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A task about a cube; or, on generalization in 3D

Abstract: The necessity of teaching such an activity as generalization was considered by Z. Krygowska in the '70s (Krygowska, 1977). The formation of this ability requires an adequate selection of non-stereotypical tasks, for which the algorithm is unknown to the person solving the task, ones in which a student is forced to search for their own method of solving the task on the basis of their knowledge. In literature, there are known studies concerning the process of generalization of students of different ages which use, at most, 2D visual patterns. However, the author still did not find any research based on tasks which examine the process of generalization in 3D. In this paper, results will be shown of using a task concerning a cube carried out in a diverse group of students from middle school, high school, and university.

1 Spatial abilities

Spatial imagination and its formation in a human brain was examined by many researchers. The most important examples of this type of research are from the years 1940-1950 (see Unal, 2005, p. 16). In literature, one can find various definitions concerning the aforementioned abilities, e.g. spatial ability, spatial skills, spatial perception, and spatial reasoning, which are contained in one notion – spatial abilities. However, a more essential difference is not the definition, but the division of this notion into subgroups.

According to Maccoby and Jacklin (1974), spatial abilities consist of two factors: non-analytical concerned with the rotation of an object in one's imagination, and analytical for all others.

McGee (1979) distinguishes two factors concerning spatial abilities, i.e. spatial visualization and spatial orientation. Tartre (1990) states that these

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two notions can be defined in such a way that if we say that an element moves fully or partially, or changes strictly in a human mind, this is spatial visualization, e.g. in the form of a task of imagining a cube with faces painted in different colours and spreading it out into a 2D grid. On the other hand, spatial orientation does not require to move anything in one's imagination. Rather, it concerns the foresight of how the surroundings change when a human being moves, e.g. when during diving or flying one would feel where the bottom of the reservoir or air surface is. Similar division and definitions of spatial abilities as presented above were also proposed by Pittalis and Christou (2010).

Moreover in (Pittalis, Christou, 2010) the authors define spatial abilities as a kind of mental activity which allows us to create spatial images and manipulate them in order to solve various practical and theoretical tasks. Further on, the authors state that the research done on this 3D-space related activity concerns the manipulation of diverse spatial objects, the construction and recognition of their 2D grids, recognizing solids and their elements, counting surfaces and volumes and recognizing properties of solids, which are all topics included in school syllabuses.

Undoubtedly, spatial visualization is a crucial function, especially regarding its wide application in fields such as: computer graphics, engineering, architecture (Baki et al., 2011). This means that teaching this ability requires special preparation both from the student as well as the teacher. In (Heggarty, Waller, 2005), the authors also emphasize the meaning of these skills as the heart of the matter of the construction and understanding of abstract spatial representations in solving mathematical problems.

Children mainly develop their spatial abilities through experimentation, initially through manipulation of objects, and further on by using computers. Such opportunities are given to them by contemporary IT, especially computer games and interactive software. Due to the fact that spatial visualization develops in a human brain over a long period of time by experimentation and manipulation by using one's hands (as emphasized in Robichaux and Guarino, 2000), it seems that crucial subjects in kindergarten as well as school for children this age should be not only mathematics, but also art and technical activities.

Although in current school syllabuses of mathematical education, not just in Poland, there are too few lessons devoted to teaching geometry, it seems that the presence of graphical software and different types of 3D computer games should balance this deficit. Such a conclusion can be deduced on the basis of observations made by Braukmann and Pedras (1993), who admit that computer software as well as physical manipulation of 3D objects help in developing spatial abilities.

As stated by Berthelat and Salin (1998), a plane is the most often used object in the representation of 3D space (as in, a plane with an additional dimension). Although the results by the authors of (Beim-Chaim, Lappan and Howang, 1989), show both children and adults have problems with drawing 3D solids by hand as well as representing on a plane those lines which in 3D are parallel or perpendicular (Parzysz, 1988).

For a deeper analysis of solids and their properties (the analysis of their elements), it is useful to represent the solids as their grids in 2D. But this procedure requires the "translation" of an object in 3D into an object in 2D by focusing on particular elements of solids as emphasized in (Pittalis, Christou, 2010). Cohen (2003) maintains that children are able to learn this procedure with the help of a teacher.

In the paper (Battista, Clements, 1996), the authors noticed that students have problems even with the simplest solid, which is the cube, mainly in situations in which it is necessary to "look inside" the object without physically cutting it, e.g. to answer the question of how many smaller cubes are there in a bigger cube or a cuboid. This research was carried out by the author in a middle school.

