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Creating action schema based on conceptual knowledge

Abstract: Much of constructivist pedagogy focuses on moving past direct instruction or "chalk and talk" to promote active construction of knowledge or teaching for understanding. In this article a group of teacher-researchers conducts two classroom investigations. The first is epistemological (structural) in nature, and the second is functional. In the first investigation, a student's efforts to employ her conceptual knowledge to guide her actions is analyzed through the lens of a conceptual framework based upon the integration of creativity theory due to Koestler and learning theory due to Piaget. The second investigation focuses on the functional aspect of a teacher's role in promoting the conceptual based solution activity observed in the first investigation. This functional effort employs visual concepts to direct student actions and is at first ineffective however, later with more transparent structure it is successful. These results suggest student efforts to create action schema and internalize structure through conceptual visuals will depend upon the cognitive gap (ZPD) between the concepts and the coordination involved with the associated actions.

Introduction

This article documents two investigations (cycles) by a team of teacher-researchers. In the first investigation we analyze a student's use of conceptual know-

Key words: teaching experiment, creativity, bisociation, abduction, reflective abstraction, action schema, constructive generalization.

ledge to construct an action schema through a theoretical framework constructed by synthesizing Koestler's mechanism of creativity 'bisociation' and Piaget's mechanism of learning 'reflective abstraction.' In the second, an instructor promotes solution activity based upon the reasoning observed in the first cycle. The first section recalls how difficult the transition from direct instruction to teaching for understanding can be. The second section outlines arguments by mathematical educators that creativity research should be extended from the genius view to ordinary students and our argument for a further extension to underserved students. The theoretical framework reviews Glasersfeld's (1995) understanding of an action schema, used to describe solution activity when a student recognizes a situation and associates it with a relevant action. Then Piaget & Garcia's (2001) notion of 'reflective abstraction' is explained as the mechanism by which action schemas are modified to accommodate new problem situations. In Glasersfeld's (1998) work, Piaget's notion of creating an actions schema through reflection upon and abstraction of an existing schema is extended to include conceptual knowledge. Glasersfeld based this extension on Peirce's (1931-1935) notion of 'abduction.' In our conceptual framework Koestler's mechanism of 'bisociation' is integrated with Glasersfeld's work as a setting for which conceptual knowledge and its associated actions are abstracted to guide student solution activity and thus create an action schema. The research hypotheses focus on the relationship between concepts and the construction of an action schema, how bisociation can be used to describe this process and how visual representations of concepts can be used to guide schema development. The methodology contains classroom transcripts of these two investigations followed by the research team's discussion and analysis.

Understanding

Bruner (1990) states that the cognitive revolution had as its aim, "...to discover and to describe formally the meanings that human beings created out of their encounters with the world, and then to propose hypotheses about what meaning making processes were implicated" (p. 2). Hiebert and Carpenter (1992) comment that, although the search for how we create meaning out of mathematics has been widely studied, a useful description for this process remains elusive:

> The goal of many research and implementation efforts in mathematics education has been to promote learning with understanding. But achieving this goal has been like searching for the Holy Grail (p. 66).

The search for understanding has two components: a functional and structural perspective. "The functional view focuses on the activity within the classroom" (Hiebert et al., 1996, p. 16). On the other hand, the structural-cognitive view is focused upon relationships between knowledge: "understanding means representing and organizing knowledge internally in ways that highlight relationship between pieces of information" (Hiebert et al., 1996, p. 17).

The word structural is used by educators who believe that understanding is achieved when new information is assimilated or accommodated into an individual's schema, Hiebert & Carpenter (1992), Skemp (1987) and Glasersfeld (1995). Seeger (2011) notes that this structural view raises the critical questions: "How do we relate things? How do we relate to things we know and how do we relate old and new things?" (p. 215). Seeger (2011) asserts that for Hiebert et al. (1997) the answer was straightforward: we relate through reflecting upon and communicating our knowledge. However, Seeger (2011) concludes that although this answer "... has an intuitive persuasiveness, it is alarmingly incomplete..." (p. 216).

The search for understanding is central to the constructivist debate about the teacher's role in promoting a creative learning environment. As noted by Simon (2013):

> Traditionally teaching and curriculum development were based upon showing and telling students what they were to learn. The assumption, often implicit, was that motivated students would take in the knowledge shared by the teacher and incorporate it into their mathematical knowledge.... Although this approach is still the most frequently used, it has been discredited as a primary approach to mathematics instruction because the results have not been good. With the loss of confidence in direct instruction came a loss in clarity in the teacher's role in promoting student learning, (p. 96).

Dubinsky (1991) eloquently expresses a widely held constructivist position on educational pedagogy that a focus on procedural skills and passing exams has limited interest in mathematics:

> Whatever is the mechanism... that moves students to make cognitive construction, to learn, it seems to us to be a very natural human drive.... We admit that this suggestion is inconsistent with the experience of most mathematics teachers, especially at the post-secondary level, where students, other than those with obvious talent in mathematics, do not seem to be interested at all. Our conjecture is that this is due to the overall approach in the traditional classroom, where the goal, as presented and defended by the teacher, is for the student to develop skills in computa-

> tional procedures, to display on examinations, and to get a good grade (p. 117).

