students who solved Exercise IA most often presented the following solution scheme

$$\sqrt[3]{\left(\frac{2}{5}-\frac{3}{7}\right)^3}+\left(\sqrt[3]{-\frac{2}{7}}\right)^3=\sqrt[3]{\left(-\frac{1}{35}\right)^3}+\left(\sqrt[3]{-\frac{2}{7}}\right)^3\stackrel{(*)}{=}-\frac{1}{35}+\left(-\frac{2}{7}\right)=-\frac{11}{35}.$$

The exact calculations of powers such as $(-\frac{1}{35})^3=-\frac{1}{39200}$ or $(-\frac{2}{7})^3=-\frac{8}{343}$ did not appear in any of the solutions. On one hand, the reason may be the large numbers in the denominators of the fractions. However, on the other hand, it seems that the simplification of the root with the power in step $(*)$ was, for students, an automatic activity. The worked out habit, which is fully justified in the case of the root of odd degree, does not always work in case of an even degree. The latter case requires an intermediate step, about which students who solve Exercise IB most often either forgot or were even unaware of them. This step is based on equality $\forall_{x\in\mathbb{R}}\sqrt{x^2}=|x|$, that students should know. It occurs in most textbooks at the previous stage of education, for example in (Dobrowolska, 2011).

$$\sqrt{\left(\frac{2}{5}-\frac{3}{7}\right)^2}+\left(\sqrt{\frac{2}{7}}\right)^2 = \sqrt{\left(\frac{14}{35}-\frac{15}{35}\right)^2}+\frac{2}{7}=\sqrt{\left(-\frac{1}{35}\right)^2}+\frac{10}{35}=\frac{1}{35}+\frac{10}{35}=\frac{11}{35}$$

Figure 3. The correct solution to Exercise IB.

Few students solving Exercise IB coped with the simplification of a root with the exponent of a power. However, the symbol of the absolute value did not appear explicitly (see Figure 3). The others were clearly unaware of the necessity of using an absolute value. The obtained negative value of the square root (see. Figure 4) is in contradiction with its definition.

$$\sqrt{\left(\frac{2}{5}-\frac{3}{7}\right)^2}+\left(\sqrt{\frac{2}{7}}\right)^2 = \frac{2}{5}-\frac{3}{7}+\frac{2}{7}=\frac{14}{35}-\frac{15}{35}+\frac{10}{35}=\frac{9}{35}$$

Figure 4. The typical incorrect solution to Exercise IB.

It should be emphasized that the solution consisting in the use of an absolute value refers to the definitional condition of nonnegativity of root but only in indirect way. A student simplifying a square with a root and obtaining the absolute value may not be aware of this fact. Therefore, it can not be inferred explicitly from the correct solutions that the student referred directly to the formal definition. However, in the case of erroneous solutions, presented in Figure 4 as an example, the interpretation may be as follows: the student could use the condition of the formal definition (evaluate a sign of the number), but he or she did not do so. Due to the specificity of the research problem and the method used, mostly incorrect solutions will be useful in analysis.

The result of comparison of student performance on Exercises IA and IB is quite surprising, since we expected that as square roots are more familiar to students at this level, the correctness of answers on IB should be higher than on IA requiring understanding of cubic roots.

The study also included individual interviews with selected participants. They aimed at a deeper analysis of the results and to draw attention to important aspects of generally incorrect solutions. The transcription of one of them regarding Exercise IB is given below.

Student:

(writes) $\sqrt{\left(\frac{2}{5}-\frac{3}{7}\right)^2}+\left(\sqrt{\frac{2}{7}}\right)^2=\frac{2}{5}-\frac{3}{7}+\frac{2}{7}=\ldots$

Teacher:

Could you write the general law that you used, simplifying the first of the roots? For convenience, note the expression $\frac{2}{5}-\frac{3}{7}$ *as* x.

Student:
(writes) $\sqrt{x^2} = x$
(corrects after a moment of reflection) $\sqrt{x^2} = |x|$
(says) *Here, the absolute value should be.*

Teacher:
How do you know that?

Student:
This is the rule.

Teacher:
Can you somehow explain this fact?

Student:
(after long thought) *I do not remember how the teacher once explained it to us.*

Teacher:
Why do you think, in the calculation, have you simplified the exponent of the power with the root, omitting the absolute value?

Student:
I do not know. Somehow reflexively.