Similar problems arise if a student is faced with the task of counting the surfaces or volumes of solids. As shown by research (for example Owens, Outhred, 2006), a student solving this type of a task uses solely the formula, completely ignoring the measure units at the same time. Battista in (Battista, 1999) emphasizes that the inner structure of a given solid should be developed in the students' brains. Then the question of "how many small cubes are in a bigger cube" helps the students understand how to calculate the volume of a solid.

2 Graphical representation of a given problem

While reviewing literature in the context of searching for the presence of tasks with which the people taking part in the study are faced with, one can notice the crucial role played by graphical representations presenting the problems given in the tasks. (Pantziara, 2009; Booth and Thomas, 2000) claim that while graphs and diagrams are very helpful for a certain group of people, they do not help the people who have problems with gathering information from tasks formulated in such a way (due to their problems with analyzing graphs) or they do not know how to use a given diagram for solving the problem occurring in the task.

As Zhang and Norman (1994) emphasize, tasks with and without graphical representation are not considered the same task for the people who are solving them.

On the other hand, when the subject draws a diagram for a task which they are yet to solve, their attention is focused on the not-yet-solved parts of the task which generates knowledge helpful for matching pieces of information which are not yet connected (Cox, 1999). Moreover, a task without any auxiliary drawing or diagram does not influence the way of thinking and of solving the task, which helps in attaining knowledge as well.

3 3D geometry and the syllabus of mathematical education in Poland

The initial notions and tasks concerning spatial geometry are introduced when teaching children in primary school (aged 9–12). That is when children are taught about cubes and cuboids. Students at this educational stage know the 2D grids of these solids through assembling them by hand, which undoubtedly makes it easier for them to learn some of the special properties of solids. At this educational stage, the students also learn about the area of a rectangle. It is usually done by counting how many small squares (of a side with the length of 1 unit) can be put into a given rectangle of sides which are natural numbers. This way, students know the natural way of multiplying numbers.

Another aspect introduced in this educational stage is the notion of the volume of a cuboid using the same method as when counting the area of a rectangle. For this purpose, the spatial model is used in the form of the wireframe of a cube and requires the children to fill it in by using specific elements which represent overlapping parts which assemble the height of the cube.

In further levels of education, more sophisticated notions are introduced, but they are omitted here as they are not suitable for this article.

4 On the process of generalization in mathematics

The necessity of teaching generalization was emphasized in the ‘70s by Z. Krygowska in (Krygowska, 1977, part 3 p. 93). Undoubtedly, a crucial aspect of the process of generalization is the proper selection of non-stereotypical tasks, so instead of tasks in which the student knows the algorithm of solving it, one should provide tasks in which the student is forced to find his/her own way of solving it based on his/her knowledge.

Attempts at advancing from stereotypical to open tasks occurred in the ‘70s of the last century (Wittmann, 1972) (see also (Krygowska, 1977)). Wittmann carried out research in which the goal was to observe students solving non-stereotypical tasks and, if needed, receiving help from the teacher. When analyzing the work of such students, he concluded that this way, the students' thinking and invention skills rise to "a higher level of energy" (Krygowska, 1977 part 3 p. 102). The consequence of this operation is the possibility of "extending" the problem of a task. One of the forms of such extension is – as Krygowska writes – the specification or generalization of a theorem obtained before. Specification – in the sense of Krygowska – is the formulation of the particular cases of the theorem, and generalization is the formulation of a theorem, the particular case of which is the given theorem. If we do not know whether a given theorem is true or false then specification can help us in helping us decide.

One of the types of tasks used in researching the process of generalization are tasks which include patterns. This topic is known in literature as "pattern generalization". In literature related to the research of this type of generalization, number patterns and visual patterns are used.

In the paper (Hale, 1981), the author suggests that recognizing, describing, and extending sequences has enormous significance because it is the introduction to formal algebra. Working with patterns – as Hale writes – also helps the students discover the regularities and relations which encourage children towards generalization.

In (Mouhayar et al. 2013), the authors mention the factors influencing the process of generalization. They list, among others, the mathematical structure of the task, presenting the task in a graphical form (visual patterns), discussion with the teacher and classmates. They also introduce different methods of generalization:

- Near generalization, which involves providing a particular, further step (for example, after three steps given, the students formulate the fourth one).
- Far generalization, which involves providing the 100th step without showing the 99 previous steps.
- The other types, which are, in the light of this article, less important. One can find them in (Mouhayar et al. 2013, p. 383).