In this article we investigate student use of conceptual knowledge to guide their actions and concept based instruction to promote conceptual based or 'relational' rather than procedural or 'instrumental' understanding (Skemp, 1987). In terms of the dichotomy between procedural and conceptual knowledge we essentially accept the 'Educational Approach' that states, conceptual knowledge is necessary for and enables use of procedures (Haapasalo & Kadijevich 2000). Our premise is that conceptual knowledge is essential for understanding of one's actions. More specifically, while one may learn procedural knowledge from rote memorization, conceptual knowledge is what gives meaning to one's actions and understanding of mathematical structure.

Creativity

In this section we review arguments that educators have made for the introduction of creativity research within the classroom with gifted and non-gifted students and the extension of this argument made in Czarnocha at al, (2016) and Prabhu (2016) that it is equally important to inspire creativity with non-gifted and underserved students.

Creativity: Genius View

Koestler (1964) analyses the creative experience of many eminent mathematicians during the creative leap of insight between the gestalt periods of incubation and illumination often referred to as 'Aha' or 'Eureka' moments. Koestler considers two essential characteristics of creativity: first, an affective component in which the individual experiences an aesthetical connection to a higher self-transcending nature, and second, a cognitive component, the synthesis (bisociation) of two frames or references previously considered independent or unrelated. Liljedahl (2013) comments that a result of Koestler's characterization of creativity is the so called genius view, that creativity is limited to "extraordinary individuals" (p. 255). Silver (1997) was an early proponent that the so called genius-view of creativity must be adapted to shift its focus to more pragmatic efforts in the mathematics classroom. Efforts to integrate creativity research in the classroom often quote Hadamard (1945), "Between the work of the student who tries to solve a problem in geometry or algebra and a work of invention, one can say that there is only the difference of degree, the difference of a level, both works being of similar nature" (p. 104).

Leikin (2009) reviews several characterizations of an individual's propensity for creativity. One pragmatic approach makes a distinction between convergent thinkers (those who can only focus on one possible solution strategy) and divergent thinkers. Another characterization of creativity is based upon gifted students' capacity for flexibility, fluency, novelty and elaboration.

Creativity: Ordinary and Underserved Students

Shriki (2011) suggests that creativity-enriched instruction should be a part of all students' learning experience:

> It is widely agreed that mathematics students of all levels should be exposed to thinking creatively and flexibly about mathematical concepts and ideas. To that end, teachers must be able to design and implement learning environments that support the development of mathematical creativity (p. 73).

Eugenio María de Hostos Community College was established in the South Bronx as an inner city (2 year) college in the City University of New York (CUNY) system to meet the educational needs of people who historically have been excluded from higher education. The college student profile indicates that the student body is composed of predominately female and minority students for whom poverty is a real concern. The effect of poverty on completion of a High School (HS) degree is studied in (Kewal Ramani et al., 2011) who note that low income children are five times more likely to drop out of High School as high income children. The negative effects of poverty and lack of family support for these students frequently continues into their college experience. Santiago & Stettne (June 2013) report that Hispanic and other minority students in the U.S. that do graduate from High School frequently attend local community college, yet their graduation rate in community colleges is a real concern. One academic barrier to underserved students is placement exams for college readiness in mathematics and the remedial non-credit courses they must pass if they fail these placement exams. As noted by Hacker (2012) the graduation rate for students placed into remedial course is very low. The U.S is not alone in this issue, a report from PISA 2012 (PISA In Focus, 36, February 2014) documents the difficulties that the European Union as well as some Asian countries have with mathematics, and the negative effects that a low socio-economic level have on the results of the PISA-exam.

Wood et al. (2006) insightfully characterize the attitude that community college students often demonstrate: mathematics is all about random rules and that they should be passive as the expert teacher explains these rules. These authors consider the lack of engagement with mathematics has led them to a state of 'learned helplessness.'

Prabhu's insight (Prabhu & Czarnocha, 2014) and (Czarnocha et al., 2016) is that students who state, 'I stuck at math' are not necessarily underserved but rather have had limited exposure to the positive affect of accessing their own creativity. For students with low self confidence engagement begins with recognition of a relevant action for a problem situation. However, the passive affect of such students often means they cannot or will not suggest any such action even when prodded by the instructor. The thesis of this article is that in such a situation creativity may begin when students access their conceptual knowledge to give meaning to appropriate actions.

Theoretical Framework

Lester (2010) credits Eisenhart (1991) for the notion of a 'conceptual framework' which he defines as:

> An argument that the concepts chosen for investigation, and any anticipated relationships between them, will be appropriate and useful given the research problem under investigation. Like theoretical frameworks, concetual frameworks are based on previous research, but conceptual frameworks are built from an array of current and possibly far-ranging sources (p. 72).

This analysis of students' attempt to create meaning for themselves of mathematics takes place within a framework that blends creativity research and theories of learning. In particular, we integrate the work of Koestler and his mechanism of creativity, 'bisociation' with learning theories of Piaget and his mechanism of learning, 'reflective abstraction.' The work of Koestler assists in the effort to 'democratize creativity,' i.e. to apply creativity in the general classroom environment, because it provides a precise mechanism in which to analyse any student's attempt to give meaning to solution activity based upon their existing conceptual knowledge and thus realize Hadamard's (1945) view that the creativity for the gifted and struggling student is similar in nature.

