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Isomorphism as generalization tool (in combinatorics)

Abstract: The article deals with constructing the discovery of the number of combinations. It stems from the approach of genetic constructivism, which works with the generic model theory, and its research goal was to develop the stages of isolated models, particularly using the phenomenon of isomorphism. Children aged 11–13 participated in the experiment during their mathematics club, in which they worked in groups at first and together at the blackboard afterwards. The results come from analyzing the experiment protocol and a video recording of part of the experiment. In this study, the authors focus primarily on a detailed analysis of individual sub-phases of isolated models under the generic model theory. The experiment highlights the importance of isomorphism as a generalization tool. The authors have identified the obstacles on the way to advanced sub-phases of isolated models. Another contribution is the division of the fourth sub-phase into two separate sub-phases, which has been achieved using the method of atomic analysis.

1 Introduction

The study presents a part of the results of the research carried out in collaboration with the Department of Mathematics with Didactics of the Pedagogical Faculty at University of Ostrava and Department of Mathematics and Mathematical Education of Faculty of Education at Charles University. More results of the collaborative research can be found in the works Jirotková, Zemanová (2013), Jirotková, Papadopoulos, Zemanová (2015). We were inspired by the results of the Prague research, which have been significantly impacting the teaching of mathematics in the Czech Republic in the past five years.

Key words: generic model theory, isolated models, combinatorics, isomorphism, generalization.

The aim of this research is to thoroughly analyze the cognitive process of a group of pupils of discovering the number of combinations of $\binom{n}{2}$ through the tool of genetic constructivism. We believe the constructive education style is more effective than traditional teaching. This belief is shared among many other authors, for example: (1) An important factor in unsatisfactory learning outcomes of these problems are the traditional teaching methods. (2) The outcomes may be significantly improved by using methods based on a constructivist approach. (Mrożek, 2015).

We selected the field of combinatorics as it had been thoroughly covered in didactics research. The reason is its suitability for experimental research with young children as stated by Maher (Maher, Powell, Uptegrove, 2010): *... learners can develop sophisticated ideas about proof and justification, generalization, isomorphism, and mathematical reasoning at an early age and can continue to refine and expand those ideas over time, developing increasingly sophisticated presentations and representations.* The type of isomorphism described by Siegler is suitable for our research objectives. Siegler (1977) defined the concept of isomorphic problems (or isomorphs) as follows: *Isomorphs are problems that are formally identical but differ in their surface structure.* Additional inspiration stems from the works of Maher and her colleagues (Maher & Martino, 1997; Maher & Speiser, 1997; Martino & Maher, 1999; Muter & Maher, 1998) and similarly to Sriraman and English (2004, p. 184), for our study we also... *used two or more related problems that were conducive to the formation of isomorphic mathematical structures.*

A detailed analysis of the rich research database based on the mathematics clubs of pupils aged 11–13 from Secondary Grammar School of Prof. Otto Wichterle, Ostrava provides new knowledge in three directions:

The first direction, which is covered in this article, concerns the cognitive process of the pupil. A new sub-phase of isolated models phases is discovered as an addition to the Theory of Generic Models. Readers can find about the Generic Models Theory in Hejný (1997). The most recent description of this tool is in Hejný (2012). The Generic Models Theory divides the cognitive process into phases, the key role being played by the generic model; at which the pupil arrives by generalizing a series of previous experiences. The material collected in our research allows to depict the phase of isolated models in great details, which has not been done in previous works. New relations that emerge in this phase are described.

The second direction focuses on the work of the teacher and their effort to change their own educational style to be more constructive. Research in this area has come a long way; see Hejný (2012), Jirotková (2012). In this regard, we depart from genetic constructivism, which seeks the optimal education strategy to develop the creativity of pupils. The theory, originally coined under Scheme Oriented Education, was renamed to genetic constructivism after as a result of methodological analyses by L. Kvasz (2016). Kvasz demonstrated that this method systematically uses the parallel between ontogeny and phylogeny, i.e. how mathematics history enables to construct a series of isolated models towards a generic model in a way to comply with the needs of ontogeny. Kvasz' study provides a deeper theoretical insight into this topic.

The third direction examines the misunderstandings in teacher-pupil interactions. Our research continues the works of Slezáková-Kratochvílová, Swoboda (2006, 2008).

2 Theoretical Background

2.1 Generic Model Theory

The cognitive process under the Generic Model Theory (GMT) comprises of five consecutive phases (Hejný, 2012). It begins with motivation, the need of the pupil to learn something, try something. Experiences saved in the pupil's consciousness as a result of their motivated activities are called isolated models (of further knowledge). Through generalization of these experiences, the consciousness of the pupil is enriched by a more general piece of information, called the generic model. Through abstraction, the generic model becomes abstract knowledge. Nearly every time this is accompanied by a change in language. While the previous description of the cognitive process had crystallization – the domestication of the new piece of knowledge in the existing knowledge network of the pupil – as the fifth and final phase, lately it has been incorporated to stretch from the second to the fourth phase (see Table 1).

Lifts		Generalization		Abstraction
Phases	Motivation	Isolated models	Generic model processual, conceptual	Abstract knowledge
		Crystallization		

Table 1.

Our experiments discovered a noteworthy phenomenon in the phase of isolated models. It is covered in greater detail in chapter 6.2. For a detailed graphic description of this phase in the current GMT see Table 2.

<table>
<tr><td colspan="2">Stages</td><td>Lifts</td><td></td><td rowspan="8">↑</td></tr>
<tr><td colspan="2">Abstract knowledge</td><td rowspan="2">Abstraction</td><td rowspan="6">Crystalisation</td></tr>
<tr><td colspan="2">Generic model</td></tr>
<tr><td rowspan="4">Isolated models</td><td>4^{th} sub-phase
Discowering the essence of “sameness”</td><td>Generalisation</td></tr>
<tr><td>3^{th} sub-phase
Isolated models referring to each other</td><td rowspan="3"></td></tr>
<tr><td>2^{th} sub-phase
Additional isolated models</td></tr>
<tr><td>1^{th} sub-phase
First tangible experience</td></tr>
<tr><td colspan="2">Motivation</td><td></td><td></td></tr>
</table>

Table 2.

It shows that each phase of isolated models is further divided into four sub-phases. We know that not every type of cognitive process includes all of these phases.

2.2 Phases of the Cognitive Process under GMT

Motivation

Psychologists distinguish between external and internal motivation; the external motivation includes the condition in which the pupil is driven to do the activity by external forces, such as fear of a bad mark, parents' demands, attempt to be popular, etc. External motivation will be simply called stimulation (from Latin *stimulo* – to sting). According to GMT stimulation is not so effective as internal motivation (from Latin *moveo* – to move).

Isolated models

The phase of isolated models is divided into four sub-phases.

First Sub-Phase:
The first isolated model forms in the pupil's consciousness.

Second Sub-Phase:
The existing isolated model is joined by one or more additional isolated models. The pupil does not see any relations between the models yet.

Third Sub-Phase:
The pupil discovers partial relations between some of the isolated models; we say that in the pupil's consciousness some of the models start to refer to each other in the form of “sameness”.

Fourth Sub-Phase:
The pupil sees this "sameness" in at least one pair of isolated models referring to each other.

Generic Model

One of the isolated models becomes dominant in the sense of referring to a higher number of isolated models and already acquires a universality quality. If these references have a procedural character (e.g. we examine a certain phenomenon for $n = 1, 2, 3, \ldots$ and the pupil discovers how to apply the knowledge of the phenomenon for certain n to understand $n + 1$), we talk about a processual generic model, if they have a conceptual character, we talk about a conceptual generic model.

Abstract Knowledge

The generic model describes a certain piece of knowledge (term, instruction, relation, procedure or situation) with specific numbers or shapes, which are understood as generic numbers or shapes. If the generic number can be substituted with a letter, the generality can be grasped directly.

Crystallization

Crystallization is a process during which the new piece of knowledge plants itself in the consciousness of the pupil; it runs constantly and its aim is to create a sufficiently dense network between individual pieces of information.

3 Research Methodology

As stated in the introduction is the aim of this research is to analyze the cognitive process of a group of pupils of discovering the number of combinations of $\binom{n}{2}$.

A teaching experiment is used as the research method. The experiment took place during multiple sessions of a mathematics club, with the participation of a varying number of pupils aged 11–13. The club took place once in two weeks. This publication focuses on two classes of the club, which took place between October and November 2014.

3.1 Experiment Description

The experiment focused on exercises revolving around 2-combinations from a given set of n elements without repetition. Pupils sometimes worked in groups and sometimes together at a blackboard. The first session was recorded on the basis of experimenter's notes. Further sessions were video recorded, and the protocol was compiled based on these recordings. Names used in this article are fictional.