In (Jurdak et al., 2014), the authors carried out research with the participation of students. Similarly to previous attempts, the research was qualitative-quantitative. The students taking part in the study were divided into several groups with regard to their age which allowed researchers to observe the process of generalization on different levels. This way, they noticed that the process is different on different levels: starting from near generalization all the way up to far generalization, in order to reach symbolic generalization. As the authors emphasize in (Jurdak et al., 2014), before publishing this article, such research was not carried out on such a wide scale. Although there were different kinds of research carried out and the different strategies of generalization used by the students were being defined, none of this was carried out with regard to the age difference of the students. As a result of the analysis of this research, the authors came to the conclusion that while near generalization is attainable by the students, far generalization is too difficult for them and does not occur at the early levels of education.

5 Methodology and analysis of research

In the research, 120 students of three levels of education took part. 50 middle school students (I and II year), 28 high school students (I and II year) and 42 university students studying mathematics (I and III academic year). The subjects came from two different regions of Poland. The subjects were to solve the tasks presented below.

Task: Imagine a cube made out of a transparent material with the sides painted in the same color. Then, cut this cube with two planes which are perpendicular to the base of the cube and pass through the centers of the opposite edges of the base and then passing through the middle of the edge of the plane perpendicular to the side edge. This way, the original cube is divided into little cubes such that every edge of the resulting cube was divided into two equal parts. Imagine that each edge of the original cube is divided in a similar table: way into three, four, and five equal parts. Fill in the following table:

No. n of parts of each edge after division	No. of cubes obtained as a result of division	How many cubes have 3 painted sides?	How many cube have 2 painted sides?	How many cube have 1 painted side?	How many cube have no painted sides?
$k = 2$					
$k = 3$					
$k = 4$					
$k = 5$					

Did you notice any regularities in this table? Formulate your observations and try to explain. Complete the table for $n = k$, where k is any natural number and explain what you obtained.

$k = n$					

This task is sourced from the book (Ciosek, Żeromska, 2013, (task 3.11.)). Contrary to the task presented by the authors, the version presented to the subjects did not have any additional drawings of the cube. The students had 40 minutes to solve this task. In the analysis of the study, the students' solutions were analyzed.

The goal of this research was to find the answer for the following questions:

1. Are there any common features for different age groups which were observed when solving the task?
2. Which features are characteristic for the given age groups (levels of education)?
3. On which level do fully-developed abilities of generalization of the problem posted in the task occur?
4. What kind of mistakes were repeated in the given age groups?
5. What kind of difficulties did the students meet when solving this task?

It is important to note that all students have the geometrical and algebraic knowledge needed to solve this task.

Because the scope of the age of the students was 13–22, the analysis of the students' solutions is divided into several groups.

6 Students aged 13-15 (the middle school)

First year students of middle school (50 people, though 12 did not try solving the task), when solving the task, used auxiliary drawings which they created properly with respect to the spatial representation of the cube on a plane but only in the case for $n = 2$ and $n = 3$. Four students provided results for $n = 2$ but these results were only partially correct (see the table below (Table 1)).

No. n of parts of each edge after division	No. of cubes obtained as a result of division	How many cubes have 3 painted sides?	How many cube have 2 painted sides?	How many cube have 1 painted side?	How many cube have no painted sides?
$k = 2$	8	8	8	8	0

Table 1. A student's answer.

This solution can mean that the students did not completely understand that the division, due to the type of cube, was not of partially disjoint sets.

One of the students generalized the result for $n = 3$, 4, 5 because he/she realized that the constant property is 8 cubes with three painted faces and it is independent of the number of edges being cut. Similar to the other students in this age group, he/she did not take into consideration the sum of cubes with different amounts of painted faces obtained after cutting which made it impossible for him/her to continue solving the task. He/she finished his/her reasoning at $n = 3$.

Two other students filled the second column in the table by providing the number of cubes obtained after cutting for $n = 2, 3, 4, 5$ as the consecutive powers of 2 i.e. 8, 16, 32, 64, which can mean that they only took into account the case for $n = 2$ and generalized this case for 2 to the power of $(n + 1)$.

Another student gave the proper answer for $n = 2$ and $n = 3$ by using an auxiliary drawing, unfortunately since he/she provided the improper answers for $n = 4$ and $n = 5$ (he/she gave random numbers 32 and 40), he/she could not generalize the task (Figure 1).