Mechanism of Creativity: Bisociation

Koestler (1964) describes the mechanism of creativity in terms of a *code* and a *matrix*, or a frame of reference: I use the term *matrix* to denote any ability, habit, or skill, any pattern of ordered behaviour governed by a *code* or fixed rules (p. 38). In Koestler's framework the code of a matrix represents the rules that dictate how and why one applies these actions:

> The *matrix* is the pattern before you, representing the ensemble of permissible moves. The *code* which governs the *matrix*... is the fixed invariable factor in a skill or habit... The two words do not refer to different entities; they refer to different aspects of the same activity (Koestler, 1964, p. 40).

For Koestler (1964), bisociation represents a "spontaneous flash of insight... which connects previously unconnected matrices of experience" (p. 45). Bisociation occurs when the concepts in a problem situation are given meaning by an association to a plane of reference that was previously unconnected. In this process the concepts exist simultaneously in both frames of reference and a new matrix-schema is created when the codes of the existing matrices synthesize to form a new code. A phenomena Koestler (1964) refers to as the discovery of 'a hidden analogy.'

Vygotsky: Instruction and Concept Development

Vygotsky (1986) focuses on the development of structural or scientific concepts, such as those of algebra which are generalized from 'spontaneous' arithmetical concepts. Vygotsky like Hiebert & Carpenter (1992) considers concept development as inherently structural. Thus for Vygotsky, conscious scientific concepts are distinct from intuitive spontaneous concepts because they are inherently embedded in a schema or hierarchy of concepts: "...a concept can become subject to conscious control only when it is part of a system" (Vygotsky, 1986, p. 171). In contrast to Koestler, Vygotsky has a strong focus on the role of education in concept development. This is due in part to his observation that when presented with assistance, some students, "could with cooperation solve problems... while other could not..." (Vygotsky, 1986, p. 187). Vygotsky labels this phenomena the child's *Zone of Proximal Development* (ZPD), defining it as "the discrepancy between the child's mental age and the level he reaches in solving problems with assistance" (p. 187). The role of education in Vygotsky's framework is to assist students to reach the upper limits of their ZPD.

Piaget: Action Schema Development

As noted by Piaget and Campbell (2001) Piaget's research and theories concern the way cognitive structures are developed and used to guide one's actions during problem-solving:

> Knowledge is fundamentally operative, it is knowledge of what to do with something under certain circumstances... For Piaget operative knowled-

> ge consists of cognitive structures... If knowledge consists of cognitive structures, then development comes down to what structures do... (p. 2).

The structure or schema that develops during problem-solving is described by Glasersfeld (1995) in his description of an action schema which contains three parts or steps:

1. Recognition of a certain situation...
2. Association of a specific activity with that kind of item...
3. Expectation of a certain result (p. 65).

In our conceptual framework Koestler's notion of matrix is similar to the Glasersfeld-Piaget notion of an action schema as both contain all actions an individual associates with a given problem situation, as well as the rules or expectations of what these actions accomplish.

Piaget employs the term assimilation to describe when an individual recognizes a situation and associates an activity that obtains the expected result. For Piaget a prerequisite for cognitive change is the failure of one's existing schema to process the situation in which case 'accommodation' or restructuring of one's schemes restores the homeostasis between the individual's internal representation and the external situation, i.e. equilibrium, Gallaher and Reid (2002).

Piaget: Reflective Abstraction

Piaget & Garcia (1989) describe the mechanism of reflective abstraction as a two-step process involving reflection and generalization of one's actions and operations. The first step is a projection of what one knows on a lower level into the new problem situation that cannot be assimilated into one's existing schema, and the second step involves coordination with the problem information to modify the original schema into a new one that can accommodate the situation:

> First, projection essentially establishes correspondences at the next higher level, associating the old contents that can be integrated within the initial structure but permitting it to be generalized. Second, these first organizations also lead to the discovery of related contents, which may not be directly assimilated into the earlier structure. This makes it necessary to transform that structure by means of a constructive process until it becomes integrated within a larger, and therefore partially novel, structure. This mode of construction, by reflective abstraction and *constructive generalization*, repeats itself indefinitely, at each successive

> level so that cognitive development is the result of the interaction of a single mechanism (pp. 2-3).

Simon et al. (2004) comment that although Piaget's notion of reflective abstraction was a significant step in describing accommodation of new knowledge and schema development, mathematics pedagogy has not benefited Piaget's contribution due to a lack of clarity about how it applies to learning in the classroom:

> The postulation of reflective abstraction was a significant contribution... describing the kind of process that can derive more advanced structures from those at a lower level... Recent efforts to ground pedagogy in Piagetian theory have been hampered, in our estimation, by the lack of a sufficiently elaborated mechanism for explaining mathematics conceptual development (p. 314).

Dubinsky: Process/Object Duality – APOS

Dubinsky (1991) studies reflective abstraction through the lens of Piaget's notion that concept development involves a cycle of actions upon concepts, and reflection upon actions to form processes and ultimately new concepts. Tall et al. (1999) note that although authors such as Piaget and Davis (1984) had earlier expressed the process/object duality of concept development, the work of Dubinsky has greatly increased interest in this phenomenon. Dubinsky describes specific forms of reflective abstraction: interiorization, coordination, generalization, reversibility and encapsulation. Interiorization involves the internalization of processes or actions, while reversibility involves reflection upon the inverse of a known process. One important contribution of the work of Dubinsky is that specific types of reflective abstraction are named and spelled out. As noted by Arnon et al. (2014), this work has resulted in the process/object model referred to as APOS or the Action-Process-Object-Schema model of concept development:

> According to Piaget and adopted by APOS theory, a concept is first conceived as an action, that is, as an externally directed transformation of a previously conceived object or objects. An action is external in the sense that each step of the transformation needs to be performed explicitly and guided by external instruction additionally, each step prompts the next, that is the steps of the action cannot be imagined and none can be skipped (p. 19).