From the conversation above it can be assumed that the student knew the indicated equality, but not in an operative form, that is, he could not apply it. It also seems that it is not fully understood by him. In addition, he admitted that his action was reflexive. It is highly probable that in the past, examples in which the students simplified the even exponent of a power with a root, could be selected in such a way that the procedure worked without any problems; the result of simplification was always a nonnegative number. From the point of view of the process of adaptation of the concept, the formation of these types of reflexes, only in the context of certain specific circumstances at the relatively early stage of mathematical awareness, seems to be very detrimental.

Other aspects of the qualitative analysis of the results obtained in the research is summarized at the end.

4.4 Part II: rules for exponents and radicals

Another important goal was to look at students' ability to operatively simplify expressions containing roots as one symptom of definitional understanding.

In test tasks the concept has been put in the context of other arithmetic operations. In such a case student can refer to the rule for order of operations, exponents and radicals. Participants of the study had to solve the following tasks.

Exercise (IIA) *Perform the following calculations*

$$2 \cdot \sqrt{3^2 + 4^2} - 4 =$$

Exercise (IIB) *Perform the following calculations*

$$\left(\sqrt{2} + \sqrt{3}\right)^2 - 5 =$$

The quantitative comparison of the results of part II is shown in Table 2.

	A	B
Number of participants	33	30
Lack of solutions	1	1
Correct solutions	20	12
Incorrect solutions	12	17

Table 2. Part II – the quantitative results.

In Exercise IIA as many as 12 out of 33 students solved the problem incorrectly, in the way presented in Figure 5. It should be emphasized that the simplification of a root with an exponent takes place against the rule for order of operation. However, the presence in the solution the term $2 \cdot (3 + 4)$ and then, using the property of distribution of multiplication over addition, is a sign that these students are most likely familiar with the rule for order of operation, but they apply it to a limited extent.

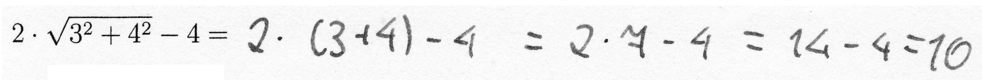

Figure 5. The typical incorrect solution to Exercise IIA (12 students).

Exercise IIB, which was correctly solved by 12 out of 30 students could be performed either by using a square of sum formula (Figure 6(a)) or by multiplying out two brackets (Figure 6(b)).

$$\left(\sqrt{2}+\sqrt{3}\right)^2-5 = 2+3+2\sqrt{6}-5 = 2\sqrt{6}$$

(a) The solution to Exercise IIB with the square of sum formula (11 students).

$$\left(\sqrt{2}+\sqrt{3}\right)^2-5 = \left(\sqrt{2}+\sqrt{3}\right)\left(\sqrt{2}+\sqrt{3}\right)-5 = 2+2\sqrt{6}+3-5 = 2\sqrt{6}$$

(b) The solution to Exercise IIB by multiplying out two brackets (1 student).

Figure 6.

An incorrect solution to Exercise IIB was found in 17 out of 30 students. Most often, it consisted in simplifying a root with an exponent against the rule for order of operations (Figure 7(a)). One of the participants solved the task as shown in Figure 7(b), in which the square sum formula was incorrectly applied. The "doubled products" was turned into the "doubled sum". In teaching practice such mistakes, consisting in the change of operations, are commonly observed. One of the most common is to perform "multiplication by 2" instead of "squaring". Such a tendency may indicate pronounced deficiencies in the mental structure associated with a given concept. Performed procedures have not yet been interiorized, therefore they are limited only to mechanical actions on symbols and not on abstract objects represented by them. Operations performed on symbolic expressions without continually having recourse to the semantic meaning of the symbols are called *deviated formalism* (Krygowska, 1988).

$$\left(\sqrt{2}+\sqrt{3}\right)^2-5 = (2+3)-5 = 5-5 = 0$$

(a) The typical incorrect solution to Exercise IIB (16 students).

$$\left(\sqrt{2}+\sqrt{3}\right)^2-5 = 2+3+2\left(\sqrt{2}+\sqrt{3}\right)-5 = 2\sqrt{2}+2\sqrt{3}$$

(b) The example of an incorrect change of operations (1 student).

Figure 7.