The first session with pupils leads to discovering the model of the number of combinations of $\binom{5}{2}$ in a semantic situation. By having different pupils represent the same number of combinations in different contexts, the class has a better chance to structurally understand the number of combinations $\binom{5}{2}$ by looking for similarities between the different contexts. Each context guides pupils to a certain semantic understanding of the number of combinations $\binom{5}{2}$. This creates a group of six isolated models and discovering how they reference each other is in fact looking for isomorphism which in this case is a powerful tool of discovering the generic model.

Our education scenario leads to two deep thoughts. The first is $\binom{n}{2}$, which is discovered from $\binom{5}{2}$. The second deep thought is isomorphism, which pupils get to know by discovering the sameness between models of the number of combinations $\binom{5}{2}$.

The shift in the pupil's mind, during which the pupil moves from intuitive to formally precise cognition, is studied in detail particularly in the works of Semadeni (2002).

3.2 Research Tool

The research tool creation was based on the works of Czech and international authors, including (Polya, 1962), (Hejný, 1990) and (Vilenkin, 1971). The six following exercises were created:

1. Pavel is organizing a party. One after other, Mirek, Michaela, Jana and Petra come to the party. Each newcomer clinks their lemonade glass with those already present. How many clinks are there going to be?
2. There is a regular pentagon ABCDE. How many unique lines are there between its vertices?
3. There is a security lock with five buttons numbered 1, 2, 3, 4, 5. It unlocks by pressing two specific buttons in any order with the others remaining intact. How many code combinations can this lock have?
4. Five teams enrolled in a football tournament. Every team played against each other one time. How many games took place?
5. Five pupils are awarded with two types of board games. Two pupils could be given board game a and the others would receive board game b. How many combinations are possible?

6. Annie wanted to draw the Olympic rings with her own colors – two rings colored in blue and the others in red. How many possible positions could she draw?

3.3 Analytic Tools

The aim of the analysis was to discover what was going on in the heads of participants during the club, including the teacher. In other words to specifically answer the question: Why that certain pupil said or did the one thing or the other? In some cases the analysis allows us to understand and describe deeper personality characteristics of some pupils. When this opportunity presents itself, we try to act on it.

Analysis methods:

- comparative – comparing one with the other,
- chronologic – how a pupil evolves,
- atomic analysis pupil's utterances are broken down to individual words and each word is analyzed independently (particularly important with utterances that are unclear).

4 Experiment

4.1 First Session

4.1.1 Events Taking Place

Pupils divided themselves into six groups:
1st group: Karlík, Klement, Květoslav;
2nd group: Ládík, Luboš, Leoš;
3rd group: Milan, Mirek, Miloš, Mojmír;
4th group: Olda, Ondra, Ota;
5th group: Pavla, Petra, Patricie, Pavlína;
6th group: Radek, René, Roman.

The teacher (experimenter) handed out a sheet to pupils with the six exercises (see Research Tool).

Exercises were assigned to each group; n-th exercise to n-th group. Some pupils had finished sooner, and were assigned another task: to think about the solution for their exercise with six elements instead of five. Looking at the GMT we can understand this as solving two isolated models ($n = 5, 6$). After solving these two exercises a pupil could reach at least 2nd sub-phase or even more. He/she could realize how can these two isolated models refer to each other.

After approximately eight minutes the teacher noticed that the majority of pupils had finished and called for group presentations. The teacher invited the first group to the black board and instructed the other groups to get acquainted with the first exercise.

4.1.2 Solution of the First Group

Karlík and Klement come to the blackboard.
Karlík01: At first there is only one person (writes 1 on the blackboard), then a second person arrives, clinks with the first (writes 2 and adds an arrow $2 \to 1$), then a third clinks their glass with the first and second (again updates the blackboard, see Figure 1), then a fourth (number 4 goes up on the blackboard, arrows from the bottom up), and the last party member (adds number 5 and draws the arrows from top to bottom; he does not add numbers showing the order of arrows).

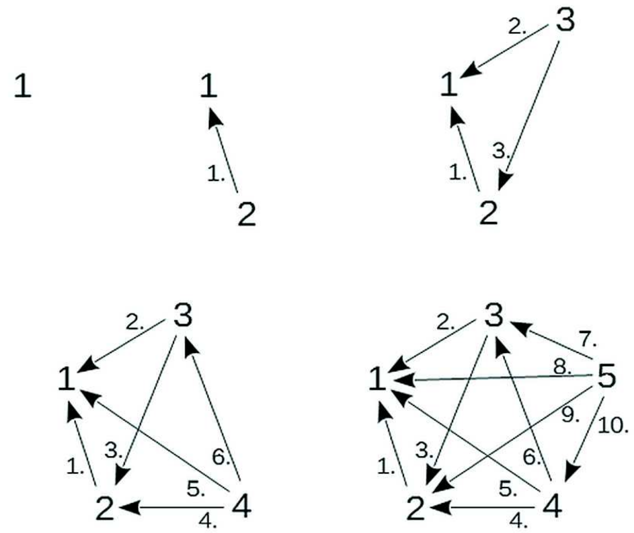

Figure 1. Schematic representation of the first exercise solution on the blackboard (numbered order of the arrows added later).

Klement01: There will be ten clinks.

Pupils from other groups agree with the solution, the first group goes back to their desks. The teacher calls the second group and instructs other pupils to look at the second exercise.

4.1.3 Solution of the Second Group

Ládík, Luboš and Leoš come to the blackboard.
Without speaking, Ládík sketches the vertices of a (regular) pentagon and connects only the adjacent vertices, see Figure 2.

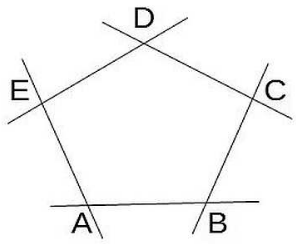

Figure 2. Schematic representation of the initial solution of the second exercise.

Luboš01: Only five lines.
Mirek01: (from his desk) Couldn't the lines be going through the pentagon as well?
Leoš01: If so, then there are ten of them.
Ládík adds the remaining lines, connecting vertices that are not adjacent.

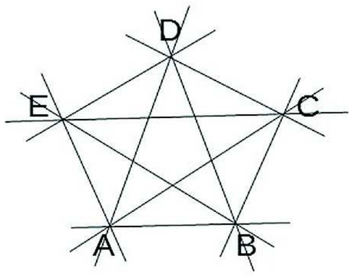

Figure 3. Schematic representation of the corrected solution of the second exercise.

Other pupils agree and the second group goes back to their desks.

Commentary:
The error by Ládík and his group was caused by inaccurate reading of the assignment and the image of a pentagon (saved in pupils' memories) which supplemented precise reading. It is possible that Leoš had considered additional lines, but may not have shared this with his group for social reasons. We consider this possibility because of the fact that Leoš immediately reacted to Mirek's remark. As a reviewer noticed: Another possible explanation of Leoš´s

remark is, that he could "just in time" think about Mirek's note "Couldn't the lines be going through the pentagon as well?" and get the quick answer because he recognized the "similarity" with the first task.

The teacher calls the third group to the blackboard and instructs other pupils to get acquainted with the third assignment.

4.1.4 Solution of the Third Group

Milan, Mirek and Miloš approach the blackboard.
Mirek01: We can press 1 and 2, or 1 and 3, or 1 and 4, or 1 and 5 (he writes the combinations on the blackboard and adds the second column without comment).
Miloš01: There are nine combinations.
Mirek02: Ah yes, there is also 4 and 5 (writes 4 and 5 to the next column), so there are ten total.

Other pupils agree and the boys go back to their desks. The teacher calls the fourth group to the blackboard and instructs the others to get to know the fourth exercise.

1 2	2 3
1 3	2 4
1 4	2 5
1 5	3 4
	3 5

Figure 4. Schematic representation of the solution of the third exercise.

4.1.5 Solution of the Fourth Group

Ondra, Olda and Ota come to the blackboard.
Olda01: Each team plays with each other, let's put it down in a table.
Ondra01: (draws the table for five teams playing one match with each other)
Olda02: No team can play with themselves (shades out the diagonal). (Then he shows the table line by line, where every team plays with each other). This cannot be, because it is already there above (points to the field in the second row and first column and shades out this field under the diagonal, continues with the shading of all fields under the diagonal).
Ondra02: There will be ten matches.

Other pupils agree and the boys go back to their desks. The teacher calls the fifth group to the blackboard and instructs the others to get to know the fifth exercise.

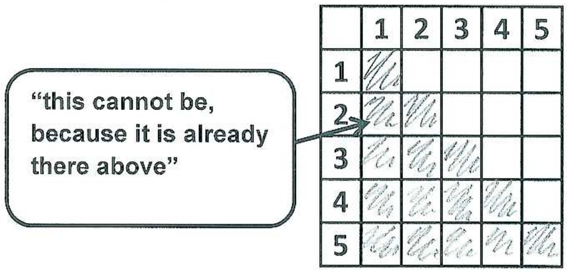

Figure 5. Schematic representation of the solution of the fourth exercise. The text "this cannot be, because it is already there above" was not put on the blackboard, Olda only said it and pointed on the board to the field in the second row and first column.