Arnon et al. (2013) describe a solver in the action conception stage learning to evaluate function variables as an individual, who "...can do little more than substitute for the variable and manipulate it... the expression acts as an external cue" (p. 19). The focus on concept development in APOS as the internalization of actions into processes and eventual transformation into concepts is an important contribution. Asiala et al. (1997) point out the actions of APOS are meant to be similar to action schema: "...what we are calling actions are closely related to Piaget's action schemes..." (p. 41).

If schema development can only begin with reflection upon actions the question that arises is, how does a solver who fails to associate an action with a situation proceed without external instruction-command? Glasersfeld (1998) grapples with this question in his analysis of the learning paradox expressed as, how does an individual form a higher order schema when there is not appropriate lower order schema to modify? In this situation Glasersfled extends reflection and abstraction of existing schema to conceptual knowledge.

Reflective Abstraction and Abduction

Von Glasersfeld (1998) and Norton (2009) review the work of Charles Peirce (1931-1935) who studied creativity and differentiates between induction from a plurality of data, deduction that arises from an established rule, and abduction – references based upon a conjecture that explains a given problem situation. Von Glasersfeld describes abductive reasoning as a three-step process: C is observed and does not assimilate into our schemes – if A is true this would explain C – thus we conjecture A. Von Glasersfeld (1998) explains the abductive process as selecting what relevant conception in phenomena C one considers to be analogous to something in our experience that we can generalize to form hypothesis A that explains C. Norton (2009) describes Peirce's notion of abduction as the process of problem-solving through the creation of hypotheses, "...a kind of reverse deduction in which one adopts a general rule in order to explain a surprising observation" (p. 5). Norton (2009) suggests that abduction can be used "to address the question of how students select particular operations to use in novel problem situations..." (p. 4). He further suggests that Peirce's notion of creative reasoning by abduction readily fits within Piaget's theory of genetic epistemology because, in order to perform an action on an object there must first be an assumed rule or principle that suggests to the solver that the action is somehow appropriate.

The Conceptual Framework

In this section we highlight for the reader those aspects of the theoretical review that we have connected to form our conceptual framework. Dubinsky (1991) and Glasersfeld (1995) both review Piaget's 'reflective abstraction' which occurs due to reflection upon internal mental actions as distinct from empirical abstraction; the inductive abstraction of natural phenomena. The view that reflection upon procedural actions leads to internalized processes and ultimately new concepts is characteristic of process/object models. Haapasalo & Kadijevich (2000) refer to the view that concepts develop is a result of reflection upon procedures as the genetic view and suggest process/object theories such as APOS support this view. Educators such as: Gray et al., 1999, Gilmore & Inglis, 2008, Tall, 1999, and Gray & Tall, 2001 have criticized this genetic view as too narrow, making the claim that reflection occurs on both concepts and actions. Gallagher & Reid (2002) place reflective abstraction within the context of an effort by Piaget to explain how an individual transforms or accommodates lower level schemes into more advanced schemes. Thus Piaget certainly meant for reflective abstraction to be on schemes, which we take to include both concepts and their associated actions. Furthermore, as we have pointed out, Asiala et al. (1997) interpret the action conception of APOS to include reflection upon the entire action schemes not only procedural actions.

One question that arises is the relationship between conceptual schemes (typically viewed as networks of concepts connected by relationships in a hierarchical structure) and action schema as portrayed by Glasersfeld. We consider that Koestler matrix covers both types of schemes and note that a thorough study of this question would connect the often separate branches of mathematics education research, concept development and problem solving.

Research Hypothesis for Investigation: One method of action schema formation occurs when solvers creates meaning through bisociation of their conceptual knowledge with problem situation. During this bisociative process the code of the conceptual knowledge is abstracted (abducted) and coordinated with the problem information to guide solution activity.

Educational Considerations

Dubinsky (1991) eloquently relates his thoughts on the traditional method of instruction that employs repeated examples to encourage students to internalize the underlying process:

> It is an article of faith with most mathematics instructors that, 'lots of examples' must be an integral part of any instructional treatment... We suggest, however, that working with examples may not help very much with the construction of concepts. Indeed ... and it is a major part of our theory that understanding mathematical ideas come from sources other than looking at many examples and 'abstracting their common features',

> which is what happens if there is only empirical abstraction. Something more is needed and we suggest it is precisely the constructive aspects of reflective abstraction... (pp. 118-119).

Dubinsky's argument is that when students view examples as external – the process of abstracting a common feature is akin to empirical abstraction, i.e. inductive reasoning. However, when the student begins to internalize the processes and relates it to her schema then accommodation and learning can take place. One related pedagogical question that arises is how effective a conceptual based lesson is compared to doing repeated examples. Simon et al. (2004) note that with repeated exercises learning the activity sequence is the goal of the instruction. In a conceptual based lesson plan this is not the case; the activity sequence has as its goal reflection upon concepts and associated activities the student can readily relate to in order to give meaning to algebraic structure.

Research Question: Can a conceptual based lesson supported by visuals assist students, guide actions, and internalize this process in an effort to give meaning to the concepts of the algebraic structure?