Mistakes that appeared in Exercises IIA and IIB may have a similar genesis. Failure to comply with the rule for order of operation is an error, the cause of which may lie much deeper. Incorrect students' solutions may indicate lack of awareness of an autonomous structure of the summands $3^2, 4^2$ (Exercise IIA) and $\sqrt{2}, \sqrt{5}$ (Exercise IIB). Therefore, the concepts exponent and root may be still unstructured. In Exercise IIA an incorrect simplification is even more unjustifiable as the calculations under the radical sign can be performed mentally.

4.5 Part III: root as an operation – a quadratic equation with two real solutions

The third part of the study concerned the square root operation associated with solving a quadratic equation with two solutions in the form $(x-b)^2 = a$, where $a > 0$ and $b \in \mathbb{R}$. Task groups A and B corresponded to different methods of solving – "mentally" (Exercise IIIA) or "algebraically" (Exercise IIIB). The tasks regarding part III were as follows.

Exercise (IIIA) *(solving mentally) Without performing the calculations, find all real numbers for which decreased by 1 and then squared are equal to 16.*

Exercise (IIIB) *(solving algebraically) A student decreased certain real number by 3, then squared the result and obtained 4. Write the equation, which illustrates a given problem and solve it. Mark the unknown as x.*

The quantification of the correctness of solved Exercises IIIA and IIIB is provided in the Table 3.

It should be noted that in group A each participant attempted to solve the task. Among the incorrect solutions to Exercise IIIA 10 out of 33 students found only one solution. The remaining 7 gave two opposite numbers: $5, (-5)$ as the answer. Apparently, these students mistakenly assumed that the set of solutions to the equation due to the even exponent is symmetric. However, it should be emphasized that all students who solved a quadratic equation mentally found at least one solution and among them 10 students – the entire set of solutions $\{5, -3\}$.

	A	B
Number of participants	33	30
Lack of solutions	0	9
Correct solutions	16	3
Incorrect solutions	17	18

Table 3. Part III – the quantitative results.

In Exercise IIIB the quadratic equation $(x-3)^2 = 4$ was to be solved algebraically. As many as 9 students did not try to solve it. Only 3 participants solved the equation correctly. They decided to transform it to the equivalent form $|x-3| = 2$, then wrote a disjunction $(x-3=2) \vee (x-3=-2)$ and finally obtained the set of solutions $\{5, 1\}$.

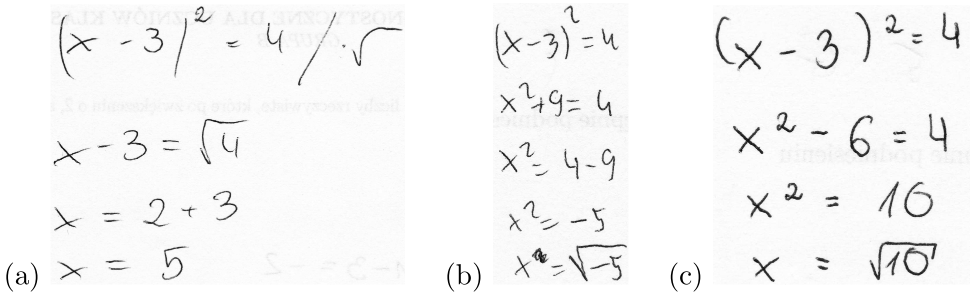

Figure 8. Examples of incorrect solutions to Exercise IIIB.

Figure 8 shows the most common incorrect solutions to Exercise IIIB. In Figure 8(a) the absolute value has been omitted; the answer of this type presented 8 out of 30 students. Attention should be paid to the incorrect use of the radical sign as regards the definitional meaning of an arithmetic root. Figure 8(b) shows incorrect squaring of $(x-3)$. The transformation of $(x-3)^2$ into x^2+9 resulted in an inconsistent equation. In the next step, the root of a negative number was calculated. In Figure 8(c) not only the expression $(x-3)$ was incorrectly squared but also, it seems, the squaring operation was changed into multiplication by 2. Additionally, in the next part, the absolute value or the second solution was omitted.

Among students solving Exercise IIIB 11 out of 30 found at least one solution, including three of them that obtained a full two-element set. The diagnosed errors indicate the inadequate operativeness associated with algebraic expressions in which exponents and roots simultaneously occur.

4.6 Part IV: root as an operation – a quadratic equation with no real solution

In part IV, in analogy to the previous one, two ways of solving were distinguished: “mentally” (Exercise IVA) or “algebraically” (Exercise IVB). The tasks in this part were related to an inconsistent quadratic equation of the form $(x-b)^2 = a$ with coefficients $a < 0$ and $b \in \mathbb{R}$.