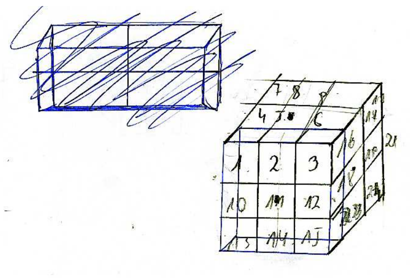


Figure 1. Student's work.

One student used the following drawing for $n = 4$ (Figure 2),

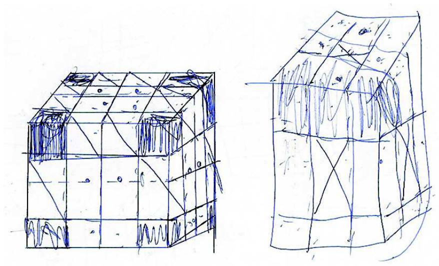


Figure 2. Student's work.

properly solving the case $n = 2$, but improperly solving $n = 3$ (he/she crossed out the number 18 and corrected it to 24).

Based on the auxiliary drawing, he/she easily noticed that there was a constant number of cubes with three painted sides, but otherwise provided improper answers. The next sheet (Figure 3).

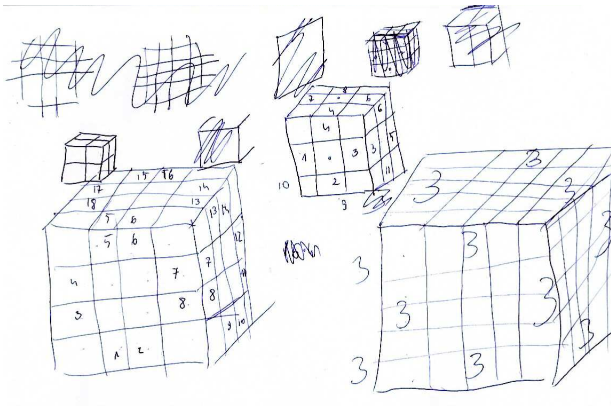


Figure 3. Student's work.

shows partially correct answers – the proper number of cubes but the lack of checking the sum of all the different cubes caused mistakes. In this case, the student did not take into account the hidden sides and the cube inside the bigger cube.

The next two sheets of answers contain mistakes except for $n = 2$ as well as the general number of cubes for $n = 3, 4, 5$, but the students seem to not have been able to generalize the problem of the task, as the remaining numbers in the table were random.

The three other sheets contained the proper case for $n = 2$ as well as the appropriate numbers in the second column, but without realizing that the sum of cubes with three painted sides is constant. In these sheets, one can notice that the students are able to take into consideration the interior of the original cube as they state that in the case of $n = 3$ only one cube has no painted sides. In these sheets, attempts at generalization are being made, as can be seen e.g. in the general number of cubes obtained through cutting is equal to k to the power of 3, which can suggest that the students thought "with layers" as they were taught to do when learning about the volume of cubes.

Two other sheets contain the wrong numbers of cubes for $n = 2$, 3, 4, 5, respectively 20, 30, 40, 50, which means that these students completely misunderstood the task.

In other sheets, the students notice that for $n = 2$, 3, 4, the number of cubes without painted sides is respectively 0, 1, 4, 9 for $n = 2$, 3, 4, 5. In this case, one can notice that these students were not able to "think spatially" as they did not notice that the interior of the original cube composed of cubes without painted sides is still a cube.

The remaining sheets contain a number of mistakes which can suggest that these students filled the table with randomly chosen numbers.

They probably did not understand the point of the task. For example, they used numbers 4, 6, 8, 10 or 8, 18, 36, 72 respectively for $n = 2, 3, 4, 5$.

In the last case, the remaining data is as follows (Table 2) which does not seem to be randomly chosen numbers as the number in the second column is equal to the sum of the numbers provided in the previous line.

No. n of parts of each edge after division	No. of cubes obtained as a result of division	How many cubes have 3 painted sides?	How many cube have 2 painted sides?	How many cube have 1 painted side?	How many cube have no painted sides?
$k = 2$	8	8	0	0	
$k = 3$	18	12	4	2	
$k = 4$	36	8	16	12	
$k = 5$	72	16	26	16	14

Table 2. A student's answer.