Methodology

We now turn our attention to the mathematics classroom to investigate and apply our conceptual framework. Thus, like Cobb's, (2011) our effort begins with direct observation of the classroom experience:

> First and foremost, I continue to believe in the importance of sustained, direct engagement with the phenomena under investigation, be it young children's understanding of measurement, the collective mathematical learning of the teacher and students in a classroom, or the learning of a professional teaching community (p. 11).

Acting simultaneously as both researchers and teachers, the collaborative effort of these authors is an example of teaching research or the synthesis of educational research with the craft of teaching to reflect upon and improve educational methodology. We present several teaching research investigations that build upon or refine the previous ones upon the team's assessment. This methodology is referred to as the Teaching Research NYCity Model in (Czarnocha & Prabhu, 2006) and (Czarnocha et al. 2016) (see Fig. 1).

All these examples concern adult students reviewing introductory algebraic thinking (inverse reasoning with the operator conception of a fraction) before taking college level mathematics. In Baker et al. (2009) it was suggested that students understand the operator construct of a fraction (taking a fraction of a quantity) as highly related to the part-whole construct and these investigations further that work.

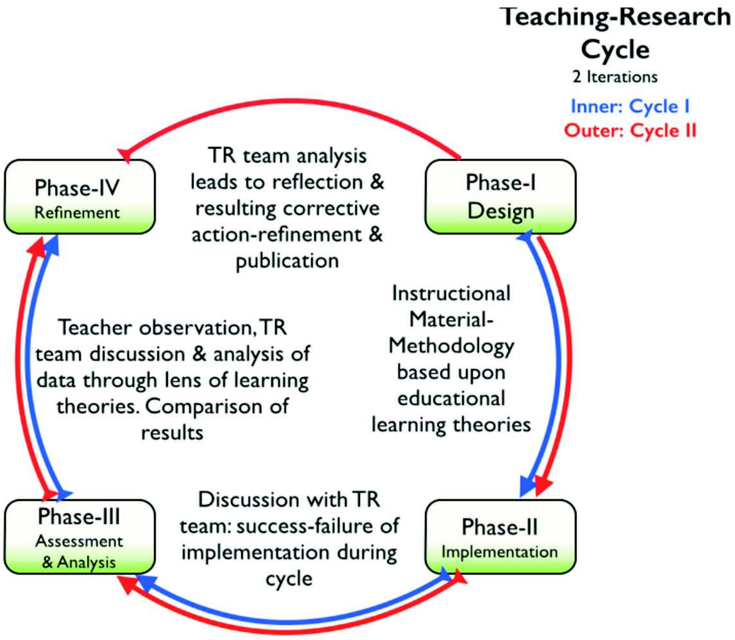


Figure 1. Teaching Research Cycle NYCity.

We present two investigations or cycles. The first, containing observations of a student engaged in solving two exercises, is structural in nature. The second investigation observes the difficulties and successes of an instructor employing a conceptual based lesson founded upon the reasoning observed in the first cycle.

First Investigation

In the first investigation adult students engage in a brief review of algebra before taking college level mathematics. The exercises involve inverse reasoning with an operator problem (given a fraction of an unknown, find the unknown – typically solved using reciprocals).

Exercise 1: Instructor-teacher 'T' asks the students, 'Three-quarters of what number is 18?' Student 'MR' provides the correct solution 24. When asked to explain she replies:

MR: First I divided: 18 by 3 to get 6

T: Why?

MR: Because 18 is three quarters so 6 is one quarter

MR: Then I added 6 + 18 = 24

T: Why did you add?

MR: To get the 4 quarters

Analysis Exercise 1

MR understands fractional parts of quarters as an encapsulated object 'Q' and makes an association between three of these objects and the problem information 18:

$$3Q = 18 \text{ – the Partial Amount.}$$

Abstracting her part-whole conception for 'taking 3 out of 4 equal parts' in this situation leads her to represent the unknown as 4 quarters:

$$4Q = \text{Unknown Total } X.$$

She then abstracts (abducts) the action of taking three out of four objects associated with her three quarters conception (leaving one object remaining) and coordinating this with the goal of finding all four objects $4Q = X$, she realizes she needs one more quarter.

Thus, the characteristics of bisociation that have been observed include: The simultaneous existence of concepts with MR's quarter conception and the problem situation ($3Q = 18$ & $4Q =$ Total X). The abstraction (abduction) of the three-quarter conception as taking 3 out of 4 objects which led to the solution activity (find Q & add) this occurs during the coordination of ($3Q = 18$, $4Q = X$, & Find X). It is difficult to determine whether there is an 'Aha' moment on the part of MR as the teacher did not observe her original work. It was also not clear to the instructor if MR is flexible enough to modify her additive reasoning to more formal multiplicative reasoning. Thus, the instructor provides MR with the following example.

Exercise 2: 'Two-fifths of some number is 500, find the number'.
This requires more generalization, in part, because fifths are a more abstract concept than quarters which are a half of a half and thus intuitive, also because using two-fifths instead of four-fifths means MR cannot readily employ an additive solution schema. The student MR once again gave the correct answer and the instructor asked her to explain.

MR: First I found one part is 250

T: How did you do that?

MR: I divided by 2

T: OK then what?