4.1.6 Solution of the Fifth Group

Two girls, Pavla and Petra, go to the blackboard.
Pavla01: We determine all possible combinations like this, because every pair gets the same game.
Petra01: Which leaves us with these options (writes on the blackboard without speaking further).

1,2	2,3	3,4
1,3	2,4	3,5
1,4	2,5	4,5
1,5		

Figure 6. Schematic representation of the solution of the fifth exercise.

Pavla02: We have ten possible combinations total.

Other pupils agree and the girls go back to their desks. The teacher calls the sixth group to the blackboard and instructs the others to get acquainted with the sixth exercise.

4.1.7 Solution of the Sixth Group

Radek (sixth grade) and René (seventh grade) come to the blackboard.
Radek01: We solved the exercise by drawing all the possibilities, there are ten.
The class protocol showed that on first glance their drawings at the desks did

not demonstrate any systematic approach. Unfortunatelly, we have no picture and less information about work of this group.

The pupils sit back at their desks and a discussion of the solutions follows.

4.1.8 Pupils' Discussion of the Solutions

Once the boys from the sixth group sat down, Ládík started a discussion on the solved exercises without the teacher giving any instruction.
Ládík01: The exercises are the same, there must be some formula for this.
Mirek03: Keep your formulas.
Teacher07: If you think they are identical, for the next class try to think why. Is it the result, solution, procedure, assignment?
Most pupils try to comprehend the teacher's task, some quietly speak to each other, but nobody speaks up. The teacher continues with another task.
Teacher08: And what about six elements? (The task applies to all groups, but the situation is different for each group, because the tasks are different. Soon enough, Klement speaks up.)
Klement02: It will be five more. (His statement is not specific to clinking glasses, it covers all exercises. For each one, Klement's answer is correct.)
Teacher09: Is that so? The others agree?
(Only at this moment, Klement takes his reasoning to his own exercise about clinking.)
Klement03: It is, because the sixth one must clink their glass with the other ones that came before him, so it's fifteen.
(Some pupils understand Klement's argument and are able to relate it to their own exercise. The teacher notices the agreement of these pupils.)
Milan01: It must be fifteen, because it was ten for two, so ten divided by two is five and three times five is fifteen.
Ládík02: But we don't have triplets, there are six.
Milan02: Oh, it must be different then.
(The class continues with different activities.)

Commentary:

The last consideration is not semantically grounded. It shows that Milan's need to semanticize structural relations is undernourished. A structural cluster of numbers and relations emerges in his mind. Ten plus five is fifteen. Ten divided by two is five and from this point Milan continues that fifteen divided by three is five or three times five is fifteen. Ládík precisely reacts to the last relation presented by Milan. He points out the absence of semantic grounding of the number three. From the quick reaction of Milan it is obvious that he is aware of his weak abilities to connect semantic and structural relations.

The first entry of Ládík has become a subject of more thorough examination and led to the results presented in chapter 6.1.2. Ládík demonstrates his assumption. I remember similar reactions from other students (secondary school and university) and the phenomenon has been confirmed by other colleagues of mine including Hejný, who realized he had missed it in his previous analyses, and was grateful for pointing it out. There are pupils who have either encountered the expected piece of knowledge beforehand or expected it based on their experiences.

4.2 Second session

4.2.1 Events Taking Place

There are fewer pupils and not all the groups are composed the same way as during the first session:
1st group: Radek, Milan;
2nd group: Karlík, Klement, Květoslav;
3rd group: Olda, Luboš, Leoš;
4th group: René, Dominik, Ondra;
5th group: Pavla, Patricie, Pavlína.

On the onset there is a discussion regarding the conclusions of solving the six combinatorics exercises from the first session. It cannot be discerned from the video recording which pupil is talking at this stage, therefore we use the generic boy or girl.
Teacher01: Are you in the same groups as last time?
Class: Yes. No.
Teacher02: Very well. What did we do last time? What were you supposed to think over?
Boy: See what those exercises have in common.
Teacher03: It was you, well at least you told me that the exercises have something in common.
Girl: The results.
Teacher04: At least they had the same results?
Boy: Formulas.
Teacher05: But we didn't use any formulas, were there any formulas?
Boy: The principle of solving them was the same.
Teacher06: So you say the principle was the same. During today's task, I will assign two exercises to each group and your task will be to find out whether the principle to solve them is identical. Do these exercises always have to have the same solution principle if they have the same results? There are various

exercises and you are to decide, if the principle to solve them is the same and convince others of your findings.

Afterwards pupils in groups selected pairs of exercises in a way to include an exercise they had already been solving during the first session.
First group – 3^{rd} and 6^{th} exercises;
Second group – 4^{th} and 6^{th} exercises;
Third group – 2^{nd} and 4^{th} exercises;
Fourth group – 5^{th} and 6^{th} exercises;
Fifth group – 5^{th} and 1^{st} exercises.

Work in groups continues for about six minutes. The teacher asks pupils of the first group to come forward to the blackboard and start by saying only which two exercises they are solving (everyone has the details of each exercise on paper in front of them).

4.2.2 First Group's Solution – Third and Sixth Exercises

Boys from the first group, Milan and Radek, come to the blackboard.

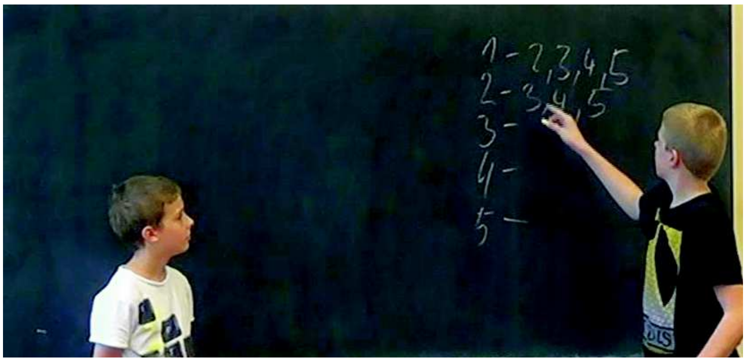

Figure 7.

Milan02: We had exercises three and six, we solved the third one by writing down numbers and their possible combinations. (On the blackboard he writes numbers from 1 to 5 below each other and dashes next to them). To number one we added two, three, four and five (after the dash he adds 2, 3, 4 and 5), to number two we added three, four and five (see Figure 7), not one, because that combination is already there (points to the line above), number three goes with numbers four and five, and number four only with five, so there are ten combinations.
Radek02: And the sixth exercise, the one with the Olympic rings, so we draw them and number them and we can write it the same way. (see Figure 8)

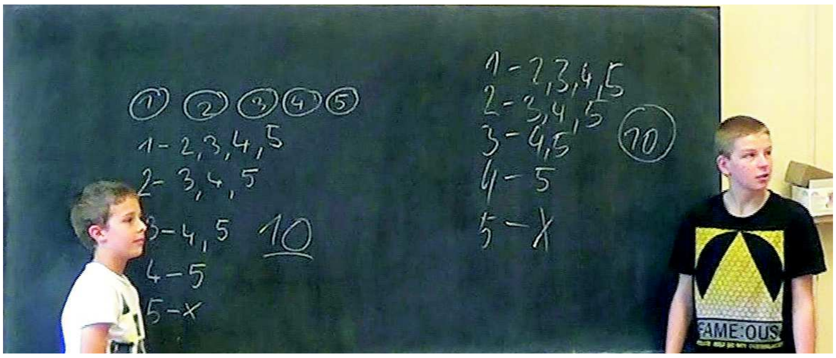

Figure 8.

Teacher14: So the principle is the same?
Class: Yes.

The teacher invites the second group to the blackboard and reminds others it is the second group who decides what happens at the blackboard.

4.2.3 Second Group's Solution – Fourth and Sixth Exercises

Karlík, Klement and Květoslav come to the blackboard.

Karlík starts drawing the table silently and then starts commenting.
Karlík01: These teams couldn't play against each other, not these two, neither these, etc. (crosses the diagonal out) and these two have already played, so that's one and these two won't work either (writes down number 1 in first line first column and crosses out the second line field of the first column, see Figure 9), same for this one (writes down number 2 in the first line of the third column and crosses out the symmetric field, then number 3, cross out of the symmetric field and number 4, crossing out the symmetric field), (crosses out all the diagonal fields and one by one adds the numbers 5 to 10).

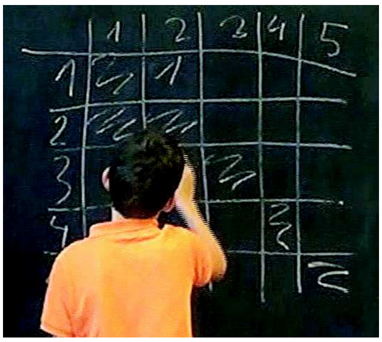

Figure 9.

Klement01: And now these can be football teams (pauses for one second) or numbers of those rings (meaning the Olympic rings), so it's the same.