Exercise (IVA) *(solving mentally) Without performing the calculations, find all real numbers which increased by 2 and then squared are equal to* (-1)*.*

Exercise (IVB) *(solving algebraically) A student increased certain real number by 2, then squared the result and obtained (-9). Write the equation, which illustrates a given problem and solve it. Mark the unknown as x.*

	A	B
Number of participants	30	33
Lack of solutions	8	1
Correct solutions	18	2
Incorrect solutions	4	30

Table 4. Part IV – the quantitative results.

In both groups, the correct solution is understood as one of the following statements: the equation is inconsistent, tasks cannot be solved, there are no such numbers, $x \in \emptyset$ etc.

The quantitative comparison of results (Table 4) shows that the students who solved the tasks mentally (Exercise IVA) in the majority did not have a problem with obtaining the correct answer. Participants solving the task algebraically (Exercise IVB), in the initial step, wrote the equation $(x+2)^2 = -9$, which was done correctly each time. Then, in very many cases (30 out of 33 students), an attempt was made to solve the equation without reflecting on whether this action made any sense. There was a clear tendency to separate critical thinking, associated with the essence of the task, from the equation that should have been solved.

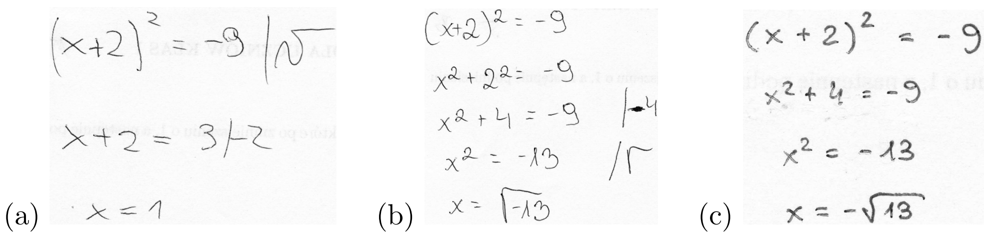

Figure 9. Examples of incorrect solutions to Exercise IVB.

Figure 9 shows the most common examples of incorrect solutions to Exercise IVB. Students tried to solve the equation by calculating a square root of a negative number to get: the positive number (Figure 9(a)), the single square root of a negative number (Figure 9(b)), the square root of a negative number from which the minus sign was taken away (Figure 9(c)).

5 Research summary

The study confirmed the presence of students' difficulties with the definitional understanding of the root concept. It should be clearly emphasized that a full analysis of the issue would require more thorough research. The observed inconsistencies relate only to selected aspects of the concept. The following is an attempt to compile some key conclusions from the study.

Main issues of definitional understanding

The first part of the research indicates a lack of compliance by students with the assumption of the definition of a square root related to its nonnegativity while performing routine and simple procedures. Students show much better operativeness associated with a cube root than with a square root. One of the reasons may be the occurrence of additional assumptions in the definition of the latter, concerning its nonnegativity, that students were not fully aware of. However, an equally important factor may be different properties of both of these roots, although both of them are designates of one concept – "root". Participants freely used the following identities ($a \in \mathbb{R}$): $\sqrt[3]{-a} = -\sqrt[3]{a}$ and $\sqrt[3]{a^3} = a$. However, they do not use the equality $\sqrt{a^2} = |a|$, perhaps due to the fact that it is not sufficiently exposed at previous stages of education.

In part II of the study many students did not refer to the rule for order of operations. Some of participants displayed the misconception that squares and radicals distribute over addition. The errors identified in this part may indicate a lack of proper conceptual structure of not only the root (Exercise IIB), but also the prerequisite notion of a positive integer exponent (Exercise IIA).

Pitta-Pantazi, Christou and Zachariades (2007) pointed out that manipulations with roots are associated with the highest level of exponential reasoning. It was shown that in order to proper understand roots students have to not only enrich the existing conceptual structure related to the exponents, but also reorganize their thinking. Incorrect understanding of the concept of root can therefore result from the student's difficulties related to the notion of exponent.