Four students established the number of unpainted sides to be 0, 1, 4 and 9 for $n = 2, 3, 4, 5$ respectively. This mistake was discussed earlier.

Only two sheets contained fully proper solutions with proper generalization (Table 3).

No. n of parts of each edge after division	No. of cubes obtained as a result of division	How many cubes have 3 painted sides?	How many cube have 2 painted sides?	How many cube have 1 painted side?	How many cube have no painted sides?
$k = n$		8	$(k - 2)12$		

Table 3. A student's answer.

7 Students aged 16-18 (high school)

The task was being solved by 28 students, but 10 did not try to solve it and only one person gave a full and proper solution.

Six respondents made the same mistake as middle school students who did not distinguish the types of cubes in the third to sixth column: for $n = 2$ answering, respectively, 8, 8, 8, 0. Fortunately, in these sheets there is proper data for $n = 3$, 4 but only in the first two columns. As shown previously, in this age group the students also used auxiliary drawings.

The next eight sheets present correct data in the first two columns as well as the generalization of the general number of cubes (as k to the power of 3). Although in one sheet the two last columns have the proper data, but it cannot be considered fully correct as the subject did not verify the sum of cubes in the appropriate columns against the total amount of cubes.

Two sheets contain incorrect data in the second column, e.g. for $n = 2$, consecutively 4, 9, 16, 25, hence k to the power of 2. Thus, we can conclude that the students were able to "think spatially" and write down only the cubes that they could see from the one side.

In this age group, the first explanations on how the students reached their solutions can be seen (on two of the sheets). These students noticed that the number of cubes with three painted sides is constant and equals eight. The number of cubes with two painted sides increases by four units. Those with one painted side increase by 20, and the cubes without painted sides increase by three. One can notice that these students verified the sum of cubes of all types against the general number of cubes.

8 Students aged 19-22 (university, faculty of mathematics)

Among first year university students, one can notice that they know how to generalize tasks, even though their reasoning and solutions are not free of mistakes.

Also, at this level the students used auxiliary drawings to represent the problem shown in the task. On two sheets, only the first two columns contain correct data, one sheet contains three proper columns, but in one of them, despite the previous steps being correct, there are a lot of mistakes for the case $n = 5$, which can suggest a lack of adhering to the deduced generalization pattern. Only one sheet contains proper answers with a full explanation (Figure 4).

$n = k$	n^3	8	$(n-2)\cdot 12$	$(n-2)(n-2)\cdot 6$	$(n-2)^3$

Rogów jest zawsze 8. Krawędzi jest 12, więc $(n-2)\cdot 12$, bo są dwa rogi. Ścian jest 6, więc $(n-2)(n-2)\cdot 6$. $(n-2)^3$ to sześcian, który powstaje w środku (niepomalowany).
(sześcianik)

(*There are always 8 corners. There are 12 edges so* $(n-1)12$*, there are two corners. There are 6 sides, hence* $(n-2)(n-2)6$*. is a cube (small cube) which is built in the middle (not painted), trans. JJ*).

Figure 4. Student's work.

Among third year students, only six people provided a partially proper solution, while others gave fully correct answers with proper generalizing patterns, out of which 16 sheets lack explanation.

Among the incorrect answers we can distinguish a lack of generalization in the last column (four sheets) or mistakes in the case of $n = 5$ with the number of cubes with one painted side. Other mistakes are related to the generalizing patterns, although one sheet contains a drawing of the 2D grid of the original cube for the case of $n = 3$ (Figure 5).