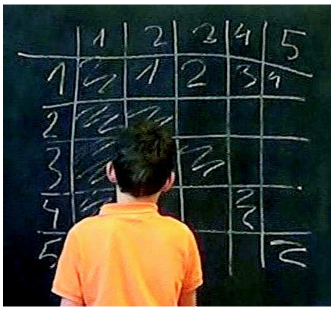

Figure 10.

Teacher17: So you say it can be either football teams or numbers of the rings, do others agree?
Class: Yes and don't wipe it off.

The teacher invites the pupils of the third group to the table.

4.2.4 Third Group's Solution – Second and Fourth Exercises

Olda, Luboš and Leoš go to the blackboard.

Olda draws a pentagon. Then creates a table next to it with numbers 1–5 and Luboš wipes off the pentagon (irregularly shaped) and draws a new one. Olda wipes off the numbers 1–5 and replaces them with letters A, B, C, D, E. Luboš adds lines inside the pentagon between the individual vertices.
Olda crosses out the fields in diagonal and the pairs with a line (see Figure 11). Teacher19: Do you understand this (to the class)? I don't understand, why is this the same?

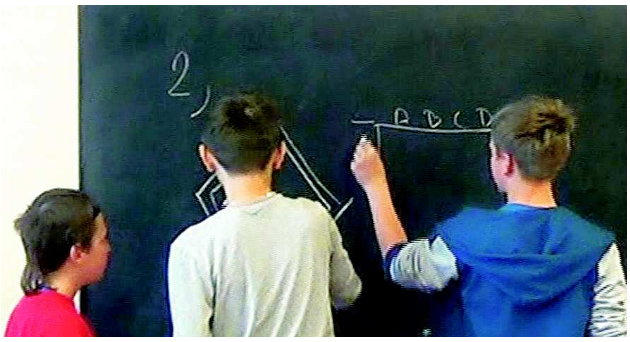

Figure 11.

How are these two connected? (Points to the pentagon and then to the table on the right).
Luboš adds letters A, B, C, D and E to the vertices.

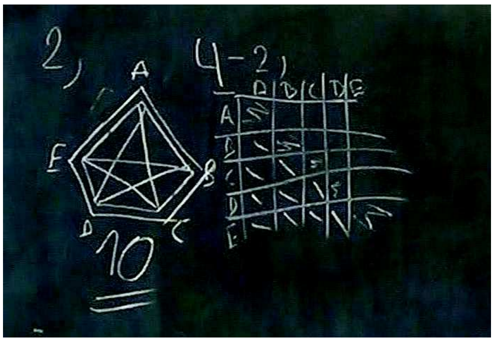

Figure 12.

The classroom starts humming.
Class: It is clear, it's the same.
Teacher20: Do you agree it's the same?
Class: Yes.

Pupils go back to their desks and the teacher invites the pupils of the fourth group.

4.2.5 Fourth Group's Solution – Fifth and Sixth Exercises

René, Dominik and Ondra come to the blackboard.

René01: There are five pupils getting two types of games, so everyone gets a game, but they can't all have the same game. So there were ten ways they could get the games, and should I draw it up?
Class: Draw it up.
René marked pupils with numbers 1–5 and tries to draw how games a and b could be distributed among them.
René added letters *a, a, a, a, b* to the first line under the numbers, and as he wrote them he read them.
Dominik01: But there are two as and two bs.
René02: There are two types of board games.
René and Dominik reread the exercise assignment. Afterwards Dominik wipes off the as in the third and fourth columns and René replaces them with bs. René then finishes the additional five lines (see Figure 13) with Dominik and Ondra helping out.

Figure 13.

René specifies the ordering of lines by adding 1^{st}, 2^{nd}, 3^{rd}, 4^{th}, etc. to the beginning and adds a 7^{th} line.
Teacher21: Boys at the back don't understand this entry, what does it mean?
Dominik02: (to René) Well explain it to them.
René03: Well there are two games.

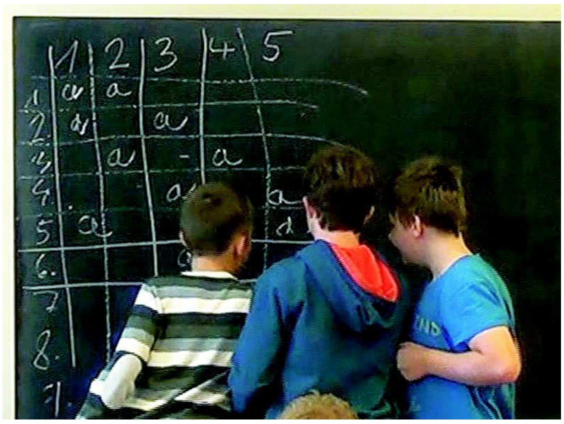

Figure 14.

The recording does not catch the quick chat between the boys. It seems to have been covering the redundancy of the *b* letters, because Dominik starts to wipe them off, then the boys add vertical and horizontal lines, and together add only the *a* letters to the table. (See Figure 14)
Teacher22: Explain the drawing.
The boys are unable to explain it, René only notes that five times two is ten.
Dominik decides to draw another table, in the horizontal heading he uses A1,

A2, B1, B2, B3.
Dominik03: Those are games *a* and *b*.

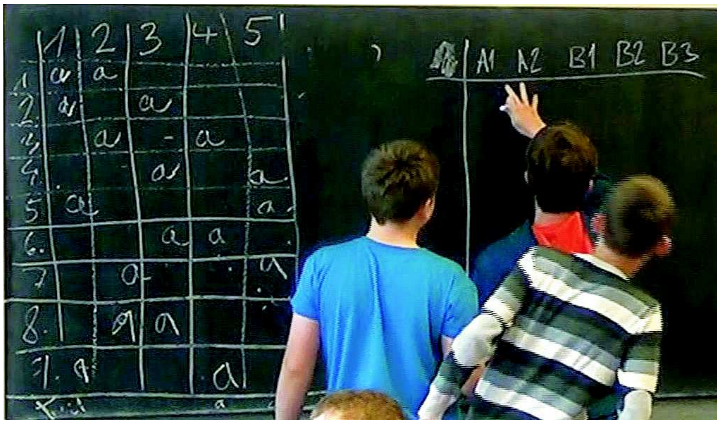

Figure 15.

Here are the pupils, so the first pupil (writes 1 in the left part of the table), second, third, fourth and fifth pupil (to the vertical heading he writes 2, 3, 4, 5 below each other and adds horizontal lines).
Dominik notes the first possibility with crosses in the diagonal line, meaning the first pupil has game A1, second pupil has A2, third pupil B1, etc.

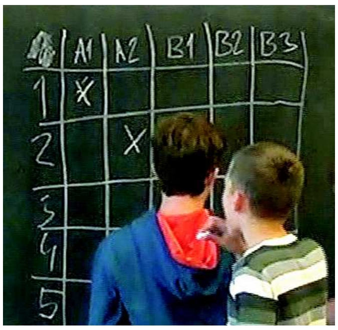

Figure 16.

Dominik then runs out of ideas how to mark the next possibility, so he wipes off the crosses and starts using numbers, i.e. he marks the first possibility with 1 in diagonal: first pupil gets an A1 game, second pupil an A2 game, third pupil a B1 game, etc., the next possibility he marks with 2s.

Figure 17.

Then Dominik tries to write 3s and the boys realize it collides and does not work, and Dominik erases the whole table and writes down number ten. The general noise covers specific words of pupils.
Teacher23: So you will think this through some more. (Towards the boys at the blackboard.)
The boys return to their desks.
Karlík03: And why you didn't do the first picture at least a bit.
René02: But that's how it simply works and it's that and it's ten possibilities (resolutely, not wanting to give more attention to their exercises).
Teacher24: So the fifth exercise, the one from the last session, you will think through for the next class.

The teacher invites girls from the fifth group to the blackboard.

4.2.6 Fifth Group's Solution – Fifth and First Exercises

Pavla, Patricie and Pavlína go to the blackboard.
Pavla01: So I draw this picture (draws numbers 1-5 in a pentagon and connects 1 with 2, 3, 4 and 5 and quietly talks to herself, then continues with number 2, 3 and 4, then counts the lines for herself first on the perimeter than on the inside).
Voice from the class: I don't get it?
Laughter starts in the class.
Teacher25: Let's not laugh at this, let's play nice.
Patricie01: (goes to explain the picture) The last newcomer clinks their glass with the other four already present. (points out to lines 5–4, 5–3, 5–2, 5–1) (See Figure 18).