In fact, the typical mistakes found in the incorrect solutions in part II, consisting in the usage of the following incorrect formulas: $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$ (Exercise IIA) and $(a+b)^2 = a^2 + b^2$ (Exercise IIB), are not limited to exponents and radicals. The problem with "linearised formulas" of the form $f(a+b) = f(a) + f(b)$ is often applied also to other concepts, for example $\sin(a+b) = \sin a + \sin b$ or $\ln(a+b) = \ln a + \ln b$. A similar result in relation to radicals was obtained in (Özkan, 2011). Özkan (2011) suggested that many of these errors derive from previous misconceptions that students have developed in prerequisite mathematical topics such as, for example, the arithmetic operations. He also proposed that these misconceptions must first be resolved in order for students to understand the concept of radicals.

Parts III and IV of the study were related to a square root operation associated with the procedure of solving a quadratic equation. Students had difficulty with determining the feasibility of a square root operation and its correct implementation. Often in the solution such an operation was denoted by the symbol $\surd$, although its definitional meaning was different. Inconsistencies with a formal definition of a root of an even degree were also manifested in failures to comply with its assumption; there were square roots of negative numbers in the solutions to tasks. These issues are discussed below.

Choice of a solution method

Based on the solutions to tasks of parts III and IV, attention was also paid to the aspect of a method of solving an equation corresponding to a word problem. It was surprising to observe that much better results associated with solving a quadratic equation or noticing its inconsistency were achieved by students performing calculations mentally (Exercises IIIA, IVA). Note that all students attempted the mental solution of Exercise IIIA, but in the Exercise IVA some students could not handle the unexpected issue a root from a negative number, hence they did not provide any solution.

The remaining participants were obliged to use an algebraical method, which first step was to write an equation with the unknown x. The second step was to find a solution to the equation. It should be emphasized that all students that were obliged to use the algebraical method and attempted to solve the task, wrote a correct equation corresponding to a word problem. Any difficulties appeared only in the second step when finding a solution to the equation. It is worth noted that the number of correct mental solutions was significantly larger than the number of the algebraic solution.

The observation shows that in the case of an algebraical method, the content of a word problem was clearly separated from the solution to an equation. It seems that the signal to start a calculation procedure resulted in forgetting the essence of the task. The loss of a holistic view to the problem may have resulted from a significant mental effort related to the performance of the calculation procedure.

Stewart and Tall (2015) on the example of a formal proof distinguished between two types of reasoning. The first of these is a *step by step* reasoning, which corresponds to tracking succeeding parts of a proof, and the second is a *comprehensive approach*, consisting in working out a broad perspective of the activities performed. The authors emphasize that a "comprehensive approach" is much more difficult, since it requires gathering many single pieces of information and creating a coherent whole (Stewart, Tall, 2015). The process of solving a word problem can also be seen through this prism. The "step by step" reasoning corresponds to the mechanical performance of a calculation procedure after writing an equation, whereas the "comprehensive approach" will require the control of all stages of solving with the essence of a word problem. In this context, it is worth emphasizing the importance of all tasks in which a solution should be found mentally. This method may be useful, especially at the initial stages of introducing a given concept, when the operativeness required for the algebraical method has not yet been developed. In addition, it engages areas of the mental structure associated with critical thinking and promotes the control of the content of the task at every stage of solving. In this way, the process of forming concepts, usually based on the interiorization of procedures, could be creatively supplemented.

Reflexes and habits

The analysis of students' solutions indicates that they have certain habits that could be grounded in their previous experiences. The first of these is a strong impulse to simplify a root with an exponent of a power, if only they appear simultaneously in a task. The student solving Exercise IB, in an interview, described it as a reflex. It should be noted that in Exercises IB and IIA, this activity was carried out in spite of small numbers and rather uncomplicated expressions, and in Exercises IIA and IIB was performed against the rule for order of operations. This type of behavior can be dictated by the fact that the pair of operations: nth power and nth root are equivalent in the hierarchy of the rule for order of operations. A similar example of simplification of the following way $\frac{2x+5}{2} = x + 5$ has been described in (Krygowska, 1988). In this case the dash indicating a fractional expression stimulates the student to automatically "simplify" the expression in the incorrect way. When facing some certain visual stimulants, such as for example the radical sign and the exponent, some students are at a loss. They have a tendency to simplify and reduce at all cost without consideration for the meaning imparted by the operation.

The desire to eliminate roots from a task may also indicate negative experiences related to this concept in the past. Often school tasks are designed in such a way that their result is a rational number. Hence, the presence of a surd number, as a result of a task, may be for students a sign of incorrectness.