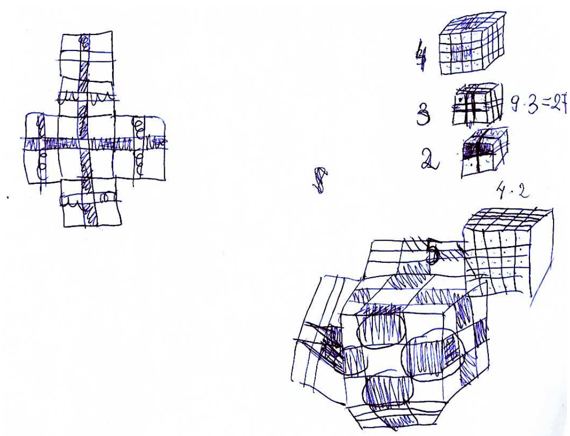
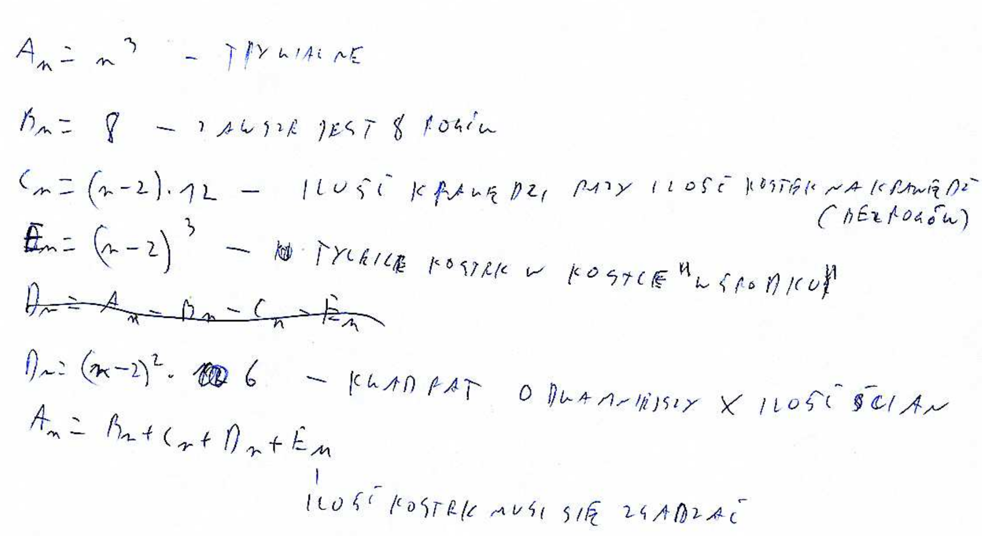


Figure 5. Student's work.

The remaining sheets contain only correct answers.

One sheet contains an explanation of what each notion means and that the sum of the particular cubes must be equal to the general number of cubes in the original cube (Figure 6).


$A_n = n^3$ – TRYWIALNE

$B_n = 8$ – ZAWSZE JEST 8 ROGÓW

$C_n = (n-2) \cdot 12$ – ILOŚĆ KRAWĘDZI RAZY ILOŚĆ KOSTEK NA KRAWĘDZI (BEZ ROGÓW)

$E_n = (n-2)^3$ – TYLKO KOSTKI W KOSTCE "W ŚRODKU"

$\cancel{D_n = A_n - B_n - C_n - E_n}$

$D_n = (n-2)^2 \cdot 6$ – KWADRAT O DWA MNIEJSZY X ILOŚĆ ŚCIAN

$A_n = B_n + C_n + D_n + E_n$

ILOŚĆ KOSTEK MUSI SIĘ ZGADZAĆ

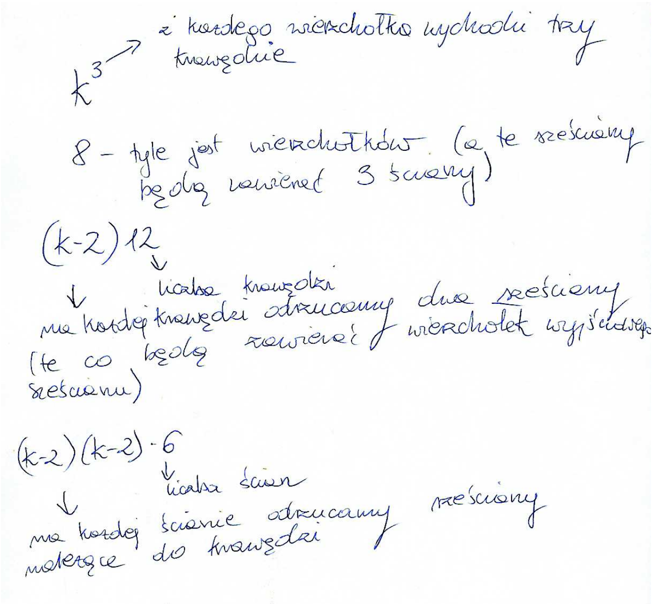
(= – *trivial there are always 8*
the number of edges times the number of cubes on edge(… unreadable…)
the number of cubes "in the middle"
the square less than two times the number of faces
+ the number of cubes must be equal, transl JJ).

Figure 6. Student's work.

Two additional sheets contain correct data but without any explanation. Another sheet is presented below (Figure 7).