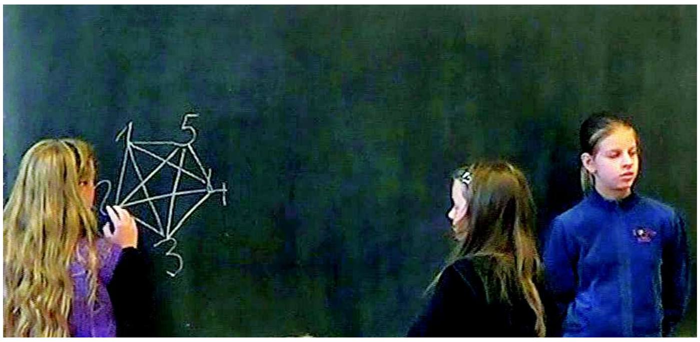

Figure 18.

In the same way the fourth clinks with the tree, the third with the two and the second with the first (pointing to the lines in the picture).
Teacher26: And the fifth exercise? How does that come into this?
Pavlína01: Well we have the pupils (draws numbers 1–5 again as pentagon vertices). First could go with the second, third, fourth and fifth (connects the first vertex with the second and then third, fourth and fifth). The second with the third, fourth and fifth, and the third with the fourth and fifth, the fourth with the fifth (step by step she connects the vertices).

Figure 19.

Radek03: (from the first group) But that's the first exercise.
Teacher27: Because there's some confusion, read the fifth assignment again.
Patricie reads the assignment.
Radek04 and others: It's not written there what they get.
Teacher28: (repeating what rings out among the classroom noise) Boys are asking how to tell from that who gets which game. You said the first with the second, so explain what that means.
Pavlína adds a above the line 1–2 and then erases it.
Teacher29: You say it's correct (to the boys of the second group) so come and explain.

Luboš04: The first could get the game with the second, third, fourth, fifth pupil and so on.
Teacher30: So what does that line mean? (points to line 1–2)
Luboš05: That they got the same game, game a.
Teacher31: So they got game a and what about the others getting game b?
Boys: They don't need to be there as they are a complement.
Teacher32: They say it doesn't need to be there. Does or does it not?
Boys from the class: It doesn't.
Teacher33: Very well then, sit down.

5 Analysis of the Experiment

5.1 Analysis of the First Session Experiment

5.1.1 First and Second Sub-Phases of Isolated Models

During the first session all pupils are at the level of the first sub-phase while solving their tasks. Each pupil solves the exercise that has been assigned to their group. This task, in their mind, is the first isolated model of the number of combinations, and they do not know of any relation to the tasks of other groups.

Pupils directly joining the explanation certainly understand their task. The following discussions among pupils point out that every pupil at least understood their task. However no conclusive evidence is available for pupils.

The pupils reach the second sub-phase the moment they understand the explanation of a pupil from another group presenting their solution to their exercise. At the time the first group shows their solution at the blackboard they are still in the first sub-phase as opposed to the other pupils, who have the opportunity to move to the second sub-phase if they understand the solution being given at the blackboard. Whether some of the pupils have already moved to any of the following sub-phases cannot be recognized at the time. It is possible that some of the pupils already see how the exercise being presented at the blackboard refers to the one they have been solving, but no pupil have displayed that.

Pupils from the second group would have been closest to reaching at least the third sub-phase at this stage, had they correctly understood their assignment. Although the solution of the first exercise was processual at first, the final representation corresponded with its concept[1]. The schematic represen-

[1]The matter of process and concept developed differently with different authors and the meaning of these terms shifted slightly. The understanding of process and concept in GMT

tations of the solution of the first and second tasks, as presented by the pupils, are strongly similar (see Figures 3 and 5). The reason why the pupils of the second group could not discern the "sameness" or "similarity" of their solution to the solution of the first group at the time the first group presented theirs at the blackboard is that they had only found five lines at first (see Figure 2).

Concerning the pupils of the third group, some of them could see the "sameness" of some elements of their solution and the solution of the first exercise. The graphical representation in this instance is completely different, but the method of how they reached the solution is identical at least in the fact that both groups are looking for pairs among five elements. Karlík says: "First there is one, then the second one comes and clinks with the first." Which is the "same" as taking number one and adding number two with the lock exercise. The processual side of the solution is similar for both exercises. To aid seeing the "sameness" by some of the pupils from the processual point of view, it would be beneficial if the numeric pairs on the lock corresponded with the order of pairs clinking their glasses at the party.

Similarly, some of the pupils from the fifth group could have discovered the "sameness"; their solution could also correspond with the one from the first exercise. In their minds the following conception of the process to solve the fifth task could form: Add the first and second pupils to game a, the other pupils obtain game b; a process the pupils from the fifth group did not take into account.

The methods of solving the first and fourth tasks refer to each other substantially less than the above mentioned comparisons. Using table is conceptual. The only reference which may have formed in the minds of the pupils of the fourth group is the following one: When going through possible matches, moving in the table in rows from left to right, top to bottom, then the first match is played by second team with the first team, which is similar to second partygoer clinking their glass with the first one, etc.

The reference to "sameness" is weakest for the pupils of the sixth group. By drawing all ten options unsystematically they could not see any "sameness" of the solution with the first task. If they listed the options by coloring rings from left to right as first and second, then first and third, first and fourth, first and fifth, then second and third and second and fourth, finally finishing with fourth and fifth, some pupil could have realized to number the rings in their mind and see the "sameness" with the first exercise solution. However, the unsystematic solution was one of the obstacles in this case which prevented

theory is inspired by the works of authors including (Dienes, 1960), (Gray, Tall, 1994) or (Sfard, 1989).

seeing the reference to the "sameness" of their solution to the first exercise solution.

After the second group presented their solution, it is highly probable that more pupils move to the second sub-phase of isolated models. These include pupils of the first group who understood the solution of the second group. However also pupils from other groups who may have not understood the solution of the first group but understood the second one are included. Based on the numerous discussions happening between pairs and triplets of pupils, it can be expected that most of the pupils are at this phase in the second sub-phase of isolated models. We do not have evidence of whether any pupil have already moved to the third sub-phase, but there are numerous reasons that increase the probability.

1. *A pupil of the first group pointed out the "sameness" of their solution with the second exercise.*
2. *A pupil of the second group, after finding the right solution to their exercise, saw that the solutions of the second and first exercises refer to each other.*
3. *A pupil of the third, fourth, fifth or sixth group, who did not see their solution referring to the first or second exercise, saw that these two exercises refer to each other.*
4. *A pupil of the third to sixth group saw that their exercise and the second exercise refer to each other.*

If a pupil left the process of solving the first exercise out of consideration, i.e. left out the succession of pairs that happen during clinking, and also replaced the arrows in chart in Figure 1 with lines, their conceptual understanding of solutions of the first and second exercises would overlap more profoundly. This pupil would almost certainly move to the third sub-phase. This situation includes pupils mentioned above in reasons 1–3.

Reason number four is a special case. If we take the corrected solution of the second group, which is graphically similar to the solution of the first exercise, we can assume that the reference between the exercises 3–6 and the second one is similar to exercises 3–6 referring to the first exercise, as has been analyzed above.

Another phenomenon that is difficult to register by the experimenter, is pupils solving the exercise at their desks differently from what they present at the blackboard. Specially if these pupils do not feel the need to show their solutions to others. It is the various angles to the solution of one exercise that could move more pupils to the next sub-phase of isolated models. None of the pupils had this need after solutions to the first and second exercises had been presented, and therefore the experiments remains to be based on the solutions demonstrated at the blackboard.

After the third group (third exercise) demonstrated their solution, the following pupils were at least in the second sub-phase:

1. pupils from the first, second and third groups who understood at least two of the three presented exercises,
2. pupils from the fourth, fifth and sixth groups, who understood at least one of the presented exercises and their own, solved at their desks.

After the third group's presentation we can distinguish eight different pairs of exercises referring to each other: (1–2), (1–3), (2–3), (1–4), (1–5), (1–6), (2–5), (2–6). We have already analyzed these. Now let us have a look at the next pairs of isolated models, which pupils are able to see “side-by-side”:

a) Solutions of the second and fourth exercises are understood conceptually by pupils. However, it is not easy to see their connection. A pupil would need to figure out that a respective line (between two vertices) corresponds with a respective match (between two respective teams), which is a fairly difficult task given the completely different graphic representations.
b) The solution of the third exercise is processual, the solution of the fourth one is conceptual. Their graphic solutions are widely different as well. A pupil could advance to the third sub-phase if they realize that selecting 1 and adding 2 to a lock combination is the “same” as selecting a match between the first and second team. Meaning that in their mind the pupil processualizes the conceptual table.
c) The solution of the fifth exercise is processual, as is the solution of the third exercise and their graphic representations are nearly “identical”. It is highly probable that some pupils of the fifth group already see the reference between their solution and the solution of the third exercise during its presentation. Possibly advancing even further as to see the “sameness” of their solution and the third exercise solution.
d) The solution of the sixth exercise by randomly drawing out all the combinations does not give any cues to the pupils to follow and discover any references to the “sameness” with the third exercise solution.

   Even after the third group presented their solution, the teacher had no indication that any of the third group pupils advanced further than the second sub-phase.

Further analyses concerning references between exercises will cover only those pairs of exercises that had not been analyzed yet. For this reason, we omit the reference of the fourth exercise to the first three exercises.