MR: I multiplied by 5 to get 1250

Analysis Exercise 2

In the first exercise MR demonstrates an abstraction of the quarter conception, and coordinates this with the problem information to guide her actions (find Q) and then add. This strategy would not work as readily in the second problem. Indeed, the conception '$5F = X$' and the information $2F = 500$ does not immediately suggest a need to find F. Thus, it appears likely to the research team that MR's first step of finding F expressed in line 1 & 2 was the result of her recognition of the similar structure between the problems. That is, she immediately understood that a sub-goal of this problem, like the first one was to find a unit amount, in this case one fifth 'F'. This is likely not only because the second problem was presented to MR immediately after she solved the first but also because as stated the information given of two-fifths does not immediately suggest that knowing one-fifth will yield the unknown total five-fifths. In this scenario, MR projects her previous schemata into the second problem situation. This projection would be considered reflective abstraction as MR realizes that addition is problematic so instead adapts a multiplicative approach, i.e. she restructures her existing schema to accommodate the new problem situation.

Summary Discussion Investigation I

The creative act of giving meaning to mathematics during schema development is demonstrated by MR in these exercises. In the first MR bisociates her quarter conception with problem information $3Q = 18$ and $4Q = X$ while simultaneously coordinating this with the goal find X to abstracts a code (Find Q and add to find $4Q = X$) that directs her solution activity. Norton (2009) reviews a solver 'Josh' who given a diagram of half of a half of an object first employs his part-whole conception incorrectly labeling a resulting piece as a one-third. Then after realizing this mistake he abducted the larger half part as equal to two of the smaller parts to obtain the correct answer one-quarter. Thus, both Josh and MR employ their part-whole conceptions specifically abducting the quarter's conception of four equal parts to guide their solution activity.

In the second problem MR recognizes that the action schema developed in the first problem is appropriate. She projects the action done in the first, 'find the unit amount through division' and then adapts this schemata through employing a multiplication rather than addition approach as a more efficient manner to find the goal. In this scenario MR reasoning is:

1. $2F = 250$, $5F = X$ and in the previous-similar problem, I found the unit so I first find F.
2. Multiply by 5 to find $5F = X$.

The shift in focus from an additive to a multiplicative approach within the context of proportional reasoning is viewed as evidence of growth in understanding (Caddle & Brizuela, 2011; Fernandez et. al., 2010). Thus, MR is projecting from an additive-lower to a multiplicative-higher structural schema as in the definition of reflective abstraction given by Piaget & Garcia (1989).

Second Investigation

The next exercises demonstrate the functional component of teaching for understanding. In these narratives the instructor acted as a mediator negotiating between the problem information and students' conceptual knowledge in an effort to engage them in the reasoning demonstrated by MR. This second investigation is with adult students who require a more extensive review of algebra before taking college level mathematics than those in the first investigation. The instructor employs visuals of the part-whole conception to encourage student engagement. Two students 'EM' and 'RR' navigate through this reasoning yet a third student 'SL' rejects it as 'baby math.'

In a prelude to this exercise the instructor 'T' introduces an example (Find $\frac{3}{4}$ of 120) in which a fraction is used as an operator. When the instructor considers that the class understands this operator principle, he assigns this next question to assess whether they can transition to inverse reasoning.

Exercise 1)
If $\frac{3}{4}$ of some number is 36 find the number.

T: How is this problem different than the first one? Silence!

EG: The answer is 27.

T: No, but if you take $\frac{3}{4}$ of 36 you do get 27.

[The class does not understand the inverse reasoning required, they have ignored the inverse structure of this problem associating it instead with the direct multiplication problem done immediately before.]

T: Class, what is wrong with taking $\frac{3}{4}$ of 36 and multiplying?

T: What are we taking $\frac{3}{4}$ of? Not the 36 but of what?

EM: Some number.

T: Correct, 36 is the result what we get, we take $\frac{3}{4}$ of another number.

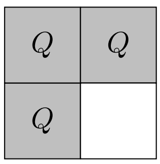
[The instructor now employs a visual image of the part-whole construct hoping it can promote the bisociation and subsequent coordination employed by MR among these students.]

T: So if 36 represents the $3Q$ the question is what represents the $4Q$'s?

RR: 48

T: No, but if you take $\frac{3}{4}$ of 36 you do get 27

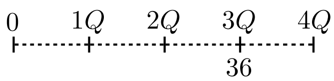
T: (writes the following)



$$Q + Q + Q = 3Q = 36$$

T: The 36 represent the 3 quarters out of the 4 quarters some number is divided up into

T: (Simultaneous writes)



T: So if 36 represents the $3Q$ the question is what represents the $4Q$'s?

RR: 48

T: Very Good, how did you do this?

RR: Well, 1 is 12

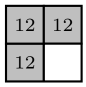
T: OK, so $1Q$ is 12

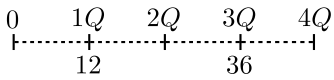
T: Good, how did you get the 12?

RR: I divided by 3

T: Good, does everyone see this?

T: (writes the following)





T: What did you do to get the 48?

RR: I multiplied by 4.

[Supported by the visual's RR follows the conceptual based reasoning of the student MR in the first cycle with multiplicative rather than additive reasoning.]

SL: Can we do this using real math? Not baby stuff!

T: Well this is real, and it is math but how can we do this problem without using pictures?

Let's go back to the problem: If $\frac{3}{4}$ of some number is 36 find the number.

T: What did we say the word 'of' translates into?

EM: Multiplication.

T: Yes, but as we have been saying, we do not multiply $\frac{3}{4}$ with 36 instead with some number we don't know. What do we use to represent this unknown number?

EM: Using X.