The students went even further in their generalization. In the case of the number of cubes in the corner of the original cube, they provided the pattern, which can suggest that they wanted to have one pattern generalize all types of cubes.

Contrary to middle and high school students, university students have performed the generalization, but obtained a pattern they described in various ways (Table 4).



(– *three edges go out of all corners*
8 – the number of corners (and these cubes will contain 3 faces)
$(k-2)12$ – *on each edge we reject two cubes (which will contain the corner of the original cube times 12 - each edge)*
$(k-2)(k-2)6$ *on each side we reject cubes belonging to the edge times 6 – number of sides, trans. JJ).*

Figure 7. Student's work.

No. n of parts of each edge after division	No. of cubes obtained as a result of division	How many cubes have 3 painted sides?	How many cube have 2 painted sides?	How many cube have 1 painted side?	How many cube have no painted sides?
$k = 2$	(31 persons) (2) (5)	8 (28 persons) (10)	$(k-2)12$ (33 persons) $2k(k-1)$ (1, but there is a mistake due to the lack of verifying the case $n = 5$)	(28 persons) $(k-2)(k-2)6$ (5)	(29 persons) (1) $(k-2)(k-2)(k-2)$ (2)

Table 4. The students' generalizing patterns.

9 Conclusions

One of the common features which one can observe in all age groups is the fact that auxiliary drawings helped the respondents in representing the problem and in providing correct answers. Although common mistakes occurred in the considered age groups, such as the not verifying the sum of cubes of each type against the general number of cubes in the original cube, these mistakes happened less the older the subjects were. This way, one can observe an increase in the number of proper solutions as well as an increase in the willingness to describe one's manner of reasoning. The crucial element is the fact that on the middle and high school level generalization did not occur, which can imply that despite the empirical introduction of the notion of the volume of a cube in primary school, this notion does not fully assimilate. With regards to the large number of mistakes in providing the number of cubes without painted sides (hidden in the interior of the cube, invisible), especially in the group of middle school students, one can conclude that these subjects had problems with their spatial imagination. The second proof for this situation is the additional fact that the students were not able to "divide" the origin cube into particular levels and increase the number of cubes of particular types, even in the case of $n = 2, 3, 4, 5$. Below is the number of proper solutions with respect to their case and age group, where M means middle school, H means high school and U means university (Table 5).

No. n of parts of each edge after division	No. of cubes obtained as a result of division	How many cubes have 3 painted sides?	How many cube have 2 painted sides?	How many cube have 1 painted side?	How many cube have no painted sides?
Age group	M / H / U	M / H / U	M / H / U	M / H / U	M / H / U
$k = 2$	0,48/0,54/0,9	0,40/0,54/0,9	0,22/0,18/0,9	0,22/0,18/0,9	0,48/0,43/0,9
$k = 3$	0,16/0,46/0,9	0,18/0,46/0,9	0,02/0,04/0,9	0,04/0,07/0,9	0,18/0,25/0,9
$k = 4$	0,12/0,44/0,9	0,18/0,44/0,9	0,02/0/0,9	0,04/0,07/0,9	0,02/0,07/ 0,9
$k = 5$	0,12/0,32/0,9	0,18/0,32/0,9	0,02/0/0,9	0,04/0,04/0,9	0,02/ 0,07/0,9
$k = n$	0,02/0,07/0,9	0,02/0,04/0,9	0,02/0,04/0,81	0,02/0,04/0,79	0,02/0,04/0,76

Table 5. The number of correct answers with respect to case and age group given as a percentage.

(The table provides data as a percentage, where the number of proper answers is divided by the number of all subjects in a particular age group. The data is approximated to two decimals after the comma).

In summation, one can conclude the following facts: the process of generalization in the groups occurred in the older group, which was composed of adults (students of the faculty of mathematics), but the process of the formation of proper reasoning was proportional to the age of the groups, which is indicated by the increasing number of correct answers. Undoubtedly, further generalization is present among adults. Due to the small number of subjects who refused to take on the task, one can conclude that such tasks are interesting for the students and encourage them to work. The task is interesting because one can solve it in their own mind without using any special tools such as computers or calculators.

Undoubtedly, this research will encourage us to continue further and deeper into this direction of research (qualitative research with regard to the so-called participant observation and possibly giving hints and directions to the subjects, especially at the earlier levels of education) by using the same or a similar task.