The analysed automation can also be influenced by popular and numerous tasks in which students simplify roots in a reflexive manner without validating different strategies of a solution, paying attention to the formal aspect, and dictated by common sense, legitimacy or feasibility of these actions. Asimilar pattern can also be seen, for example, in the context of a procedure of rationalize a denominator of a fraction. Multitude of "free the denominator of irrationality" tasks induces a strong impulse among the students to do this, even if it is not required. It often happens that a result obtained, for example in the form $\frac{1}{\sqrt{2}}$ students rate as incorrect. In their opinion, it acquires correctness only after the denominator is freed from irrationality, and thus transformed into the form $\frac{\sqrt{2}}{2}$. It should be stressed that a significant impact on this type of consciousness can be influenced also by the language aspect associated with the repeated "free of irrationality" command. It may suggest that all irrationalities are undesirable or even incorrect. Krygowska (1988) described how a student, when came across the numeric expression $x = \sqrt{9+4}$, gave signs of worry and doubt about the value $\sqrt{13}$ asking, "May I leave it at $\sqrt{13}$?". Then she came back to the previous way of solution $x = \sqrt{9+4} = 3+2 = 5$ although she had primarily assessed it as being wrong. In a confrontation with automatic action, logic and common sense often lose.

Another automatic action observed among students, which may result from their previous experiences, is to proceed to solving an equation before reflection on whether such action makes any sense. In Exercise IVB the students attempt to solve the equation which they should immediately describe as a inconsistency. This may result from a certain imperative related to the necessity to write a solution to the task and a belief that the task is solvable.

6 Final conclusions

The results of this pilot study certainly do not cover the extensive issues related to the concept of root. The wide scope of the problem indicates the necessity of analysing many contexts related to it: mathematical, pedagogical, psychological or linguistic. An attempt was made to analyse the definitional understanding of the notion. Certain aspects regarding the course of the process of forming its mental structure in terms of didactic theories were compared with a formal layer associated with its definition. All this has been referred to the way of teaching the concept in the Polish education system, bearing in mind the manner of providing a formal definition, as well as the selection of calculation procedures. The difficulties diagnosed in the didactic research indicated that the students do not respect the content of the formal definition, in particular the assumptions regarding the sign of both radicand and the number being radical. In addition, some automatic actions related to the exponent and root operations occurring in the context of their pseudo, mutually inverse, have been observed. It should be emphasized that due to the multifaceted nature of the discussed issue, it is impossible to unequivocally determine the genesis of the observed difficulties. The appearance of errors in the students' solutions may be treated as an exterior symptom of a certain negative deviation in the learning process. Our interesting positive results regarding mental problem solving need to be explored further in other contexts. This paper should serve as a starting point for a discussion on the spectrum of students' experiences related to a given concept that have been acquired in the teaching process.

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Rozumienie definicyjne pojęcia pierwiastka

S t r e s z c z e n i e

Artykuł stanowi próbkę rozważań na temat trudności uczniów związanych z poprawnym rozumieniem pojęcia pierwiastka. Problem przeanalizowano w kontekście trzech wyróżnionych składowych rozumienia pojęcia matematycznego. Pierwszą z nich stanowią mechanizmy związane z kształtowaniem pojęcia w ujęciu teorii dydaktycznych. Kolejnym czynnikiem są doświadczenia ucznia wynikające ze sposobu jego nauczania. Ostatnią gałąź stanowi formalna definicja pojęcia, której udział w sposobie rozumienia jest ściśle związany z jego naturą. Pełne rozumienie pojęcia uwzględniające ten aspekt formalny jest określane w niniejszym opracowaniu jako *rozumienie definicyjne*. Zaprezentowano wyniki badania dydaktycznego diagnozującego sposób rozumienia definicyjnego pojęcia pierwiastka przez uczniów polskiej szkoły. Wyróżnione aspekty badawcze, które odpowiadały teoretycznym podstawom zagadnienia, odniesione zostały do sposobu nauczania pojęcia, a także zestawione z warstwą formalną związaną z jego definicją. W badaniu wyróżniono pewne symptomy niepoprawnego rozumienia definicyjnego pojęcia pierwiastka. Podjęto próbę scharakteryzowania zaobserwowanych trudności oraz określenia możliwej ich genezy.

Institute of Mathematics
Pedagogical University of Cracow
Poland
e-mail: *michal.krowczynski@up.krakow.pl*