T: Good. So we have: $\frac{3}{4}X$ is 36, what does the word 'is' translate into?

EM: The equal sign.

T: Good, writes $\frac{3}{4}X = 36$, how do we solve this? Can we multiply?

EM: We flip the fraction.

T: Good, we find the reciprocal and multiply.

Analysis

In this example RR with the instructor's guidance and appropriate visuals follows the conceptual based reasoning of MR. Thus such reasoning is within the ZPD of this pre-algebra class. However, SL rejects this methodology due to presence of "baby stuff" i.e. the visuals. Clearly this method did not result in positive affect for students, nor did it promote student engagement and construction of knowledge.

The second question given after covering formal proportions was designed to initiate inverse reasoning. However, after the failure of the students to accept this structure in the first exercise the instructor's fallback plan is to employ a basic proportion.

Exercise 2)
If $\frac{3}{5}$ of some number is 150, find the number.

T: What is the difference between this problem and the earlier ones? Can I multiply the fraction with the 150?

Student: No

T: Why not?

Silence

T: Does everyone see that you multiply the $\frac{3}{5}$ with some unknown number to get 150. The 150 is the result! The question is what is the number you multiply $\frac{3}{5}$ with. So how do we write it? What notation do we use for the unknown number we multiply?

EM: X

T: Good, writes: $\frac{3}{5}X = 150$

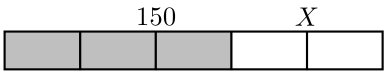
T: How do we solve it … (begins to work out problem using reciprocal)

SL: Teacher, I don't get it.

T: OK so we take 3 fifths of some number and get 150, that means we divide some number 'X' into 5 equal pieces take 3 of them and the result is 150

T: Draws the visual

3 out of 5 parts is 150, what is the total?

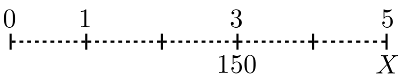


The question is how much is the total 5 parts correct?

SL: Yes, but?

[SL is the same student who rejected this approach as baby math in the first exercise, she still does not get it]

T: OK you don't see it, how about in this diagram (draws different visual – to motivate proportion)



T: Can we set up a proportion? How would it go?

SL: Sets up the proportion: $\frac{3}{5} = \frac{150}{X}$

T: Good, now how would you solve this?

SL: (works out the problem through cross multiplication)... This is a much better way.

T: Class, who can tell me why the 150 goes with the 3 and the X with the 5 in this proportion and not the other way around?

EM: Because you are taking the 150 out of the total!

Discussion Investigation II

In the first exercise the instructor employs visuals to induce the class to follow the reasoning employed by MR. However, the students rejected this approach and did not internalize the resulting action scheme even though it was clearly within their ZPD. One factor was that MR's reasoning with the operator conception requires more coordination than the part-whole proportion conception used by the instructor with EM and SL, Doyle et al. (2016). Specifically MR's bisociated information: $3Q = 18$, $4Q = X$, find X, requires more coordination during solution activity (To find all $4Q$ one must find Q and add). In contrast, the proportional reasoning used in the second exercise: 3 parts out of a total 5 is equivalent to 150 out of a total X, requires less coordination to set up the corresponding proportion and as Tossavainen (2009) points out, the more steps to coordinate, the higher the cognitive load. Also it is clear that MR demonstrated an ability to use her 'quarter conception' as an object while SL and the class required the instructor's visuals to engage effectively in such reasoning.

Conclusion

In the first investigation, MR employs her bisociated conceptual knowledge, coordinating it with the problem situation to guide her actions. We view the cognitive aspects of her bisociation as containing two components, which may occur simultaneously or one after another. The first is her ability to relate her conceptual knowledge to the situation. At this point the concepts can be thought of as existing simultaneously in her conceptual matrix and the matrix of the problem situation: ($3Q = 18$ & $4Q = X$). The second component is her abstraction (abduction) of the code that guides her solution activity. The abduction occurs on the action associated with her three quarters conception (given $3Q$ taken out of a total $4Q$ means I need the remaining Q and then I will add to find the total $4Q = X$). Thus, MR's solution activity is guided in large part by her reflection upon an action associated with her quarter's conception. We conclude that an attempt to separate reflection upon concepts and their associated actions during schema development is tenuous at best.

In the second exercise, the team's analysis suggests MR's actions were guided by her realization that this problem was similar in nature to the first. Thus, she projects the first schemata into the second problem matrix. Her reasoning is along the lines that, I will find the unit object 'one-fifth – F,' then she coordinates this with the problem goal (Find $5F = X$) to modify her schemata to a multiplicative strategy. In this scenario, her reasoning is reflective abstraction as she recognizes that this situation required her previous action schemata and because her coordination with the problem situation results in constructive generalization, i.e. a modified multiplicative schemata.

We have made a distinction between MR's reasoning in the first and second exercises. In the first, she recognizes relevant conceptual knowledge and reflects upon its associated actions to guide her solution activity. In the second, she recalls the action schema developed in the first. We note that Norton (2009) does not make such a distinction in his analysis of 'Josh' employing his part-whole conception, which Norton refers to as an elementary schema. In this framework MR's reasoning in the first exercise is also reflective abstraction. A relevant question for future research is how can conceptual schemes typically viewed as networks of concepts with associated actions be integrated with the action schemes used in problem solving.