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Zadanie o kostce czyli o uogólnianiu w przestrzeni trójwymiarowej

S t r e s z c z e n i e

Na konieczność nauczania uogólniania jako specyficznej aktywności matematycznej zwracano uwagę już w latach siedemdziesiątych ubiegłego stulecia. Jak podkreślała Zofia Krygowska, kształtowanie takiej umiejętności wymaga rozwiązywania przez uczniów szczególnego rodzaju zadań otwartych, nie zadań stereotypowych, dla których uczeń zna procedurę rozwiązania, ale takich, które zmuszą go do poszukiwania własnej metody rozwiązania. W literaturze opisywano badania procesu uogólniania regularności geometrycznych u uczniów na różnych etapach edukacji, wykorzystujące tzw. szablony wizualne w przestrzeni dwuwymiarowej. Artykuł ten dotyczy analizy rozwiązywania pewnego zadania o kostce będącej szablonem wizualnym trójwymiarowym.

Ściany kostki zostały pomalowane jednym kolorem na określoną liczbę płaszczyzn przecinających każdą ze ścian. Zadaniem badanych było podać, na ile kostek została podzielona wyjściowa kostka przy danej liczbie cięć płaszczyznami, ponadto ile tych kostek ma trzy pomalowane ściany, bądź dwie lub jedną pomalowaną ścianę, a ile nie ma żadnej pomalowanej ściany. Rozwiązanie należało podać dla n cięć, gdzie n jest dowolną liczbą naturalną.

Badaniu zostały poddane trzy grupy wiekowe: 50 uczniów gimnazjum (13–15 lat), 28 uczniów liceum (16–18 lat) oraz 42 studentów kierunku matematyka (19–22 lat). Pytania badawcze były następujące:

1. Czy są pewne wspólne cechy procesu uogólniania dla wszystkich badanych grup?

2. Które cechy są charakterystyczne dla danych grup wiekowych (z danego etapu edukacji)?
3. Na którym poziomie edukacyjnym występują w pełni rozwinięte umiejętności uogólniania?
4. Jakiego rodzaju błędy w rozwiązywaniu zadania/rozumowaniu były popełniane w danych grupach wiekowych?
5. Z jakimi rodzajami trudności zetknęli się uczniowie i studenci podczas rozwiązywania zadania?

Z badań tych wyciągnięto następujące wnioski.

Jedną z głównych cech wspólnych, zaobserwowaną we wszystkich grupach wiekowych, jest wykonywanie rysunków pomocniczych obrazujących treść zadania. Ten zabieg okazał się bardzo pomocny i zdecydowanie pomógł w podaniu poprawnych odpowiedzi. Dostrzeżono pewne podobne błędy w rozwiązaniach we wszystkich grupach wiekowych, ale zarazem też ujawnił się wzrost liczby poprawnych odpowiedzi wraz z wiekiem badanych, a także wzrost chęci werbalnego opisywania sposobu rozumowania użytego w rozwiązaniu zadania.

Badanie pokazało, że proces uogólnienia (podanie wzoru uogólniającego dla dowolnej liczby naturalnej) nie ujawnił się u uczniów gimnazjum i liceum. Uczniowie ci opierali swoje rozumowania na badaniu konkretnych przykładów, przy czym gimnazjaliści głównie badali kostkę podzieloną na 8 (lub rzadziej 27) drobniejszych kostek i nie potrafili posegregować tak powstałych części ze względu na liczbę pomalowanych ścian. Część licealistów badała wprawdzie również kostki podzielone na 64 i 125 kostek, jednak popełniała podobne błędy co gimnazjaliści. Główną trudnością, zwłaszcza wśród gimnazjalistów, było podanie liczby kostek, które nie mają ścian pomalowanych, co dowodziłoby ograniczeń wyobraźni przestrzennej. Ponadto uczniowie na tych dwóch etapach (nawet ci, którzy podali konkretne liczby rodzajów kostek powstałych przez zadane podziały) nie potrafili uogólnić uzyskanych ciągów liczbowych. W pełni rozwiązane zadania wraz z uzasadnieniem zaobserwowano tylko wśród studentów.

Pomimo trudności, jakie badanym przysporzyło rozwiązywanie tego zadania, okazało się ono dla nich interesujące i zachęcało ich do pracy, o czym świadczy relatywnie duża liczba podjętych prób rozwiązania zadania we wszystkich grupach wiekowych. Wynika stąd, że prowadzenie dalszych, pogłębionych badań jakościowych z użyciem podobnych zadań jest sensowne

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