We focus therefore on comparing the solution of the fourth group with solutions of the fifth and sixth groups who have not presented their solutions yet. The pupils of the fifth group solved their exercise in a similar way as pupils from the third group, therefore the reference between the fifth and the fourth exercises is similar to the fourth and the third. Perhaps the least referring pair of exercises is the sixth and the fourth one. The probability of discovering the "sameness" between drawing out all the options of the solution of the sixth exercise and registering all possible matches in a table (solution of the fourth task, see Figure 5) is highly unlikely for the pupils of this age.

Pupils are able to advance to the third sub-phase in connection with pairs of exercises other than the one they had been solving at their desks. Furthermore the four already presented exercises had the same result – ten. Even then, none of the pupils felt the need to point out that the exercises were the same in this aspect.

None of the pupils displays advancing to the third sub-phase even after the fifth group's solution has been presented. As mentioned above, the approaches to the solution of the third and the fifth exercises are the most closely related ones, both from the processual point of view and the way they are graphically represented (see Figures 6 and 8).

Pupils of the sixth group have only described their solution at the blackboard and other pupils agreed that the solution was correct. We do not know how certain they were. We also do not know, if any of the pupils of this group checked the solution in any way, thus being able to discover any relation with the solutions of previous exercises. It seems that to agree with the solution, the children only needed to know that the answer to the sixth exercise was ten, which is the same answer as for all the previous exercises.

At this moment of the experiment, it is likely that all pupils are at least in the second sub-phase of isolated models. Although we do not have the evidence, the probability of a pupil not understanding at least two of the presented isolated models is low. Even though we do not have evidence that some pupils have already advanced to the third or possibly even the fourth sub-phase, it is highly probable.

5.1.2 Third and Fourth Sub-Phases of Isolated Models

Judging by the 01 input of Ládík straight at the beginning of the discussion it can be concluded that he felt that some isolated models refer to each other and advanced to the third and fourth sub-phases. It is possible he had moved to the third sub-phase sooner than that, but unfortunately he did not feel the need to share it with others. Ládík already senses the "sameness" of some isolated models of the number of combinations 5 choose 2. His statement

The exercises are the same, there must be some formula for this.

demonstrates that the fourth sub-phase of isolated models contains two mental activities. It also shows that Ládík has either discovered the "sameness" or sensed it and tries to arrive at a conclusion. There are two ways to discover the sensed "sameness":

1. Discovering the sensed "sameness" by one's own examination.
2. Learning about the "sameness" from external source.

Mirek reacts to Ládík's statement with "Keep your formulas". We cannot know for sure whether Mirek has entered the third sub-phase, but he will surely belong to the group of those trying to discover the sensed "sameness" for themselves. The next input (02) from Ládík even suggests that he has already discovered the "sameness", as he understands that at least in some of the exercises pairs out of five elements were being created. We do not know, however, if he sees the "sameness" in all the exercises, or only senses it in some pairs of exercises.

Another pupil entering the discussion with an interesting insight is Klement with input 02. It cannot be judged from his statements whether he sees isomorphisms between individual isolated models of 5 choose 2. We can say with certainty, however, that he has advanced to the third, possibly even fourth sub-phase of isolated models of $\binom{n}{2}$, connected with the exercise in which friends attending a party clink their glasses with people already present (see first exercise). Klement sees the connection in that there were five friends coming in one by one and clinking their glasses with those already present. He deduced that if there are six friends then the exercise for five people applies (which he knows already, isolated model for five elements) and the sixth person clinks their glass with the other five. It is possible he can already see how this type of exercises (first exercise) would work for 7, 8, 9, …15 …friends. At this moment we do not have evidence of that.

5.2 Analysis of the Second Session Experiment

5.2.1 Third, Fourth and Fifth Sub-Phases of Isolated Models

At the beginning of the second session of the mathematics club, pupils are at various sub-phases of isolated models. Certainly there are some who have advanced to the fifth sub-phase and possibly even further. To allow other pupils to advance to this sub-phase, the aim of the first part of the class was to discover isomorphisms between selected pairs of exercises.

That some pupils are at least in the third sub-phase of isolated models is demonstrated by their statements in the introductory discussion; when asked what the exercises had in common, one girl said "the results" and a boy added that "the principle of solving them was the same".

First group

In the first group there were Radek from the sixth grade, who was solving the sixth exercise in the first session, and Milan from the seventh grade, who was solving the third exercise. Pupils may found the "sameness" in:

a) objects – one button on the lock corresponds with one Olympic ring. Numbers on the lock are clearly individualized, each number is clearly stated as a symbol and creates the given structure. The rings are individualized by their position in the design (we could describe them as upper right ring, etc. – they are not standardized, we need to create the names).

b) processes/pairs – pressing numbers 1 and 2 is the same as coloring the first and second rings. This understanding requires previous individualization of objects.

Individualization of the rings is ponderous and unsuitable for describing the required isomorphism. Therefore Radek chooses different coding of these objects, as seen in Figure 8. The inconvenient coding by position is changed to suitable coding by numbers. The rings which are assigned numbers remind us of the origin of these objects. Radek introduced the above mentioned recoding spontaneously, which demonstrates his developed communication in the field of mathematics. When Radek carried out this mental step is not precisely known. He may have realized it already towards the end of the first session, or during the joint solving of the third exercise with Milan, and there is also the possibility that Milan proposed this approach while they were solving the sixth exercise together. The significant role here was played by the coding and exploratory power of language[2].

With the new coding there was no problem to list all the ten possible combinations and by comparing the list of possibilities of the third and sixth

[2]The power of language plays a significant role in the individual phases of the cognitive process of the pupil. The power of language is universally explored in the works of professor Kvasz (2008), which the work of professor Hejný follows.

exercises, their isomorphism is clear to everyone following. At this moment, it is certain all pupils enter the fifth sub-phase.

Second group

Understanding the fourth and sixth exercises by the pupils of the second group develops in phases; at first they solve the fourth exercise using a table. After the presentation of the sixth exercise solution by the first group, the second group understands that the rings can be numbers and shows that the football table solves the sixth exercise as well.

If we compare the first and second groups, we see the following difference. With the first group, both exercises are placed next to each other, none dominates the other. The second group in contrast, although they solved the first exercise during the first session using a pentagon, at this phase (second session) the table clearly dominates. With it, it was enough to introduce the correspondence of elements, the correspondence of pairs was not mentioned, possibly because the ring exercise was well understood by the pupils. Table was what the groups knows well from real life. The sixth exercise have been significantly transformed. With the pupils of the second group, the natural metacognitive need to search among isolated models for the dominant one has appeared.

Considering the lack of information we need to consider the following:

a) pupils worked at their desks the same way as they present at the blackboard,

b) at their desks, they solved both exercises separately and then looked for ways to solve the exercises the same way, and table representation was dominant.

At the end of their presentation, a voice from the class said "*don't erase it*", which shows that a pupil have not yet adopted Klement's idea and has to ponder the argument further to advance to the fifth sub-phase in regards the "sameness" between the fourth and sixth exercises.

It is possible there are pupils in the class who have progressed further than the fourth or fifth sub-phases, but we do not have this information from any of them.

Third group

The presentation of isomorphism between the second and fourth exercises given by the pupils of the third group is completely different from the previous two groups. Pupils of the third group first solve the second and fourth exercises independently, as they were solved in the first session. For the second exercise they draw a pentagon and add the lines without marking the vertices. The fourth exercise they solve using the table. Olda first marks the objects in the table with numbers, the way he has probably learned during the first session. Then he moves to marking the teams with letters A, B, C, D and E. From this modification we can deduce they had teams marked with letters while working at the desks, and had pentagon vertices marked the same way, and realized the coding power of language. The fact they did not mark the vertices on the blackboard proved to be the critical obstacle to discovering the "sameness" between the two isolated models. The pupils at the blackboard see the "sameness" of the solution of the two exercises. In the classroom there are probably those who see the isomorphism and those who do not. Even in the second session, some pupils still do not feel the need to speak up when they do not understand something, possibly to avoid drawing attention to it. Pupils at the blackboard discover the "sameness" of objects by adding the letters A, B, C, D and E to the vertices of the pentagon as a reaction to teacher's question "*I don't understand, why is this the same?*" addressed to both the pupils at the desks and at the blackboard. After this addition there were no additional questions from the pupils, the isomorphism of the relation between the (pairs of) exercises must have been discovered by most pupils. The coding power of language helped uncover the argumentative power of language.

Adding letters to the pentagon helped other pupils move to the fourth and fifth sub-phases of isolated models.