We also note that in Norton (2009) study of 'Josh', the observed abduction takes place in the second & third step of an action schema. Instead, it occurs after 'Josh' understands that has initial action of labeling half of a half as one third is incorrect. Thus, it occurs during the period when he realizes that the half part can be measured as two of the quarter parts. In contrast, our investigation studies the first recognition step. Another relevant question for future research is the relationship between bisociation and abduction within the framework of an actions schema.

The second investigation is functional rather than theoretical in nature as an instructor attempts to induce his students by means of supporting visuals to follow the conceptual based reasoning of the student MR. The question for a mathematics teacher, as posed by Simon et al (2004) is, "How to foster the students' reinvention of particular mathematics ideas; which situation will make the mathematical idea transparent?" (p. 325).

In the first exercise, although the student RR supported by the visuals follows this reasoning, student's affect is not positive. Indeed, SL outright rejects this methodology and although the stated reason is 'this is baby math', it is much more likely that the cognitive load required was too much for the student. Thus vindicating Tossavainen (2009) statement that: "The intrinsic cognitive load cannot be modified by instructional design." The difficulty SL and the students had with the approach used by MR was due in part to the coordination involved and their difficulty employing their 'quarter conception' as an object. However, a visual diagram did allow SL and other students to associate the problem information with a proportion structure, which required less coordination.

Clearly, conceptual based lesson plans cannot modify the intrinsic load of an activity sequence nor guarantee internalization of mathematical schemes especially when students have weak object conceptions however, they can assist students create meaning for mathematical structure if the association between the conceptions and the structure is within the students' ZPD and the coordination steps are not too difficult or numerous.

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Rozwój „struktur działania" w oparciu o matematyczną wiedzę pojęciową

S t r e s z c z e n i e

Nasza praca opisuje dwa cykle badań przeprowadzonych na lekcjach matematyki przez nauczycieli-badaczy. Pierwszy cykl ma charakter badań epistemologicznych: bada proces konstruowania przez ucznia rozwiązującego zadanie sensownego postępowania na podstawie jego wiedzy pojęciowej. W drugim cyklu badamy, w jaki sposób nauczyciel próbuje uruchomić u uczniów rozumowanie oparte na wiedzy pojęciowej.

Baza pojęciowa pierwszego badania to synteza teorii rozwoju pojęć matematycznych i teorii kreatywności, skonstruowana w celu zrozumienia rozwojowych schematów myślowych uczniów. W drugim badaniu dokonujemy opartej na tej idei próby „uczenia przez zrozumienie" jako metody nauczania rozwiązywania zadań, ułatwiając przy tym rozumowanie pojęciowe wizualnymi reprezentacjami odpowiednich pojęć.

Baza metodologiczna tego etapu została stworzona przez wykorzystanie dwóch źródeł: wiedzy zawodowej nauczyciela dotyczącej przyczyn, dla których uczeń nie jest w stanie znaleźć właściwego postępowania matematycznego, by rozwiązać dany problem, oraz integracji teorii uczenia się i teorii kreatywności. Taka synteza teorii i wiedzy praktycznej jest charakterystyczna dla metodologii nauczania-badania (*teaching research*) a energia wyzwolona dzięki niej pozwala na stworzenie twórczego środowiska na lekcji matematyki.

Strukturę pojęciową uczenia się otwiera mechanizm abstrakcji refleksywnej (*reflective abstraction*), wprowadzonej przez J. Piageta, który wyjaśnia, jak schemat myślenia jest modyfikowany w czasie rozwiazywania zadania, gdy uczeń nie wie, jak się do tego zabrać. Von Glasersfeld dodał pojęcie „struktury działania" (*action schema*), skladającej się z rozpoznania, skojarzenia

(właściwych) czynności i przewidywania wyniku. Odpowiedź na pytanie, jak „struktura działania" powstaje lub wcześniejsza jest modyfikowana w procesie rozumowania, Von Glasersfeld znajduje w opisanym przez Pierce'a myśleniu nazwanym „abdukcją" (*abduction*). Bisocjacja (*bisociation*) Koestlera opisuje mianowicie syntezę pojęciową dwóch rozłącznych schematów myślenia w momencie olśnienia „Aha!", które może inspirować i motywować ucznia do szukania matematycznego postępowania rozwiązującego problem. Jest to synteza kreatywności opartej na „abdukcji" z refleksywną abstrakcją Piageta.

Analiza i wyniki przeprowadzonych badań potwierdzają hipotezę, że bisocjatywna synteza wiedzy pojęciowej ucznia z warunkami zadania może być wykorzystana do rozwiązywania zadań. Wykorzystanie wizualnej reprezentacji problemu jest ograniczone przez możliwości ucznia w operowaniu dużym zasobem pojęciowym.

Synteza teorii kreatywności z teoriami uczenia się daje wgląd w to, jak uczniowie mogą wykorzystać swoją wiedzą pojęciową w celu rozwiązania zadania. Ta integracja sugeruje także kilka nowych pytań, wykraczających poza opisane tu badania. Jednym z nich jest związek pomiędzy strukturami pojęciowymi i schematami postępowania matematycznego: Czy ten związek sam jest „strukturą działania" (*action schema*)? Jak używane są pojęcia w celu rozpoznania sytuacji, w której powinny być zastosowane? Jaka jest rola pojęć intuicyjnych? I wreszcie podstawowe pytanie: Jaki jest związek pomiędzy „bisocjacją" i „abdukcją", który tutaj był wykorzystany tylko do zrozumienia pierwszego kroku rozpoznania sytuacji matematycznej.

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