Fourth group

Pupils of the fourth group were supposed to show the isomorphism between the solutions of the fifth and sixth exercises. However, the problem arose with solving the fifth exercise. They have probably tried different approaches at their desks, of which we have no record unfortunately. We do not know whether they have discovered the solution presented at the blackboard already at their desks or made it on the spot, as suggested by the course of their presentation. The pupils solve the fifth exercise completely differently from the solution of the girls in the first session. The girls solved the exercise by counting out pairs. Apparently no pupil of the fourth group seems to remember that solution. This time, the boys tried using a table. They marked the pupils in horizontal heading with numbers 1, 2, 3, 4 and 5. Beneath that, they enumerate possibilities of pupils receiving games a and b. In the first line, René incorrectly divides the games; a result of incorrect understanding of the task. After Dominik's notice, he corrects the mistake. They enumerate six variants (add six lines) and add numbering of the lines to the column heading. Their solution suggests they do not have any system to determine the pairs. The solution strategy at this phase was to enumerate all possibilities of distributing games a and b. Two impulses elicit change in their approach. The first impulse was external – a question from the class "*I don't understand*". The second impulse was internal – we must make this more comprehensible. A new strategy is being decided. The first thought of the new strategy manifests itself in the language shift, to determine pairs they only need to assign game a. They realize that assigning game b is complementary and unnecessary. The second thought is to make the table more clear by adding vertical lines dividing the columns. The third though is to add horizontal lines dividing the lines. Then they add the remaining possibilities totaling ten options. All these steps further confirm our notion that the solution presented at the blackboard is different from their solution at the desks. There is no information whether the boys see the isomorphism with the solution of the sixth exercise.

Another external impulse presents itself when they are asked to explain their presentation. The pupils realized they were able to make their first solution more comprehensible two times and decided for a new representation. The second attempt was not successful however; pupils were content with stating that "*The correct result is ten*".

If René, Dominik and Ondra have not reached the third and higher subphases during the presentations of previous groups, they could not have reached them with their pair of exercises.

Fifth group

The approach to presenting the isomorphism between the first and the fifth exercises by the girls of the fifth group is similar to the one of the first group. Patricie and Pavlína demonstrate the "sameness" of the two exercises with a pentagon model. The strategy has changed compared to the first session, when the girls solved the fifth exercise by enumerating all possible pairs. The reasons leading to this change could be as follows:

1. The previous strategy is unwieldy and ponderous while the new one is clearer.
2. Change of strategy, because the new strategy is more suitable for the further development of the idea.
3. Change of strategy, because the new one is already the generic model.

In case of the first exercise the friends arriving at the party are represented by the vertices and the clinking by the lines between them. The girls adopted the same model for solving the first exercise as the first group during the first session. Patricie already understands the exercise conceptually because during the presentation she begins at the end; by the fifth guest clinking the other four, then the fourth with the other three, etc. To represent the other exercise, Pavlína draws another pentagon, which is identical to the first one, but the numbers are shifted. In this case, the pupils receiving the games are represented by the vertices. From this approach by the girls, we can estimate the change of strategy was driven by the second reason, and possibly the third; i.e. because they find the new strategy more suitable and it is possible it has turned into a generic model for them. Some pupils from the class however do not see the solution of the fifth exercise, but still the first, as demonstrated by Radek's statement: "*But that's the first.*" (exercise). For pupils, the obstacle is the two games mentioned in the assignment, while Pavlína only talks about pairs of pupils and not differentiating between the games received. The next input of Radek confirms that: "*It's not written there what they get.*" At this moment we can divide the pupils to three groups:

1. Those who see the solution of the fifth exercise in the second pentagon, and therefore the isomorphism of the two exercises.
2. Those who sense the solution of the fifth exercise presented on the blackboard.
3. Those who do not sense the solution of the fifth exercise.

Pavlína reacts to Radek; first she adds the letter a to the pair represented by the 1–2 line in the graph, then wipes it off. The reason being that after marking the first pair with a, Pavlína wanted to continue with another pair, but she would mark it again with letter a. Then she would have two as, which could mean that these two pairs would both get game a so she wipes it off. In the classroom there are pupils who see the solution of the fifth exercise, as is represented graphically on the blackboard. This is evident by input04 by Luboš and following statements by other pupils. They show that each line represents one way of dividing the games, i.e. game a is given to the two pupils whose numbers are connected with lines. Pupils themselves then reason that the game b does not need to be added since it is a "complement".

At the end of this series of solutions, pupils can be found in various subphases of isolated models. There are pupils able to provide isomorphisms between any two exercises, and have therefore progressed further than the fifth sub-phase of isolated models. However, there are also pupils in the fifth subphase, because they see the isomorphisms between some pairs of exercises, but only sensing it between other exercises.

5.2.2 Transition from Fourth and Fifth Sub-Phases of Isolated Models to Generic Model

Whether any of the pupils present at the beginning of the second session were already in the phase of the generic model we do not know and have no information about. At that moment we can only deduce which of the isolated models could become the generic model.

Where pupils of the first group treat both their exercises equally, pupils of the second group clearly prefer the exercise solved using a table, i.e. the fourth exercise. They also used the table to solve the sixth exercise. How well they could have used a table to solve the other exercises is not known. It is probable however that they would be successful in most cases. That way, the fourth exercise becomes their generic model.

Pupils of the third group used two different models to display the isomorphism. Which of those two could become the generic model cannot be said. It is possible that some pupils of the third group would choose the first model, while others would prefer the second one.

For pupils of the fourth group, based on their presentation, we may suppose that their generic model is the incorrect "*five times two is ten*", which they repeated several times.

Girls of the fifth group solved the first and fifth exercises with the model used by the first group during the first session. The model seems to be most accessible for them and could become their generic model.

At the end of this part of the second session some pupils (probably the majority) have already entered the phase of the generic model. However, we do not have sufficient evidence on which pupils have and which have not.

6 Discussion

6.1 Discussion of the Analysis of the First Session Experiment

Whereas the first and second sub-phases can be easily registered by the observer, the third sub-phase is more complicated. It begins at a different time with each pupil. The evidence of a pupil reaching the third sub-phase presents itself to the observer (experimenter) only when the pupil demonstrates it in some way. This is connected with how much pupils are used to sharing their speculations. Not even the shared result (10) guided them to share. If the pupils do not share their findings the experimenter ends without the evidence, but it is probable that at least some of them realize the "sameness" of exercises, but their meta-cognition does not include the fact that "sameness of situations is key to mathematical discoveries". At the same time, assertions of "same method of solution" are triggered by identical results, despite exercises not being solved in the "same" way.

Some video recordings, for example from the classes of Jitka Michnová, show that pupils feel the need to uncover "sameness" in situations in which there is none. An example from another class (from the archive of M. Hejný):

Second grade: Children build towers. The first group has two white and two black cubes – they build six different towers. The second group builds towers from five white and one red cubes – they build six different towers. One girls says: "It is the same." Teacher asks: "Is it the same?" Discussion ensues and the conclusion: "Result is the same, but the situation is not."

In cases pupils do not feel the need to share their discoveries with others, we must ask "Why is it so?" and what are the possibilities of reeducation.

The reasons why pupils do not disclose any discoveries probably include their low need to share their assumptions or new discoveries and possibly concerns of stating something incorrect. They are usually not used to that in their traditional schoolwork, and because this has been their first class of mathematics club, it will take more time for them to consider this behavior of sharing as something natural.

The abovementioned experimental experience points out a serious didactic insufficiency of the traditional mathematics teaching. The phenomena of affinity and even isomorphisms are not viewed with the same level of significance as they actually play in mathematical discoveries.

6.1.1 Obstacles to Pupils Advancing to the Third Sub-Phase of Isolated Models

One of the frequent obstacles for pupils trying to advance to the third sub-phase of isolated models is the situation when some of the isolated models are solved incorrectly, because in that scenario the models do not refer to each other, as was the case with the original solution of the second task.

Another obstacle is a different approach to the solution of an exercise. It can be a difference in approaching the exercise during its solving, e.g.:

- *process – concept,*
- *way of approaching it: graphic – table – extract,*
- *language used: abstraction of specific elements in only one of the pair of exercises.*

Sometimes the pupil may approach the solution of two exercises in a processual way and even use the same notation for both cases (e.g. listing out pairs) and still not advance to the third sub-phase of isolated models. The obstacle that prevents the referencing of isolated models to each other may in this case be the reality that from the processual point of view, one of the exercises is solved "in reverse order" from the other. While solving the first exercise begins with one pair clinking their glasses (second with first), then two pairs (third with first and second), then three pairs (fourth with first, second and third) then four pairs (fifth with first, second, third and fourth); in the fifth exercise (see Figure 6) four pairs are created in the first column, then three pairs in the second column, then two pairs in the third column, and then the last pair is added to that.

6.1.2 Two Parts of the Fourth Sub-Phase of Isolated Models

A more detailed examination of pupils' thought processes uncovered that the fourth sub-phase includes two different mental activities. It was uncovered by the atomic analysis of the statement commented on in detail previously. The summary of the statement:

The exercises are the same, there must be some formula for this.

The first part of the statement demonstrates that the pupil realizes the sameness of the isolated models. The second part implies that the sameness could be acquired from an external source. For that reason we propose to divide the fourth sub-phase included in Table 2 into two new sub-phases (See Table 3):

- 4^{th} sub-phase – the pupil senses that the experienced isolated models are the same, but does not see this "sameness" clearly yet; however, they feel the need to investigate the situation further and discover the "sameness"; the need to discover the "sameness" of isolated models brings them either to one's own further and deeper research, or to trying to receive this knowledge from an external source.
- 5^{th} sub-phase – the pupil discovers the sensed "sameness"; either by their own research or from an external source.

These two possible executions of the fifth sub-phase:

a) the pupils discover the sameness on their own,
b) the knowledge comes from an external source,

lead us to a more detailed research of the fact that the knowledge comes from an external source.

Frequently, the adoption of knowledge from an external source, even for the genetic constructivism based teaching, takes place when a slower pupil is still in the second sub-phase of isolated models for instance, but their classmate happily announces the general piece of knowledge. In this moment, the mind of the first pupil experiences a jump between where they advanced on their own and what came from the outside. Decisive for the quality of the final knowledge adoption is the amount of additional work that the pupil spends in the phase of isolated models in order to overcome this jump.

The abovementioned phenomenon has not been discovered before, probably because the experimenter was not granted with the chance to register it. We were lucky in that regard that our experiment demonstrated this phenomenon in a single statement from a pupil. The technique of atomic analysis, requiring every acoustic fragment of pupils' statements to be examined for what actually took place in their minds, discovered this phenomenon.

<table>
<tr><th colspan="2">Stages</th><th>Lifts</th><th></th></tr>
<tr><td colspan="2">Abstract knowledge</td><td rowspan="2">Abstraction</td><td rowspan="7">Crystalisation</td></tr>
<tr><td colspan="2">Generic model</td></tr>
<tr><td rowspan="5">Isolated models</td><td>5th sub-phase
Discowering the sameness or acquiring the knowledge from an external source</td><td rowspan="2">Generalisation</td></tr>
<tr><td>4th sub-phase
Sensing the "sameness" and the need to uncover or get the knowledge from external source</td></tr>
<tr><td>3th sub-phase
Isolated models referring to each other</td><td rowspan="3"></td></tr>
<tr><td>2th sub-phase
Additional isolated models</td></tr>
<tr><td>1th sub-phase
First tangible experience</td></tr>
<tr><td colspan="2">Motivation</td><td></td><td></td></tr>
</table>

Table 3. The cognitive process including the fourth and fifth sub-phases of isolated models.

6.2 Discussion of the Analysis of the Second Session Experiment

During the experiment we have evidenced several key elements that lead to the discovery of the isomorphism between two isolated models:

- Change of the coding language.
- The ability to "fit" other isolated models into the dominant one.

- Comprehensible representation of the isomorphism – communication problem (third group: adding letters to pentagon vertices, fifth group: marking the game with a – game b is a complement).

The challenge of discovering isomorphisms among isolated models is often grounded in the communication tool of the pupil. For the pupil to be able to discover the isomorphism there must often be a change of the way objects are coded. The pupil may found this change on their own, but they can also receive it from their classmates, a textbook, etc. During classroom discussions, different pupils usually offer different ways of coding and the pupil decides which way is the most accessible for them. The ability to discover but also select from several effective ways of coding points to a developed level of communication in mathematics. Discovering effective coding of objects often reveals isomorphism among other elements of models, such as the relations between objects.

Another evidenced phenomenon of discovering isomorphisms among isolated models is the dominance of one of the isolated models. This model often becomes the generic model. The pupil solves the exercise using a familiar method, e.g. from real life experience. They sense the next exercise is solved in "the same" way and try to "fit" the solution of the new exercise on the previous one, of which they are certain is correct. The solution of the first exercise becomes the dominant isolated model.

Another obstacle in discovering isomorphisms lies in the area of communication. In a group of pupils there may be those who see the isomorphism immediately, but also those who do not. One of the reasons why some pupils do not see it is the insufficient representation of isomorphism, of objects for example. Sometimes it is enough to reveal only the isomorphism among objects (adding the same coding of objects), and the isomorphism between relations surfaces on its own. This situation can arise even when a pupil senses the isomorphism between relations, but cannot see it between objects. In some cases, it is necessary to recode relations in some of the models so that other pupils are able to discover the isomorphism.

7 Conclusion

The first contribution of this study is the enhancement of the Generic Model Theory with a more detailed elaboration of the phases of isolated models. The atomic analysis highlighted the significance of the "sameness", or isomorphism, between the solutions of two semantically different but structurally identical exercises. It allowed to better understand the transitions between individual sub-phases of isolated models, especially from the second to the third. It was demonstrated that the obstacle in discovering the "sameness" lies in the seemingly incompatible solution strategies – one exercise is solved processually, the other conceptually. Other obstacles include the way of approaching the exercise and the language used.

The second contribution of this study is the division of the original fourth sub-phase of isolated models into two separate sub-phases. The key to this discovery was the evidence that for some pupils the perception of "sameness" of exercises happens in two steps: the pupils first sense the exercises are the "same" and then see the "sameness" by assigning objects and also relations. In the new fourth sub-phase, there is the pupil who senses the provided isolated models are identical. They advance to the fifth sub-phase in the moment they realize the sensed "sameness"; either by their own reasoning or through external source.

In addition to the results mentioned in this study, the research delivered interesting results connected to genetic constructivism, specifically in the field of the teacher's work when managing discussion. These results will be presented in future works.

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Izomorfizm jako narzędzie uogólniania (w kombinatoryce)

S t r e s z c z e n i e

Studium przedstawia część wyników badań realizowanych we współpracy między Katedrą Matematyki i Dydaktyki Wydziału Pedagogiki Uniwersytetu Ostrawskiego a Katedrą Matematyki i Dydaktyki Matematyki Wydziału Pedagogiki Uniwersytetu Karola w Pradze. Inspiracją dla nas były wyniki badań praskich, które w okresie ostatnich 5 lat w sposób zdecydowany wpłynęły na nauczanie matematyki w Republice Czeskiej.

Celem badań jest szczegółowe przeanalizowanie procesu poznawczego odkrywania liczby kombinacji $\binom{n}{2}$, prowadzonego na dwóch grupach uczniów z wykorzystaniem teorii konstruktywizmu genetycznego. Jesteśmy przekonani, że nauka przebiegająca w stylu konstruktywistycznym jest bardziej efektywna niż nauka tradycyjna. Podobny pogląd podziela wielu innych autorów.

Jako obszar badań została wybrana kombinatoryka, ponieważ w literaturze dydaktycznej jest to obszar szeroko opracowywany. Przyczyna tego stanu rzeczy leży w jej przydatności do prowadzenia badań eksperymentalnych już od najmłodszych lat.

Badania zostały przeprowadzone na uczestnikach kółka matematycznego dla uczniów w wieku 11–13 lat z Gimnazjum im. Wichterla w Ostrawie. Szczegółowa analiza bogatej bazy danych dostarcza wiedzy odnoszącej się do trzech obszarów:

Obszar pierwszy, któremu poświęcone jest niniejsze studium, dotyczy procesu poznawczego ucznia. Ujawniony został nowy podetap etapu modeli izolowanych, jako przyczynek do teorii modelu generycznego. Teoria modelu generycznego dzieli proces poznawczy na etapy, wśród których kluczową rolę

odgrywa tzw. etap modelu generycznego. Uczeń osiąga ten etap poprzez uogólnienia serii uprzednich doświadczeń. Materiał uzyskany w naszych badaniach pozwala na szczegółowe zdefiniowanie etapu modeli izolowanych, które w dotychczasowych pracach nie zostały tak szczegółowo opracowane. Są opisane nowe prawidłowości, które przejawiają się na tym etapie.

Obszar drugi jest zorientowany na pracę nauczyciela, na jego dążenia do zmiany własnego stylu edukacyjnego na styl ukierunkowany na konstruktywizm. Badania tego obszaru są już bardzo zaawansowane. Tu wychodzimy z konstruktywizmu genetycznego, który poszukuje optymalnej strategii edukacji dla rozwoju kreatywności uczniów. Teorii, pierwotnie opracowanej pod nazwą Scheme Oriented Education, w efekcie analiz metodologicznych przeprowadzonych przez L. Kvasza nadano nazwę konstruktywizmu genetycznego. Kvasz wykazał, jak w tej metodzie systematycznie jest wykorzystywana paralelność pomiędzy ontogenezą a filogenezą, tzn. jak historia matematyki umożliwia konstruowanie serii modeli izolowanych w kierunku do modelu generycznego w taki sposób, aby odpowiadało to wymogom ontogenezy. Studium Kvasza udostępnia czytelnikowi głębszy teoretyczny pogląd dotyczący tej problematyki.

Trzeci obszar bada zjawisko niezrozumienia w interakcji nauczyciel-uczeń. Wydaje się, że nasze badania nawiązują do prac Slezáková-Kratochvílová, Swoboda.

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